Generalizations of the Kerr-Newman solution

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1.1 ICRANet Participants

- Donato Bini
- Andrea Geralico
- Roy P. Kerr
- Hernando Quevedo
- Jorge A. Rueda
- Remo Ruffini

1.2 Ongoing collaborations

- Medeu Abishev (Kazakh National University - KazNU, Kazakhstan)
- Kuantay Boshkayev (Kazakh National University - KazNU, Kazakhstan)
- Antonio C. Gutierrez (University of Bolivar, Colombia)
- Orlando Luongo (University of Naples, Italy)
- Daniela Pugliese (Silesian University in Opava, Czech Republic)
- Saken Toktarbay (Kazakh National University - KazNU, Kazakhstan)

1.3 Students

- Viridiana Pineda (UNAM PhD, Mexico)
- Pedro Sánchez (UNAM PhD, Mexico)
- Bakytzhan Zhami (KazNU PhD, Kazakhstan)

2 Brief description

One of the most important metrics in general relativity is the Kerr-Newman solution that describes the gravitational and electromagnetic fields of a rotating charged mass. For astrophysical purposes, however, it is necessary to take into account the effects due to the moment of inertia of the object. To attack this problem we investigate new exact solutions of Einstein-Maxwell equations which posses an infinite set of gravitational and electromagnetic multipole moments and contain the Kerr-Newman solution as special case.

We review the problem of describing the gravitational field of compact stars in general relativity. We focus on the deviations from spherical symmetry which are expected to be due to rotation and to the natural deformations of mass distributions. We assume that the relativistic quadrupole moment takes into account these deviations, and consider the class of axisymmetric static and stationary quadrupolar metrics which satisfy Einstein's equations in empty space and in the presence of matter represented by a perfect fluid. We formulate the physical conditions that must be satisfied for a particular spacetime metric to describe the gravitational field of compact stars. We present a brief review of the main static and axisymmetric exact solutions of Einstein's vacuum equations, satisfying all the physical conditions. We discuss how to derive particular stationary and axisymmetric solutions with quadrupolar properties by using the solution generating techniques which correspond either to Lie symmetries and Bäckund transformations of the Ernst equations or to the inverse scattering method applied to Einstein's equations. As for interior solutions, we argue that it is necessary to apply alternative methods to obtain physically meaningful solutions, and review a method which allows us to generate interior perfect-fluid solutions.

We apply the Hartle formalism to study equilibrium configurations in the framework of Newtonian gravity. This approach allows one to study in a simple manner the properties of the interior gravitational field in the case of static as well as stationary rotating stars in hydrostatic equilibrium. It is shown that the gravitational equilibrium conditions reduce to a system of ordinary differential equations which can be integrated numerically. We derive all the relevant equations up to the second order in the angular velocity. Moreover, we find explicitly the total mass, the moment of inertia, the quadrupole moment, the polar and equatorial radii, the eccentricity and the gravitational binding energy of the rotating body. We also present the procedure to calculate the gravitational Love number. We test the formalism in the case of white dwarfs and show its compatibility with the known results in the literature.

We investigate the equilibrium configurations of uniformly rotating white dwarfs, using Chandrasekhar and Salpeter equations of state in the framework of Newtonian physics. The Hartle formalism is applied to integrate the field equation together with the hydrostatic equilibrium condition. We consider the equations of structure up to the second order in the angular velocity, and compute all basic parameters of rotating white dwarfs to test the so-called moment of inertia, rotational Love number and quadrupole moment (I-Love-Q) relations. We found that the I-Love-Q relations are also valid for white dwarfs regardless of the equation of state and nuclear composition. In addition, we show that the moment of inertia, quadrupole moment and eccentricity (I-Q-e) relations are valid as well.

3 Introduction

It is hard to overemphasize the importance of the Kerr geometry not only for general relativity itself, but also for the very fundamentals of physics. It assumes this position as being the most physically relevant rotating generalization of the static Schwarzschild geometry. Its charged counterpart, the Kerr-Newman solution, representing the exterior gravitational and electromagnetic fields of a charged rotating object, is an exact solution of the Einstein-Maxwell equations.

Its line element in Boyer–Lindquist coordinates can be written as

$$\begin{aligned} ds^2 &= \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta d\varphi)^2 \\ &\quad - \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}[(r^2 + a^2)d\varphi - adt]^2 \\ &\quad - \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2}dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 \,, \end{aligned} \tag{3.0.1}$$

where M is the total mass of the object, $a = J/M$ is the specific angular momentum, and Q is the electric charge. In this particular coordinate system, the metric functions do not depend on the coordinates t and ϕ, indicating the existence of two Killing vector fields $\xi^I = \partial_t$ and $\xi^{II} = \partial_\varphi$ which represent the properties of stationarity and axial symmetry, respectively.

An important characteristic of this solution is that the source of gravity is surrounded by two horizons situated at a distance

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \tag{3.0.2}$$

from the origin of coordinates. Inside the interior horizon, r_-, a ring singularity is present which, however, cannot be observed by any observer situated outside the exterior horizon. If the condition $M^2 < a^2 + Q^2$ is satisfied, no horizons are present and the Kerr–Newman spacetime represents the exterior field of a naked singularity.

Despite of its fundamental importance in general relativity, and its theoretical and mathematical interest, this solution has not been especially useful for describing astrophysical phenomena, first of all, because observed astrophysical objects do not possess an appreciable net electric charge. Furthermore, the limiting Kerr metric takes into account the mass and the rotation, but does not consider the moment of inertia of the object. For astrophysical applications it is, therefore, necessary to use more general solutions with higher multipole moments which are due not only to the rotation of the body but also to its shape. This means that even in the limiting case of a static spacetime, a solution is needed that takes into account possible deviations from spherically symmetry.

4 The general static vacuum solution

In general relativity, stationary axisymmetric solutions of Einstein's equations [1] play a crucial role for the description of the gravitational field of astrophysical objects. In particular, the black hole solutions and their generalizations that include Maxwell fields are contained within this class.

This type of exact solutions has been the subject of intensive research during the past few decades. In particular, the number of know exact solutions drastically increased after Ernst [2] discovered an elegant representation of the field equations that made it possible to search for their symmetries. These studies lead finally to the development of solution generating techniques [1] which allow us to find new solutions, starting from a given seed solution. In particular, solutions with an arbitrary number of multipole moments for the mass and angular momentum were derived in [3] and used to describe the gravitational field of rotating axially symmetric distributions of mass.

The first analysis of stationary axially symmetric gravitational fields was carried out by Weyl [4] in 1917, soon after the formulation of general relativity. In particular, Weyl discovered that in the static limit the main part of the vacuum field equations reduces to a single linear differential equation. The corresponding general solution can be written in cylindrical coordinates as an infinite sum with arbitrary constant coefficients. A particular choice of the coefficients leads to the subset of asymptotically flat solutions which is the most interesting from a physical point of view. In this section we review the main properties of stationary axisymmetric gravitational fields. In particular, we show explicitly that the main field equations in vacuum can be represented as the equations of a nonlinear sigma model in which the base space is the 4-dimensional spacetime and the target space is a 2-dimensional conformally Euclidean space.

4.1 Line element and field equations

Although there exist in the literature many suitable coordinate systems, stationary axisymmetric gravitational fields are usually described in cylindric coordinates (t, ρ, z, φ). Stationarity implies that t can be chosen as the time coordinate and the metric does not depend on time, i.e. $\partial g_{\mu\nu}/\partial t = 0$. Consequently, the corresponding timelike Killing vector has the components δ^μ_t. A second Killing vector field is associated to the axial symmetry with respect to the axis $\rho = 0$. Then, choosing φ as the azimuthal angle, the metric satisfies the conditions $\partial g_{\mu\nu}/\partial\varphi = 0$, and the components of the corresponding spacelike Killing vector are δ^μ_φ.

Using further the properties of stationarity and axial symmetry, together with the vacuum field equations, for a general metric of the form $g_{\mu\nu} = g_{\mu\nu}(\rho, z)$, it is possible to show that the most general line element for this type of gravitational fields can be written in the Weyl-Lewis-Papapetrou form as [4, 5, 6]

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] , \tag{4.1.1}$$

where f, ω and γ are functions of ρ and z, only. After some rearrangements which include the introduction of a new function $\Omega = \Omega(\rho, z)$ by means of

$$\rho\partial_\rho\Omega = f^2\partial_z\omega , \qquad \rho\partial_z\Omega = -f^2\partial_\rho\omega , \tag{4.1.2}$$

the vacuum field equations $R_{\mu\nu} = 0$ can be shown to be equivalent to the following set of partial differential equations

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \partial_z^2 f + \frac{1}{f}[(\partial_\rho\Omega)^2 + (\partial_z\Omega)^2 - (\partial_\rho f)^2 - (\partial_z f)^2] = 0 , \tag{4.1.3}$$

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Omega) + \partial_z^2\Omega - \frac{2}{f}\left(\partial_\rho f\,\partial_\rho\Omega + \partial_z f\,\partial_z\Omega\right) = 0 , \tag{4.1.4}$$

$$\partial_\rho\gamma = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right] , \tag{4.1.5}$$

$$\partial_z\gamma = \frac{\rho}{2f^2}\left(\partial_\rho f\,\partial_z f + \partial_\rho\Omega\,\partial_z\Omega\right) . \tag{4.1.6}$$

It is clear that the field equations for γ can be integrated by quadratures, once f and Ω are known. For this reason, the equations (4.1.3) and (4.1.4) for f and Ω are usually considered as the main field equations for stationary axisymmetric vacuum gravitational fields. In the following subsections we will focus on the analysis of the main field equations, only. It is interesting to mention that this set of equations can be geometrically interpreted in the context of nonlinear sigma models [7].

Let us consider the special case of static axisymmetric fields. This corresponds to metrics which, apart from being axially symmetric and independent of the time coordinate, are invariant with respect to the transformation

$\varphi \to -\varphi$ (i.e. rotations with respect to the axis of symmetry are not allowed). Consequently, the corresponding line element is given by (4.1.1) with $\omega = 0$, and the field equations can be written as

$$\partial_\rho^2 \psi + \frac{1}{\rho} \partial_\rho \psi + \partial_z^2 \psi = 0 \, , \quad f = \exp(2\psi) \, , \tag{4.1.7}$$

$$\partial_\rho \gamma = \rho \left[(\partial_\rho \psi)^2 - (\partial_z \psi)^2 \right] \, , \quad \partial_z \gamma = 2\rho \partial_\rho \psi \, \partial_z \psi \, . \tag{4.1.8}$$

We see that the main field equation (4.1.7) corresponds to the linear Laplace equation for the metric function ψ.

4.2 Static solution

The general solution of Laplace's equation is known and, if we demand additionally asymptotic flatness, we obtain the Weyl solution which can be written as [4, 1]

$$\psi = \sum_{n=0}^{\infty} \frac{a_n}{(\rho^2 + z^2)^{\frac{n+1}{2}}} P_n(\cos\theta) \, , \qquad \cos\theta = \frac{z}{\sqrt{\rho^2 + z^2}} \, , \tag{4.2.1}$$

where a_n $(n = 0, 1, ...)$ are arbitrary constants, and $P_n(\cos\theta)$ represents the Legendre polynomials of degree n. The expression for the metric function γ can be calculated by quadratures by using the set of first order differential equations (4.1.8). Then

$$\gamma = - \sum_{n,m=0}^{\infty} \frac{a_n a_m (n+1)(m+1)}{(n+m+2)(\rho^2 + z^2)^{\frac{n+m+2}{2}}} \left(P_n P_m - P_{n+1} P_{m+1} \right) \, . \tag{4.2.2}$$

Since this is the most general static, axisymmetric, asymptotically flat vacuum solution, it must contain all known solution of this class. In particular, one of the most interesting special solutions which is Schwarzschild's spherically symmetric black hole spacetime must be contained in this class. To see this, we must choose the constants a_n in such a way that the infinite sum (4.2.1) converges to the Schwarzschild solution in cylindric coordinates. But, or course, this representation is not the most appropriate to analyze the interesting physical properties of Schwarzchild's metric.

In fact, it turns out that to investigate the properties of solutions with multipole moments it is more convenient to use prolate spheroidal coordinates

(t, x, y, φ) in which the line element can be written as

$$ds^2 = f dt^2 - \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right]$$

where

$$x = \frac{r_+ + r_-}{2\sigma}\,, \quad (x^2 \geq 1), \quad y = \frac{r_+ - r_-}{2\sigma}\,, \quad (y^2 \leq 1) \tag{4.2.3}$$

$$r_\pm^2 = \rho^2 + (z \pm \sigma)^2\,, \quad \sigma = const\,, \tag{4.2.4}$$

and the metric functions are f, ω, and γ depend on x and y, only. In this coordinate system, the general static solution which is also asymptotically flat can be expressed as

$$f = \exp(2\psi)\,, \quad \psi = \sum_{n=0}^{\infty}(-1)^{n+1} q_n P_n(y) Q_n(x)\,, \quad q_n = const$$

where $P_n(y)$ are the Legendre polynomials, and $Q_n(x)$ are the Legendre functions of second kind. In particular,

$$P_0 = 1, \quad P_1 = y, \quad P_2 = \frac{1}{2}(3y^2-1)\,, ...$$

$$Q_0 = \frac{1}{2}\ln\frac{x+1}{x-1}\,, \quad Q_1 = \frac{1}{2}x\ln\frac{x+1}{x-1} - 1\,,$$

$$Q_2 = \frac{1}{2}(3x^2-1)\ln\frac{x+1}{x-1} - \frac{3}{2}x\,, ...$$

The corresponding function γ can be calculated by quadratures and its general expression has been explicitly derived in [8]. The most important special cases contained in this general solution are the Schwarzschild metric

$$\psi = -q_0 P_0(y) Q_0(x)\,, \quad \gamma = \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2}\,,$$

and the Erez-Rosen metric [9]

$$\psi = -q_0 P_0(y) Q_0(x) - q_2 P_2(y) Q_2(x)\,, \quad \gamma = \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2} + ...$$

In the last case, the constant parameter q_2 turns out to determine the quadrupole moment. In general, the constants q_n represent an infinite set of parameters that determines an infinite set of mass multipole moments.

5 Stationary generalization

The solution generating techniques [12] can be applied, in particular, to any static seed solution in order to obtain the corresponding stationary generalization. One of the most powerful techniques is the inverse method (ISM) developed by Belinski and Zakharov [13]. We used a particular case of the ISM, which is known as the Hoenselaers–Kinnersley-Xanthopoulos (HKX) transformation to derive the stationary generalization of the general static solution in prolate spheroidal coordinates.

5.1 Ernst representation

In the general stationary case ($\omega \neq 0$) with line element

$$ds^2 = f(dt - \omega d\varphi)^2$$
$$- \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right]$$

it is useful to introduce the the Ernst potentials

$$E = f + i\Omega\,, \quad \xi = \frac{1-E}{1+E}\,,$$

where the function Ω is now determined by the equations

$$\sigma(x^2-1)\Omega_x = f^2\omega_y\,, \quad \sigma(1-y^2)\Omega_y = -f^2\omega_x\,.$$

Then, the main field equations can be represented in a compact and symmetric form:

$$(\xi\xi^*-1)\left\{[(x^2-1)\xi_x]_x + [(1-y^2)\xi_y]_y\right\} = 2\xi^*[(x^2-1)\xi_x^2 + (1-y^2)\xi_y^2]\,.$$

This equation is invariant with respect to the transformation $x \leftrightarrow y$. Then, since the particular solution

$$\xi = \frac{1}{x} \to \Omega = 0 \to \omega = 0 \to \gamma = \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2}$$

represents the Schwarzschild spacetime, the choice $\xi^{-1} = y$ is also an exact solution. Furthermore, if we take the linear combination $\xi^{-1} = c_1 x + c_2 y$ and introduce it into the field equation, we obtain the new solution

$$\xi^{-1} = \frac{\sigma}{M} x + i \frac{a}{M} y \, , \; \sigma = \sqrt{M^2 - a^2} \, ,$$

which corresponds to the Kerr metric in prolate spheroidal coordinates.

In the case of the Einstein-Maxwell theory, the main field equations can be expressed as

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\xi = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\xi \, ,$$

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\mathcal{F} = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\mathcal{F}$$

where ∇ represents the gradient operator in prolate spheroidal coordinates. Moreover, the gravitational potential ξ and the electromagnetic $\mathcal{F}$ Ernst potential are defined as

$$\xi = \frac{1 - f - i\Omega}{1 + f + i\Omega} \, , \quad \mathcal{F} = 2\frac{\Phi}{1 + f + i\Omega} \, .$$

The potential Φ can be shown to be determined uniquely by the electromagnetic potentials A_t and A_φ One can show that if ξ_0 is a vacuum solution, then the new potential

$$\xi = \xi_0 \sqrt{1 - e^2}$$

represents a solution of the Einstein-Maxwell equations with effective electric charge e. This transformation is known in the literature as the Harrison transformation [10]. Accordingly, the Kerr–Newman solution in this representation acquires the simple form

$$\xi = \frac{\sqrt{1 - e^2}}{\frac{\sigma}{M} x + i \frac{a}{M} y} \, , \qquad e = \frac{Q}{M} \, , \qquad \sigma = \sqrt{M^2 - a^2 - Q^2} \, .$$

In this way, it is very easy to generalize any vacuum solution to include the case of electric charge. More general transformations of this type can be used in order to generate solutions with any desired set of gravitational and electromagnetic multipole moments [11].

5.2 Representation as a nonlinear sigma model

Consider two (pseudo)-Riemannian manifolds (M, γ) and (N, G) of dimension m and n, respectively. Let M be coordinatized by x^a, and N by X^μ, so that the metrics on M and N can be, in general, smooth functions of the corresponding coordinates, i.e., $\gamma = \gamma(x)$ and $G = G(X)$. A harmonic map is a smooth map $X : M \to N$, or in coordinates $X : x \longmapsto X$ so that X becomes a function of x, and the X's satisfy the motion equations following from the action [14]

$$S = \int d^m x \sqrt{|\gamma|}\, \gamma^{ab}(x)\, \partial_a X^\mu\, \partial_b X^\nu\, G_{\mu\nu}(X)\,, \tag{5.2.1}$$

which sometimes is called the "energy" of the harmonic map X. The straightforward variation of S with respect to X^μ leads to the motion equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b \left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu \right) + \Gamma^\mu_{\ \nu\lambda}\, \gamma^{ab}\, \partial_a X^\nu\, \partial_b X^\lambda = 0\,, \tag{5.2.2}$$

where $\Gamma^\mu_{\ \nu\lambda}$ are the Christoffel symbols associated to the metric $G_{\mu\nu}$ of the target space N. If $G_{\mu\nu}$ is a flat metric, one can choose Cartesian-like coordinates such that $G_{\mu\nu} = \eta_{\mu\nu} = diag(\pm 1, ..., \pm 1)$, the motion equations become linear, and the corresponding sigma model is linear. This is exactly the case of a bosonic string on a flat background in which the base space is the 2-dimensional string world-sheet. In this case the action (5.2.1) is usually referred to as the Polyakov action [16].

Consider now the case in which the base space M is a stationary axisymmetric spacetime. Then, γ^{ab}, $a, b = 0, ..., 3$, can be chosen as the Weyl-Lewis-Papapetrou metric (4.1.1), i.e.

$$\gamma_{ab} = \begin{pmatrix} f & 0 & 0 & -f\omega \\ 0 & -f^{-1}e^{2k} & 0 & 0 \\ 0 & 0 & -f^{-1}e^{2k} & 0 \\ -f\omega & 0 & 0 & f\omega^2 - \rho^2 f^{-1} \end{pmatrix}. \tag{5.2.3}$$

Let the target space N be 2-dimensional with metric $G_{\mu\nu} = (1/2)f^{-2}\delta_{\mu\nu}$, $\mu, \nu = 1, 2$, and let the coordinates on N be $X^\mu = (f, \Omega)$. Then, it is straightforward to show that the action (5.2.1) becomes

$$S = \int \mathcal{L}\, dt d\varphi d\rho dz\,, \qquad \mathcal{L} = \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2 \right]\,, \tag{5.2.4}$$

and the corresponding motion equations (5.2.2) are identical to the main field equations (4.1.3) and (4.1.4).

Notice that the field equations can also be obtained from (5.2.4) by a direct variation with respect to f and Ω. This interesting result was obtained originally by Ernst [2], and is the starting point of what today is known as the Ernst representation of the field equations.

The above result shows that stationary axisymmetric gravitational fields can be described as a $(4 \to 2)-$nonlinear harmonic map, where the base space is the spacetime of the gravitational field and the target space corresponds to a 2-dimensional conformally Euclidean space. A further analysis of the target space shows that it can be interpreted as the quotient space $SL(2,R)/SO(2)$ [15], and the Lagrangian (5.2.4) can be written explicitly [17] in terms of the generators of the Lie group $SL(2,R)$. Harmonic maps in which the target space is a quotient space are usually known as nonlinear sigma models [14].

The form of the Lagrangian (5.2.4) with two gravitational field variables, f and Ω, depending on two coordinates, ρ and z, suggests a representation as a harmonic map with a 2-dimensional base space. In string theory, this is an important fact that allows one to use the conformal invariance of the base space metric to find an adequate representation for the set of classical solutions. This, in turn, facilitates the application of the canonical quantization procedure. Unfortunately, this is not possible for the Lagrangian (5.2.4). Indeed, if we consider γ^{ab} as a 2-dimensional metric that depends on the parameters ρ and z, the diagonal form of the Lagrangian (5.2.4) implies that $\sqrt{|\gamma|}\gamma^{ab} = \delta^{ab}$. Clearly, this choice is not compatible with the factor ρ in front of the Lagrangian. Therefore, the reduced gravitational Lagrangian (5.2.4) cannot be interpreted as corresponding to a $(2 \to n)$-harmonic map. Nevertheless, we will show in the next section that a modification of the definition of harmonic maps allows us to "absorb" the unpleasant factor ρ in the metric of the target space, and to use all the advantages of a 2-dimensional base space.

Notice that the representation of stationary fields as a nonlinear sigma model becomes degenerate in the limiting case of static fields. Indeed, the underlying geometric structure of the $SL(2,R)/SO(2)$ nonlinear sigma models requires that the target space be 2-dimensional, a condition which is not satisfied by static fields. We will see below that by using a dimensional extension of generalized sigma models, it will be possible to treat the special static case, without affecting the underlying geometric structure.

The analysis performed in this section for stationary axisymmetric fields can be generalized to include any gravitational field containing two commuting Killing vector fields [1]. This is due to the fact that for this class of gravitational fields it is always possible to find the corresponding Ernst representation in which the Lagrangian contains only two gravitational variables which depend on only two spacetime coordinates.

5.3 Representation as a generalized harmonic map

Consider two (pseudo-)Riemannian manifolds (M, γ) and (N, G) of dimension m and n, respectively. Let x^a and X^μ be coordinates on M and N, respectively. This coordinatization implies that in general the metrics γ and G become functions of the corresponding coordinates. Let us assume that not only γ but also G can explicitly depend on the coordinates x^a, i.e. let $\gamma = \gamma(x)$ and $G = G(X, x)$. This simple assumption is the main aspect of our generalization which, as we will see, lead to new and nontrivial results.

A smooth map $X : M \to N$ will be called an $(m \to n)-$generalized harmonic map if it satisfies the Euler-Lagrange equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b \left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^{\mu}_{\ \nu\lambda}\,\gamma^{ab}\,\partial_a X^\nu \partial_b X^\lambda + G^{\mu\lambda}\gamma^{ab}\,\partial_a X^\nu\,\partial_b G_{\lambda\nu} = 0\,, \tag{5.3.1}$$

which follow from the variation of the generalized action

$$S = \int d^m x \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a\,X^\mu \partial_b X^\nu G_{\mu\nu}(X, x)\,, \tag{5.3.2}$$

with respect to the fields X^μ. Here the Christoffel symbols, determined by the metric $G_{\mu\nu}$, are calculated in the standard manner, without considering the explicit dependence on x. Notice that the new ingredient in this generalized definition of harmonic maps, i.e., the term $G_{\mu\nu}(X, x)$ in the Lagrangian density implies that we are taking into account the "interaction" between the base space M and the target space N. This interaction leads to an extra term in the motion equations, as can be seen in (5.3.1). It turns out that this interaction is the result of the effective presence of the gravitational field.

Notice that the limiting case of generalized linear harmonic maps is much more complicated than in the standard case. Indeed, for the motion equations (5.3.1) to become linear it is necessary that the conditions

$$\gamma^{ab}(\Gamma^{\mu}_{\ \nu\lambda}\ \partial_b X^\lambda + G^{\mu\lambda}\,\partial_b G_{\lambda\nu})\partial_a X^\nu = 0\,, \tag{5.3.3}$$

be satisfied. One could search for a solution in which each term vanishes separately. The choice of a (pseudo-)Euclidean target metric $G_{\mu\nu} = \eta_{\mu\nu}$, which would imply $\Gamma^{\mu}_{\ \nu\lambda} = 0$, is not allowed, because it would contradict the assumption $\partial_b G_{\mu\nu} \neq 0$. Nevertheless, a flat background metric in curvilinear coordinates could be chosen such that the assumption $G^{\mu\lambda}\partial_b G_{\mu\nu} = 0$ is fulfilled, but in this case $\Gamma^{\mu}_{\ \nu\lambda} \neq 0$ and (5.3.3) cannot be satisfied. In the general case of a curved target metric, conditions (5.3.3) represent a system of m first order nonlinear partial differential equations for $G_{\mu\nu}$. Solutions to this system would represent linear generalized harmonic maps. The complexity of this system suggests that this special type of maps is not common.

As we mentioned before, the generalized action (5.3.2) includes an interaction between the base space N and the target space M, reflected on the fact that $G_{\mu\nu}$ depends explicitly on the coordinates of the base space. Clearly, this interaction must affect the conservation laws of the physical systems we attempt to describe by means of generalized harmonic maps. To see this explicitly we calculate the covariant derivative of the generalized Lagrangian density

$$\mathcal{L} = \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X,x)\ , \tag{5.3.4}$$

and replace in the result the corresponding motion equations (5.3.1). Then, the final result can be written as

$$\nabla_b \widetilde{T}_a{}^b = -\frac{\partial \mathcal{L}}{\partial x^a} \tag{5.3.5}$$

where $\widetilde{T}_a{}^b$ represents the canonical energy-momentum tensor

$$\widetilde{T}_a{}^b = \frac{\partial \mathcal{L}}{\partial(\partial_b X^\mu)}(\partial_a X^\mu) - \delta_a^b \mathcal{L} = 2\sqrt{\gamma}G_{\mu\nu}\left(\gamma^{bc}\partial_a X^\mu\,\partial_c X^\nu - \frac{1}{2}\delta_a^b \gamma^{cd}\partial_c X^\mu\,\partial_d X^\nu\right) . \tag{5.3.6}$$

The standard conservation law is recovered only when the Lagrangian does not depend explicitly on the coordinates of the base space. Even if we choose a flat base space $\gamma_{ab} = \eta_{ab}$, the explicit dependence of the metric of the target space $G_{\mu\nu}(X,x)$ on x generates a term that violates the standard conservation law. This term is due to the interaction between the base space and the target space which, consequently, is one of the main characteristics of the generalized harmonic maps introduced in this work.

An alternative and more general definition of the energy-momentum tensor is by means of the variation of the Lagrangian density with respect to the metric of the base space, i.e.

$$T_{ab} = \frac{\delta \mathcal{L}}{\delta \gamma^{ab}}\ . \tag{5.3.7}$$

A straightforward computation shows that for the action under consideration here we have that $\widetilde{T}_{ab} = 2T_{ab}$ so that the generalized conservation law (5.3.5) can be written as

$$\nabla_b T_a{}^b + \frac{1}{2}\frac{\partial \mathcal{L}}{\partial x^a} = 0\ . \tag{5.3.8}$$

For a given metric on the base space, this represents in general a system of m differential equations for the "fields" X^μ which must be satisfied "on-shell".

If the base space is 2-dimensional, we can use a reparametrization of x to choose a conformally flat metric, and the invariance of the Lagrangian density under arbitrary Weyl transformations to show that the energy-momentum tensor is traceless, $T_a{}^a = 0$.

In Section 5.1 we described stationary, axially symmetric, gravitational fields as a $(4 \to 2)-$nonlinear sigma model. There it was pointed out the convenience of having a 2-dimensional base space in analogy with string theory. Now we will show that this can be done by using the generalized harmonic maps defined above.

