

NEW RESULTS

Recent works by Wen-Biao Han have concerned the radiation emitted by a spinning body orbiting a black hole. This is a very interesting result which fits other research lines in this report (see section “Symmetries in General Relativity” included in the present report.). Here follows the draft version of his recently accepted paper by Physical Review D.

Gravitational Radiations from a Spinning Compact Object around a Supermassive Kerr Black Hole in Circular Orbit

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Abstract

The gravitational waves and energy radiations from a spinning compact object with stellar mass in a circular orbit in the equatorial plane of a supermassive Kerr black hole are investigated in this paper. The effect how the spin acts on energy and angular momentum fluxes is discussed in detail. The calculation results indicate that the spin of a small body should be considered in waveform-template production for the upcoming gravitational wave detections. It is clear that when the direction of spin axes is the same as the orbital angular momentum (“positive” spin), spin can decrease the energy fluxes which radiate to infinity. For antirection spin (“negative”), the energy fluxes to infinity can be enlarged. And the relations between fluxes (both infinity and horizon) and spin look like quadratic functions. From frequency shift due to spin, we estimate the wave-phase accumulation during the inspiraling process of the particle. We find that the time of particle inspiral into the black hole is longer for positive spin and shorter for negative compared with the nonspinning particle. Especially, for extreme spin value, the energy radiation near the horizon of the extreme Kerr black hole is much more than that for the nonspinning one. And consequently, the maximum binding energy of the extreme spinning particle is much larger than that of the nonspinning particle.

I. INTRODUCTION

It is a popular opinion that at the center of our Galaxy and many galaxies there are supermassive black holes [1]-[4]. And extreme mass ratio inspiral (EMRI): a compact stellar-mass object (black hole, neutron star or white dwarf) spiraling into the supermassive black hole in galactic nuclei is one of the most important sources of gravitational waves for NASA’s Laser Interferometer Space Antenna (LISA) [5]. People estimated that LISA should see about 50 such events of EMRIs at distances out to redshifts $z \gtrsim 1$ [6][7][8], and the signal-to-noise ratios of these sources are around 10-100 [9]. Furthermore, the gravitational signals from EMRIs can also offer very useful information to study the mass distribution of galactic nuclei and the basic physics near the horizon of black holes, such as the no-hair theorem.

For the motives mentioned above, we should have enough accurate and numerous waveform templates to match the upcoming observational data in the future. A very universal method to calculate the gravitational wave templates is the black hole perturbation technology developed by Teukolsky [10][11]. The Teukolsky equation determines the (linearized) evolution of a perturbation to the Kerr spacetime. There are many published papers investigating gravitational radiations and orbital evolution. The Teukolsky equation is solved in these papers by post-Newtonian expansion ([12][13] and references inside) and numerical simulation ([14]-[18] and references inside).

Usually, the study of EMRIs did not contain spins of the small mass particles. Because for the spinning particle in the Kerr spacetime, the dynamical equation is non-integrable, different from the spinless one. The motion of spinning particles is much more difficult to work out. Only a few papers considered gravitational waves from the spinning particles for some special orbits by using the Teukolsky equation. Mino, Shibata and Tanaka first investigated gravitational waves induced by a spinning particle falling into a rotating black hole [19]. Then Tanaka et al. researched the case of particle with small spin in circular orbits around a Kerr black hole [20].

On the other hand, when the spin is measured by terms of μM , the spin parameter is

$$S \leq \frac{\mu^2}{\mu M} = \frac{\mu}{M} \ll 1, \quad (1)$$

where μ, M are the mass of the small object and central black hole respectively. People think that the spins of small objects are so tiny that they can be ignored without obvious influence.

But small spin can produce considerable accumulated effect when we observe the secular evolution of EMRIs, and can also make the phase of gravitational waves different from the spinless case. The importance of spin was emphasized by Burko who considered a spinning compact object in a quasicircular, planar orbit around a Schwarzschild black hole [21][22]. The effect of spin is at most marginally relevant for signal detection [23]. And we will see that the results of this paper can prove the spin of small body cannot be ignored.

Furthermore, for the bigger value of spin, perhaps the first order perturbation method is not accurate enough, but the result can also offer some useful information on intermediate-mass-ratio inspirals (IMRIs). Especially, when spin achieves extreme value ($S \sim 1$), some very interesting phenomena appear. For example, in the Schwarzschild or Kerr background, researchers found chaotic behavior of extreme spinning particles[24]-[27]. Even without chaos, extreme spin can also produce some distinct orbital characteristics [27]. Obviously, gravitational signals of chaotic orbits would be much different from those of regular ones. References [28] and [29] by using quadruple radiation formula calculated gravitational waveforms of chaotic systems.

In the present paper, as the beginning of future research, we study the gravitation radiation of spinning particles in circular orbits on an equatorial plane by calculating the Teukolsky equation. In Ref. [20], the authors used the post-Newtonian expansion and assumed very small spin to keep the circular orbits. Differing from them, we calculate the Teukolsky equation numerically and make the particles be restricted on equatorial plane exactly. So we can get the results in a strong field where the small objects are near the black holes ($r \sim M$). The signals from such a strong-field area are easier for the detectors to catch. And also we can study gravitational radiation and orbital evolution when the value of spin becomes much bigger.