Consider a $(2 \to 2)-$generalized harmonic map. Let $x^a = (\rho, z)$ be the coordinates on the base space M, and $X^\mu = (f, \Omega)$ the coordinates on the target space N. In the base space we choose a flat metric and in the target space a conformally flat metric, i.e.

$$\gamma_{ab} = \delta_{ab} \qquad \text{and} \qquad G_{\mu\nu} = \frac{\rho}{2f^2}\delta_{\mu\nu} \qquad (a, b = 1, 2;\ \mu, \nu = 1, 2). \tag{5.3.9}$$

A straightforward computation shows that the generalized Lagrangian (5.3.4) coincides with the Lagrangian (5.2.4) for stationary axisymetric fields, and that the equations of motion (5.3.1) generate the main field equations (4.1.3) and (4.1.4).

For the sake of completeness we calculate the components of the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$. Then

$$T_{\rho\rho} = -T_{zz} = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right], \tag{5.3.10}$$

$$T_{\rho z} = \frac{\rho}{2f^2}\left(\partial_\rho f\, \partial_z f + \partial_\rho\Omega\ \partial_z\Omega\right). \tag{5.3.11}$$

This tensor is traceless due to the fact that the base space is 2-dimensional. It satisfies the generalized conservation law (5.3.8) on-shell:

$$\frac{dT_{\rho\rho}}{d\rho} + \frac{dT_{\rho z}}{dz} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\rho} = 0\,, \tag{5.3.12}$$

$$\frac{dT_{\rho z}}{d\rho} - \frac{dT_{\rho\rho}}{dz} = 0\,. \tag{5.3.13}$$

Incidentally, the last equation coincides with the integrability condition for the metric function k, which is identically satisfied by virtue of the main field equations. In fact, as can be seen from Eqs.(4.1.5,4.1.6) and (5.3.10,5.3.11), the components of the energy-momentum tensor satisfy the relationships $T_{\rho\rho} = \partial_\rho k$ and $T_{\rho z} = \partial_z k$, so that the conservation law (5.3.13) becomes an identity. Although we have eliminated from the starting Lagrangian (5.2.4) the variable k by applying a Legendre transformation on the Einstein-Hilbert Lagrangian (see [17] for details) for this type of gravitational fields, the formalism of generalized harmonic maps seems to retain the information about k at the level of the generalized conservation law.

The above results show that stationary axisymmetric spacetimes can be represented as a $(2 \to 2)-$generalized harmonic map with metrics given as in (5.3.9). It is also possible to interpret the generalized harmonic map given above as a generalized string model. Although the metric of the base space M is Euclidean, we can apply a Wick rotation $\tau = i\rho$ to obtain a Minkowski-like structure on M. Then, M represents the world-sheet of a bosonic string in which τ is measures the time and z is the parameter along the string. The string is "embedded" in the target space N whose metric is conformally flat and explicitly depends on the time parameter τ. We will see in the next section that this embedding becomes more plausible when the target space is subject to a dimensional extension. In the present example, it is necessary to apply a Wick rotation in order to interpret the base space as a string world-sheet. This is due to the fact that both coordinates ρ and z are spatial coordinates. However, this can be avoided by considering other classes of gravitational fields with timelike Killing vector fields; examples will be given below.

The most studied solutions belonging to the class of stationary axisymmetric fields are the asymptotically flat solutions. Asymptotic flatness imposes conditions on the metric functions which in the cylindrical coordinates used here can be formulated in the form

$$\lim_{x^a\to\infty} f = 1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \omega = c_1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \Omega = O\left(\frac{1}{x^a}\right) \tag{5.3.14}$$

where c_1 is an arbitrary real constant which can be set to zero by appropriately choosing the angular coordinate φ. If we choose the domain of the spatial coordinates as $\rho \in [0, \infty)$ and $z \in (-\infty, +\infty)$, from the asymptotic flatness conditions it follows that the coordinates of the target space N satisfy the boundary conditions

$$\dot{X}^\mu(\rho, -\infty) = 0 = \dot{X}^\mu(\rho, \infty) , \quad X'^\mu(\rho, -\infty) = 0 = X'^\mu(\rho, \infty) \tag{5.3.15}$$

where the dot stands for a derivative with respect to ρ and the prime represents derivation with respect to z. These relationships are known in string theory [16] as the Dirichlet and Neumann boundary conditions for open strings, respectively, with the extreme points situated at infinity. We thus conclude that if we assume ρ as a "time" parameter for stationary axisymmetric gravitational fields, an asymptotically flat solution corresponds to an open string with endpoints attached to $D-$branes situated at plus and minus infinity in the $z-$direction.

5.4 Dimensional extension

In order to further analyze the analogy between gravitational fields and bosonic string models, we perform an arbitrary dimensional extension of the target space N, and study the conditions under which this dimensional extension does not affect the field equations of the gravitational field. Consider an $(m \to D)-$generalized harmonic map. As before we denote by $\{x^a\}$ the coordinates on M. Let $\{X^\mu, X^\alpha\}$ with $\mu = 1,2$ and $\alpha = 3,4,...,D$ be the coordinates on N. The metric structure on M is again $\gamma = \gamma(x)$, whereas the metric on N can in general depend on all coordinates of M and N, i.e. $G = G(X^\mu, X^\alpha, x^a)$. The general structure of the corresponding field equations is as given in (5.3.1). They can be divided into one set of equations for X^μ and one set of equations for X^α. According to the results of the last section, the class of gravitational fields under consideration can be represented as a $(2 \to 2)-$generalized harmonic map so that we can assume that the main gravitational variables are contained in the coordinates X^μ of the target space. Then, the gravitational sector of the target space will be contained in the components $G_{\mu\nu}$ $(\mu, \nu = 1, 2)$ of the metric, whereas the components $G_{\alpha\beta}$ $(\alpha, \beta = 3, 4, ..., D)$ represent the sector of the dimensional extension.

Clearly, the set of differential equations for X^μ also contains the variables X^α and its derivatives $\partial_a X^\alpha$. For the gravitational field equations to remain unaffected by this dimensional extension we demand the vanishing of all the terms containing X^α and its derivatives in the equations for X^μ. It is easy to show that this can be achieved by imposing the conditions

$$G_{\mu\alpha} = 0\,, \quad \frac{\partial G_{\mu\nu}}{\partial X^\alpha} = 0\,, \quad \frac{\partial G_{\alpha\beta}}{\partial X^\mu} = 0\,. \tag{5.4.1}$$

That is to say that the gravitational sector must remain completely invariant under a dimensional extension, and the additional sector cannot depend on the gravitational variables, i.e., $G_{\alpha\beta} = G_{\alpha\beta}(X^\gamma, x^a)$, $\gamma = 3, 4, ..., D$. Furthermore, the variables X^α must satisfy the differential equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\alpha\right) + \Gamma^\alpha_{\ \beta\gamma}\,\gamma^{ab}\,\partial_a X^\beta \partial_b X^\gamma + G^{\alpha\beta}\gamma^{ab}\,\partial_a X^\gamma\,\partial_b G_{\beta\gamma} = 0\,. \tag{5.4.2}$$

This shows that any given $(2 \to 2)-$generalized map can be extended, without affecting the field equations, to a $(2 \to D)-$generalized harmonic map.

It is worth mentioning that the fact that the target space N becomes split in two separate parts implies that the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$ separates into one part belonging to the gravitational sector and a second one following from the dimensional extension, i.e. $T_{ab} = T_{ab}(X^\mu, x) + T_{ab}(X^\alpha, x)$. The generalized conservation law as given in (5.3.8) is satisfied by the sum of both parts.

Consider the example of stationary axisymmetric fields given the metrics (5.3.9). Taking into account the conditions (5.4.1), after a dimensional extension the metric of the target space becomes

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\rho}{2f^2} & 0 & \cdots & 0 \\ 0 & 0 & G_{33}(X^\alpha, x) & \cdots & G_{3D}(X^\alpha, x) \\ . & . & \cdots & \cdots & \cdots \\ 0 & 0 & G_{D3}(X^\alpha, x) & \cdots & G_{DD}(X^\alpha, x) \end{pmatrix} . \tag{5.4.3}$$

Clearly, to avoid that this metric becomes degenerate we must demand that $\det(G_{\alpha\beta}) \neq 0$, a condition that can be satisfied in view of the arbitrariness of the components of the metric. With the extended metric, the Lagrangian density gets an additional term

$$\begin{aligned} \mathcal{L} = \quad & \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right] \\ & + \left(\partial_\rho X^\alpha \partial_\rho X^\beta + \partial_z X^\alpha \partial_z X^\beta\right) G_{\alpha\beta} \, , \end{aligned} \tag{5.4.4}$$

which nevertheless does not affect the field equations for the gravitational variables f and Ω. On the other hand, the new fields must be solutions of the extra field equations

$$\left(\partial_\rho^2 + \partial_z^2\right) X^\alpha + \Gamma^\alpha{}_{\beta\gamma} \left(\partial_\rho X^\beta \partial_\rho X^\gamma + \partial_z X^\beta \partial_z X^\gamma\right) \tag{5.4.5}$$

$$+ \, G^{\alpha\gamma}\left(\partial_\rho X^\beta \partial_\rho G_{\beta\gamma} + \partial_z X^\beta \partial_z G_{\beta\gamma}\right) = 0 \, . \tag{5.4.6}$$

An interesting special case of the dimensional extension is the one in which the extended sector is Minkowskian, i.e. for the choice $G_{\alpha\beta} = \eta_{\alpha\beta}$ with additional fields X^α given as arbitrary harmonic functions. This choice opens the possibility of introducing a "time" coordinate as one of the additional dimensions, an issue that could be helpful when dealing with the interpretation of gravitational fields in this new representation.

The dimensional extension finds an interesting application in the case of static axisymmetric gravitational fields. As mentioned in Section 4.1, these fields are obtained from the general stationary fields in the limiting case $\Omega = 0$ (or equivalently, $\omega = 0$). If we consider the representation as an $SL(2,R)/SO(2)$ nonlinear sigma model or as a $(2 \to 2)-$generalized harmonic map, we see immediately that the limit $\Omega = 0$ is not allowed because the target space becomes 1-dimensional and the underlying metric is undefined. To avoid this degeneracy, we first apply a dimensional extension and only then calculate the limiting case $\Omega = 0$. In the most simple case of an extension with $G_{\alpha\beta} = \delta_{\alpha\beta}$, the resulting $(2 \to 2)-$generalized map is described by the metrics $\gamma_{ab} = \delta_{ab}$ and

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{5.4.7}$$

where the additional dimension is coordinatized by an arbitrary harmonic function which does not affect the field equations of the only remaining gravitational variable f. This scheme represents an alternative method for exploring static fields on nondegenerate target spaces. Clearly, this scheme can be applied to the case of gravitational fields possessing two hypersurface orthogonal Killing vector fields.

Our results show that a stationary axisymmetric field can be represented as a string "living" in a D-dimensional target space N. The string world-sheet is parametrized by the coordinates ρ and z. The gravitational sector of the target space depends explicitly on the metric functions f and Ω and on the parameter ρ of the string world-sheet. The sector corresponding to the dimensional extension can be chosen as a $(D-2)-$dimensional Minkowski spacetime with time parameter τ. Then, the string world-sheet is a 2-dimensional flat hypersurface which is "frozen" along the time τ.

5.5 The general solution

If we take as seed metric the general static solution, the application of two HXK transformations generates a stationary solution with an infinite number of gravitoelectric and gravitomagnetic multipole moments. The HKX method is applied at the level of the Ernst potential from which the metric functions can be calculated by using the definition of the Ernst potential E and the field equations for γ. The resulting expressions in the general case are quite cumbersome. We quote here only the special case in which only an arbitrary quadrupole parameter is present. In this case, the result can be written as

$$
\begin{aligned}
f &= \frac{R}{L} e^{-2qP_2Q_2} , \\
\omega &= -2a - 2\sigma \frac{\mathcal{M}}{R} e^{2qP_2Q_2} , \\
e^{2\gamma} &= \frac{1}{4}\left(1 + \frac{M}{\sigma}\right)^2 \frac{R}{x^2 - y^2} e^{2\hat{\gamma}} ,
\end{aligned} \tag{5.5.1}
$$

where

$$
\begin{aligned}
R &= a_+a_- + b_+b_- , \qquad L = a_+^2 + b_+^2 , \\
\mathcal{M} &= \alpha x(1-y^2)(e^{2q\delta_+} + e^{2q\delta_-})a_+ + y(x^2-1)(1 - \alpha^2 e^{2q(\delta_+ + \delta_-)})b_+ , \\
\hat{\gamma} &= \frac{1}{2}(1+q)^2 \ln\frac{x^2-1}{x^2-y^2} + 2q(1-P_2)Q_1 + q^2(1-P_2)\bigg[(1+P_2)(Q_1^2 - Q_2^2) \\
&\quad + \frac{1}{2}(x^2-1)(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2')\bigg] .
\end{aligned} \tag{5.5.2}
$$

Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind respectively. Furthermore

$$
\begin{aligned}
a_\pm &= x(1 - \alpha^2 e^{2q(\delta_+ + \delta_-)}) \pm (1 + \alpha^2 e^{2q(\delta_+ + \delta_-)}) , \\
b_\pm &= \alpha y(e^{2q\delta_+} + e^{2q\delta_-}) \mp \alpha(e^{2q\delta_+} - e^{2q\delta_-}) , \\
\delta_\pm &= \frac{1}{2}\ln\frac{(x \pm y)^2}{x^2-1} + \frac{3}{2}(1 - y^2 \mp xy) + \frac{3}{4}[x(1-y^2) \mp y(x^2-1)] \ln\frac{x-1}{x+1} ,
\end{aligned}
$$

the quantity α being a constant

$$
\alpha = \frac{\sigma - M}{a} , \qquad \sigma = \sqrt{M^2 - a^2} . \tag{5.5.3}
$$

The physical significance of the parameters entering this metric can be clarified by calculating the Geroch-Hansen [18, 19] multipole moments

$$
M_{2k+1} = J_{2k} = 0 , \quad k = 0, 1, 2, ... \tag{5.5.4}
$$

$$
M_0 = M , \quad M_2 = -Ma^2 + \frac{2}{15} qM^3 \left(1 - \frac{a^2}{M^2}\right)^{3/2} , ... \tag{5.5.5}
$$

$$
J_1 = Ma , \quad J_3 = -Ma^3 + \frac{4}{15} qM^3 a \left(1 - \frac{a^2}{M^2}\right)^{3/2} , ... \tag{5.5.6}
$$

The vanishing of the odd gravitoelectric (M_n) and even gravitomagnetic (J_n) multipole moments is a consequence of the symmetry with respect to the equatorial plane. From the above expressions we see that M is the total mass of the body, a represents the specific angular momentum, and q is related to the deviation from spherical symmetry. All higher multipole moments can be shown to depend only on the parameters M, a, and q.

We analyzed the geometric and physical properties of the above solution. The special cases contained in the general solution suggest that it can be used to describe the exterior asymptotically flat gravitational field of rotating body with arbitrary quadrupole moment. This is confirmed by the analysis of the motion of particles on the equatorial plane. The quadrupole moment turns out to drastically change the geometric structure of spacetime as well as the motion of particles, especially near the gravitational source.

We investigated in detail the properties of the Quevedo-Mashhoon (QM) spacetime which is a generalization of Kerr spacetime, including an arbitrary quadrupole. Our results show [20] that a deviation from spherical symmetry, corresponding to a non-zero electric quadrupole, completely changes the structure of spacetime. A similar behavior has been found in the case of the Erez-Rosen spacetime. In fact, a naked singularity appears that affects the ergosphere and introduces regions where closed timelike curves are allowed. Whereas in the Kerr spacetime the ergosphere corresponds to the boundary of a simply-connected region of spacetime, in the present case the ergosphere is distorted by the presence of the quadrupole and can even become transformed into non simply-connected regions. All these changes occur near the naked singularity which is situated at $x = 1$, a value that corresponds to the radial distance $r = M + \sqrt{M^2 - a^2}$ in Boyer-Lindquist coordinates. In the limiting case $a/M > 1$, the multipole moments and the metric become complex, indicating that the physical description breaks down. Consequently, the extreme Kerr black hole represents the limit of applicability of the QM spacetime.

Since standard astrophysical objects satisfy the condition $a/M < 1$, we can conclude that the QM metric can be used to describe their exterior gravitational field. Two alternative situations are possible. If the characteristic radius of the body is greater than the critical distance $M + \sqrt{M^2 - a^2}$, i.e. $x > 1$, the exterior solution must be matched with an interior solution in order to describe the entire spacetime. If, however, the characteristic radius of the body is smaller than the critical distance $M + \sqrt{M^2 - a^2}$, the QM metric describes the field of a naked singularity.

6 Quadrupolar metrics

6.1 Introduction

It is well known that the Kerr solution describes the exterior gravitational field of a mass M with specific angular momentum $a = J/M$. It is asymptotically flat and reduces to the Minkowski metric in the limit $M = 0$ and $a = 0$, and to the Schwarzschild metric in the limit $a = 0$. The Kerr spacetime is characterized by the presence of a curvature singularity determined by the equation

$$r^2 + a^2 \cos^2 \theta = 0 \tag{6.1.1}$$

which corresponds to a ring located on the equatorial plane $\theta = \pi/2$. This ring singularity, however, cannot be observed from outside because it is covered by a horizon located on a sphere of radius

$$r_h = m + \sqrt{m^2 - a^2}\,. \tag{6.1.2}$$

Since no information can be extracted from behind the horizon, an external observer will never be aware about the existence of the ring singularity. In this sense, the singularity can be considered as non-existing for observers located outside the horizon. The Kerr spacetime can be therefore interpreted as describing the exterior gravitational field of a rotating black hole. Furthermore, the black hole uniqueness theorems [21] state that the Kerr spacetime is the most general vacuum solution that corresponds to a black hole. In other words, to describe a black hole, we only need two parameters, namely, mass and angular momentum.

In the case $a^2 > m^2$, no horizon exists and the ring singularity becomes naked. However, several studies [22, 23, 24] show that in realistic situations, where astrophysical objects are surrounded by accretion disks, a Kerr naked singularity is an unstable configuration that rapidly decays into a Kerr black hole. Furthermore, it now seems to be well established that in generic situations a gravitational collapse cannot lead to the formation of a Kerr naked singularity. These results seem to indicate that rotating Kerr naked singularities do not exist in Nature. Again, these results corroborate that the Kerr spacetime describes rotating black holes.

From an astrophysical point of view, black holes belong to the class of compact objects which include also neutron stars and white dwarfs. The question arises whether the Kerr metric can also be used to describe the exterior gravitational field of neutron stars and white dwarfs. To try to answer this question, let us recall that from the point of view of general relativity, the gravitational field of a compact source should be described by a complete Riemannian differential manifold, i.e., it should include an exterior metric and an interior metric as well. Let us suppose for a moment that the Kerr spacetime describes the exterior field of all compact objects, and consider the interior counterpart. In the case of black holes, in which the matter content of the original star has collapsed to form a curvature singularity, we argue that it is not possible to find the interior counterpart within the framework of classical general relativity. Indeed, since all the information about the internal structure of a black hole is located inside the singularity, where the classical theory is not valid any more, we should apply an alternative theory that must take into account the effects of gravity under extreme pressures and densities, as intuitively expected at the singularities. Such a theory could be quantum gravity which, in the best case, is still under construction. This argument implies that the quantum interior counterpart of the Kerr metric is well beyond our reach in the short term.

Consider now the interior field of neutron stars and white dwarfs. An interior metric should describe an equilibrium structure, probably a fluid, bounded by a surface of zero pressure and matched across this surface to the exterior Kerr metric. The search for such an interior solution has been conducted for over 50 years, and not even a single physical meaningful solution has been found to date. Many arguments can be found to explain this negative result, especially, regarding the relatively simple models used to describe the internal structure of such compact stars. Nevertheless, if we consider a more elaborated internal model, the mathematical complexity of the field equations and the matching conditions usually increases as well, implying that the possibility of solving the problem decreases. This is probably the reason why the search for physically meaningful interior solutions has not been very successful. In our opinion, the simplest solution to this problem is to assume that the Kerr metric does not describe the exterior field of rotating compact objects, but black holes. This is exactly the working hypothesis we will assume henceforth.

The question arises: What metric should we use to describe the exterior field of neutron stars and white dwarfs? The black hole uniqueness theorems [21] sheds some light on how to look for an answer to this question. In fact, black holes are described by only the mass m and the angular momentum J. From the point of view of the multipole structure of exact vacuum solutions (for a review see, for instance, [25]), this is equivalent to saying that only the lowest multipoles are present in black holes, namely, the mass monopole m and the angular-momentum dipole J. Then, it seems reasonable to include higher moments in order to describe the exterior field of compact objects, other than black holes. The simplest choice to begin with is the mass quadrupole. Consequently, we assume in this work that to describe the exterior field of neutron stars and white dwarfs, we need a vacuum metric with three physical parameters, namely, mass m, angular momentum J, and quadrupole q[1]. From a physical point of view, it is also reasonable to consider the mass quadrupole as an additional parameter, because it represents the natural deviations of a mass distribution from the ideal spherical symmetry. In other words, we assume that in the case of neutron stars an white dwarfs, it is not possible to neglect the gravitational field generated by the quadrupole, whereas in the case of black holes, the uniqueness theorems prove that the quadrupole is zero.

On the other hand, since the uniqueness theorems are valid only in the case of mass and angular momentum, with the Kerr metric as the only exact solution, there must exist several exact solutions with mass, angular momentum and quadrupole. The main goal of this work is to present a review and a brief description of the main exact vacuum solutions of Einstein equations with mass quadrupole.

This chapter is organized as follows. In Sec. 6.3, we focus on static gravitational sources. We present the most important properties that a metric should satisfy in order to describe the exterior field of a compact source. We present the explicit form of the metrics that, to our knowledge, have been used in general relativity to describe the field of static mass distributions. In Sec. 6.4, we study the rotating generalizations of the quadrupolar metrics. Then, in Sec. 6.5, we describe the uninspiring situation in the case of interior solutions. Finally, in Sec. 7.8, we discuss the situation in general and comment on the open problems regarding the description of the gravitational field of neutron stars and white dwarfs.

6.2 The gravitational field of compact stars

To describe the exterior gravitational field it is necessary to obtain exact solutions of Einstein's equations in empty space. Since Einstein's field equations are in general difficult to handle, especially when the aim is to obtain physically meaningful solutions, it is necessary to assume the validity of certain physical conditions about the problem under consideration. We assume that the gravitational field of compact stars do not change drastically in time so

[1]Of course, one could also include higher multipoles like the mass octupole, the angular-momentum quadrupole, etc. However, we limit ourselves here to the lowest non-ignorable multipole which is the mass quadrupole.

that stationarity can be adopted. In general, we know from observations that this condition is satisfied in most astrophysical objects. Moreover, the assumption of stationarity does not exclude the possibility of rotation which is an important characteristic of all known compact stars. To consider the deviations of the mass distribution from spherical symmetry, we will assume the existence of an axis of symmetry which for the sake of simplicity is supposed to coincide with the axis of rotation. Moreover, to take into account the deformations of the mass distribution with respect to the axis of symmetry, we will consider only the quadrupole moment.

The above assumptions imply that we must focus our analysis on the case of stationary axisymmetric gravitational fields. The corresponding line element in the case of empty space is known as the Weyl-Lewis-Papapetrou [26, 27, 28] line element that in cylindrical coordinates can be written as in Eq.(4.1.1). The corresponding field equations have been presented previously in Section 4. Due to the implementation of solution generating techniques the number of exact stationary solutions has increased enormously. In fact, given a particular static solution, it is possible to generate, in principle, an infinite number of stationary solutions which contain the seed static solution as a particular case. It is therefore necessary to establish some criteria to classify the solutions. In particular, we are interested in the conditions that must imposed in order for a given solution to be physically meaningful.

6.2.1 Physical conditions

One can find many stationary axisymmetric solutions of Einstein's equations, but not all them are necessarily suitable to describe the gravitational field of compact stars. Several physical conditions must be imposed which can be described as follows.

i) The spacetime must be asymptotically flat. This means that far away from the source the gravitational field should be negiglible small, and can be described approximately by the Minkowski metric.

ii) The spacetime must be elementary flat, i.e., the axis of symmetry must be free of conical singularities. This property means that the coordinate φ is a well-defined angle coordinate that can be used to represent the rotation of the compact star.

iii) The spacetime must be free of singularities outside the surface of the star. Curvature singularities can exist inside the surface where the vacuum solution is not valid any more and, instead, an interior solution should exist that "covers" the singularity.

iv) The spacetime must be free of horizons in order to be in accordance with the black hole uniqueness theorems.

v) The solution must reduce to the Minkowski metric in the limiting case when the mass monopole vanishes, independently of the values of the remaining parameters. This condition guarantees that there are no rotations and no deviations from spherical symmetry without the presence of a physical mass distribution.

vi) The solution must be matched with a physically meaningful interior solution across the surface of the star where the pressure and the density of the interior configuration should vanish.

The fulfillment of these conditions represents the real challenge for describing the gravitational field of compact stars. Whereas there many metrics that can be used to represent the exterior field, the interior counterparts are still unknown.

6.3 Static quadrupolar metrics

The simplest case of a multipolar spacetime is described by the Schwarzschild metric which possesses only the mass monopole. Birkhoff's theorem [1] guarantees that this metric is unique. Furthermore, from a physical point of view, one expects that a dipole moment can be made to vanish by an appropriate coordinate transformation which, in the Newtonian limit, corresponds to locating the origin of spatial coordinates on the center of mass of the object. The next interesting configuration consists of a mass with quadrupole moment. In this case, no uniqueness theorem exists and, therefore, we can expect that Einstein's equations permit the existence of several solutions describing such a gravitational system. Indeed, several exact solutions are known.

Weyl [26] found the most general static axisymmetric asymptotically flat solution in cylindrical coordinates. The set of parameters a_n entering this solution essentially determines the set of mass multipoles M_n as computed by using the Geroch-Hansen definition [31, 32, 33], for instance. Then, a configuration composed of a mass and a quadrupole can be written as

$$\frac{1}{2}\ln f = \frac{a_0}{(\rho^2+z^2)^{1/2}} + \frac{a_2}{(\rho^2+z^2)^{3/2}} P_2(\cos\theta) \tag{6.3.1}$$

The first term is called the Chazy-Curzon metric [1] and describes the field of two particles located along the symmetry axis with a curvature singularity among them, i.e., it corresponds to a strut located along the axis. The second term can be considered as representing a quadrupole deformation of the strut. Far away from the source, the Chazy-Curzon metric leads to the Newtonian potential of a point particle. One could therefore expect that in the Newtonian limit the second term generates a quadrupole moment. From a physical point of view, one would expect that close to a non-rotating compact star with no quadrupole, the metric is spherically symmetric. We see that the above Weyl metric does not satisfy this condition. We therefore conclude that it cannot be used to describe the exterior field of compact stars.

To our knowledge, Erez and Rosen [9] found the first quadrupolar metric which reduces to the Schwarzschild metric in the limit of vanishing quadrupole. In prolate spheroidal coordinates, it can be expressed as (q_2 is a constant)

$$\ln f = \ln \frac{x-1}{x+1} + q_2(3y^2-1)\left[\frac{1}{4}(3x^2-1)\ln\frac{x-1}{x+1} + \frac{3}{2}x\right] , \tag{6.3.2}$$

$$\begin{aligned}\gamma = \quad & \frac{1}{2}(1+q_2)^2 \ln\frac{x^2-1}{x^2-y^2} - \frac{3}{2}q_2(1-y^2)\left(x\ln\frac{x-1}{x+1}+2\right) \\ & +\frac{9}{16}q_2^2(1-y^2)\left[x^2+4y^2-9x^2y^2-\frac{4}{3}\right. \\ & +x\left(x^2+7y^2-9x^2y^2-\frac{5}{3}\right)\ln\frac{x-1}{x+1} \\ & \left.+\frac{1}{4}(x^2-1)(x^2+y^2-9x^2y^2-1)\ln^2\frac{x-1}{x+1}\right] .\end{aligned} \tag{6.3.3}$$

In the limiting case $q_2 \to 0$, the Erez-Rosen metric reduces to the Schwarzschild metric, as expected for a compact star. In general, this solution is asymptotically flat and free of singularities outside the spatial region determined by $x = 1$, which in the case of vanishing quadrupole corresponds to the Schwarzschild radius. It also satisfies the condition of elementary flatness. From this point of view, the Erez-Rosen solution satisfies all the conditions to describe the exterior field of a deformed mass with quadrupole moment. However, no interior solution is known that could be matched with the exterior metric on the surface of the body.