As mentioned above, the strong spin of particle can produce chaotic orbits in some parameter range. But the circular orbits studied in this paper, can be determined completely and off course are nonchaotic. This ensures the stability of numerical results. In the future, we will discuss the gravitational waves and radiation reaction of chaotic systems in another paper.

This paper is organized as follows. In the next section, we first use the Papapetrou equation to deduce the orbital parameters of the spinning particle in the equatorially circular orbit. Second, we introduce the Teukolsky-Sasaki-Nakamura formalisms briefly and our numerical method. In Sec. IV we present our results in detail. Finally, our conclusion and discussion are given in the last section.

We use units $G = c = 1$ and the metric signature $(-, +, +, +)$. Distance and time are measured by the central black hole mass M , and the energy of the particle is measured by μ , the mass of itself. We measure the orbital angular momentum and spin of the particle in unit of μM .

II. CIRCULAR ORBIT OF A SPINNING PARTICLE

The equations of motion of a spinning test particle in a curved spacetime were given first by Papapetrou [30], and then reformulated more clearly form by Dixon [31], In our units the can be written as

$$\frac{dx^\mu}{d\tau} = v^\mu, \quad (2)$$

$$\frac{dp^\mu}{d\tau} = -\left(\frac{1}{2}R^*_{\nu\alpha\beta}p^\alpha S^\beta + \Gamma_{\nu\alpha}^\mu p^\alpha\right)v^\nu, \quad (3)$$

$$\frac{dS^\mu}{d\tau} = -(p^\mu R^*_{\nu\alpha\beta\gamma}S^\alpha p^\beta S^\gamma + \Gamma_{\nu\alpha}^\mu S^\alpha)v^\nu. \quad (4)$$

Where, v^μ is the four-velocity, p^μ is the linear momentum and S^μ the spin vector. The latter two must satisfy

$$p^\nu p_\nu = -\mu^2, \quad S^\nu S_\nu = S^2, \quad (5)$$

where S represents spin magnitude. We define

$$R^*_{\mu\nu\alpha\beta} = \frac{1}{2}R_{\mu\nu}^{\rho\sigma} \epsilon_{\rho\sigma\alpha\beta}, \quad (6)$$

and $\epsilon^{\rho\sigma\alpha\beta}$ is the Levi-Civita tensor.

Because of spin, the motion of particle does not follow the geodesic, then the p^μ is no longer parallel to v^μ . If following Dixon, choosing the rest frame of the particles center of mass, we can get one of the spin supplementary conditions,

$$p^\mu S_\mu = 0, \quad (7)$$

and the relation between four-velocity and linear momentum

$$v^\mu = u^\mu + \frac{2S^{\mu\nu} R_{\nu\alpha\beta\gamma} u^\alpha S^{\beta\gamma}}{4 + R_{\alpha\beta\gamma\delta} S^{\alpha\beta} S^{\gamma\delta}}, \tag{8}$$

where $u^\nu = p^\nu/\mu$ and spin tensor is

$$S^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} S_\alpha u_\beta. \tag{9}$$

As with the spinless particle, there are two conservations, energy and total angular momentum which includes orbit and spin angular momentum

$$E = -p_t + \frac{1}{2} g_{t\mu,\nu} S^{\mu\nu}, \tag{10}$$

$$J_z = p_\phi - \frac{1}{2} g_{\phi\mu,\nu} S^{\mu\nu}. \tag{11}$$

Another constant, the Carter constant Q for the nonspinning particle, does not exist for a spinning one anymore.

Usually, solving Eq. (2-4) is difficult. By assuming the spinning particle moving in circular orbit on the equatorial plane of a Kerr black hole, the problem can be simplified greatly. In this assumption, the only nonzero component of spin must be $s \equiv S^\theta = -S/r$. Then, using the condition of circular orbit $dp^r/d\tau = 0$, combining Eq.(3) and the relation (8), after a long calculation, we get the angular frequency

$$\Omega_\phi \equiv d\phi/dt = \frac{\sum_{i=0}^2 a_i s^i \mp \sqrt{\sum_{i=0}^4 b_i s^i}}{\sum_{i=0}^2 c_i s^i}, \tag{12}$$

where s^i presents the i th power of s . The upper sign refers to prograde and the lower to retrograde orbits and

$$a_0 = 2Mar, \quad a_1 = -(3r^2 + 6a^2)M, \quad a_2 = (4M^2a + 3Mar), \quad b_0 = 4Mr^5, \tag{13}$$

$$b_1 = -12Mar^4, \quad b_2 = 13M^2r^4, \quad b_3 = -6M^2ar^3, \quad b_4 = (9a^2 - 8Mr)M^2r^2, \tag{14}$$

$$c_0 = 2(Ma^2r - r^4), \quad c_1 = -6Ma(a^2 + r^2), \quad c_2 = (4M^2a^2 + 6Ma^2r + 2Mr^3). \tag{15}$$