Gutsunayev and Manko [34] derived the following exact static solution (A_2 is a constant)

$$\ln f = \ln\frac{x-1}{x+1} + 2A_2\frac{x(x^2-3x^2y^2+3y^2-y^4)}{(x^2-y^2)^3} , \tag{6.3.4}$$

$$\begin{aligned}\gamma = \quad & \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2} + \frac{A_2(1-y^2)}{2(x^2-y^2)^4}[3(1-5y^2)(x^2-y^2)^2 \\ & +8y^2(3-5Y^2)(x^2-y^2)+24y^4(1-y^2)] + \frac{A_2^2(1-y^2)}{8(x^2-y^2)^8} \\ & \times[-12(1-14y^2+25y^4)(x^2-y^2)^5+3(3-153y^2 \\ & +697y^4-675y^6)(x^2-y^2)^4 \\ & +32y^2(9-105y^2+259y^4-171y^6)(x^2-y^2)^3 \\ & +32y^4(45-271y^2+451y^4-225y^6)(x^2-y^2)^2 \\ & +2304y^6(1-4y^2+5y^4-2y^6)(x^2-y^2) \\ & +1152y^8(1-3y^2+3y^4-y^6)]\,. \end{aligned} \tag{6.3.5}$$

Although at first glance these two solutions look quite different, it is possible to show [35] that if we choose the parameters as

$$A_2 = \frac{1}{15}q_2\,, \tag{6.3.6}$$

the quadrupole moment of both metrics coincide, but differences appear at the level of the 2^4-pole moment.

A different quadrupolar metric was derived by Hernández-Pastora and Martí [36] which is also given in prolate spheroidal coordinates as:

$$\ln f = \ln\frac{x-1}{x+1} + \frac{5}{4}B_2\left\{\frac{3}{4}[(3x^2-1)(3y^2-1)-4]\ln\frac{x-1}{x+1} - \frac{2x}{x^2-y^2} + \frac{3}{2}x(3y^2-1)\right\}. \tag{6.3.7}$$

As in the previous cases, the corresponding γ function can be calculated by quadratures by using the explicit for of f only. The resulting expression is quite complicated. We refer to the original paper for the explicit expression. In the above solution, the constant parameter B_2 essentially determines the quadrupole moment of the mass distribution.

Recently, in [37], the multipole moment structure of the above solutions with free parameters q_2, A_2 and B_2 was investigated in detail with the result that the Geroch quadrupole moment of all three metrics can be made to coincide by choosing the free parameters appropriately. On the other hand, it is known that in general relativity, stationary and axisymmetric vacuum spacetimes can be completely characterized by their multipolar structure and if two spacetimes have the same moments, then they represent essentially the same spacetime. If then follows that the above metrics with free parameters q_2, A_2 and B_2 are in fact the same spacetime, if we consider only the quadrupole moment. However, if higher moments are taken into account, differences appear that make the three metrics different from a physical point of view.

As can be seen from the above expressions, the explicit form of the known quadrupolar metrics is not simple, usually making them difficult to be analyzed. In a recent work [38], we proposed an alternative solution as the simplest generalization of the Schwarzschild solution which contains a quadrupole parameter q. In spherical coordinates, it has the simple and compact expression

$$ds^2 = \left(1-\frac{2m}{r}\right)^{1+q} dt^2 - \left(1-\frac{2m}{r}\right)^{-q} \times \left[\left(1+\frac{m^2\sin^2\theta}{r^2-2mr}\right)^{-q(2+q)}\left(\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right)+r^2\sin^2\theta d\varphi^2\right] \tag{6.3.8}$$

This solution is obtained from the Schwarzschild metric by applying a Zipoy-Voorhees transformation [39, 40]. In the literature, for notational reasons this solution is known as the $\delta-$metric or as the $\gamma-$metric [41]. Instead, we propose to use the term quadrupole metric ($q-$metric) to emphasize the role of the parameter q which determines the quadrupole moment. Indeed, a straightforward computation of the Geroch multipole moments leads to a monopole

$$M_0 = (1+q)m \tag{6.3.9}$$

and a quadrupole

$$M_2 = -\frac{m^3}{3}q(1+q)(2+q). \tag{6.3.10}$$

If $q = 0$, we obtain the limiting case of the Schwarzschild metric. Moreover, the free parameters m and q can be chosen in such a way that the quadrupole moment M_2 is negative (oblate objects) or positive (prolate objects). Furthermore, one can easily show that this solution satisfies all physical conditions mentioned in the previous section for exterior solutions. This implies that it can be used to describe the exterior gravitational field of static compact stars. A detailed analysis of the circular motion of test particles around a compact object described by the $q-$metric shows that the presence of the quadrupole parameter can drastically change the physical behavior of test particles, and the obtained effects corroborate the interpretation of q as determining the deviation of the mass distribution from spherical symmetry [43].

6.4 Stationary quadrupolar metrics

All the solutions presented in the previous section do not take into account an important characteristic of compact objects, namely, the rotation. Realistic exact solutions should contain at least one additional parameter that could be interpreted as rotation. In terms of multipole moments, this means that the angular-momentum dipole should be nonzero. The first exact solution with a non-trivial angular-momentum dipole was discovered by Kerr in 1963 [**?**]. Soon after, Ernst proposed a general representation for stationary and axisymmetric vacuum and electrovacuum spacetimes (see Section 5.1) that allowed researchers in this field to derive a new type of internal symmetries of the field equations. As a result, some solution generating techniques [1] were developed whose main objective is to generate new solutions from known ones.

The first generating methods such as the Kerr-Schild Ansatz, the complex Newman-Janis Ansatz, and the Hamilton-Jacobi separability procedure were limited to generate only the (charged) Kerr-NUT (Newman-Unti-Tamburino) class of stationary solutions. Nevertheless, the simple and compact Ernst representation was used by Tomimatsu and Sato and Yamazaki and Hori to find exact solutions with a particular functional dependence for the Ernst potential. Furthermore, Ernst developed two generating methods that were generalized by Kinnersley [1].

All the early methods were based on particular symmetries of the field equations. The discovery of Lie symmetries of the Ernst representation in the late seventies determined the starting point for the development of modern solution generating techniques. All the symmetry transformations of the field equations involve in general an infinite dimensional group of transformations. One of the main difficulties was to isolate only those transformations that preserve asymptotic flatness and do not generate unphysical curvature singularities *a priori*. Finally, Hoenselaers, Kinnersley and Xanthopoulos found subgroups of the Geroch group which preserve asymptotic flatness and can easily be extrapolated by purely algebraic methods.

Particular cases of Bäcklund transformations of the Ernst equations were found by Harrison and Neugebauer. Bäcklund transformations were first used to generate asymptotically flat solutions, using the Minkowski metric as seed solution. In general, it can be shown that the generated solution is asymptotically flat, if this is also a property of the seed solution.

A different method was proposed by Belinsky and Zakharov in which the nonlinear Einstein field equations are represented as a linear eigenvalue problem which can be solved by means of the inverse scattering method. This method allows one to generate solitonic solutions, one of which corresponds to the Kerr-NUT solution.

All the above methods imply several detailed procedures with quite complicated calculations. A particularly simple and different method was developed by Sibgatullin [44] in which only the value of the Ernst potential on the axis of symmetry is required in order to calculate the general form of the potential from which the corresponding metric can be calculated. Suppose that the Ernst potential in cylindrical coordinates is given as an arbitrary function $e(z)$ on the axis $\rho = 0$. Then, the Ernst potential for the entire spacetime can be calculated as

$$E(\rho, z) = \frac{1}{\pi} \int_{-1}^{+1} \frac{e(\xi)\mu(\sigma)}{\sqrt{1-\sigma^2}} d\sigma \,, \tag{6.4.1}$$

where $\xi = z + i\rho\sigma$ and the unknown function $\mu(\sigma)$ satisfies the singular integral equation

$$\int_{-1}^{+1} \frac{\mu(\sigma)[e(\xi) + e^*(\eta^*)]}{(\sigma - \tau)\sqrt{1-\sigma^2}} d\sigma = 0 \,, \tag{6.4.2}$$

and the normalizing condition

$$\int_{-1}^{+1} \frac{\mu(\sigma)}{\sqrt{1-\sigma^2}} d\sigma = \pi \,, \tag{6.4.3}$$

where $\eta = z + i\rho\tau$, and the asterisk represents complex conjugation. This method has been used to generate several stationary and axisymmetric solutions [45, 46] which satisfy all the conditions to describe the exterior field of neutron stars and are in accordance with a series of observations. These solutions are characterized by a finite number of parameters which are interpreted in terms of multipoles. For instance, the most general solution of this class has six parameters and is determined on the axis by the Ernst potential [46]

$$e(z) = \frac{z^3 - (m + ia)z^2 - kz + is}{z^3 + (m - ia)z^2 - kz + is} \,, \tag{6.4.4}$$

which contains four parameters. An additional function corresponding to the electromagnetic potential on the axis contains the two remaining parameters. In the case of vanishing electromagnetic field ($s = 0$) and rotation ($a = 0$), this solution reduces to a particular static Tomimatsu-Sato solution which can be shown to be equivalent to the $q-$metric with $q = 1$, so that the quadrupole moment is entirely determined by the mass monopole. In the stationary case ($a \neq 0$), the mass quadrupole is $M_2 = -1/4m(m^2 - a^2)$ and depends on the rotation parameter and the mass monopole. This indicates that deviations from spherical symmetry are due to rotation only and there is no parameter that could be changed in order to modify the deviations. In the case of all the static metrics mentioned above, there is always a free quadrupole parameter (q_2, A_2, B_2 or q) that is responsible for the deviations. This can be interpreted as an indication that metrics with arbitrary quadrupole could describe more general configurations of compact stars.

Most stationary and axisymmetric solutions in empty space have been obtained by using the solution generating methods mentioned above. In particular, a rotating generalization of the Erez-Rosen metric was obtained in [47, 48, 49]. The explicit form of the metric functions is given in Eqs.(5.5.1). It can be shown that this solution satisfies all the physical conditions mentioned in the previous section. Therefore, it can be used to describe the exterior field of compact stars.

In the previous section, we presented the $q-$metric as the simplest generalization of the Schwarzschild metric which contains a free quadrupole parameter. Therefore, it can be expected that the stationary generalizations of the $q-$metric should also have a simple representation. To show this, we apply a particular Lie transformation to the Ernst potential

$$E = \left(\frac{x-1}{x+1}\right)^{1+q} \tag{6.4.5}$$

of the $q-$metric in prolate spheroidal coordinates. To obtain the explicit form of the new stationary Ernst potential, we use the solution generating techniques that allow us to generate stationary solutions from a static solution. The procedure involves several differential equations which must be solved under the condition of asymptotic flatness. Here, we only present the final expression for the new Ernst potential [50]

$$E = \left(\frac{x-1}{x+1}\right)^{q} \frac{x-1+(x^2-1)^{-q}d_+}{x+1+(x^2-1)^{-q}d_-} \,, \tag{6.4.6}$$

where

$$d_\pm = \alpha^2(1\pm x)h_+h_- + i\alpha[y(h_+ + h_-) \pm (h_+ - h_-)] \,, \tag{6.4.7}$$

$$h_\pm = (x\pm y)^{2q} \,, \quad x = \frac{r}{m} - 1 \,, \quad y = \cos\theta \,. \tag{6.4.8}$$

The new parameter α is introduced by the Lie transformation. As expected, we obtain the $q-$metric in the limiting case $\alpha = 0$. The behavior of the Ernst potential shows that this new solution is asymptotically flat. The corresponding metric functions corroborate this result. Furthermore, the behavior of the new potential near the axis, $y = \pm 1$, shows that the spacetime is free of singularities outside a spatial region determined by the radius $x_s = \frac{m}{\sigma}$, which in the case of vanishing α, corresponds to the exterior singularity situated at $r_s = 2m$. The expression for the Kretschmann scalar shows that the outermost singularity is situated at $x_s = \frac{m}{\sigma}$. Inside this singular hypersurface, several singular structures can appear that depend on the value of q and σ.

The coordinate invariant multipole moments as defined by Geroch and Hansen [31, 32, 33] can be found by using a procedure proposed in [25] that allows us to perform the computations directly from the Ernst potential. In the limiting case $q = 0$, with $\alpha = \frac{\sigma - m}{a}$, the resulting multipoles are

$$M_{2k+1} = J_{2k} = 0\ , \quad k = 0, 1, 2, ... \tag{6.4.9}$$

$$M_0 = m\ , \quad M_2 = -ma^2\ , ... \tag{6.4.10}$$

$$J_1 = ma\ , \quad J_3 = -ma^3\ , ... \tag{6.4.11}$$

which are exactly the mass M_n and angular J_n multipole moments of the Kerr solution. In the general case of arbitrary q parameter, we obtain the following multipole moments

$$M_0 = m + \sigma q\ , \tag{6.4.12}$$

$$M_2 = \frac{7}{3}\sigma^3 q - \frac{1}{3}\sigma^3 q^3 + m\sigma^2 - m\sigma^2 q^2 - 3\, m^2 \sigma\, q - m^3\ , \tag{6.4.13}$$

$$J_1 = ma + 2a\sigma q\ , \tag{6.4.14}$$

$$J_3 = -\frac{1}{3}\, a(-8\,\sigma^3 q + 2\,\sigma^3 q^3 - 3\, m\sigma^2 + 9\, m\sigma^2 q^2 + 12\, m^2 \sigma\, q + 3\, m^3)\ . \tag{6.4.15}$$

The even gravitomagnetic and the odd gravitoelectric multipoles vanish identically because the solution is symmetric with respect to the equatorial plane $y = 0$. Moreover, higher odd gravitomagnetic and even gravitoelectric multipoles are all linearly dependent since they are completely determined by the parameters m, a, σ and q.

In this section, we have seen that there are several exact solutions with quadrupole moment that can be used to describe the exterior field of compact stars. This is in accordance with the black hole uniqueness theorems because the presence of the quadrupole invalidates the conditions under which the theorems have been proved. On the other hand, all the quadrupolar solutions must contain naked singularities, also as a consequence of the black hole uniqueness theorems. In the case of the stationary $q-$metric and the rotating Erez-Rosen spacetime, we have shown explicitly that the naked singularities are located inside or on the Schwarzschild radius which in compact stars is always located inside the surface of the star. We do not know if this is also true in the case of other quadrupolar metrics mentioned in this section. Suppose, for instance, that a particular quadrupolar metric has a singularity at a distance of say $15km$ from the center of a source with a mass of $2M_\odot$. Then, this metric cannot be used to represent the exterior field of an isolated neutron star whose radius is about $11.5km$, i.e., the singularity is located outside the surface of the neutron star where the spacetime should be vacuum. Nevertheless, such a solution can still be a candidate to describe the exterior field, for instance, of a white dwarf of mass $1.1M_\odot$ whose radius is of the order of thousand kilometers, so that the curvature singularity could be located inside the star.

The above discussion is related to the conditions that a general solution must satisfy in order to become physically meaningful. Indeed, if a curvature singularity is present, it should be possible to "cover" it by an interior solution that can be matched with an exterior solution across the surface of the star. In our opinion, the problem of solving the matching conditions in the presence of a physically meaningful interior solution is one of the most important challenges of modern relativistic astrophysics in general relativity. It is also an important conceptual problem since general relativity, as a theory of gravity, should be able to describe physical configurations like compact stars in which the gravitational field plays an important role. We will consider this issue in the next section.

6.5 Interior quadrupolar metrics

The problem of finding an interior solution for a stationary and axisymmetric spacetime is still open. Even in the case of vanishing quadrupole, the problem is still not completely solved. Indeed, in the case of a perfect fluid with constant energy density, an interior Schwarzschild solution can be obtained analytically, but its physical properties do not allow us to use it to describe the interior field of a spherically symmetric compact star because it violates causality, i.e., a sound wave propagates inside the star with superluminal velocity. Other spherically symmetric interior solutions are usually non-physical or cannot be matched with the exterior Schwarzschild metric [1]. In the case of quadrupolar metrics, the situation is quite similar. The only rigidly rotating perfect-fluid solution, containing the Kerr spacetime in the vacuum limit, is the Wahlquist metric [51, 52] which, however, is characterized by an unphysical equation of state $(\rho + 3p = const.)$. Moreover, in the slow rotation approximation, the zero pressure surface corresponds to a prolate ellipsoid rather than an oblate ellipsoid, as expected from a physical point of view. Other solutions with quadrupole represent anisotropic fluids [53, 54, 55] which, however, either they do not satisfy the energy conditions [53, 54] or either the boundary surface of zero pressure cannot be fixed because the hydrostatic pressure cannot be isolated from the other stresses [55].

All the interior solutions mentioned above have been obtained by analyzing carefully the corresponding field equations and, as we have seen, the results are not very satisfactory. In view of this situation, we believe that it is necessary to apply a different approach. We propose to develop solution generating techniques for interior spacetimes. Indeed, the discovery of Lie symmetries, Bäcklund transformations and the inverse scattering method in the Ernst equations represented a radical change in the search for exterior stationary and axisymmetric solutions. We believe that a similar approach could be useful also in the case of interior solutions.

To illustrate the problem of finding interior solutions, we first consider the case of spherically symmetric spacetimes. To this end, let us consider the following line element in spherical coordinates

$$ds^2 = e^{\phi(r)}dt^2 - \frac{dr^2}{1-\frac{2m(r)}{r}} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \ . \tag{6.5.1}$$

We choose a perfect fluid as the physical model for the interior gravitational field. Then, Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi[(\rho + p)u_\mu u_\nu - pg_{\mu\nu}] \tag{6.5.2}$$

reduce to

$$\frac{d\phi}{dr} = \frac{m + 4\pi r^3 p}{r(r-2m)} \ , \quad m = 4\pi\rho r^2 \ . \tag{6.5.3}$$

In addition, there is a second order differential equation which is equivalent to the energy-momentum conservation law $T^{\mu\nu}_{\ \ ;\mu} = 0$. In this case, it can be written as the Tolman-Oppenheimer-Volkoff equation

$$\frac{dp}{dr} = -\frac{(p+\rho)(m + 4\pi r^3 p)}{r(r-2m)} \ . \tag{6.5.4}$$

We see that we have only three equations for determining four unknowns (ϕ, m, p, and ρ). To close the system of differential equations, it is necessary to impose an additional condition which is usually taken as the equation of state $p = p(\rho)$. In particular, one can use the barotropic equation of state $p = w\rho$, where w is the constant barotropic factor. Many barotropic solutions are known in the literature [1] which, however, usually are either not related to realistic equations of state or show a singular behavior at the level of the pressure or energy density. To obtain more realistic solutions, we propose to start from a physically realistic energy density, for instance. Indeed, suppose that the energy density is given *a priori* by the polynomial equation [56]

$$\rho(r) = \rho_c - c_1 r - c_2 r^2 - c_3 r^3 \ , \tag{6.5.5}$$

where c_1, c_2, and c_3 are real constants and ρ_c is the energy density at the center of the body. Then, the mass function can be integrated explicitly and we obtain

$$m(r) = \frac{\pi}{15} r^3 (20\rho_c - 15c_1 r - 12c_2 r^2 - 10c_3 r^3) \ . \tag{6.5.6}$$

Clearly, the above particular Ansatz allows us to obtain a realistic behavior for the energy density, provided the constants are chosen appropriately. For instance, at the surface of the sphere $r = R$ we demand that the energy density vanishes, $\rho(r = R) = 0$, and so we obtain

$$\rho_c = c_1 R + c_2 R^2 + c_3 R^3 \ , \tag{6.5.7}$$

which establishes an algebraic relationship between the free constants. The mass function $m(r)$ is then determined by the free constants only. Moreover, we impose the physical condition that the total mass

$$M = \int_0^R m(r) dr \tag{6.5.8}$$

coincides with the mass of the exterior Schwarzschild metric which implies a boundary condition for the function $\phi(r)$, namely

$$e^{2\phi(r=R)} = 1 - \frac{2M}{R} \ . \tag{6.5.9}$$

The procedure consists now in solving the differential equations for $\phi(r)$ and $p(r)$ with the boundary conditions specified above. We did not success in finding analytic solutions and, therefore, we integrate the system of differential equations numerically. To this end, it is necessary to impose additional boundary conditions as follows. At the center and at the surface of the sphere, the pressure must satisfy the boundary conditions

$$p_R \equiv p(R) = 0 \ , \quad p(r = 0) \equiv p_c < \infty \ . \tag{6.5.10}$$

Moreover, we demand that the pressure is a well behaved function inside the sphere, i.e.,

$$0 < p(r) < \infty \quad \text{for} \quad 0 \le r \le R \ , \tag{6.5.11}$$

which means that the pressure function should be free of singularities inside the sphere.

The method consists now in integrating numerically the equations for the total mass M and for the pressure p, under the conditions mentioned above. The goal is to find values for the constants c_1, c_2 and c_3 such that M is positive and $p(r)$ is positive and free of singularities. In fact, it turns out that there are several intervals of values in which all conditions are satisfied. The particular

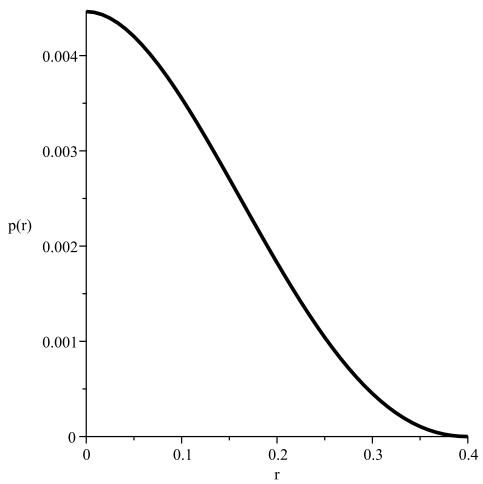

Figure 6.1: Behavior of the pressure inside a spherically symmetric body of radius $R = 0.4$. All the conditions for the pressure to be physically meaningful are satisfied.

simple choice

$$c_1 = \frac{1}{2}\,, \quad c_2 = \frac{1}{6}\,, \quad c_3 = \frac{1}{24} \tag{6.5.12}$$

with the particular radius value

$$R = 0.4 \tag{6.5.13}$$

leads to boundary values

$$\rho_c = 0.2293\,, \quad M = 0.002633\,, \quad \phi(R) = -0.0066\,. \tag{6.5.14}$$

Then, the integration of the differential equation for the pressure is straightforward. In Fig. 6.1, we illustrate the behavior of the pressure. The graphic shows that everywhere inside the sphere, the pressure has a very physical and realistic behavior. The corresponding function for the energy density shows also a physical behavior as demanded *a priori* with the polynomial Ansatz and the chosen values for the constants c_1, c_2 and c_3.

The differential equation for the function $\phi(r)$ can also be integrated and its behavior is represented in Fig. 6.2. It can be seen that this function is well behaved inside the sphere. Moreover, the value at the boundary $R = 0.4$, together with the value of the total mass, matches exactly the corresponding metric function for the exterior Schwarzschild solution.

An important condition that must be satisfied by any interior solution is the Buchdahl limit [1] which, in principle, can be associated with the Chan-

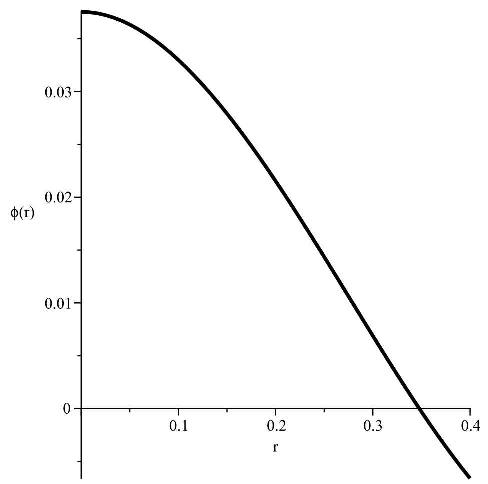

Figure 6.2: Numerical integration of the metric function $\phi(r)$ for a sphere of radius $R = 0.4$ and energy density constants $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{6}$, $c_3 = \frac{1}{24}$.

drasekhar limit about the maximum mass of compact stars. An analysis of the differential equations that determine the spherically symmetric case under consideration here shows that in order to avoid unstable configurations, which could lead to a collapse of the sphere, it is necessary that the condition

$$\frac{M}{R} < \frac{4}{9} \tag{6.5.15}$$

be satisfied. In fact, for a mass-to-radius ratio with $\frac{M}{R} \geq \frac{4}{9}$, the gravitational collapse is imminent and the staticity condition of the mass distribution is no longer valid. So, Buchdahl's limit is an essential requirement for a solution to be physically meaningful. From the boundary values obtained above, it is easy to see that this requirement is satisfied at the surface of the body. However, it could be that the behavior of the mass function inside the star violates Buchdahl's limit for a specific value of the radial coordinate, leading to an internal instability. To corroborate the stable behavior inside the body, we plot in Fig. 6.3 the behavior of the mass function for all values of the radial coordinate. We can see that inside the sphere the mass-to-radius ratio is everywhere less than the limiting value $\frac{4}{9}$, indicating that no instabilities can occur. This result reinforces the physical interpretation of the numerical solution presented here.

The simple example for a static perfect-fluid sphere as a source of a compact star shows that it is possible to find physically meaningful solutions of the interior field equations. But it also shows that it is very difficult to integrate analytically the resulting differential equations. We started from a par-

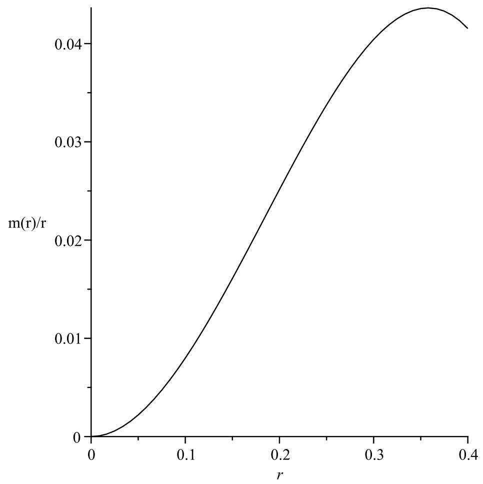

Figure 6.3: Buchdahl's limit inside the compact sphere of radius $R = 0.4$. The condition $m(r)/r < 4/9 = 0.44$ is satisfied everywhere inside the sphere.

ticular polynomial Ansatz for the energy density which guarantees a meaningful physical behavior. This allows us to integrate the mass function, but the pressure and the remaining metric function $\phi(r)$ cannot be integrated analytically. A numerical analysis seems to be always necessary. For this reason we believe that the standard method of solving directly the field equations should be complemented by a solution generating technique, similar to the methods used for obtaining exterior solutions.

We now turn back to the study of quadrupolar interior metrics. As mentioned above, all the known solutions are either unphysical or they cannot be matched with the exterior Kerr metric. To attack this problem, we propose to consider the mass quadrupole as an additional degree of freedom and to analyze the symmetry properties of the field equations in the presence of matter. To begin with, we have considered first the case of static quadrupole metrics with a perfect fluid as the source of gravity. If we consider the spherically symmetric line element analyzed above, and try to generalize it to include the case of axisymmetric fields, it turns out to be convenient to use the following line element [57]

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}\left[e^{2\gamma}\left(\frac{dr^2}{h} + d\theta^2\right) + \mu^2 d\varphi^2\right] , \tag{6.5.16}$$

where $\psi = \psi(r,\theta)$, $\gamma = \gamma(r,\theta)$, $\mu = \mu(r,\theta)$, and $h = h(r)$. A detailed analysis of the Einstein equations with an energy-momentum tensor represented by a perfect fluid shows that the resulting set of differential equations can be split into two systems in a manner which resembles the splitting in the case of vacuum spacetimes. Indeed, the main field equations can be written as

$$\frac{\mu_{,rr}}{\mu} + \frac{\mu_{,\theta\theta}}{h\mu} + \frac{h_{,r}\mu_{,r}}{2h\mu} = \frac{16\pi}{h} p e^{2\gamma - 2\psi} , \tag{6.5.17}$$

$$\psi_{,rr} + \frac{\psi_{,\theta\theta}}{h} + \left(\frac{h_{,r}}{2h} + \frac{\mu_{,r}}{\mu}\right)\psi_{,r} + \frac{\mu_{,\theta}\psi_{,\theta}}{h\mu} = \frac{4\pi}{h}(3p+\rho)e^{2\gamma-2\psi} . \tag{6.5.18}$$

Moreover, the metric function γ is determined by two first order differential equations

$$\gamma_{,r} = \frac{1}{h\mu_{,r}^2 + \mu_{,\theta}^2}\left\{\mu\left[\mu_{,r}\left(h\psi_{,r}^2 - \psi_{,\theta}^2\right) + 2\mu_{,\theta}\psi_{,\theta}\psi_{,r} + 8\pi\mu_{,r}\bar{p}\right] + \mu_{,\theta}\mu_{,r\theta} - \mu_{,r}\mu_{,\theta\theta}\right\} , \tag{6.5.19}$$

$$\gamma_{,\theta} = \frac{1}{h\mu_{,r}^2 + \mu_{,\theta}^2}\left\{\mu\left[\mu_{,\theta}\left(\psi_{,\theta}^2 - h\psi_{,r}^2\right) + 2h\mu_{,r}\psi_{,\theta}\psi_{,r} - 8\pi\mu_{,\theta}\bar{p}\right] + h\mu_{,r}\mu_{r\theta} + \mu_{,\theta}\mu_{,\theta\theta}\right\} , \tag{6.5.20}$$

where

$$\bar{p} = p e^{2\gamma - 2\psi} . \tag{6.5.21}$$

The equations for γ can be integrated by quadratures once the main field equations (6.5.17) and (6.5.18) are solved, and the pressure $\bar{p}$ is given *a priori* as an independent function. Notice that if we introduce the differential equations (6.5.17)-(6.5.20) into the original Einstein equations, a second order differential equation for γ is obtained

$$\gamma_{,rr} + \frac{\gamma_{,\theta\theta}}{h} + \psi_{,r}^2 + \frac{\psi_{,\theta}^2}{h} + \frac{h_{,r}\gamma_{,r}}{2h} = \frac{8\pi}{h}\bar{p} , \tag{6.5.22}$$

which must also be satisfied. However, a straightforward computation shows that this equation is identically satisfied if the two first-order differential equations (6.5.19) and (6.5.20) for γ and the conservation equation for the parameters of the perfect fluid

$$p_{,r} = -(\rho + p)\psi_{,r} , \quad p_{,\theta} = -(\rho + p)\psi_{,\theta} , \tag{6.5.23}$$

are satisfied. The conservation equations resemble the Tolman-Oppenheimer-Volkov relation for the spherically symmetric case.

We see that the particular choice of the above line element leads to a splitting of the field field equations into two separated sets of equations, and to a generalization of the Tolman-Oppenheimer-Volkov equation for the case of two spatial coordinates. This is an important advantage when trying to perform the integration of the main field equations. Indeed, in this manner we found a series of relatively simple approximate solutions with non-trivial quadrupole moment. The presentation and physical investigation of those solutions requires several detailed analysis. A byproduct of such analysis was the discovery of certain symmetries of the field equations for a perfect fluid which can be used to generate new solutions from known ones by using the procedure described below.

Suppose that an exact interior solution of Einstein's equations (6.5.17)-(6.5.20) for the static axisymmetric line element (6.5.16) is given explicitly by means of the functions

$$h_0 = h_0(r),\ \mu_0 = \mu_0(r,\theta),\ \psi_0 = \psi_0(r,\theta),\ \gamma_0 = \gamma_0(r,\theta), \tag{6.5.24}$$

$$\bar{p}_0 = \bar{p}_0(r,\theta),\ \bar{\rho}_0 = \bar{\rho}_0(r,\theta)\ , \tag{6.5.25}$$

where we have introduced the notation

$$\bar{p}_0 = p_0 e^{2\gamma_0 - 2\psi_0}\ , \quad \bar{\rho}_0 = \rho_0 e^{2\gamma_0 - 2\psi_0}\ , \tag{6.5.26}$$

and p_0 and ρ_0 are also known functions. Then, for any arbitrary real values of the constant parameter δ, a class of new solutions of the field equations (6.5.17)-(6.5.20) can be obtained explicitly from the functions

$$h = h_0(r),\ \mu = \mu_0(r,\tilde{\theta}),\ \psi = \delta\psi_0(r,\tilde{\theta}), \tag{6.5.27}$$

$$\bar{p} = \delta\bar{p}_0(r,\tilde{\theta}),\ \bar{\rho} = \delta\bar{\rho}_0(r,\tilde{\theta}),\ \tilde{\theta} = \frac{\theta}{\sqrt{\delta}}\ , \tag{6.5.28}$$

$$\gamma(r,\tilde{\theta}) = \delta^2\gamma_0(r,\tilde{\theta}) + (\delta^2 - 1)\int \frac{\nu_{\tilde{\theta}}}{h_0 + \nu^2} dr + 8\pi\delta(1-\delta)\int \frac{\frac{\mu_0}{\mu_{0,r}}\bar{p}_0}{h_0 + \nu^2} dr + \kappa\ , \tag{6.5.29}$$

where κ is an arbitrary real constant and

$$\nu = \frac{\mu_{0,\tilde{\theta}}}{\mu_{0,r}}\ . \tag{6.5.30}$$

To illustrate the application of this solution generating method, let us consider the spherically symmetric Schwarzschild solution which describes the interior field of a perfect-fluid sphere of radius R and total mass m. The corresponding line element can be written as

$$ds^2 = \left[\frac{3}{2}f(R) - \frac{1}{2}f(r)\right]^2 dt^2 - \frac{dr^2}{f^2(r)} - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\ , \tag{6.5.31}$$

with

$$f(r)=\sqrt{1-\frac{2mr^2}{R^3}}\ . \tag{6.5.32}$$

The physical parameters of the perfect fluid are the constant density ρ_0 and the pressure p_0, which is a function of the radial coordinate r only

$$p_0=\rho_0\frac{f(r)-f(R)}{3f(R)-f(r)}\ . \tag{6.5.33}$$

We now consider the interior Schwarzschild metric as the seed solution (6.5.24) for the general transformation (6.5.27). A straightforward comparison with the general line element (6.5.16) yields

$$e^{\psi_0}=\frac{3}{2}f(R)-\frac{1}{2}f(r)\ ,\ h_0=r^2f^2(r)\ ,\ \mu_0=r\sin\theta\, e^{\psi_0}\ ,\ e^{\gamma_0}=re^{\psi_0}\ . \tag{6.5.34}$$

According to the procedure described above, the new solution can be obtained from Eq.(6.5.34) by multiplying the corresponding metric functions with the new parameter δ. Then, the new line element can be represented as

$$ds^2=e^{2\delta\psi_0}dt^2-e^{-2\delta\psi_0}\left[e^{2\gamma}\left(\frac{dr^2}{r^2f^2(r)}+d\tilde{\theta}^2\right)+r^2e^{2\psi_0}\sin^2\tilde{\theta}d\varphi^2\right]\ , \tag{6.5.35}$$

where the new function γ is given by

$$\gamma=\delta^2\gamma_0+\int\frac{(1-\delta^2)+8\pi\delta(1-\delta)\sin^2\tilde{\theta}r^2p_0}{rf^2(r)(1+r\psi_{0\,r})\sin^2\tilde{\theta}+\frac{r}{1+r\psi_{0\,r}}\cos^2\tilde{\theta}}dr+\kappa\ , \tag{6.5.36}$$

with

$$\psi_{0\,r}=\frac{2mr}{R^3f(r)[3f(R)-f(r)]}\ . \tag{6.5.37}$$

Moreover, the physical parameters of the perfect-fluid source are

$$\rho=\delta\rho_0e^{2\gamma_0-2\gamma+2(\delta-1)\psi_0}\ \ ,\ \ p=\delta p_0e^{2\gamma_0-2\gamma+2(\delta-1)\psi_0}\ , \tag{6.5.38}$$

from which the equation of state

$$p=\frac{p_0}{\rho_0}\rho \tag{6.5.39}$$

can be obtained. This is clearly not a barotropic equation of state since the seed pressure p_0 depends explicitly on the radial coordinate r. Nevertheless, it can be interpreted as a generalized barotropic equation of state $p=w(r)\rho$.

Interestingly, the physical parameters of the perfect fluid are axisymmetric, but the equation of state preserves spherical symmetry in the sense that the generalized barotropic factor depends on the radial coordinate only.

Notice that the new function γ depends explicitly on the new coordinate $\tilde{\theta}$, in contrast to the seed metric function γ_0 which depends on the radial coordinate r only. This proves that the new solution is not spherically symmetric, but axisymmetric. Notice also that the density and pressure of the new solution are functions of the angular coordinate too, as expected for an axisymmetric mass distribution. It is expected that the obtained deviations from spherical symmetry are related to the quadrupole moment of the perfect fluid; however, a more detailed investigation is necessary to define an interior quadrupole which should be related to the exterior quadrupole.

6.6 Remarks

In this chapter, we presented a review of the problem of describing the interior and exterior gravitational field of compact objects in general relativity, which include black holes and compact stars (white dwarfs and neutron stars). To take into account rotation and deformation of the mass distribution, we consider stationary and axisymmetric solutions of Einstein's equations with quadrupole moment. We formulate the physical conditions which, in our opinion, should be satisfied by a Riemannian manifold in order to represent the interior and exterior gravitational field of compact objects.

We review the main static solutions in which the quadrupole is represented by a free parameter. We argue that the $q-$metric represents the simplest generalization of the Schwarzschild solution with a quadrupole parameter. We then present a particular generalization of the Erez-Rosen metric which includes a rotational parameter, and reduces to the Kerr metric in absence of the quadrupole parameter. In addition, we present the Ernst potential of a stationary $q-$metric which turns out to be represented by a quite simple expression.

We notice that in this review, we limited ourselves to the study of the mass quadrupole as additional parameter only. In general and in more realistic situations, it is necessary to consider also the electromagnetic field. Fortunately, the solution generating techniques have been developed also for Einstein-Maxwell equations as well and, therefore, the generalization of the vacuum solutions presented in this review to include electromagnetic multipoles is straightforward.

We argue that the interior counterpart of the exterior Kerr metric cannot be found in general relativity because it is directly related to a curvature singularity at which the classical theory breaks down. Probably, a quantum description of gravity is necessary in order to understand the interior field of a black hole. In the case of compact stars, however, we argue that general relativity should allow the existence of spacetimes which describe both the interior and exterior gravitational field. In view of the precarious situation regarding physically meaningful interior solutions, we propose to study the symmetries of the field equations in order to develop solution generating techniques. We present a particularly simple method which allows us to generate new static and axisymmetric perfect-fluid solutions from known solutions.

Summarizing, we propose to apply a different strategy to search for interior physically meaningful solutions of Einstein's equations. Firstly, we propose to include the quadrupole as an additional degree of freedom and, secondly, we propose to investigate the symmetry properties of the field equations in the presence of matter.

7 Rotating gravitation fields in the Newtonian limit

7.1 Introduction

In physics, rotation may introduce many changes in the structure of any system. In the case of celestial objects such as stars and planets, rotation plays a crucial role. Rotation does not only change the shape of the celestial objects but also influences the processes occurring inside stars, i.e., it may accelerate or decelerate thermonuclear reactions under certain conditions, it changes the gravitational field outside the objects and it is one of the main factors that determines the lifespan of all stars (giant stars, main sequence, white dwarfs, neutron stars, etc.) [58, 59, 60, 61, 62, 63, 64].

For instance, let us consider a white dwarf. A non-rotating white dwarf has a limiting mass of $1.44M_{\odot}$ which is well-known as the Chandrasekhar limit [65]. The central density and pressure corresponding to this limit define the evolution of white dwarfs. If the white dwarf rotates, then due to the centrifugal forces the central density and pressure decrease [66, 67]. In order to recover the initial values of the central density and pressure of a rotating star one needs to add extra mass. Here we see that a rotating star with the same values for central density and pressure, as those of a non-rotating star, possesses a larger mass [68].

In this work, we derive the equations describing the equilibrium configurations of slowly rotating stars within Hartle's formalism [69]. The advantage of this approach is that it allows us to consider in a simple way the influence of the rotation on the internal properties of the gravitational source. In fact, we will see that the complexity of the differential equations, which govern the dynamical properties of equilibrium configurations, is reduced to a high degree. When solving this kind of problems in celestial mechanics, astronomy and astrophysics, it is convenient to consider the internal structure of stars and planets as being described by a fluid. In the case of slow rotation, we derive equations that are valid for any fluid up to the second order in the angular velocity.

As a result we obtain the equations defining the main parameters of the rotating equilibrium configuration such as the mass, radius, moment of inertia, gravitational potential, angular momentum and quadrupole moment as functions of the central density and angular velocity (rotation period). Furthermore, we show how to calculate the ellipticity and by means of it the gravitational Love number [70]. In turn, these parameters are of great importance in defining the evolution of a star.

In order to pursue all these issues in detail we revisit the Hartle formalism in classical physics for a slowly rotating configuration as the calculation of its equilibrium properties is much more simpler, because then the rotation can be considered as a small perturbation of an already-known non-rotating configuration. We therefore will consider in this paper a rotating configuration under the following conditions [69]:

- A one-parameter equation of state is specified, $p = p(\rho)$, where p is the pressure and ρ is the density of matter [71].
- A static equilibrium configuration is calculated using this equation of state and the classical equation of hydrostatic equilibrium for spherical symmetry.
- Axial and reflection symmetry. The configuration is symmetric about a plane perpendicular to the axis of rotation.
- A uniform angular velocity sufficiently slow so that the changes in pressure, energy density, and gravitational field are small.
- Slow rotation. This requirement implies that the angular velocity Ω of the star
$$\Omega^2 \ll \frac{GM}{a^3}, \tag{7.1.1}$$
where M is the mass of the unperturbed configuration, a is its radius, G is the gravitational constant. Consequently, the condition in Eq. (7.1.1) also implies
$$\Omega \ll \frac{c}{a}, \tag{7.1.2}$$
where c is the speed of light.
- The Newtonian field equations are expanded in powers of the angular velocity and the perturbations are calculated by retaining only the first- and second-order terms.

In this chapter, the equations necessary to investigate this issue are obtained explicitly. The problem of describing rotating configurations in Newtonian gravity has been investigated in many articles and textbooks [71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81]. Here, we present a different approach. Indeed, we will use the Hartle formalism, which was originally proposed in general relativity, to consider a non-relativistic Newtonian configuration. The very intuitive coordinate approach of Hartle's formalism is utilized in order to handle the Newtonian gravity equation and the corresponding equilibrium condition. It is worth mentioning that the Hartle formalism has been recently extended for relativistic configurations up to the fourth order terms in angular velocity [82].

This chapter is organized as follows. In Sec. 7.2, we present the conditions under which the rotating compact object becomes a configuration in hydrostatic equilibrium. Moreover, we show that the use of a particular coordinate, which is especially adapted to describing the deformation due to the rotation, together with an expansion in spherical harmonics reduces substantially the system of differential equations up to the level that they can be integrated by quadratures. In Sec. 7.3, we derive expressions for the main physical quantities of the rotating object. A summary of the method to be used to find explicit numerical solutions by using our formalism is presented in Sec. 7.4. In Sec. 7.5 we apply the formalism to rotating white dwarfs in Newtonian physics and in Sec. 7.6 we show the procedure of calculating the Keplerian angular velocity and the scaling law for the physical quantities describing rotating configurations. In Sec. 7.7 we compare and contrast the results of this work with similar works in the literature.

7.2 Slowly rotating stars in Newtonian gravity

In Newtonian gravitational theory the equilibrium configuration of uniformly rotating stars are determined by the solution of the three equations of Newtonian hydrostatic equilibrium [66, 67, 69]. These are (1) the Newtonian field equation:

$$\nabla^2\Phi(r,\theta) = 4\pi G\rho(r,\theta); \tag{7.2.1}$$

where Φ is gravitational potential and ρ is the density of a fluid mass rotating with a uniform angular velocity Ω;

(2) the equation of state that shows the relationship between pressure p and density ρ is assumed to have a one-parameter form

$$p = p(\rho); \tag{7.2.2}$$

(3) the equation of hydrostatic equilibrium for uniformly rotating configurations which can be written as

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi\,, \tag{7.2.3}$$

with

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\Omega} \times \vec{r} \,. \tag{7.2.4}$$

For uniform rotation (Ω=constant) we have that

$$\frac{d\vec{v}}{dt} = \vec{\Omega} \times \vec{v} = \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\frac{1}{2}\vec{\nabla}(\vec{\Omega} \times \vec{r})^2 \,. \tag{7.2.5}$$

Therefore, substituting this expression in (7.2.3), we obtain

$$-\frac{1}{2}\vec{\nabla}(\vec{\Omega} \times \vec{r})^2 = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi \tag{7.2.6}$$

or

$$\frac{dp}{\rho} - \frac{1}{2}d(\vec{\Omega} \times \vec{r})^2 + d\Phi = 0 \,, \tag{7.2.7}$$

which can be reexpressed in terms of its first integral

$$\int_0^p \frac{dp(r,\theta)}{\rho(r,\theta)} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta + \Phi(r,\theta) = \text{const}, \tag{7.2.8}$$

where r is the radial coordinate and θ is the polar coordinate of the rotating configuration.

The main task now is to expand the equations of Newtonian hydrostatics in powers of Ω^2. The solution to the first term of the expansion is given by $\Phi^{(0)}$, $p^{(0)}$, and $\rho^{(0)}$ in the absence of rotation. Then, it is necessary to find the equations which govern the second-order terms. It is expected that the resulting differential equations can be integrated in terms of the known non-rotating solution.

7.2.1 Coordinates

An important point to be considered is the choice of the coordinate system in which the expansions in powers of Ω are carried out. As pointed out by Hartle in 1967 [69], one should be very careful when considering perturbation near the surface of the star. In fact, a simple expansion of the density as a function of the old polar coordinates r, θ is not valid throughout the star as the surface of the configuration will be displaced from its non-rotating position and the perturbation in the density may be finite where the unperturbed density vanishes. Therefore, following Hartle's approach [69] we select a coordinate transformation such that the density of the star in terms of the new

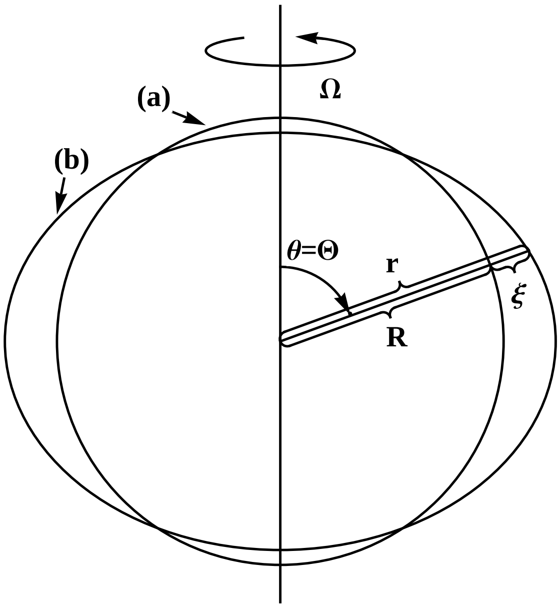

Figure 7.1: Definition of the coordinates R, Θ, and the displacement $\xi(R,\Theta)$. The surface (a) is the surface of constant density $\rho(R)$ in the non-rotating configuration. The surface (b) is the surface of constant density $\rho(R)$ in the rotating configuration (reproduced from [69]).

radial coordinate is the same as in the static configuration:

$$\rho[r(R,\Theta),\Theta] = \rho(R) = \rho^{(0)}(R). \tag{7.2.9}$$

Thus, the relationships between the old coordinates (r,θ) and the new coordinates (R,Θ) are given by

$$\theta \;=\; \Theta, \qquad r = R + \xi(R,\Theta) + O(\Omega^4)\,. \tag{7.2.10}$$

The function $r(R,\Theta)$ then replaces the density as a function to be calculated in the rotating configuration. These definitions are given pictorially in Fig. 7.1.

Following Hartle's formalism, we are always free to consider the rotating configuration as a perturbation of a non-rotating configuration with the same central density. Consequently, in the R,Θ coordinate system, the density (7.2.9) and pressure are known functions of R

$$p[r(R,\Theta),\Theta] = p(R) = p^{(0)}(R) \tag{7.2.11}$$

related by the one-parametric equation of state.

7.2.2 Spherical harmonics

The expansion of r in terms of Ω^2 is given by equation (7.2.10) and the expansion of the gravitational potential Φ can be represented as

$$\Phi(R,\Theta) \approx \Phi^{(0)}(R) + \Phi^{(2)}(R,\Theta) + O(\Omega^4) \tag{7.2.12}$$

where $\Phi^{(0)}(R)$ is the spherical part of the potential and $\Phi^{(2)}(R,\Theta)$ is the perturbed part. Calculating the Taylor expansion in terms of the new the coordinates, we obtain

$$\begin{aligned} \Phi(r,\theta) &= \Phi(R+\xi,\Theta) \approx \Phi(R,\Theta) + \xi\frac{d\Phi(R,\Theta)}{dR} + O(\Omega^4) \\ &\approx \Phi^{(0)}(R) + \xi\frac{d\Phi^{(0)}(R)}{dR} + \Phi^{(2)}(R,\Theta) + O(\Omega^4). \end{aligned} \tag{7.2.13}$$

In order to simplify the equations we expand the functions ξ and $\Phi^{(2)}$ in spherical harmonics

$$\xi(R,\Theta) = \sum_{l=0}^{\infty}\xi_l(R)P_l(\cos\Theta), \quad \Phi^{(2)}(R,\Theta) = \sum_{l=0}^{\infty}\Phi_l^{(2)}(R)P_l(\cos\Theta), \tag{7.2.14}$$

where $P_l(\cos\Theta)$ are the Legendre polynomials.

Now let us perform the computations in detail taking the polar axis to be the axis of rotation. Using the expressions for the Legendre polynomials $P_0(\cos\Theta) = 1$ and $P_2(\cos\Theta) = \frac{1}{2}(3\cos^2\Theta - 1)$, it is easy to show that

$$\sin^2\Theta = \frac{2}{3}[P_0(\cos\Theta) - P_2(\cos\Theta)], \tag{7.2.15}$$

From here we see that l accepts only two values, namely, 0 and 2. The equations for $\xi_l(R)$, $\Phi_l^{(2)}(R)$, with $l \geq 4$ are thus independent of Ω and their solution is

$$\xi_l = 0, \qquad \Phi_l^{(2)} = 0, \qquad l \geq 4. \tag{7.2.16}$$

Rewriting the condition of hydrostatic equilibrium (7.2.8) in coordinates (R,Θ) and expanding it in spherical harmonics by using Eqs. (7.2.9), (7.2.12), and (7.2.14), we get

$$\int_0^p \frac{dp^{(0)}(R)}{\rho(R)} - \frac{1}{3}\Omega^2 R^2 [P_0(\cos\Theta) - P_2(\cos\Theta)] + \Phi^{(0)}(R) \qquad (7.2.17)$$
$$+ \sum_{l=0}^{n} \Phi_l^{(2)}(R) P_l(\cos\Theta) + \sum_{l=0}^{n} \xi_l(R) P_l(\cos\Theta) \frac{d\Phi^{(0)}(R)}{dR} = \text{const}\,.$$

We now collect the terms proportional to $\sim\ \Omega^0$ and Ω^2 with $l = 0, 2$ and obtain:

$$\int_0^p \frac{dp^{(0)}(R)}{\rho(R)} + \Phi^{(0)}(R) = \text{const}^{(0)}, \qquad (7.2.18)$$

$$-\frac{1}{3}\Omega^2 R^2 + \Phi_0^{(2)}(R) + \xi_0(R) \frac{d\Phi^{(0)}(R)}{dR} = \text{const}_0^{(2)}, \qquad (7.2.19)$$
$$\frac{1}{3}\Omega^2 R^2 + \Phi_2^{(2)}(R) + \xi_2(R) \frac{d\Phi^{(0)}(R)}{dR} = 0, \qquad (7.2.20)$$

where const is defined as const $= \text{const}^{(0)} + \text{const}_0^{(2)} P_0(\cos\Theta)$, so that $\text{const}^{(0)}$ and $\text{const}_0^{(2)}$ are found from the matching between the interior and exterior solutions. The first of the above equations corresponds to the Newtonian hydrostatic equation for a static configuration.

Using the same procedure, the Newtonian field equation becomes

$$\nabla^2 \Phi(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi(r,\theta)}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi(r,\theta)}{\partial \theta}\right) \qquad (7.2.21)$$
$$= \nabla_r^2 \Phi(r,\theta) + \frac{1}{r^2}\nabla_\theta^2 \Phi(r,\theta) \approx \nabla_r^2 \Phi^{(0)}(r) + \nabla_r^2 \Phi_0^{(2)}(r)$$
$$+ \nabla_r^2 \Phi_2^{(2)}(r) P_2(\cos\theta) + \frac{1}{r^2}\nabla_\theta^2 \Phi_2^{(2)}(r) P_2(\cos\theta) = 4\pi G \rho(r,\theta).$$

Since the functions $\Phi_0^{(2)}$ and $\Phi_2^{(2)}$ are already proportional to Ω^2, we can directly write them in (R,Θ) coordinates. However $\nabla_r^2 \Phi^{(0)}(r) \approx \nabla_R^2 \Phi^{(0)}(R) + \xi(R,\Theta)\frac{d}{dR}\nabla_R^2 \Phi^{(0)}(R)$. Thus

$$\nabla^2 \Phi(r,\theta) = \nabla_R^2 \Phi^{(0)}(R) + \xi(R,\Theta)\frac{d}{dR}\nabla_R^2 \Phi^{(0)}(R) \qquad (7.2.22)$$
$$+ \nabla_R^2 \Phi_0^{(2)}(R) + \nabla_R^2 \Phi_2^{(2)}(R) P_2(\cos\Theta) + \frac{1}{R^2}\nabla_\Theta^2 \Phi_2^{(2)}(R) P_2(\cos\Theta) = 4\pi G \rho(R).$$

Taking into account that $\xi(R,\Theta) = \xi_0(R) + \xi_2(R) P_2(\cos\Theta)$ and collecting the corresponding terms, we obtain the Newtonian field equations of both static and rotating configurations:

$$\nabla_R^2 \Phi^{(0)}(R) = 4\pi G\rho(R), \tag{7.2.23}$$

$$\xi_0(R)\frac{d}{dR}\nabla_R^2 \Phi^{(0)}(R) + \nabla_R^2 \Phi_0^{(2)}(R) = 0, \tag{7.2.24}$$

$$\xi_2(R)\frac{d}{dR}\nabla_R^2 \Phi^{(0)}(R) + \nabla_R^2 \Phi_2^{(2)}(R) - \frac{6}{R^2}\Phi_2^{(2)}(R) = 0. \tag{7.2.25}$$

The differential equations for $\Phi_0^{(2)}(R)$, $\Phi_2^{(2)}(R)$, $\xi_0(R)$, and $\xi_2(R)$, which establish the relation between mass and central density for rotating star and determine the shape of the star, will now be given in forms suitable for solving these problems.

7.3 Physical properties of the model

The above description of the rotating equilibrium configuration allows us to derive all the main quantities that are necessary for establishing the physical significance and determining the physical properties of the rotating source. In this section, we will derive all the equations that must be solved in order to find the values of all the relevant quantities.

7.3.1 Mass and Central Density

The total mass of the rotating configuration is given by the integral of the density over the volume,

$$M_{tot} = \int_V \rho(r,\theta) dV = \int_V \rho(r,\theta) r^2 dr \sin\theta d\theta d\phi \, .$$

To proceed with the computation of the integral, we use formula (7.2.10) and obtain the relationship

$$\begin{aligned} r^2 dr &= (R+\xi)^2(dR + d\xi) \approx R^2\left(1+\frac{2\xi}{R}\right)\left(1+\frac{d\xi}{dR}\right)dR \\ &\approx \left(1+\frac{2\xi}{R}+\frac{d\xi}{dR}\right)R^2 dR \, , \end{aligned} \tag{7.3.1}$$

which implies that

$$\begin{aligned} M_{tot} &= \int_V \rho(R) R^2 dR \sin\Theta d\Theta d\phi \\ &+ \int_V \rho(R) R^2 \left(\frac{2\xi(R,\Theta)}{R} + \frac{d\xi(R,\Theta)}{dR} \right) dR \sin\Theta d\Theta d\phi \,. \end{aligned} \quad (7.3.2)$$

Performing the integration within the range of angles $0 < \Theta < \pi$ and $0 < \phi < 2\pi$ and using the identities

$$\int_0^\pi \sin\Theta d\Theta = 2, \qquad \int_0^\pi P_2(\cos\Theta) \sin\Theta d\Theta = 0\,, \quad (7.3.3)$$

one finds that the change in mass $M^{(2)}$ of the rotating configuration from the non-rotating one can be written as

$$M_{tot}(R) = M^{(0)}(R) + M^{(2)}(R), \quad (7.3.4)$$

$$M^{(0)}(R) = 4\pi \int_0^R \rho(R) R^2 dR, \quad (7.3.5)$$

$$\begin{aligned} M^{(2)}(R) &= 4\pi \int_0^R \rho(R) R^2 \left(\frac{2\xi_0(R)}{R} + \frac{d\xi_0(R)}{dR} \right) dR \\ &= 4\pi \xi_0(R) \rho(R) R^2 \Big|_0^R + 4\pi \int_0^R \left(-\xi_0(R) \frac{d\rho(R)}{dR} \right) R^2 dR \,. \end{aligned} \quad (7.3.6)$$

The last integral is obtained via integration by parts. On the surface of the star $R = a$ the density vanishes and the first term in the last integral $\xi_0(R)\rho(R)R^2\Big|_0^a$ vanishes too. So

$$M^{(2)}(a) = 4\pi \int_0^a \left(-\xi_0(R) \frac{d\rho(R)}{dR} \right) R^2 dR \,. \quad (7.3.7)$$

Here we have used the following expressions that follow from the field equations and definitions of the masses

$$\nabla^2 \Phi^{(0)}(R) = 4\pi G \rho(R), \quad (7.3.8)$$

$$\frac{d}{dR} \nabla^2 \Phi^{(0)}(R) = 4\pi G \frac{d\rho(R)}{dR}, \quad (7.3.9)$$

$$\frac{dM^{(0)}(R)}{dR} = 4\pi R^2 \rho(R), \quad (7.3.10)$$

$$\frac{dM^{(2)}(R)}{dR} = 4\pi \left(-\xi_0(R) \frac{d\rho(R)}{dR} \right) R^2 \,. \quad (7.3.11)$$

Using the condition that $\Phi^{(0)}(R) \to \text{const}^{(0)}$ and $\Phi_0^{(2)}(R) \to \text{const}_0^{(2)}$, as $R \to 0$, and taking into account (7.2.24) the masses of both configurations can be expressed as

$$\frac{GM^{(0)}(R)}{R^2} = \frac{d\Phi^{(0)}(R)}{dR}, \tag{7.3.12}$$

$$\frac{GM^{(2)}(R)}{R^2} = \frac{d\Phi_0^{(2)}(R)}{dR}. \tag{7.3.13}$$

It is convenient to display the $l = 0$ equation in a form in which it resembles the equation of hydrostatic equilibrium. To do this, we define

$$p_0^*(R) = \xi_0(R)\frac{d\Phi^{(0)}(R)}{dR}\,. \tag{7.3.14}$$

Moreover, taking derivative of (7.2.19) and taking into account (7.3.13), we obtain

$$-\frac{dp_0^*(R)}{dR} + \frac{2}{3}\Omega^2 R = \frac{GM^{(2)}(R)}{R^2}\,. \tag{7.3.15}$$

The above equation along with

$$\frac{dM^{(2)}(R)}{dR} = 4\pi R^2\rho(R)\frac{d\rho(R)}{dp(R)}p_0^*(R), \tag{7.3.16}$$

show the balance between the pressure, centrifugal, and gravitational forces per unit mass in the rotating star. The latter expression was obtained by using (7.2.18).