This result coincides with the approximative expression Eq. (4.26) in Ref.(3) when $s \ll 1$. Obviously, Eq. (12) reduces to the angular frequency of the spinless particle if the spin $s = 0$,

$$\Omega_\phi \equiv d\phi/dt = \pm \frac{M^{1/2}}{r^{3/2} \pm aM^{1/2}}, \tag{16}$$

As mentioned before, because of the spin, the four-velocity is not parallel with linear momentum as before. From Eq. (8), relation between them is

$$v^t = u^t \left[1 - \frac{M(3a^2 + r^2)s^2}{\mu^2 r^3} \right] + \frac{3Ma(a^2 + r^2)}{\mu^2 r^3} s^2 u^\phi, \tag{17}$$

$$v^\phi = u^\phi \left[1 + \frac{M(3a^2 + 2r^2)s^2}{\mu^2 r^3} \right] - \frac{3Mas^2 u^t}{\mu^2 r^3}. \tag{18}$$

We can clearly see that the difference between u^μ and v^μ is the s^2 order. And from the energy (10) and total angular momentum (11), we can get,

$$u_t = -\frac{E}{\mu} + \left(\frac{z_1 E/\mu - X J_z}{\mu + z_1 s} \right) s, \tag{19}$$

$$u_\phi = \frac{J_z}{\mu} + \left(\frac{z_2 J_z/\mu + Y E}{\mu - z_2 s} \right) s, \tag{20}$$

where,

$$X = \frac{\frac{M}{r^2}}{\mu - \frac{Ma}{r^2}s}, \quad (21)$$

$$Y = \frac{(r - \frac{Ma^2}{r^2})}{\mu + \frac{Ma}{r^2}s}, \quad (22)$$

$$z_1 = \frac{M}{r^2} \left[a - \frac{(r - \frac{Ma^2}{r^2})s}{\mu - \frac{Ma}{r^2}s} \right], \quad (23)$$

$$z_2 = \frac{M}{r^2} \left[a + \frac{(r - \frac{Ma^2}{r^2})s}{\mu + \frac{Ma}{r^2}s} \right]. \quad (24)$$

Furthermore, the effective 4-velocity components

$$u^t = u_0^t + \left\{ \left[\frac{a^2}{r^2} - \frac{(r^2 + a^2)^2}{\Delta r^2} \right] \mathcal{X} + \frac{a}{r^2} \left(1 - \frac{r^2 + a^2}{\Delta} \right) \mathcal{Y} \right\} s, \quad (25)$$

$$u^\phi = u_0^\phi + \left\{ \left[1 - \frac{a^2}{\Delta} \right] \mathcal{Y} + \frac{a^2}{r^2} \left(1 - \frac{r^2 + a^2}{\Delta} \right) \mathcal{X} \right\} s, \quad (26)$$

where u_0^t, u_0^ϕ are the components without spin, and \mathcal{X}, \mathcal{Y} just are the coefficients before s in Eq.(19,20).

For confirming a circular orbit, first the Kerr parameter a and spin magnitude S are fixed. Then we give a radius r_0 , from which

$$\left. \frac{\partial U(r)}{\partial r} \right|_{r_0} = 0, \quad E = U(r_0), \quad (27)$$

can determine the total angular momentum J_z and energy E . Where $U(r)$ is the so-called effective potential,

$$U(J_z, a, S, r) = \frac{B}{A} J_z + \sqrt{\left(\frac{B^2}{A^2} - \frac{C}{A} \right) J_z^2 + \frac{\mu^2 Z^2}{A}}, \quad (28)$$

and the related coefficients are

$$Z = \sqrt{\Delta} \left(1 - \frac{Ms^2}{r\mu^2} \right)^2 / \left[1 - \left(\frac{Mas}{r^2\mu} \right)^2 \right], \quad (29)$$

$$A = \left[\Delta + (r^2 + a^2) \frac{2M}{r} \right] (\mu - z_2 s)^2 - \left(1 - \frac{2M}{r} \right) (\mu + z_1 s)^2 Y^2 s^2 + \frac{4Ma}{r} Z Y s, \quad (30)$$

$$B = \left(1 - \frac{2M}{r} \right) (\mu + z_1 s)^2 Y s - \left[\Delta + (r^2 + a^2) \frac{2M}{r} \right] (\mu - z_2 s)^2 X s - \frac{2Ma}{r} Z (1 + X Y s^2), \quad (31)$$

$$C = \left[\Delta + (r^2 + a^2) \frac{2M}{r} \right] (\mu - z_2 s)^2 X^2 s^2 - \left(1 - \frac{2M}{r} \right) (\mu + z_1 s)^2 + \frac{4Ma}{r} Z X s \quad (32)$$

Finally the four-momentum components p^t, p^ϕ can be calculated from Eq.(25,26).

Now, we deduce the energy-momentum tensor of a spinning particle in curved spacetime. From Dixon's classical literature [32], we can write down the energy-momentum tensor,

$$\begin{aligned} T^{\alpha\beta}(x) &= \mu \int d\tau \left\{ \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} u^{(\alpha}(x, \tau) v^{\beta)}(x, \tau) - \nabla_\gamma \left(S^{\gamma(\alpha}(x, \tau) v^{\beta)}(x, \tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \right) \right\} \\ &= \mu \int d\tau \left\{ \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} [u^{(\alpha} v^{\beta)} - \Gamma_{\gamma\delta}^{(\alpha} v^{\beta)} S^{\gamma\delta} + \Gamma_{\gamma\delta}^{(\alpha} S^{\beta)\gamma} v^{\delta}] - \frac{\partial_\gamma [S^{\gamma(\alpha} v^{\beta)} \delta^{(4)}(x - z(\tau))] 1}{\sqrt{-g}} \right\} \\ &\equiv \mu \int d\tau \left\{ \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} U^{\alpha\beta} - \frac{[\partial_\gamma V^{\gamma\alpha\beta} \delta^{(4)}(x - z(\tau))] }{\sqrt{-g}} \right\} \equiv \mu \mathcal{T}^{\alpha\beta}. \end{aligned} \quad (33)$$

where $z(\tau)$ is the world line of the spinning particle. For convenience, we write $U^{\alpha\beta} = u^{(\alpha} v^{\beta)} + U_{(s)}^{\alpha\beta}$. Considering there are only two independent components of the spin tensor in the circular orbit[29],

$$S^{tr} = -S^{rt} = s u_\phi, \quad S^{r\phi} = -S^{\phi r} = s u_t, \quad (34)$$

the nonzero components of $U_{(s)}^{\alpha\beta}$ are

$$\begin{aligned}
U_{(s)}^{tt} &= \Gamma_{rt}^t u_\phi u^t + \Gamma_{r\phi}^t u_\phi u^\phi, \\
U_{(s)}^{t\phi} &= \frac{1}{2}(\Gamma_{rt}^\phi u_\phi u^t + \Gamma_{r\phi}^\phi u_\phi u^\phi - \Gamma_{rt}^t u_t u^t - \Gamma_{r\phi}^t u_t u^\phi), \\
U_{(s)}^{rr} &= \Gamma_{\phi t}^r u_t u^t + \Gamma_{\phi\phi}^r u_t u^\phi - \Gamma_{tt}^r u_\phi u^t - \Gamma_{t\phi}^r u_\phi u^\phi, \\
U_{(s)}^{\phi\phi} &= -\Gamma_{rt}^\phi u_t u^t + \Gamma_{r\phi}^\phi u_t u^\phi.
\end{aligned} \tag{35}$$

And the nonzero components of $V^{\gamma\alpha\beta}$ are

$$\begin{aligned}
V^{trt} &= \frac{1}{2}u_\phi v^t s, & V^{tr\phi} &= \frac{1}{2}u_\phi v^\phi s; \\
V^{\phi rt} &= -\frac{1}{2}u_t v^t s, & V^{\phi r\phi} &= -\frac{1}{2}u_t v^\phi s; \\
V^{rt\phi} &= \frac{1}{2}(u_t v^t - u_\phi v^\phi) s.
\end{aligned} \tag{36}$$

Now, we project this energy-momentum tensor onto the Newman-Penrose null tetrad[33],

$$n_\alpha = \frac{1}{2} \left(\frac{\Delta}{\Sigma}, 1, 0, -\frac{a\Delta \sin^2 \theta}{\Sigma} \right), \tag{37}$$

$$\bar{m}_\alpha = \frac{\rho}{\sqrt{2}} (ia \sin \theta, 0, \Sigma, -i(r^2 + a^2) \sin \theta). \tag{38}$$

It can be,

$$T_{nn} = n_\alpha n_\beta T^{\alpha\beta}, \quad T_{n\bar{m}} = n_\alpha \bar{m}_\beta T^{\alpha\beta}, \quad T_{\bar{m}\bar{m}} = \bar{m}_\alpha \bar{m}_\beta T^{\alpha\beta}, \tag{39}$$

where $\rho = (r - ia \cos \theta)^{-1}$. So,

$$T_{ab} = \mu C_{ab} \tag{40}$$

where C_{ab} can be expanded as

$$C_{nn} = \frac{1}{4} \left[\frac{\Delta^2}{\Sigma^2} \mathcal{T}^{00} + \mathcal{T}^{11} + \frac{a^2 \Delta^2 \sin^4 \theta}{\Sigma^2} \mathcal{T}^{33} + 2 \frac{\Delta}{\Sigma} \mathcal{T}^{01} - 2 \frac{a \Delta^2 \sin^2 \theta}{\Sigma^2} \mathcal{T}^{03} - 2 \frac{a \Delta \sin^2 \theta}{\Sigma} \mathcal{T}^{13} \right], \tag{41}$$