7.3.2 The Shape of the Star and Numerical Integration

If the surface of the non-rotating star has radius a, then equations (7.2.9) and (7.2.14) show that the equation for the surface of the rotating star has the form

$$r(a,\Theta) = a + \xi_0(a) + \xi_2(a)P_2(\cos\Theta). \tag{7.3.17}$$

The value of $\xi_0(a)$ is already determined in the $l = 0$ calculation

$$\xi_0(a) = \frac{a^2}{GM}p_0^*(a), \tag{7.3.18}$$

where $M = M^{(0)}(a)$ is the mass of the non-rotating configuration. However, the determination of $\xi_2(R)$ from $l = 2$ equations is not straightforward. So far, we have the $l = 2$ equations (7.2.20) and (7.2.25) representing the hydrostatic equilibrium and the field equation, respectively. From (7.2.20) we obtain the expression

$$\xi_2(R) = -\frac{R^2}{GM(R)}\left\{\frac{1}{3}\Omega^2 R^2 + \Phi_2^{(2)}(R)\right\}, \tag{7.3.19}$$

which we insert into (7.2.25) and get

$$\nabla_R^2 \Phi_2^{(2)}(R) - \frac{6}{R^2}\Phi_2^{(2)}(R) = \frac{4\pi R^2}{M(R)}\left\{\frac{1}{3}\Omega^2 R^2 + \Phi_2^{(2)}(R)\right\}\frac{d\rho(R)}{dR}, \tag{7.3.20}$$

where $M(R) = M^{(0)}(R)$ denotes the non-rotating mass. In order to solve the latter equation numerically, one needs to rewrite it as first-order linear differential equations. To this end, we introduce new functions $\varphi = \Phi_2^{(2)}$ and χ so that Eq. (7.3.20) generates the system

$$\frac{d\chi(R)}{dR} = -\frac{2GM(R)}{R^2}\varphi(R) + \frac{8\pi}{3}\Omega^2 R^3 G\rho(R), \tag{7.3.21}$$

$$\frac{d\varphi(R)}{dR} = \left(\frac{4\pi R^2\rho(R)}{M(R)} - \frac{2}{R}\right)\varphi(R) - \frac{2\chi(R)}{GM(R)} + \frac{4\pi}{3M(R)}\rho(R)\Omega^2 R^4. \tag{7.3.22}$$

The above equations can be solved by quadratures. The computation of the solution can be performed numerically by integrating outward from the origin. At the origin the solution must be regular. An examination of the equations shows that, as $R \to 0$,

$$\varphi(R) \to AR^2, \qquad \chi(R) \to BR^4, \tag{7.3.23}$$

where A and B are any constants related by

$$B + \frac{2\pi}{3}G\rho_c A = \frac{2\pi}{3}G\rho_c\Omega^2 \tag{7.3.24}$$

and ρ_c is the value of the density in the center of the star. The remaining constant in the solution is determined by the boundary condition that $\varphi(R) \to 0$ at large values of R. The constant is thus determined by matching the interior solution with the exterior solution which satisfies this boundary condition.

In the exterior region, the solutions of the equations (7.3.21) and (7.3.22) are

$$\varphi_{ex}(R) = \frac{K_1}{R^3}, \qquad \chi_{ex}(R) = \frac{K_1 G M^{(0)}}{2R^4}. \tag{7.3.25}$$

The interior solution to the equations (7.3.21) and (7.3.22) may be written as the sum of a particular solution and a homogeneous solution. The particular solution may be obtained by integrating the equations outward from the center with any values of A and B which satisfy (7.3.24). The homogeneous solution is then obtained by integrating the equations

$$\frac{d\chi_h(R)}{dR} = -\frac{2GM(R)}{R^2}\varphi_h(R), \tag{7.3.26}$$

$$\frac{d\varphi_h(R)}{dR} = \left(\frac{4\pi R^2 \rho(R)}{M(R)} - \frac{2}{R}\right)\varphi_h(R) - \frac{2\chi_h(R)}{GM(R)}, \tag{7.3.27}$$

with A and B related now by

$$B + \frac{2\pi}{3} G\rho_c A = 0 \tag{7.3.28}$$

The general solution may then be written as

$$\varphi_{in}(R) = \varphi_p(R) + K_2\varphi_h(R), \qquad \chi_{in}(R) = \chi_p(R) + K_2\chi_h(R). \tag{7.3.29}$$

By matching (7.3.25) and (7.3.29) at $R = a$, the constants K_1 and K_2 can be determined. Thus, $\varphi_{in}(R)$ is determined and $\xi_2(R)$ can be easily calculated from

$$\xi_2(R) = -\frac{R^2}{GM(R)}\left\{\frac{1}{3}\Omega^2 R^2 + \varphi_{in}(R)\right\}. \tag{7.3.30}$$

7.3.3 Moment of Inertia

Similarly to the total mass of the star, the total moment of inertia can be calculated as

$$I_{tot} = \int_V \rho(r,\theta)(r\sin\theta)^2 dV = \int_V \rho(r,\theta) r^4 dr \sin^3\theta d\theta d\phi\,. \tag{7.3.31}$$

Using the definition of the radial coordinate r, we find the expression

$$\begin{aligned} r^4 dr &= (R+\xi)^4 (dR + d\xi) \approx R^4 \left(1 + \frac{4\xi}{R}\right)\left(1 + \frac{d\xi}{dR}\right) dR \\ &\approx \left(1 + \frac{4\xi}{R} + \frac{d\xi}{dR}\right) R^4 dR\,, \end{aligned} \tag{7.3.32}$$

which allows us to rewrite the moment of inertia as

$$I_{tot} = \int_V \rho(R) R^4 dR \sin^3\Theta d\Theta d\phi \tag{7.3.33}$$

$$+ \int_V \rho(R) R^4 \left(\frac{4\xi(R,\Theta)}{R} + \frac{d\xi(R,\Theta)}{dR}\right) dR \sin^3\Theta d\Theta d\phi\,.$$

Performing the integration within the range $0 < \Theta < \pi$ and $0 < \phi < 2\pi$, we obtain

$$I_{tot}(R) = I^{(0)}(R) + I^{(2)}(R), \tag{7.3.34}$$

$$I^{(0)}(R) = \frac{8\pi}{3}\int_0^R \rho(R) R^4 dR, \tag{7.3.35}$$

$$I^{(2)}(R) = \frac{8\pi}{3}\int_0^R \rho(R) R^4 \left(\frac{d\xi_0(R)}{dR} - \frac{1}{5}\frac{d\xi_2(R)}{dR} + \frac{4}{R}\left[\xi_0(R) - \frac{1}{5}\xi_2(R)\right]\right) dR \tag{7.3.36}$$

$$= \frac{8\pi}{3}\left[\xi_0(R) - \frac{1}{5}\xi_2(R)\right]\rho(R) R^4\Big|_0^R + \frac{8\pi}{3}\int_0^R \left(-\left[\xi_0(R) - \frac{1}{5}\xi_2(R)\right]\frac{d\rho(R)}{dR}\right) R^4 dR, \tag{7.3.37}$$

where the last expression has been obtained via integration by parts and we have used the integrals

$$\int_0^\pi \sin^3\Theta d\Theta = \frac{4}{3}, \qquad \int_0^\pi P_2(\cos\Theta)\sin^3\Theta d\Theta = -\frac{4}{15}. \tag{7.3.38}$$

When $R = a$ the expression for the change in moment of inertia due to rotation becomes

$$I^{(2)}(a) = \frac{8\pi}{3}\int_0^a \left(-\left[\xi_0(R) - \frac{1}{5}\xi_2(R)\right]\frac{d\rho(R)}{dR}\right) R^4 dR \tag{7.3.39}$$

In the corresponding limit, our results coincide with the definition of the moment of inertia for slowly rotating relativistic stars as given in [108]. Notice that, knowing the value of the moment of inertia, one can easily calculate the total angular momentum of the rotating stars

$$J_{tot} = J^{(0)} + J^{(2)}, \tag{7.3.40}$$

where $J^{(0)} = I^{(0)}\Omega$ is the angular momentum of the spherical configuration and $J^{(2)} = I^{(2)}\Omega$ is the change of the angular momentum due to rotation and deformation.

7.3.4 Gravitational binding energy and rotational kinetic energy

The total gravitational binding energy of a rotating configuration can be calculated in analogy to the total mass and moment of inertia as

$$W_{tot} = \frac{1}{2}\int_V \Phi(r,\theta)\rho(r,\theta)dV = \frac{1}{2}\int_V \Phi(r,\theta)\rho(r,\theta)r^2 dr \sin\theta d\theta d\phi, \tag{7.3.41}$$

where the radial coordinate r can be expressed in coordinates R, Θ by using Eq. (7.2.10), the density is simply given by Eq. (7.2.9) and the gravitational potential by Eq. (7.2.13). Then, we rewrite Eq. (7.3.41) as

$$\begin{aligned} W_{tot} &= \frac{1}{2}\int_V \left[\Phi^{(0)}(R) + \Phi^{(0)}(R)\left(\frac{2\xi(R,\Theta)}{R} + \frac{d\xi(R,\Theta)}{dR}\right)\right]\rho(R)d\mathcal{V} \\ &+ \frac{1}{2}\int_V \left(\xi(R,\Theta)\frac{d\Phi^{(0)}(R)}{dR} + \Phi^{(2)}(R,\Theta)\right)\rho(R)R^2 d\mathcal{V} + O(\Omega^4), \\ & d\mathcal{V} = R^2 dR \sin\Theta d\Theta d\phi, \end{aligned} \tag{7.3.42}$$

Expanding the functions $\Phi^{(2)}(R,\Theta)$ and $\xi(R,\Theta)$ in spherical harmonics, according to Eq. (7.2.14), and integrating the above expression in the range

$0 < \Theta < \pi$, $0 < \phi < 2\pi$, taking into account Eq. (7.3.3), we obtain

$$\begin{aligned}
W_{tot}(R) &= W^{(0)}(R) + W^{(2)}(R) && (7.3.43)\\
W^{(0)}(R) &= 2\pi \int_0^R \Phi^{(0)}(R)\rho(R)R^2 dR, && (7.3.44)\\
W^{(2)}(R) &= 2\pi \int_0^R \left[\xi_0(R) \frac{d\Phi^{(0)}(R)}{dR} + \Phi_0^{(2)}(R) \right. \\
&\quad \left. + \Phi^{(0)}(R) \left(\frac{2\xi_0(R)}{R} + \frac{d\xi_0(R)}{dR} \right) \right] \rho(R) R^2 dR && (7.3.45)\\
&= 2\pi \xi_0(R) \Phi^{(0)}(R)\rho(R)R^2 \Big|_0^R \\
&- 2\pi \int_0^R \left(\xi_0(R)\Phi^{(0)}(R) \frac{d\rho(R)}{dR} - \Phi_0^{(2)}(R)\rho(R) \right) R^2 dR,
\end{aligned}$$

where $W^{(0)}$ is the gravitational binding energy of the static configuration and $W^{(2)}$ is the change in the gravitational binding energy due to rotation. On the surface $W^{(2)}$ becomes

$$W^{(2)}(a) = -2\pi \int_0^a \left(\xi_0(R)\Phi^{(0)}(R) \frac{d\rho(R)}{dR} - \Phi_0^{(2)}(R)\rho(R) \right) R^2 dR. \quad (7.3.46)$$

In the second-order approximation in Ω, the rotational kinetic energy can be written as

$$T = \frac{J_{tot}\Omega}{2} \approx \frac{J^{(0)}\Omega}{2} + O(\Omega^4) = \frac{I^{(0)}\Omega^2}{2} + O(\Omega^4). \quad (7.3.47)$$

The ratio of the rotational kinetic energy to the binding energy allows one to investigate the stability of rotating configurations.

7.3.5 Quadrupole Moment

The Newtonian potential $\Phi(R, \Theta)$ outside the star will be written as before as (see Eq. (7.2.12))

$$\Phi(R, \Theta) = \Phi^{(0)}(R) + \Phi_0^{(2)}(R) + \Phi_2^{(2)}(R) P_2(\cos\Theta), \quad (7.3.48)$$

where

$$\Phi^{(0)}(R) = -\frac{GM^{(0)}}{R}, \tag{7.3.49}$$

$$\Phi_0^{(2)}(R) = -\frac{GM^{(2)}}{R}, \tag{7.3.50}$$

$$\Phi_2^{(2)}(R) = \frac{K_1}{R^3}. \tag{7.3.51}$$

In view of (7.3.4), equation (7.3.48) can be written as follows

$$\Phi(R,\Theta) = -\frac{GM_{tot}}{R} + \frac{K_1}{R^3}P_2(\cos\Theta), \tag{7.3.52}$$

It follows that the constant K_1 determines the mass quadrupole moment Q of the star as $K_1 = GQ$. For a vanishing K_1 we recover the non-rotating configuration. Moreover, according to Hartle's definition $Q > 0$ represents an oblate object and $Q < 0$ corresponds to a prolate object.

7.3.6 Ellipticity and Gravitational Love Number

The quantity defined by

$$\epsilon(R) = -\frac{3}{2R}\xi_2(R), \tag{7.3.53}$$

is the ellipticity of the surface of constant density labeled by R. We use this expression and (7.3.19), and eliminate $\Phi_2^{(2)}$ from (7.3.20), to obtain the following equation for $\epsilon(R)$:

$$\frac{M(R)}{R}\frac{d^2\epsilon(R)}{dR^2} + \frac{2}{R}\frac{dM(R)}{dR}\frac{d\epsilon(R)}{dR} + \frac{2dM(R)}{dR}\frac{\epsilon(R)}{R^2} - \frac{6M(R)\epsilon(R)}{R^3} = 0, \tag{7.3.54}$$

or equivalently in a compact form

$$\frac{d}{dR}\frac{1}{R^4}\frac{d}{dR}\left[\epsilon(R)M(R)R^2\right] = 4\pi\epsilon(R)\frac{d\rho(R)}{dR}\,. \tag{7.3.55}$$

This equation is equivalent to Clairaut's equation. Here both $M(R)$ and $\rho(R)$ are known functions of R. The ellipticity must be regular at small values of R, and equation (7.3.55) shows that it approaches a constant at $R = 0$. With this boundary condition, equation (7.3.55) may be integrated to find the shape of $\epsilon(R)$. To find the magnitude of $\epsilon(R)$ one needs to use (7.3.30). The procedure for considering the boundary condition at the surface given in the previous section, together with the condition of regularity at the origin and the differential equation (7.3.55), uniquely determine the ellipticity of the surfaces of constant density as a function of the coordinate R.

It is easy to show that equation (7.3.54) can be written in the form given in Ref. [73]

$$R^2\frac{d^2\epsilon(R)}{dR^2}+6\frac{\rho(R)}{\rho_m(R)}\left[R\frac{d\epsilon(R)}{dR}+\epsilon(R)\right]=6\epsilon(R), \tag{7.3.56}$$

where

$$\rho_m(R)=\frac{3M(R)}{4\pi R^3} \tag{7.3.57}$$

is the average mass density. By introducing a new function as

$$\eta_2(R)=\frac{R}{\epsilon(R)}\frac{d\epsilon(R)}{dR}\,, \tag{7.3.58}$$

Eq. (7.3.56) reduces to the well known Clairaut-Radau equation [70]

$$R\frac{d\eta_2(R)}{dR}+6\mathfrak{D}(R)[\eta_2(R)+1]+\eta_2(R)[\eta_2(R)-1]=6, \tag{7.3.59}$$

where

$$\mathfrak{D}(R)=\frac{\rho(R)}{\rho_m(R)} \tag{7.3.60}$$

encodes the relevant information about the structure of the body. The differential equation is integrated outward from $R=0$, with the boundary conditions $\mathfrak{D}(0)=1$ and $\eta(R=0)=0$, up to $R=a$, obtaining the value $\eta_2(R=a)$. The Love number is then given by

$$k_2=\frac{3-\eta_2(a)}{2[2+\eta_2(a)]} \tag{7.3.61}$$

In the astronomical and celestial mechanics literature, the dimensionless quantity k_2 is called "apsidal constant", because it controls the size of tidal and rotational deformations of stars in close binary systems, which lead to observable perturbations in the "line of apsides". Sometimes, the quantity defined as $\lambda = 2k_2a^5/(3G)$ is called the Love number. In this work, however, we use k_2 as the Love number; it is named after the British geophysicists A.E.H. Love (1863-1940), who introduced it early in the 20th century [70]. The Love number characterizes the rigidity and the susceptibility of the body's shape to changes in response to a rotational deformation or to a tidal potential. For a rigid body $k_2 = 0$, meaning that a rigid body cannot change its shape. In classical physics, the tidal and rotational Love numbers coincide with each other. However, in general relativity due to the Lense-Thirring effect they differ. The knowledge of the Love numbers has a wide range of applications in the astrophysical context. Namely, using the Love numbers, one can simulate the motion of binary systems with tidal interactions, and estimate the correct values of the orbital parameters. The Love numbers are directly related to the quadrupole moment and the moment of inertia of a deformed object; hence, knowing the Love numbers, one can establish the relationship between the quadrupole moment and the moment of inertia through the I-Love-Q relations [83, 84]. Finally, Love numbers are involved in the expansions of the exterior gravitational potential for deformed objects. Consequently, whenever one considers gravitational interactions between astrophysical objects, the Love numbers play a central role in these processes [70].

Note that once $\xi_2(R)$ is known, then $\epsilon(R)$ is also known from Eq. (7.3.53), and we have

$$\eta_2(a) = \frac{a}{\epsilon(a)}\frac{d\epsilon(R)}{dR}|_{R=a} = \frac{a}{\xi_2(a)}\frac{d\xi_2(R)}{dR}|_{R=a} - 1 \tag{7.3.62}$$

One can see from here that $\eta_2(a)$ does not depend on the angular velocity of the star, neither does the Love number.

7.4 Summary

Our results show that it is possible to write explicitly all the differential equations that determine the behavior of a slowly rotating compact object. For a better presentation of the results obtained in preceding sections, we summarize the steps that must be followed to integrate the resulting equations.

7.4.1 The static case

To determine the relation between mass and central density, one must proceed as follows. (1) Specify the equation of state $p = p(\rho)$ (polytrope, tabulated, etc.). (2) Choose the value of the central density $\rho(R = 0) = \rho_c$. Calculate the mass and pressure from the Newtonian field equation and the equation of hydrostatic equilibrium with the regularity condition at the center $M^{(0)}(R = 0) = 0$

$$\begin{cases} \frac{dM^{(0)}(R)}{dR} = 4\pi R^2 \rho(R), \\ \frac{dp^{(0)}(R)}{dR} = -\rho(R)\frac{GM^{(0)}(R)}{R^2}. \end{cases} \tag{7.4.1}$$

The gravitational potential of the non-rotating star is obtained as

$$\frac{d\Phi^{(0)}(R)}{dR} = \frac{GM^{(0)}(R)}{R^2} = -\frac{1}{\rho(R)}\frac{dp^{(0)}(R)}{dR} \,. \tag{7.4.2}$$

On the surface, the pressure must vanish $p^{(0)}(R = a) = 0$.

The solution of Eq. (7.4.1) gives the mass, pressure and density profile inside the star and, in turn, the density profile allows us to calculate the moment of inertia from Eq. (7.3.35).

In order to determine the correct value of the internal gravitational potential for the static configuration $\Phi^{(0)}(R)$, one should calculate const$^{(0)}$ by matching the potential with its external counterpart on the spherical surface. The exterior potential is given by Eq. (7.3.49). By employing the matching condition

$$\Phi_{in}^{(0)}(R)|_{R=a} \quad = \quad \Phi_{ex}^{(0)}(R)|_{R=a}, \tag{7.4.3}$$

along with Eq. (7.2.18), one can determine const$^{(0)}$ as

$$\text{const}^{(0)} = \int_0^a \frac{1}{\rho(R)}\frac{dp^{(0)}(R)}{dR} dR + \Phi_{ex}^{(0)}(a). \tag{7.4.4}$$

Knowing the value of const$^{(0)}$, the correct expression for the internal potential is

$$\Phi^{(0)}(R) = \Phi_{in}^{(0)}(R) = \text{const}^{(0)} - \int_0^R \frac{1}{\rho(R)}\frac{dp^{(0)}(R)}{dR} dR. \tag{7.4.5}$$

7.4.2 The rotating case: $l = 0$ Equations

Select the value of the angular velocity of the star. For instance, take as a test value the Keplerian orbit with

$$\Omega_{test} = \Omega = \sqrt{\frac{GM^{(0)}(a)}{a^3}} \tag{7.4.6}$$

Integrate the coupled equations

$$\begin{cases} \frac{dp_0^*(R)}{dR} = \frac{2}{3}\Omega^2 R - \frac{GM^{(2)}(R)}{R^2}, \\ \frac{dM^{(2)}(R)}{dR} = 4\pi R^2 \rho(R) \frac{d\rho(R)}{dp(R)} p_0^*(R), \end{cases} \tag{7.4.7}$$

out from the origin with boundary conditions

$$p_0^*(R) \to \frac{1}{3}\Omega^2 R^2, \qquad M^{(2)}(R) \to 0. \tag{7.4.8}$$

These boundary conditions guarantee that the central density of the rotating and non-rotating configurations are the same.

In addition, to calculate $\mathrm{const}_0^{(2)}$ from Eq. (7.2.19), one makes use of the matching condition for $\Phi_0^{(2)}(R)$

$$\Phi_{0in}^{(2)}(R)|_{R=a} = \Phi_{0ex}^{(2)}(R)|_{R=a}, \tag{7.4.9}$$

where $\Phi_{0ex}^{(2)}(R)$ is given by Eq. (7.3.50). Hence

$$\mathrm{const}_0^{(2)} = -\frac{1}{3}\Omega^2 a^2 + \Phi_{0ex}^{(2)}(a) + p_0^*(a) \tag{7.4.10}$$

and eventually

$$\Phi_0^{(2)}(R) = \Phi_{0in}^{(2)}(R) = \mathrm{const}_0^{(2)} + \frac{1}{3}\Omega^2 R^2 - p_0^*(R). \tag{7.4.11}$$

To calculate the gravitational binding energy and its correction due to rotation, one should integrate Eq. (7.3.44) and Eq. (7.3.46) by using Eq. (7.4.5) and Eq. (7.4.11).

7.4.3 The rotating case: $l = 2$ equations

Particular Solution

Integrate the equations

$$\begin{cases} \frac{d\chi(R)}{dR} = -\frac{2GM(R)}{R^2}\varphi(R) + \frac{8\pi}{3}\Omega^2 R^3 G\rho(R) \\ \frac{d\varphi(R)}{dR} = \left(\frac{4\pi R^2 \rho(R)}{M(R)} - \frac{2}{R}\right)\varphi(R) - \frac{2\chi(R)}{GM(R)} + \frac{4\pi}{3M(R)}\rho\Omega^2 R^4 \end{cases}$$

outward from the center with arbitrary initial conditions satisfying the equations, as $R \to 0$

$$\varphi(R) \to AR^2, \quad \chi(R) \to BR^4, \quad B + \frac{2\pi}{3} G\rho_c A = \frac{2\pi}{3} G\rho_c \Omega^2, \qquad (7.4.12)$$

where A and B are constants subject to the above algebraic relation. Therefore, we can freely select only one of the constants. Set, for instance, $A = 1$ and define B from the above equation. This determines a particular solution $\varphi_p(R)$ and $\chi_p(R)$.

Homogeneous Solution

Integrate the homogeneous equations

$$\begin{cases} \frac{d\chi_h(R)}{dR} = -\frac{2GM(R)}{R^2}\varphi_h(R) \\ \frac{d\varphi_h(R)}{dR} = \left(\frac{4\pi R^2 \rho(R)}{M(R)} - \frac{2}{R}\right)\varphi_h(R) - \frac{2\chi_h(R)}{GM(R)} \end{cases}$$

outward from the center with arbitrary initial conditions satisfying the equations, as $R \to 0$

$$\varphi_h(R) \to AR^2, \quad \chi_h(R) \to BR^4, \quad B + \frac{2\pi}{3} G\rho_c A = 0 \qquad (7.4.13)$$

This determines a homogeneous solution $\varphi_h(R)$ and $\chi_h(R)$. Thus, the interior solution is

$$\varphi_{in}(R) = \varphi_p(R) + K_2\varphi_h(R), \quad \chi_{in}(R) = \chi_p(R) + K_2\chi_h(R) \qquad (7.4.14)$$

Matching with an Exterior Solution

The exterior solution is given as

$$\varphi_{ex}(R) = \frac{K_1}{R^3}, \qquad \chi_{ex}(R) = \frac{K_1 GM^{(0)}}{2R^4}. \qquad (7.4.15)$$

By matching (7.4.15) and (7.4.14) at $R = a$,

$$\varphi_{ex}(R = a) = \varphi_{in}(R = a), \qquad \chi_{ex}(R = a) = \chi_{in}(R = a)\,, \qquad (7.4.16)$$

the constants K_1 and K_2 can be obtained.

Moment of inertia, eccentricity and Love number

Once functions $\xi_0(R)$ and $\xi_2(R)$ are known from

$$\begin{aligned}\xi_0(R) &= \frac{R^2}{GM(R)} p_0^*(R), \\ \xi_2(R) &= -\frac{R^2}{GM(R)} \left\{ \frac{1}{3}\Omega^2 R^2 + \varphi_{in}(R) \right\}.\end{aligned} \tag{7.4.17}$$

one can easily calculate the perturbation of the moment of inertia and other important quantities.

The surface of the rotating configuration is described by the the polar r_p and equatorial r_e radii that are determined from the relationships

$$r(a,\Theta) = a + \xi_0(a) + \xi_2(a) P_2(\cos\Theta), \tag{7.4.18}$$

$$r_p = r(a,0) = a + \xi_0(a) + \xi_2(a), \tag{7.4.19}$$

$$r_e = r(a,\pi/2) = a + \xi_0(a) - \xi_2(a)/2 . \tag{7.4.20}$$

In addition, the eccentricity is defined as

$$\text{eccentricity} = \sqrt{1 - \frac{r_p^2}{r_e^2}} \tag{7.4.21}$$

and determines completely the matching surface.

In terms of function $\xi_2(R)$ one can easily calculate ellipticity $\epsilon(R)$, function $\eta_2(R)$, hence the gravitational Love number k_2.