$$\begin{aligned}
C_{n\bar{m}} &= \frac{\rho}{2\sqrt{2}} \left[\frac{ia\Delta \sin \theta}{\Sigma} (\mathcal{T}^{00} + (a^2 + r^2) \sin^2 \theta \mathcal{T}^{33}) + ia \sin \theta \mathcal{T}^{01} + \Delta \mathcal{T}^{02} - \right. \\
&\quad \left. \frac{i\Delta \sin \theta}{\Sigma} (r^2 + a^2 + a^2 \sin^2 \theta) \mathcal{T}^{03} + \Sigma \mathcal{T}^{12} - i(r^2 + a^2) \sin \theta \mathcal{T}^{13} - a \Delta \sin^2 \theta \mathcal{T}^{23} \right],
\end{aligned} \tag{42}$$

$$\begin{aligned}
C_{\bar{m}\bar{m}} &= \frac{\rho^2}{2} \left[-a^2 \sin^2 \theta \mathcal{T}^{00} + \Sigma^2 \mathcal{T}^{22} - (r^2 + a^2)^2 \sin^2 \theta \mathcal{T}^{33} + 2ia\Sigma \sin \theta \mathcal{T}^{02} + \right. \\
&\quad \left. 2a(r^2 + a^2) \sin^2 \theta \mathcal{T}^{03} - 2i\Sigma(r^2 + a^2) \sin \theta \mathcal{T}^{23} \right].
\end{aligned} \tag{43}$$

For circular orbits on the equatorial plane, the above equations can be greatly simplified.

III. GRAVITATIONAL RADIATION AND RADIATION REACTION

A. The Teukolsky-Sasaki-Nakamura formalism

In this paper, we use the Teukolsky equation to calculate gravitational radiation and radiation reaction. The Teukolsky formalism considers perturbation on the Weyl curvature scale ψ_4 , which can be decomposed in the frequency domain [10],

$$\psi_4 = \rho^4 \int_{-\infty}^{+\infty} d\omega \sum_{lm} R_{lm\omega}(r) {}_{-2}S_{lm}^{a\omega}(\theta) e^{im\phi} e^{-i\omega t}, \tag{44}$$

where $\rho = -1/(r - ia \cos \theta)$. The function $R_{lm\omega}(r)$ satisfied the radial Teukolsky equation

$$\Delta^2 \frac{d}{dr} \left(\frac{1}{\Delta} \frac{dR_{lm\omega}}{dr} \right) - V(r)R_{lm\omega} = -\mathcal{T}_{lm\omega}(r), \tag{45}$$

where $\mathcal{T}_{lm\omega}(r)$ is the source term, which will be given below, and the potential is

$$V(r) = -\frac{K^2 + 4i(r - M)K}{\Delta} + 8i\omega r + \lambda, \tag{46}$$

where $K = (r^2 + a^2)\omega - ma$, $\lambda = E_{lm} + a^2\omega^2 - 2am\omega - 2$. The spin-weighted angular function ${}_{-2}S_{lm}^{a\omega}(\theta)$ obeys the following equation,

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d {}_{-2}S_{lm}^{a\omega}}{d\theta} \right) + \left[(a\omega)^2 \cos^2 \theta + 4a\omega \cos \theta - \left(\frac{m^2 - 4m \cos \theta + 4}{\sin^2 \theta} \right) + E_{lm} \right] {}_{-2}S_{lm}^{a\omega} = 0. \tag{47}$$

The radial Teukolsky Eq.(45) has the general solution

$$R_{lm\omega}(r) = \frac{R_{lm\omega}^\infty(r)}{2i\omega B_{lm\omega}^{in} D_{lm\omega}^\infty} \int_{r_+}^r dr' \frac{R_{lm\omega}^H(r') \mathcal{T}_{lm\omega}(r')}{\Delta(r')^2} + \frac{R_{lm\omega}^H(r)}{2i\omega B_{lm\omega}^{in} D_{lm\omega}^\infty} \int_r^\infty dr' \frac{R_{lm\omega}^\infty(r') \mathcal{T}_{lm\omega}(r')}{\Delta(r')^2}, \tag{48}$$

where the $R_{lm\omega}^\infty(r)$ and $R_{lm\omega}^H(r)$ are two independent solutions of the homogeneous Teukolsky equation. They are chosen to be the purely ingoing wave at the horizon and purely outgoing wave at infinity respectively,

$$\begin{aligned} R_{lm\omega}^H &= B_{lm\omega}^{hole} \Delta^2 e^{-ipr^*}, \quad r \rightarrow r_+ \\ R_{lm\omega}^H &= B_{lm\omega}^{out} r^3 e^{i\omega r^*} + r^{-1} B_{lm\omega}^{in} r e^{-i\omega r^*}, \quad r \rightarrow \infty; \end{aligned} \tag{49}$$

and

$$\begin{aligned} R_{lm\omega}^\infty &= D_{lm\omega}^{out} e^{ipr^*} + \Delta^2 D_{lm\omega}^{in} r e^{-ipr^*}, \quad r \rightarrow r_+ \\ R_{lm\omega}^\infty &= r^3 D_{lm\omega}^\infty e^{-i\omega r^*}, \quad r \rightarrow \infty, \end{aligned} \tag{50}$$

where $k = \omega - ma/2Mr_+$ and r^* is the ‘‘tortoise coordinate’’. The solution (48) must be purely ingoing at horizon and purely outgoing at infinity. That is,

$$R_{lm\omega}(r \rightarrow \infty) = Z_{lm\omega}^H r^3 e^{i\omega r^*}, \tag{51}$$

$$R_{lm\omega}(r \rightarrow r_+) = Z_{lm\omega}^\infty \Delta^2 e^{-ipr^*}. \tag{52}$$

The complex amplitudes $Z_{lm\omega}^{H,\infty}$ are defined as

$$\begin{aligned} Z_{lm\omega}^H &= \frac{1}{2i\omega B_{lm\omega}^{in}} \int_{r_+}^r dr' \frac{R_{lm\omega}^H(r') \mathcal{T}_{lm\omega}(r')}{\Delta(r')^2}, \\ Z_{lm\omega}^\infty &= \frac{B_{lm\omega}^{hole}}{2i\omega B_{lm\omega}^{in} D_{lm\omega}^\infty} \int_r^\infty dr' \frac{R_{lm\omega}^\infty(r') \mathcal{T}_{lm\omega}(r')}{\Delta(r')^2}. \end{aligned} \tag{53}$$

The particle in this paper is in circular orbit on the equatorial plane, thus the particle’s motion is described only as the harmonic of the frequency Ω_ϕ (12), and defined

$$\omega_m = m\Omega_\phi. \tag{54}$$

Then $Z_{lm\omega}^{H,\infty}$ are decomposed as

$$\begin{aligned} Z_{lm\omega}^H &= Z_{lm}^H \delta(\omega - \omega_m), \\ Z_{lm\omega}^\infty &= Z_{lm}^\infty \delta(\omega - \omega_m). \end{aligned} \tag{55}$$

And the amplitudes $Z_{lm}^{H,\infty}$ fully determine the energy and angular momentum fluxes of gravitational radiations.

B. Energy, angular momentum fluxes and waveforms

From the Weyl scalar ψ_4 , we can extract the two polarization components h_+ , h_\times of the transverse-traceless metric perturbation at $r \rightarrow \infty$,

$$\psi_4(r \rightarrow \infty) \rightarrow \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times). \quad (56)$$

The energy flux in gravitational waves, from the Isaacson stress-energy tensor[39] is

$$\left(\frac{dE}{dAdt}\right)_{r \rightarrow \infty}^{\text{rad}} = \frac{1}{16\pi} \left\langle \left(\frac{\partial h_+}{\partial t}\right)^2 + \left(\frac{\partial h_\times}{\partial t}\right)^2 \right\rangle, \quad (57)$$

then we can get the fluxes at infinity as follow

$$\begin{aligned} \left(\frac{dE}{dt}\right)_{r \rightarrow \infty}^{\text{rad}} &= \sum_{lm} \frac{|Z_{lm}^H|^2}{4\pi\omega_m^2}, \\ \left(\frac{dJ_z}{dt}\right)_{r \rightarrow \infty}^{\text{rad}} &= \sum_{lm} \frac{m|Z_{lm}^H|^2}{4\pi\omega_m^3}. \end{aligned} \quad (58)$$

The energy and angular momentum fluxes at the horizon can be calculated by the below formula[11],

$$\begin{aligned} \left(\frac{dE}{dt}\right)_{r \rightarrow r_+}^{\text{rad}} &= \sum_{lm} \alpha_{lm} \frac{|Z_{lm}^\infty|^2}{4\pi\omega_m^2}, \\ \left(\frac{dJ_z}{dt}\right)_{r \rightarrow r_+}^{\text{rad}} &= \sum_{lm} \alpha_{lm} \frac{m|Z_{lm}^\infty|^2}{4\pi\omega_m^3}. \end{aligned} \quad (59)$$

Where the coefficient α_{lm} is given as

$$\alpha_{lm} = \frac{256(2Mr_+)p(p^2 + 4\epsilon^2)(p^2 + 16\epsilon^2)\omega_m^3}{|C_{lm}|^2}, \quad (60)$$

with $\epsilon = \sqrt{M^2 - a^2}/4Mr_+$, and

$$\begin{aligned} |C_{lm}|^2 &= [(\lambda + 2)^2 + 4am\omega_m - 4a^2\omega_m^2](\lambda^2 + 36ma\omega_m - 36a^2\omega_m^2) \\ &+ (2\lambda + 3)(96a^2\omega_m^2 - 48ma\omega_m)144\omega_m^2(M^2 - a^2). \end{aligned} \quad (61)$$

The total change in energy and angular momentum of the compact object is

$$\dot{E} = -(\dot{E}^H + \dot{E}^\infty), \quad \dot{J}_z = -(\dot{J}_z^H + \dot{J}_z^\infty). \quad (62)$$