7.5 An example: White dwarfs

In this section, we study an example of the formalism presented in the preceding sections to test the applicability of the method. To appreciate the validity of our results, we consider a very realistic case, namely, white dwarfs whose equation of state at zero temperature is given by the Chandrasekhar relationships [85]

$$\begin{aligned}\varepsilon &= \rho c^2 = \frac{32}{3}\left(\frac{m_e}{m_n}\right)^3 K_n \left(\frac{\bar{A}}{Z}\right) x^3, \\ p &= \frac{4}{3}\left(\frac{m_e}{m_n}\right)^4 K_n \left[x(2x^2-3)\sqrt{1+x^2} + 3\ln(x+\sqrt{1+x^2}) \right] .\end{aligned} \tag{7.5.1}$$

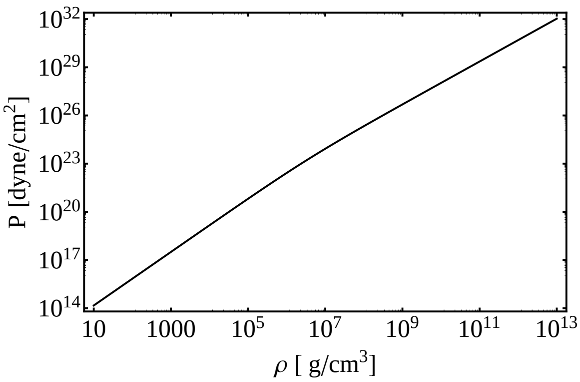

Figure 7.2: Pressure versus density for the Chandrasekhar equation of state.

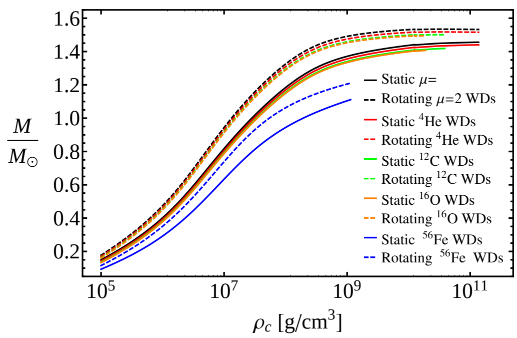

Figure 7.3: Total mass and central density relation obtained from the Chandrasekhar equation of state.

This means that the energy density $\varepsilon = \varepsilon(R)$ is determined by the nuclei, while the pressure $p = p(R)$ is determined by the degenerate electronic gas. Here $\bar{A}$ and Z are the average atomic weight and atomic number of the corresponding nuclei; $K_n = (m_n^4 c^5)/(32\pi^2\hbar^3)$ and $x = x(R) = p_e(R)/(m_e c)$ with $p_e(R)$, m_e, m_n, and $\hbar$ being the Fermi momentum, the mass of the electron, the mass of the nucleon and the reduced Planck constant, respectively. Here we consider the particular case $\bar{A}/Z = 2$. The behavior of the above equation of state is illustrated in Fig. 7.2 for the case of a degenerate electronic gas. Although the Chandrasekhar equation of state has been derived upon the basis of a phenomenological, physical approach, we see that it can be modeled with certain accuracy by means of a polytropic equation of state $p \propto \rho^\alpha$ with α =constant

In Fig. 7.3, we plot the behavior of the total mass as a function of the central density for a static star and for a rotating star with our test angular velocity. It is clear that for a given central density the value of the total mass is larger in

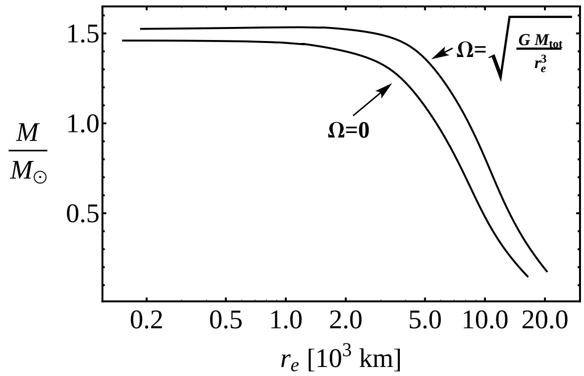

Figure 7.4: Total mass and equatorial radius relation for Chandrasekhar equation of state.

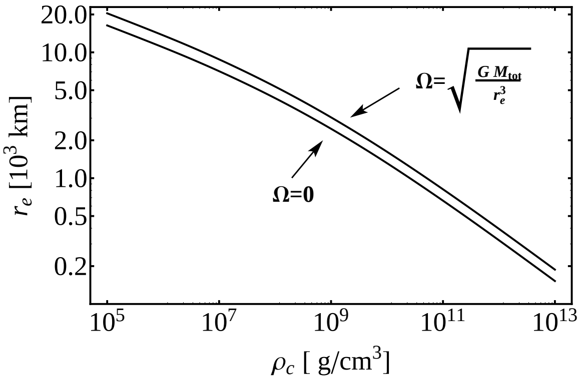

Figure 7.5: Equatorial radius versus central density. Note that $\Omega \to 0$ as $r_e \to a$.

the case of a rotating object than for a static body. This is in accordance with the physical expectations based upon other alternative studies [74, 75, 76, 77, 78, 79, 80, 81]. A similar behavior takes place when we explore the mass as a function of the equatorial radius, as shown in Fig. 7.4.

The relationship between the central density and the equatorial radius is illustrated in Fig. 7.5. As expected, the equatorial radius diminishes as the density increases, and it is larger in the case of a rotating body. In the limit of vanishing angular velocity, the equatorial radius approaches the value of the static radius a.

The moment of inertia depends also on the central density and on the value of the angular velocity, as illustrated in Fig. 7.6. For each value of the angular velocity, there is particular value of the central density at which the moment of inertia acquires a maximum. The value of the moment of inertia at the maxima increases as the angular velocity increases. For very large values of the central density, the gap between the moment of inertia of static and ro-

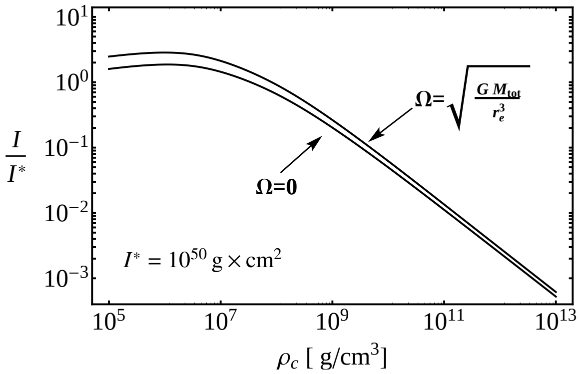

Figure 7.6: Total moment of inertia versus central density.

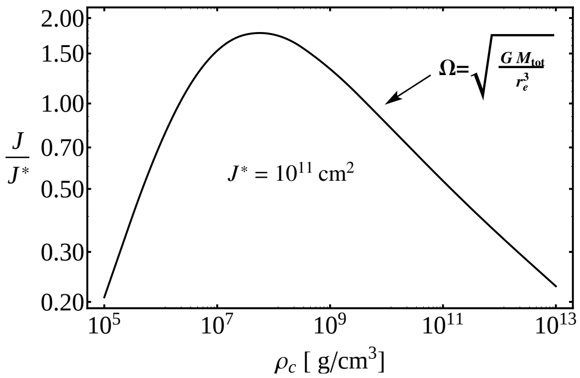

Figure 7.7: Angular momentum versus central density.

tating configurations is narrowing. Notice, however, that this happens for values close to or larger than 10^{11} g/cm^3 which should be considered as unphysical because they are larger than the critical value $\rho_c \sim 1.37 \times 10^{11}$ g/cm^3 at which the equation of state under consideration can no longer be applied because of the inverse β-decay process for white dwarfs consisting of helium ions. It should be mentioned that, in general, the inverse β-decay instability is affected by the rotation indirectly, since the main ingredient for the onset of the β-instability is the value of the density at the center of the white dwarf. Rotation affects the central density and, in turn, it affects the β-instability. Nevertheless, we are considering in all our plots the interval $(10^5 - 10^{13})$ g/cm^3 for the sake of generality.

The total angular momentum as a function of the central density is given in Fig. 7.7. With increasing central density, first, it increases up to its maximum value and then it decreases. The dimensionless angular momentum or the spin parameter is shown in Fig. 7.8. It turns out that for white dwarfs the spin

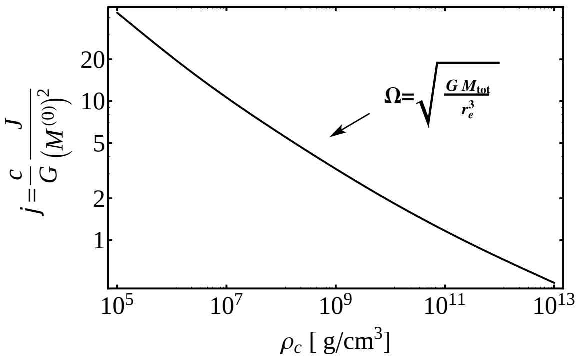

Figure 7.8: Dimensionless angular momentum $j = (cJ)/(GM^2)$ versus central density.

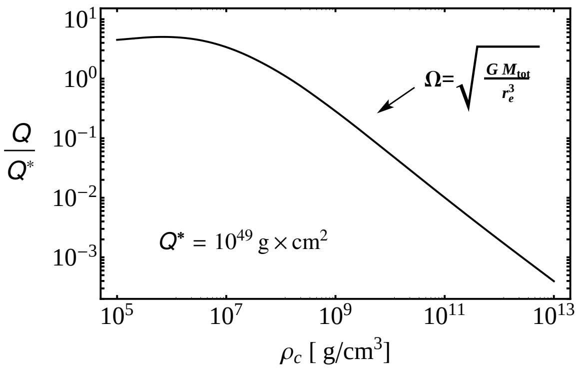

Figure 7.9: Mass quadrupole moment versus central density.

parameter is quite large for small central densities and it is less than unity for central densities larger than the inverse β-decay density for helium white dwarfs. Thus, if we assume that a white dwarf collapses into a neutron star at the inverse β-decay instability density and the spin parameter is conserved, then the spin parameter will be in agreement with the theoretical upper limits for neutron stars [86, 87, 88, 89].

In Figs. 7.9 and 7.10 we plot the quantities which determine the shape of the surface where the interior solution is matched with the exterior one, namely, the quadrupole moment and the eccentricity. Obviously, both quantities vanish in the limiting case of vanishing rotation. The quadrupole possesses a maximum at a certain value of the central density which coincides with the position of the maximum of the moment of inertia.

The quadrupole moment normalized by the Kerr quadrupole moment as a function of the central density is illustrated in Fig. 7.11. One can see here that the quadrupole moment for white dwarfs is always larger than the Kerr quadrupole.

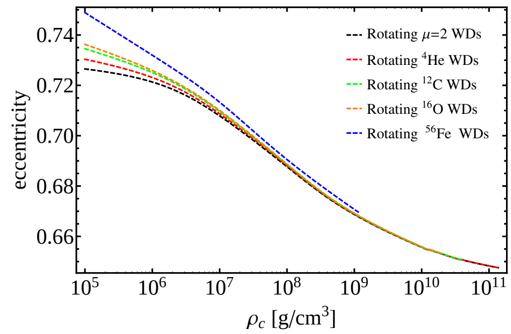

Figure 7.10: Eccentricity versus central density of rotating configurations.

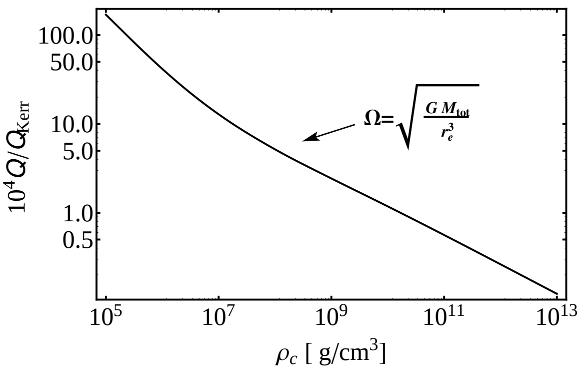

Figure 7.11: Mass quadrupole moment over the Kerr quadrupole moment $Q_{Kerr} = J^2/(c^2M)$ versus central density.

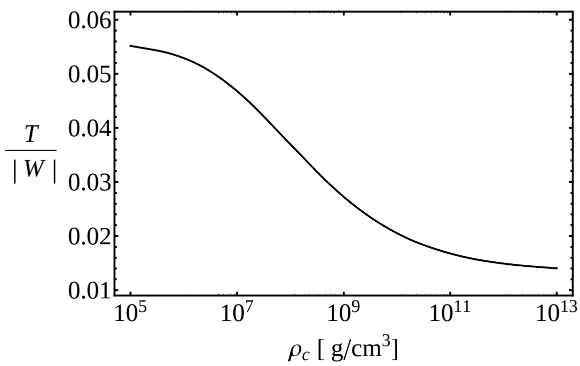

Figure 7.12: Rotational kinetic energy over gravitational binding energy $T/|W|$ versus central density of rotating configurations.

The ratio of the rotational kinetic energy to the gravitational binding energy as a function of the central density is shown in Fig. 7.12. The values of the ratio and the eccentricity are well below the upper limits for secular and dynamical instabilities. According to Ref. [90], the conditions (values) $T/|W| = 0.14$, $e = 0.81$ and $T/|W| = 0.25$, $e = 0.95$, for rigidly rotating fluids (liquids) with uniform density the so-called Maclaurin spheroids, are considered to be the upper limits for secular and dynamical instabilities, respectively.

The secular instability determines the instability of rotating configurations with respect to small perturbations. In fact, in order to investigate the secular instability of realistic objects Friedman et al. [91] formulated the method of turning points. According to this method, one needs to calculate constant angular momentum sequences, compute the mass-central density relations, and estimate the maximum mass. The maximum mass is the indicator (turning point) for the onset of the secular instability. For different values of the angular momentum there are different maximum masses and by joining all the points for the maxima one obtains the secular instability curve. In the mass-central density diagram the configurations on the left hand side of the maxima are considered to be secularly stable, and the configurations on the right hand side are unstable configurations.

We constructed the constant angular momentum sequences and found that the maximum mass is reached for configurations with central densities beyond our range of consideration. This fact allows us to state that, unlike their relativistic counterparts, the Newtonian uniformly rotating white dwarfs are secularly stable [61, 63].

The ellipticity of the rotating deformed star is illustrated as a function of the central density in Fig. 7.13. On the surface of the star the ellipticity shows similar behavior as the eccentricity and as density increases it decreases. Thus, the star becomes more compact and more spherical.

The dependence of function η_2 is shown as a function of the spherical radius a in Fig. 7.14. As the radius increases the function decreases. The function η_2 is necessary to calculate the Love number. Finally, in Figs. 7.15 and 7.16 we depict the gravitational Love number as a function of the central density and spherical radius, respectively. For increasing central density the Love number decreases. This implies that with the increasing central density or decreasing radius white dwarfs become less susceptible to rotational and tidal deformations, since $k_2 = 0$ for a rigid body. It should be mentioned that the values for the Love number in agreement with those presented in Ref. [92].

Notice that in all the plots, the selected values for the central mass and the equatorial radius are in accordance with the expected values for white dwarfs. We conclude that the results obtained from the numerical integration

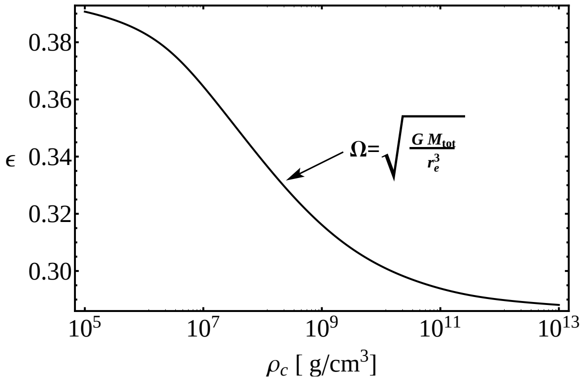

Figure 7.13: Ellipticity versus central density of rotating configurations.

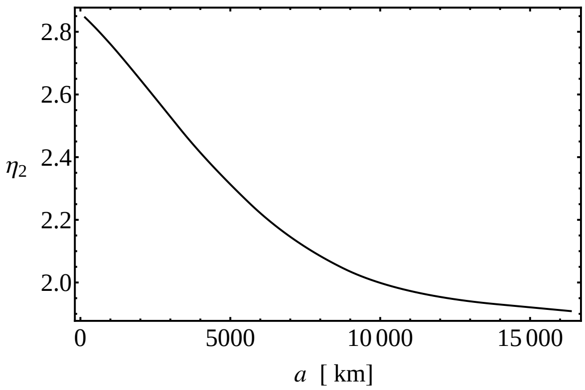

Figure 7.14: Function η_2 versus spherical radius a of static configurations.

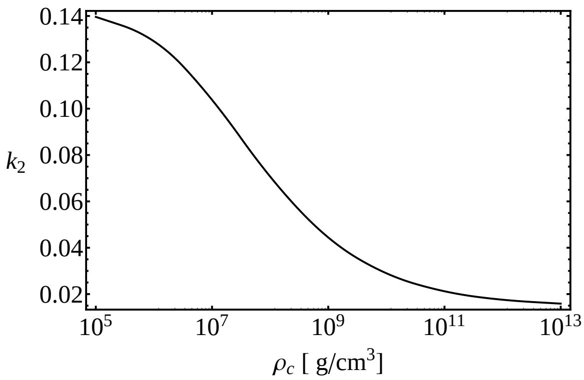

Figure 7.15: Love number versus central density of static configurations.

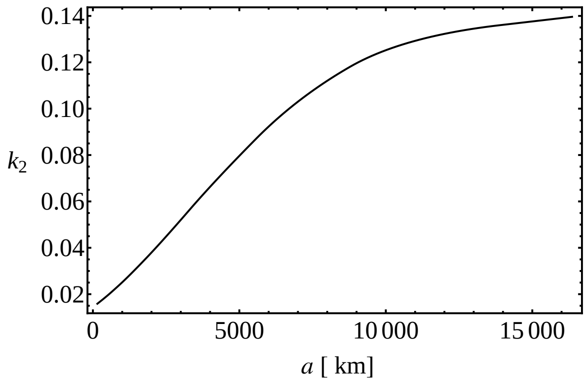

Figure 7.16: Love number versus spherical radius of static configurations.

of the differential equations derived by using the approach proposed in this work are consistent with the physical expectations, when restricted to the region in which the formalism can be applied [75, 76, 77, 78, 79, 80, 81, 93, 94, 95].

7.6 The mass-shedding limit and scaling law

In this section we will discuss about some technical details related to the computation of the Keplerian mass-shedding limit of any rotating configurations and the scaling law for physical quantities that can be rescaled without additional numerical integrations for various objectives.

The mass-shedding limit. It is well known that the velocity of particles on the equator of the star cannot exceed the Keplerian velocity of free particles, computed at the same location. At this limit, particles on the star's surface remain bound to the star only because of a balance between gravitational and centrifugal forces. The evolution of a star rotating at the Keplerian rate is accompanied by a loss of mass, thus becoming unstable. The Keplerian angular velocity in Newtonian physics is determined as follows

$$\Omega_{Kep} = \sqrt{\frac{GM_{tot}}{r_e^3}} \tag{7.6.1}$$

where G is the gravitational constant, M_{tot} is the total mass of the rotating configuration and r_e is the equatorial radius. This is the critical angular velocity at which rotational shedding will occur, and it is thus an upper bound on those angular velocities for which the assumption of slow rotation could be valid.

In order to estimate this quantity correctly, one needs to select a test value of the angular velocity, for example, in our computations we used $\Omega_{test} = \sqrt{GM^{(0)}/a^3}$. Usually $\Omega_{test} > \Omega_{Kep}$, hence one needs to decrease the values of Ω_{test} gradually and estimate Ω_{Kep} successively, until $\Omega_{test} = \Omega_{Kep}$ with a given precision. For that purpose, in practice, it is convenient to use the shooting method [96].

If we express $\Omega_{Kep} = \kappa\Omega_{test}$, then the value of multiplicative factor κ can be estimated from the above procedure by numerical integration and the results are shown in Table 7.1.

Table 7.1: The values of multiplicative factor κ for different values of the central density.

ρ [g/cm^3]	10^5	10^6	10^7	10^8	10^9	10^{10}	10^{11}	10^{12}	10^{13}
κ	0.781	0.780	0.777	0.770	0.762	0.755	0.751	0.748	0.747

As one can see from the Table, indeed $\Omega_{test} > \Omega_{Kep}$ and this results are in agreement with the ones in the literature [75, 76, 77, 78, 79, 80, 81]. It should be stressed that the Keplerian angular velocity allows one to estimate the maximum rotation rate (the minimum rotation period) and the maximum rotating mass of stars. Moreover it allows us to determine the stability region of a rotating star, inside which all rotating configurations can exist (see [61] for details).

Although the data presented in Table 7.1 seem to indicate that the value of κ is close to 0.75, it is important to note that this multiplicative factor is not the same at any density. In fact, it depends on the equation of state of the white dwarf matter and for different central densities with different nuclear compositions, it accepts distinct values. Even in the case of neutron stars the parameter κ possesses various values for a variety of the models used to construct the equations of state. For example, in [97], this parameter has been calculated for a particular equation of state and a fixed range of values for the central density, leading to the result that the parameter varies in the range (0.63-0.67). So, in general, there is no reason for this parameter to be independent of the density in the case of white dwarfs, as well.

In all our computations, we used the value $\Omega_{Kep} = \sqrt{GM_{tot}/r_e^3}$. Of course, this is a rough approximation. Nevertheless, as it was pointed out by Berti et al. [98] (see also [99]), Hartle's approach with Ω_{Kep} gives only small differences (errors) with respect to the exact numerical methods used in general relativity for neutron stars. In the case of white dwarfs, the errors should be even smaller than for neutron stars since white dwarfs rotate more slowly than neutron stars (see Appendix D of Ref. [61]). Moreover, we compare in Table 7.2 our results with other more sophisticated approaches used in the literature; our results are in good agreement with other works.

Scaling law. The scaling procedure is used in order to rescale all the known values of physical quantities for different objectives. For instance, the angular momentum J is directly proportional to Ω, hence there is always the possibility for the following scaling law $J_{new}/\Omega_{new} = J_{old}/\Omega_{old}$ to be held. This means that knowing the old value of the angular momentum J_{old} for the given angular velocity Ω_{old}, one can easily evaluate a new value of the angular momentum J_{new} for a given new angular velocity Ω_{new} without reintegrating the structure equations. The same is true for all the physical quantities which are directly proportional to the second order of the angular velocity Ω^2. The following quantities are subject to scaling: $M^{(2)}_{new}/\Omega^2_{new} = M^{(2)}_{old}/\Omega^2_{old}$, $Q_{new}/\Omega^2_{new} = Q_{old}/\Omega^2_{old}$, $I^{(2)}_{new}/\Omega^2_{new} = I^{(2)}_{old}/\Omega^2_{old}$ and $W^{(2)}_{new}/\Omega^2_{new} = W^{(2)}_{old}/\Omega^2_{old}$. From a practical point of view it is very convenient to make use of the scaling law for various computational goals.

On the other hand, a careful examination of the scaling law for the quadrupole moment shows some resemblance with the I-Love-Q relations. Indeed, Q/Ω^2 is a constant quantity which is a function of the central density only, and it can be identified with the gravitational Love number λ. In the Ω^2 approximation, expressing Ω in terms of $I^{(0)}$ through $J^{(0)} = I^{(0)}\Omega$ and normalizing Q to the Kerr quadrupole moment, one obtains the I-Love-Q relations as shown in Refs. [83, 84]. Of course, it would be interesting to check these relations also for white dwarfs. In this case, in accordance to the generally accepted theoretical models for white dwarfs, one needs to analyze at least two different equations of state with several nuclear compositions. This requires a detailed analysis which will be presented in a separate chapter.

7.7 Comparison with other results in the literature

In this section, we compare and contrast our results with other known results. The configurations of uniformly rotating white dwarfs have been intensively investigated since their first theoretical descriptions. Here we present previous results obtained in this context. For example, in Table 7.2, we show some results for the maximum static and rotating masses computed by several authors in Newtonian gravity, the post-Newtonian approximation and in general relativity. Regardless of the approximation, approach, treatment, theory and numerical codes, the results are very similar. These maximum masses for rotating white dwarfs are to be compared with the ones found in this work using the Chandrasekhar equation of state with $\mu = 2$. In New-

Table 7.2: Maximum static and rotating masses of white dwarfs. NP stands for Newtonian physics, PN stands for Post-Newtonian weak field ($1/c^2$ approximation), GR stands for general relativity, $\mu = 2$ indicates the average molecular weight in the Chandrasekhar equation of state and $n = 3$ stands for the polytrope index.

Treatment/EoS	$M_{max}^{J=0}/M_{\odot}$	$M_{max}^{J\neq 0}/M_{\odot}$	Approximation	Formalism
NP/ $\mu = 2$	1.437	1.474	slow $\sim \Omega^4$	Chandrasekhar
NP/ $n = 3$	1.437	1.487	rapid	Monaghan-Roxburgh
PN/ $\mu = 2$	1.417	1.482	slow $\sim \Omega^2$	Durney-Roxburgh
NP/ $\mu = 2$	1.435	1.513 (1.506)	slow $\sim \Omega^2$ (Ω^4)	Sedrakyan-Chubaryan
NP/ $\mu = 2$	1.437	1.516	rapid	Chandrasekhar
NP/ $\mu = 2$	1.459	1.534	slow $\sim \Omega^2$	Hartle
GR/ $\mu = 2$	1.405	1.478	slow $\sim \Omega^2$	Sedrakyan-Chubaryan
GR/ $\mu = 2$	1.429	1.516	slow $\sim \Omega^2$	Hartle

ton's gravity, the maximum static mass is $1.459 M_{\odot}$ and for a rotating mass we obtain $1.534 M_{\odot}$. The difference appears already at the level of the static configuration and this, in turn, translates to the rotating configurations.

For the sake of clarity, it should be mentioned that the structure equations describing uniformly rotating configurations, as derived in general relativity by Hartle in [69], reduce identically to the equations shown in this work for $c \to \infty$. Correspondingly, the structure equations formulated by Sedrakyan and Chubaryan [101] in general relativity also reduce to their Newtonian counterparts in the corresponding limiting case [79]. The mathematical and physical equivalence of these two formalisms has been recently shown in Ref. [102] by comparing the exterior Hartle-Thorne [103] and the Sedrakyan Chubaryan [101] solutions. Thus, the difference in numbers of these two approaches in Table 7.2 can be due to the numerical integrations, only.

7.8 Remarks

In the present work, we have revisited the Hartle formalism to describe in Newtonian gravity the structure of rotating compact objects under the condition of hydrostatic equilibrium. We use a particular set of polar coordinates that is especially constructed to take into account the deformation of the source under rotation. Moreover, we use an expansion in terms of spherical harmonics and consider all the equations only up to the second order in the angular velocity. The main point is that these assumptions allow us to reduce the problem to a system of ordinary differential equations, instead of partial differential equations. As a consequence, we derive all the equations explicitly and show how to perform their numerical integration. Numerical solutions for particular equations of state and the analysis of the stability of the resulting configurations will be discussed in a subsequent work.

In addition, the formalism developed here allows us to find explicit expressions for the main physical quantities that determine the properties of the rotating configuration. In particular, we derived the equation which determines the relation between mass and central density, and showed that it takes the form of an equation of hydrostatic equilibrium. It enforces the balance of pressure, gravitational, and centrifugal forces correctly to order Ω^2. In this approximation, the surfaces of constant density are spheroids whose ellipticity varies from zero at the center of the star up to the values which describe the shape of the star at the surface. The ellipticity, as a function of the radius, turns out to be determined by the Clairaut's differential equation. The equations which determine the relation between mass and central density and those which determine the ellipticity are systems of ordinary differential equations whose solution may be obtained by numerical integration. Furthermore, we also derived analytic expressions for the quadrupole moment and moment of inertia of the source. In addition we derived and integrated the expressions for the rotational kinetic energy and gravitational binding energy. We constructed $T/|W|$ ratio versus central density and checked its consistency with the instability criteria for secular and dynamical instabilities of the Maclaurin spheroids. Finally, we obtained the Clairaut-Radau equation from the Clairaut equation and calculated the gravitational Love number, which indicates rotational or tidal response to the exterior field.

We have tested the formalism developed here by using the Chandrasekhar equation of state for white dwarfs. All the derived physical quantities are in accordance with the results in the literature. This result reinforces the validity of the assumptions and approximations applied in this work to formulate a method that takes into account the rotation in the context of hydrostatic equilibrium in Newtonian gravity. Eventually, on top of everything the procedure of computing the Keplerian mass-shedding angular velocity along with the scaling law of physical quantities have been presented for rotating configurations.

In view of recent works [83, 84] on the so-called I-Love-Q relations in neutron stars and quark stars, it would be interesting to investigate these relations in white dwarf stars [104].

It would be interesting to consider also the case of differential rotation. However, this implies the generalization of the entire Hartle formalism to include an arbitrary rotation function. This is a task for future works.