From the energy loss and angular momentum loss of radiation, we can calculate the radiation reaction. There are two key points we can use to write down the orbit modification in an easy way: one is that circular orbits remain circular [35]-[37]; the other one is that S remains constant at first order [38] under adiabatic radiation reaction. The averaged rate of change of the radius is

$$\dot{r} = \left\{ \dot{E} - \left[\frac{B}{A} + \frac{(\frac{B^2}{A^2} - \frac{C}{A})J_z}{E - \frac{B}{A}J_z} \right] J_z \right\} / \left\{ \left(\frac{B}{A} \right)' J_z + \frac{(\frac{B^2}{A^2} - \frac{C}{A})' J_z^2 + \mu^2 (\frac{Z^2}{A})'}{2(E - \frac{B}{A}J_z)} \right\}. \quad (63)$$

It is obviously that spin can contribute to \dot{r} in the first order. Finally, the gravitational waveform can be calculated by

$$h_+(\theta, \phi, t) - ih_\times(\theta, \phi, t) = \sum_{lm} \frac{1}{\omega_m^2} Z_{lm}^H {}_{-2}S_{lm}^{a\omega_m}(\theta) e^{i(m\phi - \omega t)}. \quad (64)$$

C. The Sasaki-Nakamura equation

As mentioned in the above two subsections, for getting $Z_{lm}^{H,\infty}$, we should integrate the homogenous version of Eq.(45). But there is a difficulty when one numerically integrates Eq.(45) due to the long-range nature of the potential $V(r)$ in (45). In order to solve this problem, Sasaki and Nakamura developed the Sasaki-Nakamura function $X(r)$, governed by a short-ranged potential, to replace the Teukolsky function $R(r)$ [34]. The Sasaki-Nakamura equation reads as

$$\frac{d^2 X_{lm\omega}}{dr^{*2}} - F(r) \frac{dX_{lm\omega}}{dr^*} - U(r) X_{lm\omega} = 0. \quad (65)$$

The functions $F(r)$, $U(r)$ can be found in Ref.[34]. The Sasaki-Nakamura equation also admits two asymptotic solutions,

$$X_{lm\omega}^H = e^{-ipr^*}, \quad r \rightarrow r_+, \quad (66)$$

$$X_{lm\omega}^H = A_{lm\omega}^{\text{out}} \bar{P}(r) e^{i\omega r^*} + A_{lm\omega}^{\text{in}} P(r) e^{-i\omega r^*}, \quad r \rightarrow \infty; \quad (67)$$

and

$$X_{lm\omega}^\infty = C_{lm\omega}^{\text{out}} \bar{P}(r) e^{ipr^*} + C_{lm\omega}^{\text{in}} P(r) e^{-ipr^*}, \quad r \rightarrow r_+, \quad (68)$$

$$X_{lm\omega}^\infty = \bar{P}(r) e^{-i\omega r^*}, \quad r \rightarrow \infty, \quad (69)$$

where $P(r)$, $\bar{P}(r)$ can be found in Refs.[40], [41].

The solution of Eq.(65) is transformed to the solution of the Teukolsky equation by

$$R_{lm\omega}^{H,\infty} = \frac{1}{\eta} \left[\left(\alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta X_{lm\omega}^{H,\infty}}{\sqrt{r^2 + a^2}} - \frac{\beta}{\Delta} \frac{d}{dr} \frac{\Delta X_{lm\omega}^{H,\infty}}{\sqrt{r^2 + a^2}} \right]. \quad (70)$$

The relations between the coefficients of the Sasaki-Nakamura function and the Teukolsky function are

$$B_{lm\omega}^{\text{in}} = -\frac{A_{lm\omega}^{\text{in}}}{4\omega^2}, \quad B_{lm\omega}^{\text{hole}} = \frac{1}{d_{lm\omega}}, \quad D_{lm\omega}^\infty = -\frac{4\omega^2}{c_0}, \quad (71)$$

and the functions α , β , η , and $d_{lm\omega}$ are given clearly in Ref.[34] or in Appendix B of [14].

D. The source term

Now, we follow Ref.[14] to calculate the source term which is the left part of the Teukolsky equation (45). For the circular orbit on the equatorial plane ($\theta = \pi/2$),

$$\mathcal{T}_{lm\omega}(r) = \int dt \Delta^2 \{ (A_{nn0} + A_{n\bar{m}0} + A_{\bar{m}\bar{m}0}) \delta(r - r_0) + \partial_r [(A_{n\bar{m}1} + A_{\bar{m}\bar{m}1}) \delta(r - r_0)] + \partial_r^2 [A_{\bar{m}\bar{m}2} \delta(r - r_0)] \}. \quad (72)$$