8 Inertia and quadrupole relations for white dwarfs

8.1 Introduction

White dwarfs are the end-product of evolution of all stars with initial masses up to $9M_\odot$. As the most common endpoint of stellar evolution, white dwarf stars account for around 97% of all evolved stars. Therefore, their properties and distribution contain abundant information about star formation history and evolution of galaxies. In 1933, Chandrasekhar [74] formulated the theory of white dwarfs by using Newtonian gravity and the condition of hydrostatic equilibrium together with an equation of state corresponding to a degenerate Fermi gas. He solved the differential equations numerically and found the limiting mass of $1.44M_\odot$ (Chandrasekhar limit). This study was followed by a number of analysis in which non-rotating and rotating equilibrium configurations were considered [75, 76, 77, 66, 79]. Slightly modifications of the Chandrasekhar limit have been found, depending on the rotation and the equation of state.

As compared to Newtonian stars, the conditions determining the hydrostatic equilibrium of a star changes drastically when the full system of Einstein's equations is taken into account. In fact, in general relativity, the Newtonian equation of hydrostatic equilibrium for compact stars becomes modified to what is known as Tolman-Oppenheimer-Volkoff equation. A general analysis of spherically symmetric compact stars, taking into account general relativistic effects, was performed in [105]. By considering an equation of state valid for both non-relativistic and ultra-relativistic electron velocities, it was found that in the case of white dwarfs, general relativistic effects lead only to small perturbations of the values obtained in Newtonian gravity. It follows that it is justified to use Newton's theory to investigate the essential part of the physical properties of white dwarfs. In this work, we will follow this hypothesis.

Although many approaches can be used to investigate the properties of compact stars in Newtonian gravity, a particularly useful method was proposed recently [106, 107] in which the relativistic Hartle formalism is applied explicitly to solve the corresponding set of differential equations. Hartle's formalism has been widely used in the scientific community to describe relativistic objects such as neutron stars, quark stars, and other exotic objects [69, 108, 68]. One of the advantages of its application is that it is very intuitive from a geometric and a physical point of view, because it uses coordinates that are especially adapted to the shape and dynamics of rotating bodies. When applied to describe equilibrium configurations in Newtonian gravity, Hartle's formalism transforms the dynamic equations to be solved into a system of ordinary differential equations. Moreover, it allows us to derive in detail all physically relevant quantities such as the total rotating mass M_{tot}, equatorial r_e and polar r_p radii, angular momentum J, eccentricity e, ellipticity ϵ, rotational Love number λ, moment of inertia I, and the quadrupole moment Q. All these parameters play a pivotal role in the investigation of the stability and the lifespan of white dwarfs [109, 110, 63, 93, 94, 95].

In this chapter, we investigate the effects that rotation causes in the structure of white dwarfs; namely, we construct the I-Q, I-e, Q-e, I-Love and Love-Q relations, and show that they are universal and independent of the equations of state for white dwarfs. We focus on the case of slow and rigid rotation. We integrate the equations of structure for slowly rotating white dwarfs numerically by using the Chandrasekhar and Salpeter equations of state [65, 71, 111, 112, 85]. In addition, we analyze the stability properties of white dwarfs against the mass-shedding limit, the inverse β-decay instability, and the secular instabilities [109].

8.2 Equations of structure

The method used to construct models for uniformly and slowly rotating relativistic stars is summarized briefly here. For details and derivations the reader may refer to [69, 106, 107], the notation of which we follow. The main idea consists in considering a spherically symmetric non-rotating compact object as starting point. Then, the rotation is considered up to the second order as a small perturbation of the non-rotating model.

8.2.1 A non-rotating stellar model

For a given value of the central density, the non-rotating equilibrium configuration is determined by integrating with respect to the radial coordinate, R, the Newtonian equation of hydrostatic equilibrium for the pressure $p^{(0)}(R)$, and the mass interior to a given radius, $M^{(0)}(R)$:

$$\begin{cases} \frac{dp^{(0)}(R)}{dR} = -\rho^{(0)}(R)\frac{GM^{(0)}(R)}{R^2}, \\ \frac{dM^{(0)}(R)}{dR} = 4\pi R^2\rho^{(0)}(R), \end{cases} \tag{8.2.1}$$

where G is the gravitational constant. The integration is performed outwards, starting at the star's center, $R = 0$, where $M^{(0)}(R = 0) = 0$, $\rho(R = 0) = \rho_c$ is the given central density, and $p^{(0)}(R = 0) = p_c^{(0)} = p^{(0)}(\rho_c)$ is determined by the equation of state. The radius of the spherical surface of the star, a, is that value of R at which $p^{(0)}(R)$ drops to zero; and the value $M^{(0)}(a)$ is the star's total static (non-rotating) mass.

The internal gravitational potential of a non-rotating star is determined by integrating outwards from the center to the surface of the star the equation

$$\frac{d\Phi_{in}^{(0)}(R)}{dR} = \frac{GM^{(0)}(R)}{R^2}. \tag{8.2.2}$$

The external gravitational potential is given by

$$\Phi_{ex}^{(0)}(R) = -\frac{GM^{(0)}(a)}{R}, \tag{8.2.3}$$

with the boundary condition $\Phi^{(0)}(\infty) = 0$.

Finally, the moment of inertia of the static configuration is determined from the following expression:

$$I^{(0)}(a) = \frac{8\pi}{3}\int_0^a \rho(R)R^4 dR. \tag{8.2.4}$$

8.2.2 A rotating stellar model

In the case of rigid and slow rotation, we can apply the approximate method proposed originally by Hartle in [69], and applied to Newtonian gravity in Refs. [106, 107]. First, we introduce the new radial coordinate $r(R,\Theta) = R + \xi(R,\Theta)$, where $\xi(R,\Theta)$ is a function that takes into account, up to the second order in the angular velocity Ω, the deviations from spherical symmetry. Note, that $\Theta = \theta$. For simplicity, the deviations are assumed to preserve axial symmetry, with the rotation axis oriented along the symmetry axis, and equatorial symmetry so that in many quantities the odd powers of Ω can be neglected. Within this approximation, it is possible to assume that the density and the pressure, in terms of the new radial coordinate, have the same numerical values as in the static configuration, i.e.,

$$\rho(r,\theta) = \rho(R) = \rho^{(0)}(R)\,, \quad p(r,\theta) = p(R) = p^{(0)}(R)\,. \tag{8.2.5}$$

Then, all the relevant quantities are Taylor expanded up to the second order in Ω, leading to a system of partial differential equations for the Newtonian potential and the pressure. The important point now is that due to the axial symmetry all the perturbations can be expanded in spherical harmonics, i.e, Legendre polynomials, $P_l(\cos\theta)$, which leads to a crucial simplification of the dynamic equations, namely, they all become ordinary differential equations. Indeed, Newton's equation for the gravitational potential $\Phi(R,\theta) = \Phi^{(0)}(R) + \Phi_0^{(2)}(R) + \Phi_2^{(2)}(R)P_2(\cos\theta)$ splits into three ordinary equations that can be expressed as

$$\nabla^2\Phi^{(0)}(R) = 4\pi G\rho(R)\,, \qquad \text{(spherical)} \tag{8.2.6}$$

$$\xi_0(R)\tfrac{d}{dR}\nabla^2\Phi^{(0)}(R) + \nabla^2\Phi_0^{(2)} = 0\,, \qquad (l=0)\,, \tag{8.2.7}$$

$$\xi_2(R)\tfrac{d}{dR}\nabla^2\Phi^{(0)}(R) + \nabla^2\Phi_2^{(2)}(R) - \tfrac{6}{R^2}\Phi_2^{(2)}(R) = 0\,, \quad (l=2)\,, \tag{8.2.8}$$

where ∇ is the gradient operator in coordinates R and θ, and the deformation function $\xi(R,\theta)$ has been decomposed as $\xi(R,\theta) = \xi_0(R) + \xi_2(R)P_2(\cos\theta)$. Furthermore, the equilibrium condition leads to the following equations

$$\int_0^p \tfrac{dp^{(0)}(R)}{\rho(R)} + \Phi^{(0)}(R) = \text{const}^{(0)}\,, \qquad \text{(spherical)}\,, \tag{8.2.9}$$

$$\xi_0(R)\tfrac{d\Phi^{(0)}(R)}{dR} + \Phi_0^{(2)}(R) - \tfrac{1}{3}\Omega^2R^2 = \text{const}_0^{(2)}\,, \quad (l=0)\,, \tag{8.2.10}$$

$$\xi_2(R)\tfrac{d\Phi^{(0)}(R)}{dR} + \Phi_2^{(2)}(R) + \tfrac{1}{3}\Omega^2R^2 = 0\,. \qquad (l=2)\,, \tag{8.2.11}$$

where const is defined as const $=$ const$^{(0)}$ $+$ const$_0^{(2)}P_0(\cos\Theta)$, so that const$^{(0)}$ and const$_0^{(2)}$ are found from the matching between the interior and exterior solutions. All the physically relevant quantities for a rigidly and slowly rotating star should be derived from this set of ordinary differential equations. For instance, the total mass of the star $M_{tot} = \int \rho(r,\theta)\sin\theta dr d\theta d\phi$ gets a contribution from the deformation function and becomes

$$\begin{aligned} M_{tot}(R) &= M^{(0)}(R) + M^{(2)}(R) \\ &= 4\pi\int_0^R \rho(R)R^2dR \\ &\quad -4\pi\int_0^R \xi_0(R)\frac{d\rho(R)}{dR}R^2dR + 4\pi\xi_0(R)\rho(R)R^2\Big|_0^R\,, \end{aligned} \tag{8.2.12}$$

which can be expressed equivalently as

$$\frac{d\Phi^{(0)}(R)}{dR} = \frac{GM^{(0)}(R)}{R^2} , \quad \frac{d\Phi_0^{(2)}(R)}{dR} = \frac{GM^{(2)}(R)}{R^2} . \tag{8.2.13}$$

8.2.3 Central density and angular velocity

For slow rotation, once the equation of state is specified, there is a unique equilibrium configuration for each choice of the central density and angular velocity. The small perturbations away from a non-rotating equilibrium configuration are all proportional to the angular velocity or to its square. Consequently, for a given central density, all the models of different angular velocities can be obtained from a single model by *scaling*. In this work, the results are given in graphical form for the angular velocity satisfying

$$\Omega = \Omega_{Kep} = \sqrt{\frac{GM_{tot}}{r_e^3}} \tag{8.2.14}$$

where $M_{tot} = M_{tot}(a)$ is the total mass of the rotating configuration and r_e is the equatorial radius. This is the critical angular velocity at which rotational shedding will occur, and it is thus an upper bound on those angular velocities for which the assumption of slow rotation could be valid.

Knowing the value of the moment of inertia $I^{(0)}$ and the angular velocity Ω, one can determine the angular momentum of a spherical star by

$$J = I^{(0)}(a)\Omega + O(\Omega^3) , \tag{8.2.15}$$

where a is the spherical (unperturbed) radius of the star.

Having chosen a value of the angular velocity for each value of the central density, one constructs a sequence of equilibrium models by integrating the Newtonian equations of structure for a sequence of central densities.

8.2.4 Spherical deformations of the star

To determine the deformations, we first must choose a particular value for the angular velocity of the star. For instance, as a test value we can take

$$\Omega = \sqrt{\frac{GM^{(0)}(a)}{a^3}} . \tag{8.2.16}$$

The spherical part of the rotational deformation is calculated by integrating the $l = 0$ equations of hydrostatic equilibrium for the change of mass $M^{(2)}$ and the pressure perturbation function p_0^* which is defined as

$$p_0^* = \xi_0(R)\frac{d\Phi^{(0)}(R)}{dR} . \tag{8.2.17}$$

The hydrostatic equilibrium condition for the pressure is obtained by differentiating Eq.(8.2.10) and using Eq.(8.2.13). Then, we obtain the balance equations

$$\begin{cases} \frac{dp_0^*(R)}{dR} = \frac{2}{3}\Omega^2 R - \frac{GM^{(2)}(R)}{R^2}, \\ \frac{dM^{(2)}(R)}{dR} = 4\pi R^2 \rho(R)\frac{d\rho(R)}{dp(R)} p_0^*(R) , \end{cases} \tag{8.2.18}$$

which must be integrated out from the origin with boundary conditions

$$p_0^*(R) \to \frac{1}{3}\Omega^2 R^2, \qquad M^{(2)}(R) \to 0 . \tag{8.2.19}$$

These boundary conditions guarantee that the central density of the rotating and non-rotating configurations are the same [69]. The system of equations (8.2.18) represents the balance between the pressure, centrifugal and gravitational forces of rotating configurations. Consequently the total mass of the star with central density ρ_c and angular velocity Ω is

$$M_{tot} = M^{(0)}(a) + M^{(2)}(a), \tag{8.2.20}$$

where a is again the radius of the spherical configuration.

8.2.5 Quadrupolar deformations of the star

The quadrupolar part of the rotational deformation is calculated by integrating the $l = 2$ equation of the field equations (8.2.8) together with the condition for hydrostatic equilibrium (8.2.11). This is equivalent to a second-order differential equation that can be split into two first-order inhomogeneous differential equations

$$\begin{cases} \frac{d\chi(R)}{dR} = -\frac{2GM(R)}{R^2}\varphi(R) + \frac{8\pi}{3}\Omega^2 R^3 G\rho(R) \\ \frac{d\varphi(R)}{dR} = \left(\frac{4\pi R^2\rho(R)}{M(R)} - \frac{2}{R}\right)\varphi(R) - \frac{2\chi(R)}{GM(R)} + \frac{4\pi}{3M(R)}\rho(R)\Omega^2 R^4 \end{cases} \tag{8.2.21}$$

to be integrated outward from the center of the star with arbitrary initial conditions, satisfying the relationships

$$\varphi(R) \to AR^2, \quad \chi(R) \to BR^4, \quad B + \frac{2\pi}{3} G\rho_c A = \frac{2\pi}{3} G\rho_c \Omega^2, \tag{8.2.22}$$

where A and B are constants. Set, for example, $A = 1$ and define B from the above algebraic equation; this determines particular solutions $\varphi_p(R)$ and $\chi_p(R)$. Then, the homogeneous solution should be considered by integrating the homogeneous equations

$$\begin{cases} \frac{d\chi_h(R)}{dR} = -\frac{2GM(R)}{R^2}\varphi_h(R) \\ \frac{d\varphi_h(R)}{dR} = \left(\frac{4\pi R^2\rho(R)}{M(R)} - \frac{2}{R}\right)\varphi_h(R) - \frac{2\chi_h(R)}{GM(R)} \end{cases} \tag{8.2.23}$$

outward from the center with arbitrary initial conditions, satisfying the relationships

$$\varphi_h(R) \to AR^2, \quad \chi_h(R) \to BR^4, \quad B + \frac{2\pi}{3} G\rho_c A = 0\,. \tag{8.2.24}$$

If we set, for instance, $A = 1$ and take B as given by the above equation, we obtain homogeneous solutions $\varphi_h(R)$ and $\chi_h(R)$. Thus, the interior solution is determined by the sum of the particular and the homogeneous solutions

$$\varphi_{in}(R) = \varphi_p(R) + K_2\varphi_h(R), \quad \chi_{in}(R) = \chi_p(R) + K_2\chi_h(R), \tag{8.2.25}$$

where K_2 is the constant to be determined from the matching with the exterior solutions.

8.2.6 Matching with the Exterior Solutions

The exterior solutions of (8.2.21) are given by

$$\varphi_{ex}(R) = \frac{K_1}{R^3}, \qquad \chi_{ex}(R) = \frac{K_1 GM^{(0)}}{2R^4}, \tag{8.2.26}$$

where K_1 is an integration constant to be determined from the matching with the interior solutions. Indeed, the matching of (7.4.15) with (7.4.14) at $R = a$ leads to the conditions

$$\varphi_{ex}(R = a) = \varphi_{in}(R = a), \qquad \chi_{ex}(R = a) = \chi_{in}(R = a), \tag{8.2.27}$$

from which the constants K_1 and K_2 can be determined.

8.2.7 Polar and equatorial radii and eccentricity

The surface of the rotating configuration, and the polar r_p and equatorial r_e radii are given by

$$\begin{aligned} r(a,\Theta) &= a+\xi_0(a)+\xi_2(a)P_2(\cos\Theta), &(8.2.28)\\ r_p &= r(a,0)=a+\xi_0(a)+\xi_2(a), &(8.2.29)\\ r_e &= r(a,\pi/2)=a+\xi_0(a)-\xi_2(a)/2, &(8.2.30)\end{aligned}$$

where $\xi_0(R)$ and $\xi_2(R)$ are determined from Eqs.(8.2.18) and (7.4.14), respectively. Then,

$$\begin{aligned} \xi_0(R) &= \frac{R^2}{GM^{(0)}(R)}p_0^*(R), &(8.2.31)\\ \xi_2(R) &= -\frac{R^2}{GM^{(0)}(R)}\left\{\frac{1}{3}\Omega^2R^2+\varphi_{in}(R)\right\}, &(8.2.32)\end{aligned}$$

where $M^{(0)}(R)$ is the static mass.

The eccentricity is defined as

$$e=\sqrt{1-\left(\frac{r_p}{r_e}\right)^2} \qquad (8.2.33)$$

On the other hand, one can write the eccentricity in terms of the function $\xi_2(a)$ as follows

$$e=\sqrt{-\frac{3\xi_2(a)}{a}}\,, \qquad (8.2.34)$$

an expression that is convenient for further analysis which will be presented below.

8.2.8 Ellipticity and Gravitational Love Number

The quantity defined by

$$\epsilon(R)=-\frac{3}{2R}\xi_2(R), \qquad (8.2.35)$$

is the ellipticity of the surface of constant density labeled by R. The equation for $\epsilon(R)$ is known as the Clairaut differential equation:

$$\frac{M(R)}{R}\frac{d^2\epsilon(R)}{dR^2}+\frac{2}{R}\frac{dM(R)}{dR}\frac{d\epsilon(R)}{dR}+\frac{2dM(R)}{dR}\frac{\epsilon(R)}{R^2}-\frac{6M(R)\epsilon(R)}{R^3}=0\,. \qquad (8.2.36)$$

Here both $M(R)$ and $\rho(R)$ are known functions of R. The ellipticity must be regular at small values of R, and Eq.(7.3.54) shows that it approaches a constant at $R = 0$. With this boundary condition, Eq.(7.3.54) may be integrated to find the shape of $\epsilon(R)$. To find the magnitude of $\epsilon(R)$ one needs to use (8.2.32).

By comparing the expressions for the eccentricity (8.2.34) and ellipticity (7.3.53), we see that they are interrelated by means of the function ξ_2. The only difference is that the eccentricity is usually determined only on the surface of the rotating configurations, whereas the ellipticity can be considered from the center to the surface as a function of the radial coordinate.

It is easy to show that Eq.(7.3.54) can be written in the form given in Ref.[73]

$$R^2\frac{d^2\epsilon(R)}{dR^2} + 6\frac{\rho(R)}{\rho_m(R)}\left[R\frac{d\epsilon(R)}{dR} + \epsilon(R)\right] = 6\epsilon(R), \tag{8.2.37}$$

where

$$\rho_m(R) = \frac{3M(R)}{4\pi R^3} \tag{8.2.38}$$

is the average mass density. By introducing the new function

$$\eta_2(R) = \frac{R}{\epsilon(R)}\frac{d\epsilon(R)}{dR}\,, \tag{8.2.39}$$

Eq.(7.3.56) reduces to the well known Clairaut-Radau equation [70]

$$R\frac{d\eta_2(R)}{dR} + 6\mathcal{D}(R)[\eta_2(R)+1] + \eta_2(R)[\eta_2(R)-1] = 6, \tag{8.2.40}$$

where

$$\mathcal{D}(R) = \frac{\rho(R)}{\rho_m(R)} \tag{8.2.41}$$

encodes the relevant information about the structure of the body. The differential equation is integrated outward from $R = 0$, with the boundary conditions $\mathcal{D}(0) = 1$ and $\eta(R = 0) = 0$, up to $R = a$, obtaining the value $\eta_2(R = a)$.

Note, on the other hand, that if $\xi_2(R)$ is known from Eq.(8.2.32), then $\epsilon(R)$ can also be determined from Eq.(7.3.53), so that it is not necessary to integrate Eq.(7.3.59). Accordingly, we obtain

$$\eta_2(a) = \frac{a}{\epsilon(a)}\frac{d\epsilon(R)}{dR}\bigg|_{R=a} = \frac{a}{\xi_2(a)}\frac{d\xi_2(R)}{dR}\bigg|_{R=a} - 1\,. \tag{8.2.42}$$

One can see from here that $\eta_2(a)$ does not depend on the angular velocity of the star as $\xi_2(R) \sim \Omega^2$ and $d\xi_2(R)/dR \sim \Omega^2$, hence their ratio does not depend on Ω. Therefore, in this work, we will use the quantity

$$k_2 = \frac{3 - \eta_2(a)}{2[2 + \eta_2(a)]}\,, \tag{8.2.43}$$

as the rotational apsidal constant which allows us to introduce the rotational Love number λ as

$$\lambda = \frac{2a^5}{3G} k_2 \tag{8.2.44}$$

where a is the static (spherical) radius as before.

The rotational Love number can be written in the dimensionless form as

$$\bar{\lambda} = \frac{c^{10}}{G^4} \frac{\lambda}{M^5} = \frac{2}{3} \left(\frac{c^2 a}{GM} \right)^5 k_2 \tag{8.2.45}$$

where M is the total static mass.

8.2.9 Quadrupole Moment

The Newtonian potential $\Phi(R, \Theta)$ outside the star $R > a$ will be written as

$$\Phi(R, \Theta) = -\frac{GM_{tot}}{R} + \frac{GQ}{R^3} P_2(\cos\Theta), \tag{8.2.46}$$

where Q is the mass quadrupole moment of the star. For a vanishing Q one recovers the static configuration. Moreover, according to Hartle's definition $Q > 0$ represents an oblate object and $Q < 0$ corresponds to a prolate object [69]. One can define the dimensionless quadrupole moment as

$$\bar{Q} = \frac{c^2 Q}{J^2/M}. \tag{8.2.47}$$

Knowing the fact that the quadrupole moment Q and the square of the angular momentum J^2 are proportional to Ω^2 one defines $\bar{Q}$ as a function that does not depend on the angular velocity at all. This will have some practical importance.

8.2.10 Total Moment of Inertia

The total moment of inertia of a rotating configuration is determined as the sum of the moment of inertia of a static star and the change in the moment of inertia due to rotation and deformation

$$I_{tot}(a) = I^{(0)}(a) + I^{(2)}(a) , \tag{8.2.48}$$

where the moment of inertia of the non-rotating star is determined as before as in Eq. (8.2.4) and its change due to rotation is given by

$$\begin{aligned} I^{(2)}(a) &= \frac{8\pi}{3}\int_0^a \rho(R)R^4\left(\frac{d\xi_0}{dR} - \frac{1}{5}\frac{d\xi_2}{dR} + \frac{4}{R}\left[\xi_0 - \frac{1}{5}\xi_2\right]\right)dR \\ &= \frac{8\pi}{3}\int_0^a \left(\left[\frac{1}{5}\xi_2(R) - \xi_0(R)\right]\frac{d\rho(R)}{dR}\right)R^4 dR. \end{aligned} \tag{8.2.49}$$

In the corresponding limit, our results coincide with the definition of the moment of inertia for slowly rotating relativistic stars as given in [113].

It is convenient to define the total dimensionless moment of inertia as

$$\bar{I}_{tot}(a) = \bar{I}^{(0)}(a) + \bar{I}^{(2)}(a) = \left(\frac{c^2}{G}\right)^2 \frac{I_{tot}(a)}{M^3} = \left(\frac{c^2}{G}\right)^2 \frac{I^{(0)}(a)}{M^3} + \left(\frac{c^2}{G}\right)^2 \frac{I^{(2)}(a)}{M^3}. \tag{8.2.50}$$

8.3 Stability criteria for rotating white dwarfs

8.3.1 Inverse β-decay instability

The inverse β-decay instability allows us to determine the critical density which in turn defines the onset of instability for a white dwarf to collapse into a neutron star. The inverse β-decay instability is crucial both for static and rotating configurations. It represents one of the boundaries of the stability region in rotating white dwarfs [109].

It is known that a white dwarf might become unstable against the inverse β-decay process $(Z, A) \to (Z-1, A)$ through the capture of ultra-relativistic electrons, when the central density increases. In order to trigger such a process, the electron Fermi energy (with the rest-mass subtracted off) must be larger than the mass difference between the initial (Z, A) and final $(Z-1, A)$ nucleus. We denote this threshold energy as ϵ_Z^β. Usually, the condition $\epsilon_{Z-1}^\beta < \epsilon_Z^\beta$ is satisfied and therefore the initial nucleus undergoes two successive decays, i.e., $(Z, A) \to (Z-1, A) \to (Z-2, A)$ (see e.g. [111, 67]). Some of the possible decay channels in white dwarfs with the corresponding known experimental threshold energies ϵ_Z^β are listed in Table 8.1. The electrons in the white dwarf may eventually reach the threshold energy to trigger a given decay at some critical density $\rho_{\rm crit}^\beta$. Since the electrons are responsi-

Decay	ϵ_Z^β (MeV)	$\rho_{\rm crit}^\beta$ (g/cm^3)
^{4}He $\rightarrow^3$ H + $n \rightarrow 4n$	20.596	1.37×10^{11}
^{12}C $\rightarrow^{12}$B $\rightarrow^{12}$Be	13.370	3.88×10^{10}
^{16}O $\rightarrow^{16}$N $\rightarrow^{16}$C	10.419	1.89×10^{10}
^{56}Fe $\rightarrow^{56}$Mn $\rightarrow^{56}$Cr	3.695	1.14×10^9

Table 8.1: Onset for the inverse β-decay of ^{4}He, ^{12}C, ^{16}O and ^{56}Fe. The experimental values of the threshold energies ϵ_Z^β have been taken from Table 1 of [115]; see also [116, 67].

ble for the internal pressure of the white dwarf, configurations with $\rho > \rho_{\rm crit}^\beta$ become unstable due to the softening of the equation of state, as a result of the electron capture process (see [114, 111] for details). In Table 8.1, for each threshold energy ϵ_Z^β, the critical density $\rho_{\rm crit}^\beta$ given by the Salpeter equation of state is shown (see also [85] for more details).

8.3.2 Mass-shedding limit

The velocity of particles on the equator of the star cannot exceed the Keplerian velocity of free particles, computed at the same location. At this limit, particles on the star's surface remain bound to the star only because of a balance between gravity and centrifugal forces. The evolution of a star rotating at the Keplerian rate is accompanied by a loss of mass, thus becoming unstable.

The Keplerian angular velocity is determined as follows

$$\Omega_{Kep} = \sqrt{\frac{GM_{tot}}{r_e^3}} \,. \tag{8.3.1}$$

In order to compute this quantity correctly, one needs to select a test value for the angular velocity, for example $\Omega_{test} = \sqrt{GM^{(0)}/a^3}$. Usually, $\Omega_{test} > \Omega_{Kep}$, hence one needs to decrease gradually the values of Ω_{test} and estimate Ω_{Kep} successively, until $\Omega_{test} = \Omega_{Kep}$ with a given precision. For that purpose, in practice it is convenient to use the shooting method.

The Keplerian angular velocity allows us to estimate the maximum angular velocity, mass, moment of inertia, and other parameters of the star. Moreover, it allows us to determine the stability region inside which rotating configurations can exist.

8.3.3 The turning-point criterion and secular axisymmetric instability

Friedman et al. formulated a turning-point method to locate the points where secular instability sets in for uniformly rotating relativistic stars. Along a sequence of rotating stars with fixed angular momentum and increasing central density, the onset of secular axisymmetric instability is given by

$$\left(\frac{\partial M(\rho_c, J)}{\partial \rho_c}\right)_J = 0. \tag{8.3.2}$$

Thus, the configurations on the right hand side of the maximum mass of a J-constant sequence are secularly unstable. After the secular instability sets in, the configuration evolves quasi-stationarily until it reaches a point of dynamical instability where gravitational collapse should take place (see Stergioulas 2003). The secular instability boundary thus separates stable from unstable stars. It is worth stressing here that the turning point of a constant J sequence is a sufficient but not a necessary condition for secular instability; therefore, it establishes an absolute upper bound for the mass (at constant J).

8.4 Equations of state for cold white dwarfs

In this section, we consider the Chandrasekhar [65, 71] and the Salpeter [111, 112] equations of state at zero temperature to describe the white dwarf matter. The Chandrasekhar relationship is given by

$$\begin{aligned} \varepsilon_{Ch} &= \rho c^2 = \frac{32}{3}\left(\frac{m_e}{m_n}\right)^3 K_n \left(\frac{\bar{A}}{Z}\right) x^3, \\ p_{Ch} &= \frac{4}{3}\left(\frac{m_e}{m_n}\right)^4 K_n \left[x(2x^2-3)\sqrt{1+x^2} + 3\ln(x+\sqrt{1+x^2})\right] \end{aligned} \tag{8.4.1}$$

This means that the energy density $\varepsilon_{Ch} = \varepsilon_{Ch}(R)$ is determined only by the nuclei, while the pressure $p_{Ch} = p_{Ch}(R)$ is determined by means of the degenerate electronic gas. Here $\bar{A}$ and Z are the average atomic weight and atomic number of the corresponding nuclei; $K_n = (m_n^4 c^5)/(32\pi^2\hbar^3)$ and $x = x(R) = p_e(R)/(m_e c)$ with $p_e(R)$, m_e, m_n, and $\hbar$ being the Fermi momentum, the mass of the electron, the mass of the nucleon, and the reduced Planck constant, respectively. Here we consider the particular case for the average molecular weight $\bar{A}/Z = \mu = 2$. In our diagrams, we will refer to $\mu = 2$ as the Chandrasekhar equation of state.

The extension of the Chandrasekhar approximation has been developed by

Salpeter. Here, for the sake of simplicity, we will define the energy density exactly as in the case of Chandrasekhar [111]. However, the electron-electron and electron-nuclei Coulomb interactions and the Thomas-Fermi corrections, which consider deviations of the electron charge distribution from uniformity, have been accounted when constructing the expression for the pressure of the degenerate white dwarf matter [112],

$$\begin{aligned} \varepsilon_{Sal} &= \varepsilon_{Ch}, \\ p_{Sal} &= p_{Ch} + p_C + p_{TF}\,, \end{aligned} \tag{8.4.2}$$

where $p_C + p_{TF}$ are the contributions to the pressure due to the Coulomb interactions and the Thomas-Fermi corrections. This sum is given by

$$\begin{aligned} p_C + p_{TF} &= -m_e c^2 \left(\frac{m_e c}{\hbar}\right)^3 \left[\frac{\alpha Z^{2/3}}{10\pi^2} \left(\frac{4}{9\pi}\right)^{1/3} x^4 \right. \\ &\left. + \frac{162}{175} \frac{(\alpha Z^{2/3})^2}{9\pi^2} \left(\frac{4}{9\pi}\right)^{2/3} \frac{x^5}{\sqrt{1+x^2}} \right], \end{aligned} \tag{8.4.3}$$

where $\alpha = 1/137.036$ is the fine structure constant.

The Salpeter equation of state allows one to take into account the nuclear composition of the white dwarf matter. We have chosen white dwarfs consisting of pure helium ^{4}He, carbon ^{12}C, oxygen ^{16}O and iron ^{56}Fe, according to Table 8.1. The behavior of the Chandrasekhar and Salpeter equations of state is illustrated in Figs. 8.1 and 8.2 in terms of the mass-central density and mass-radius relations. As one can see from these figures, the equations of state display different features, especially, in the case of white dwarfs composed of pure iron.

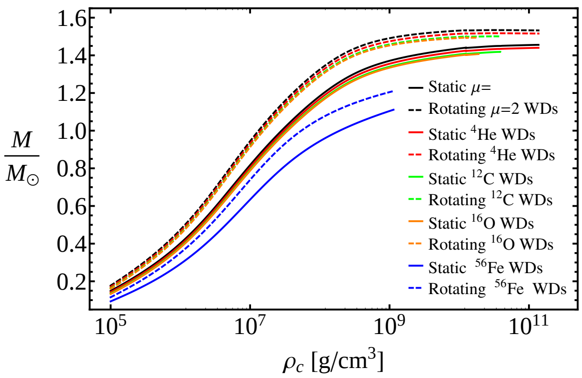

Figure 8.1: Mass versus central density.

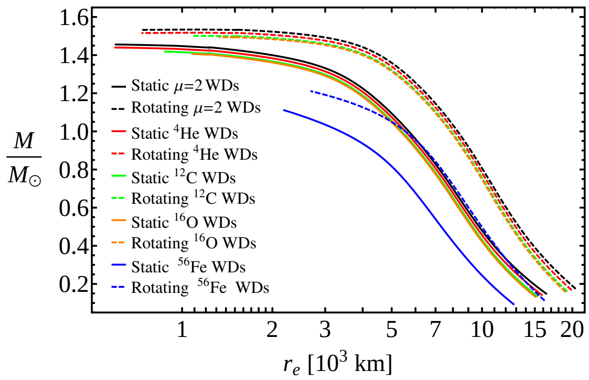

Figure 8.2: Mass versus radius.

8.5 Results and discussion

In equilibrium, a rotating star attains a balance between pressure, gravitational and centrifugal forces. In classical physics, the magnitude of the centrifugal force is determined by the angular velocity Ω of the fluid relative to a distant observer. We adopt this angular velocity Ω as the mass shedding or the Keplerian angular velocity for the sake of generality.

In Fig. 8.1, the mass of a white dwarf is shown as a function of the central density. The mass is given in units of one solar mass and the central density is given in g/cm^3. We have selected two equations of state: the Chandrasekhar equation of state with average molecular weight μ=2, and the Salpeter equation of state for pure helium ^{4}He, carbon ^{12}C, oxygen ^{16}O and iron ^{56}Fe white dwarfs, as limiting cases. All solid curves indicate non-rotating (static) white dwarfs, whereas all dashed curves indicate rotating white dwarfs at the mass shedding rate. As expected, rotating white dwarfs have larger masses with respect to their static counterparts. In all our computations we restricted the maximum values of the central density to the values of inverse β-decay density to fulfill the stability condition of white dwarfs [85, 109]. In addition, by using the turning point method, we investigated secular instability of the white dwarfs. It turned out that in classical physics all uniformly rotating white dwarfs are stable against axisymmetric secular instabilities.

Fig. 8.2 shows the mass and equatorial radius relation. The equatorial radius for a static case reduces to the static radius. All legends in the plot are the same as in Fig. 7.3 and hereafter we keep these legends in all our plots. Depending on the equation of state and chemical composition, white dwarfs display different mass-radius relations. This explains the variety of observed white dwarfs. Nowadays, we have data for more than thirty two thousand white dwarfs and all of them have diverse characteristics [117, 118, 119, 120,

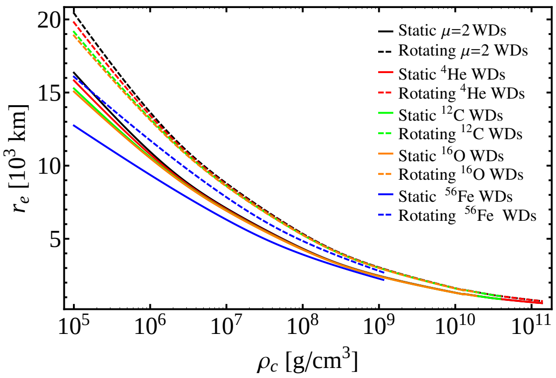

Figure 8.3: Radius versus central density.

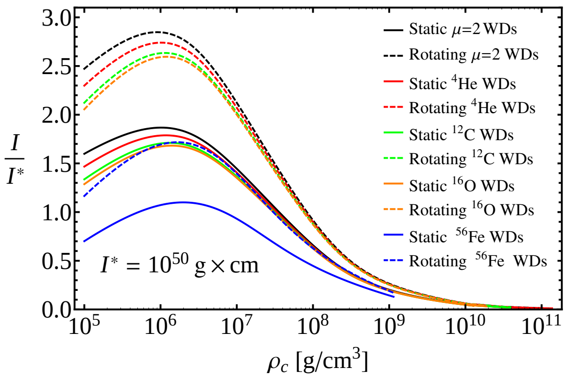

Figure 8.4: Moment of inertia versus central density.

121].

Fig. 8.3 shows the radius and central density relations of both rotating (equatorial radius) and non-rotating (static radius) white dwarfs. For increasing central densities, the gap between equatorial and static radii is narrowed. Thus, white dwarfs become more gravitationally bound and spherical.

Fig. 8.4 illustrates the behavior of the normalized moment of inertia in terms of the static mass and the square of the static radius as a function of the central density. For lower densities, the difference in the moment of inertia between rotating and static white dwarfs is quite large. However, for higher densities the difference becomes smaller. A different representation of the moment of inertia in terms of the total mass is given in Fig. 8.5, where one can see a similar behavior.

The total mass as a function of the angular momentum for rotating white dwarfs is presented in Fig. 8.6. The dependence between mass and angular momentum is nonlinear. This is obvious for large values of the angular mo-

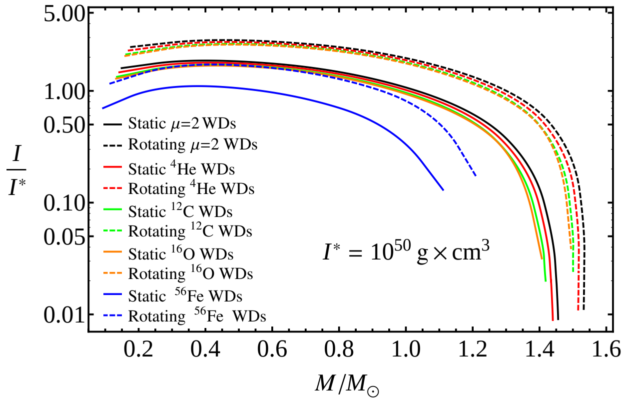

Figure 8.5: Moment of inertia versus mass.

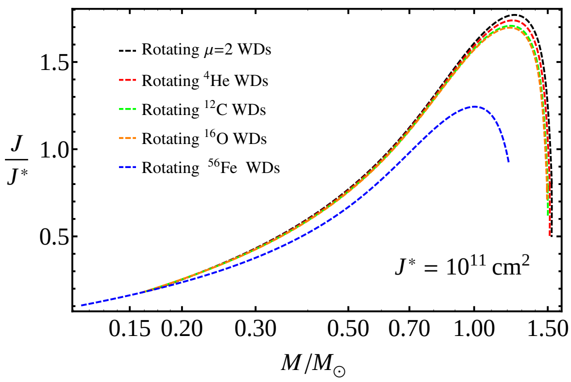

Figure 8.6: Mass versus angular momentum.

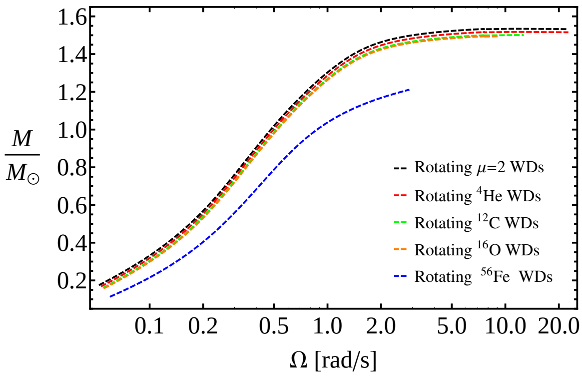

Figure 8.7: Mass in units of solar mass versus Keplerian angular velocity.

mentum. This is a very important fact, especially when one deals with its astrophysical implications.

The rotating mass as a function of the angular velocity is presented in Fig. 8.7. As one can see, due to their compactness, massive white dwarfs can reach higher values for the angular velocity than less massive stars. In addition, Fig. 8.7 shows the maximum rotation rate for a given mass.

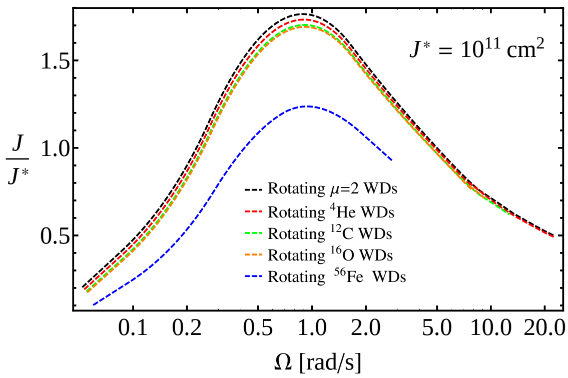

Figure 8.8: Angular momentum versus angular velocity.

The angular momentum as a function of the angular velocity along the Keplerian sequence is given in Fig. 8.8. It should be noted that the angular momentum is not always directly proportional to the angular velocity. Depending on the mass, one can observe either spin-up or spin-down effects. White dwarfs having mass close to the Chandrasekhar mass limit can experience both spin-up and spin-down effects [109, 110].

The eccentricity of rotating white dwarfs as a function of the central density is shown in Fig. 8.9. For higher densities, the eccentricity decreases and

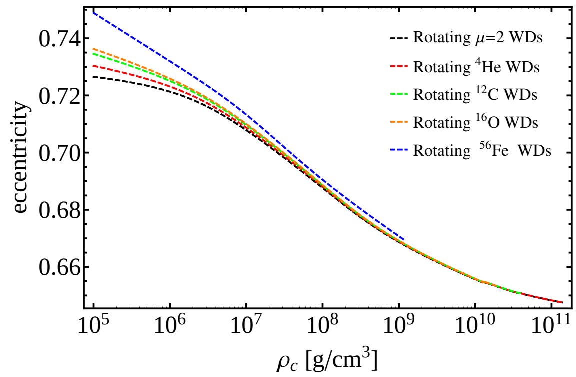

Figure 8.9: Eccentricity versus central density.

vice versa. Thus, white dwarfs with increasing central density become more spherical as they approach their maximum mass.

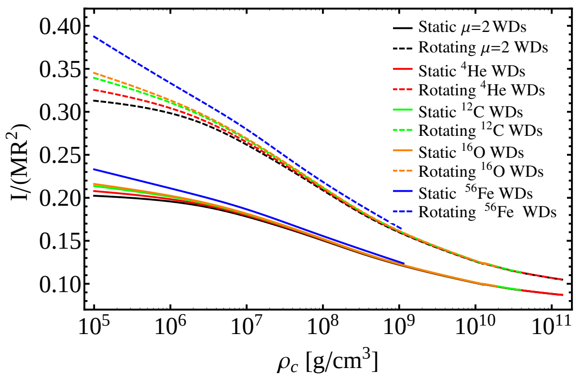

Figure 8.10: Normalized moment of inertia versus central density.

The normalized moment of inertia is shown as a function of the central density in Fig.8.10. The behavior of both rotating and static moments of inertia is similar to the eccentricity.

The normalized quadrupole moment as a function of the central density is shown in Fig. 8.11. Here the normalized quadrupole moment shows the same behavior as the eccentricity and the moment of inertia. So, for more massive white dwarfs it decreases as the radius decreases.

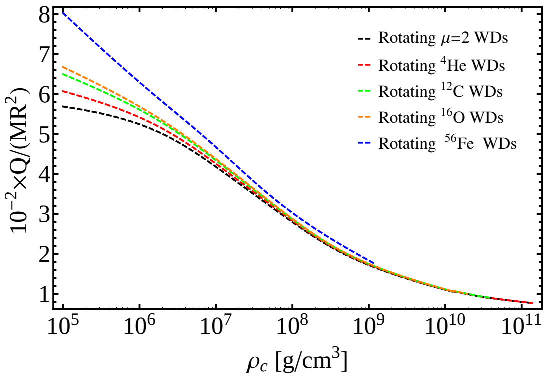

Figure 8.11: Normalized quadrupole moment versus central density.

8.6 I-Love-Q and I-Q-e relations for white dwarfs

First, we notice that an inspection of Figs. 8.9, 8.10 and 8.11 shows an intriguing correlation between the moment of inertia and the quadrupole moment as functions of the eccentricity of rotating white dwarfs. We therefore analyze the behavior of the moment of inertia as a function of the eccentricity in Fig. 8.12, where the moment of inertia is divided by 10^{22} cm^3 for the sake of convenience. As expected, the moment of inertia increases with the eccentricity until it reaches a maximum value which depends on the angular velocity and composition of the star. This graphic does not present any especial peculiarities. However, the situation changes completely, if instead we consider the normalized moment of inertia $I/(10MR^2)$ with respect to the static configuration. The result is presented in Fig. 8.13 for static and rotating stars. It shows an universal behavior for the I-e relation, independently of the equation of state and chemical composition of the white dwarf.

Consider now the quadrupole moment; in Fig. 8.14, we show its behavior as a function of the eccentricity. The value of the quadrupole is bounded, reaching a maximum value which depends on the chemical composition of the star. Furthermore, in Fig. 8.15, we consider the normalized quadrupole $Q/(100MR^2)$, where M and R correspond to the static configuration. The universality of this Q-e relation is evident because no dependence can be observed neither from the equation of state nor from the chemical composition of the white dwarf.

We now consider the quadrupole moment as a function of the moment of inertia. This behavior is illustrated in Figs. 8.16 and 8.17. Similar to the previous cases for the I-e and Q-e relations, the Q-I relation displays the same trend, i.e., only for the normalized quantities one can establish the same universality, independently of the equation of state.

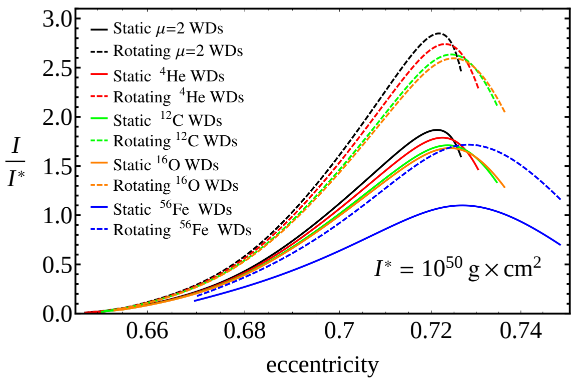

Figure 8.12: Moment of inertia versus eccentricity.

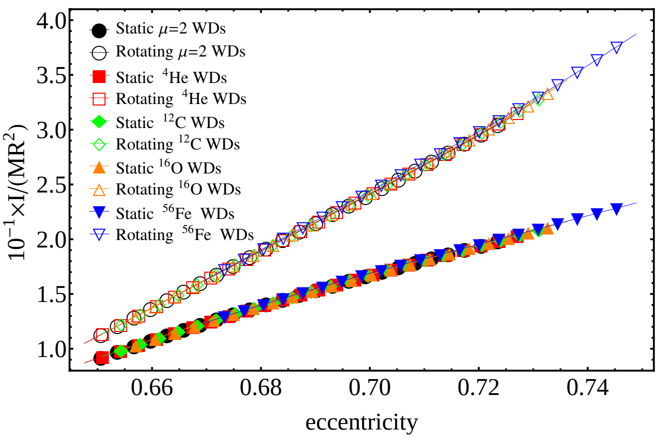

Figure 8.13: Normalized moment of inertia versus eccentricity.

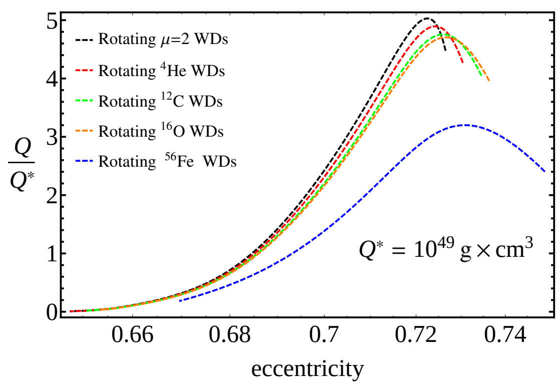

Figure 8.14: Quadrupole moment versus eccentricity.

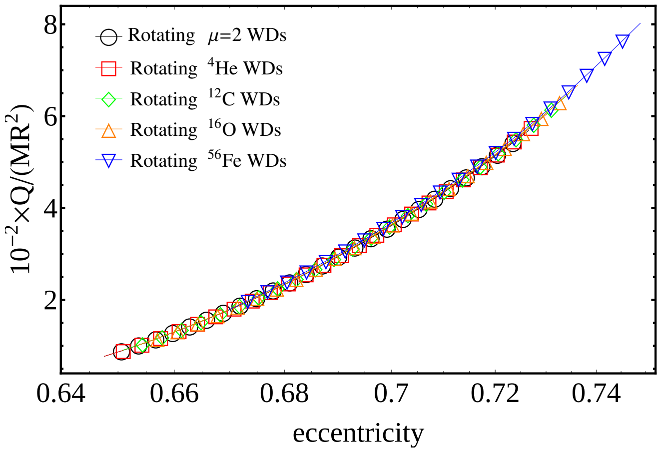

Figure 8.15: Quadrupole moment normalized to the static mass and corresponding radius squared versus eccentricity.

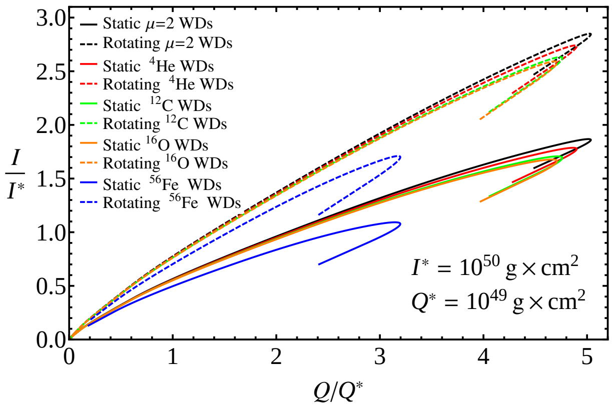

Figure 8.16: Quadrupole moment versus moment of inertia.

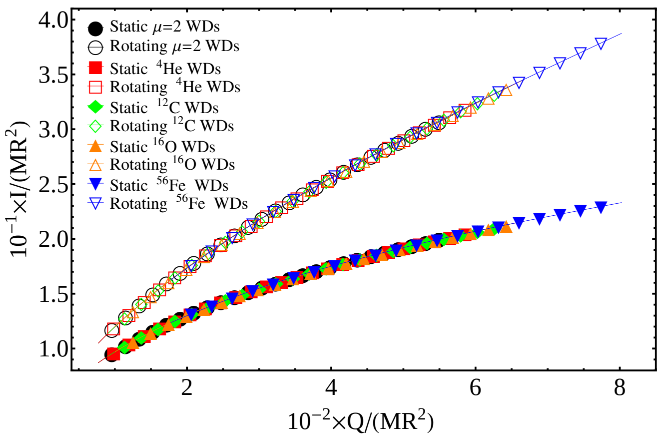

Figure 8.17: Normalized quadrupole moment versus normalized moment of inertia.

Finally, we investigate the I-Love-Q relations in Figs. 8.18, 8.19 and 8.20 which represent the dependence of the dimensionless moment of inertia $\bar{I}$, the dimensionless Love number $\bar{\lambda}$, and the dimensionless quadrupole moment $\bar{Q}$ on each other. We see that the I-Love-Q relations proposed in [83] for relativistic objects are also true for white dwarf stars in classical physics.

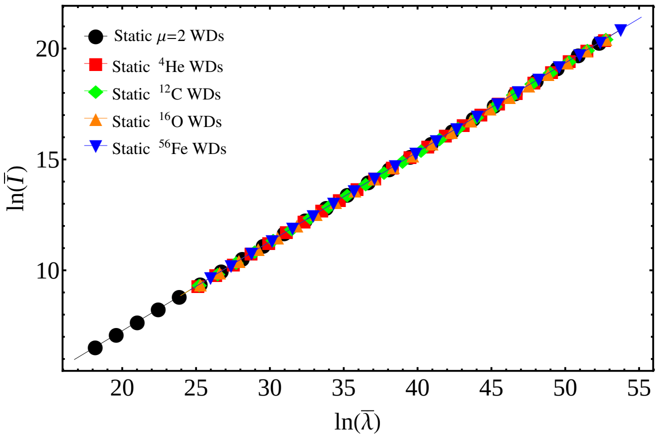

Figure 8.18: Dimensionless moment of inertia versus dimensionless Love number.

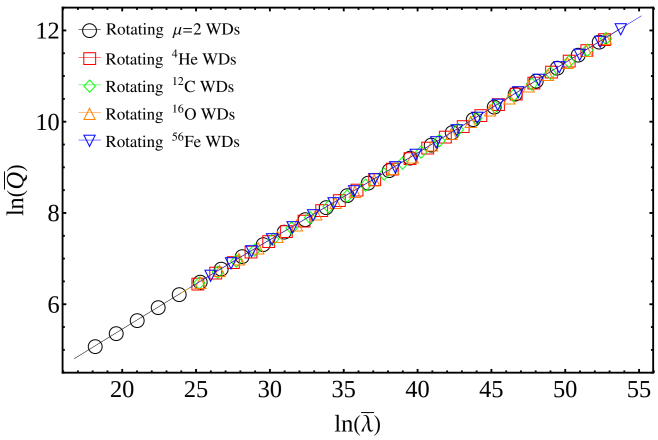

Figure 8.19: Dimensionless quadrupole moment versus dimensionless Love number.

The relation between the dimensionless quantities is given as

$$\bar{\lambda} = \bar{I}^{(0)}\bar{Q}, \tag{8.6.1}$$

where $\bar{I}^{(0)}$ is the moment of inertia of the static configuration. If we write it in physical units

$$\lambda = \frac{Q}{\Omega^2}, \tag{8.6.2}$$

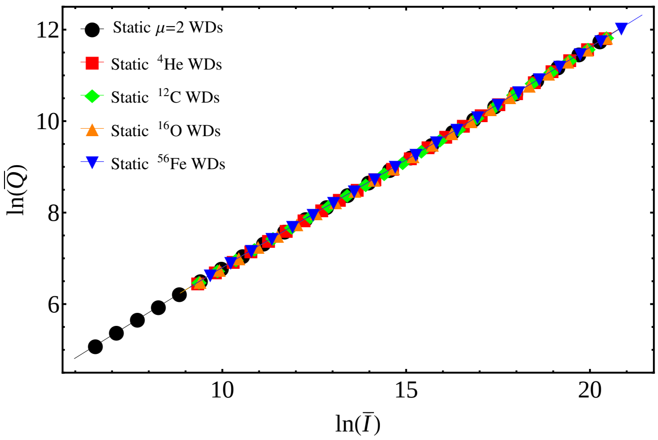

Figure 8.20: Dimensionless quadrupole moment versus dimensionless moment of inertia.

where Ω is the angular velocity of the star, this equation is equivalent to Eq.(2.253) of [70], namely,

$$GI_{02} = -\frac{2}{3}\sqrt{\frac{5}{4\pi}}k_2\Omega^2 a^5, \tag{8.6.3}$$

where $-I_{02}\sqrt{4\pi/5} = Q$ is the quadrupole moment, $(2/3)k_2a^5/G = \lambda$ is the rotational Love number, though it does not directly depend on Ω. Although this relation is very well-known in Newtonian physics, it is astonishing that it holds also in relativistic regimes [83].

Regarding the relation between the quadrupole moment and eccentricity, it can be confronted with Eq.(2.260) of [70], i.e.,

$$J_2 = \frac{2}{3}\frac{k_2}{1+2k_2}e^2, \tag{8.6.4}$$

where $J_2 = Q/(Ma^2)$ is the dimensionless quadrupole defined as the normalization of the quadrupole moment in terms of the static mass and the square of the static radius. It then follows that there exist Q-e , I-Q and hence I-e-Q relations.

8.7 Remarks

We numerically integrated the underlying differential equations in order to determine the structure of slowly rotating classical white dwarfs in hydrostatic equilibrium. In particular, the relations for the mass, the shapes of rotating stars, the moment of inertia, and the quadrupole moment were established as functions of the central density and angular velocity for the Chandrasekhar and Salperter equations of state. All these quantities play a fundamental role in describing the equilibrium configurations of uniformly rotating main sequence stars as well as massive stars. The results we obtained for the interdependence between mass, radius, density, etc. are in agreement with other works related to rigidly rotating white dwarfs in Newtonian physics [74, 75, 76, 77, 79].

Furthermore, we considered all the stability criteria for rotating white dwarfs and investigated only configurations that are stable against the mass-shedding limit, inverse β-decay instability and the axisymmetric secular instability. Unlike general relativity, in classical physics all white dwarfs, even those composed of light elements such as helium ^{4}He and carbon ^{12}C, are stable against secular instabilities.

In addition, by solving the Radau-Clairaut equation or, equivalently, by determining the function ξ_2 for the deviation from spherical symmetry, we calculated the rotational Love number and corroborated the I-Love-Q relations for rotating white dwarfs. In doing this, we worked with the Chandrasekhar and Salpeter equations of state, selecting different nuclear compositions for the latter. In all diagrams, the white dwarfs composed of pure iron ^{56}Fe have very particular distinctive features. Nonetheless, this distinction disappeared in the I-Love-Q relation. We also investigated the dependence of the moment of inertia and the quadrupole in terms of the eccentricity. We found that the I-Q-e relation is universal because it does not depend on the equation of state and on the chemical composition of the white dwarf.

From the astrophysical standpoint, I-Love-Q-e relations are key points as just knowing only two of the parameters, one can extract the rest basic quantities. It would be interesting to investigate in the relativistic regime the I-Q-e relations and test other relations to be independent of the equation of state. This will be the issue of future studies.

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