All functions in the above equation are given as below:

$$A_{nn0} = -\frac{2\mu\rho^{-3}\bar{\rho}^{-1}C_{nn}}{\Sigma\Delta^2\dot{t}} [L_1^\dagger L_2^\dagger \mathcal{S} + 2ia\rho L_2^\dagger \mathcal{S}], \quad (73)$$

$$A_{n\bar{m}0} = -\frac{2\mu\sqrt{2}\rho^{-3}C_{n\bar{m}}}{\Sigma\Delta\dot{t}} \left[\left(\frac{iK}{\Delta} - \rho - \bar{\rho} \right) L_2^\dagger \mathcal{S} + i \left(\frac{iK}{\Delta} + \rho + \bar{\rho} \right) a\mathcal{S}(\rho - \bar{\rho}) \right], \quad (74)$$

$$A_{\bar{m}\bar{m}0} = \frac{\mu\mathcal{S}\rho^{-3}\bar{\rho}C_{\bar{m}\bar{m}}}{\Sigma\dot{t}} \left[\frac{K^2}{\Delta^2} + 2i\rho\frac{K}{\Delta} + i\partial_r \left(\frac{K}{\Delta} \right) \right], \quad (75)$$

$$A_{n\bar{m}1} = -\frac{2\mu\sqrt{2}\rho^{-3}C_{n\bar{m}}}{\Sigma\Delta\dot{t}} [L_2^\dagger \mathcal{S} + ia\rho(\rho - \bar{\rho})\mathcal{S}], \quad (76)$$

$$A_{\bar{m}\bar{m}1} = \frac{2\mu\mathcal{S}\rho - 3\bar{\rho}C_{\bar{m}\bar{m}}}{\Sigma\dot{t}} \left(\rho - \frac{iK}{\Delta} \right), \quad (77)$$

$$A_{\bar{m}\bar{m}2} = -\frac{\mu\mathcal{S}\rho^{-3}\bar{\rho}}{\Sigma\dot{t}} C_{\bar{m}\bar{m}}, \quad (78)$$

where \mathcal{S} just is shorthand of ${}_{-2}\mathcal{S}_{lm}^{a\omega}[\theta(t)]$. The projected energy-momentum tensor components are

$$C_{ab} = U_{ab} + i\omega V_{ab}^t - imV_{ab}^\phi + V_{ab}^r \frac{\partial}{\partial r}, \quad (79)$$

where ab represents $nn, n\bar{m}, \bar{m}\bar{m}$, and U_{ab}, V_{ab}^γ can be calculated from $U^{\alpha\beta}, V^{\gamma\alpha\beta}$ using Eq. (41)-(43). We can find that for the spinning particle there are three additional terms in the above equation compared to the nonspinning one. And

$$L_2^\dagger \mathcal{S} = a\omega \sin\theta \mathcal{S} - \sum_{k=l_{\min}}^{\infty} b_k [(k-1)(k+2)]^{1/2} {}_{-1}Y_{km}(\theta), \quad (80)$$

$$L_1^\dagger L_2^\dagger \mathcal{S} = \sum_{k=l_{\min}}^{\infty} b_k [(k-1)k(k+1)(k+2)]^{1/2} {}_0Y_{km}(\theta) + 2a\omega \sin\theta L_2^\dagger \mathcal{S} - (a\omega \sin\theta)^2 \mathcal{S}. \quad (81)$$

Where b_l represents the coefficients of the eigenvector of \mathcal{S} , ${}_0Y_{lm}(\theta)$ and ${}_{-1}Y_{lm}(\theta)$ are spin-weighted spherical harmonics. Of course, we should choose $\theta = \pi/2$ in our case.

Now, we can directly write down Z_{lm}^H, Z_{lm}^∞ . Taking the source term (72) into Eq.(53), then using Eq.(55), we have

$$Z_{lm}^H = \frac{\pi}{i\omega_m B_{lm}^{\text{in}}} I_{lm}^H(r_0), \quad (82)$$

$$Z_{lm}^\infty = -\frac{\pi}{4i\omega_m^3 d_{lm\omega} B_{lm}^{\text{in}}} I_{lm}^\infty(r_0), \quad (83)$$

where,

$$I_{lm}^{H,\infty}(r_0) = R_{lm\omega}^{H,\infty}(r_0)[A_{nn0} + A_{n\bar{m}0} + A_{\bar{m}\bar{m}0}] - \frac{dR_{lm\omega}^{H,\infty}}{dr} \Big|_{r_0} [A_{n\bar{m}1} + A_{\bar{m}\bar{m}1}] + \frac{d^2 R_{lm\omega}^{H,\infty}}{dr^2} \Big|_{r_0} A_{\bar{m}\bar{m}2}. \quad (84)$$

E. Numerical method

Now, we introduce simply our numerical method for calculating the gravitational radiation of the spinning particle inspiraling Kerr black hole. First, we need determine the orbital parameters of the particle. Setting Kerr parameter a , spin magnitude S and orbit radius, we solve Eq.(27) to get energy and angular momentum of the spinning particle. Then based on the discussions in Sec. II, it is easy to get the angular frequency, four-velocity and linear momentum.

Second, we numerically integrate the Sasaki-Nakamura equation to get Z_{lm}^H, Z_{lm}^∞ in the loop on $l = 2, 3, 4, \dots$ and $m = -l, \dots, l$. But considering the symmetry of $Z_{lm}^{H,\infty}$ about harmonics m , we only need a loop on m from 1 to l . Usually (based on the experience of many calculations done before) it is enough for obtaining satisfying accuracy to loop on 2 up to 8.

Then from Eqs.(58), (59) and (62), we can calculate the energy and angular momentum loss due to the gravitational radiation. Finally, the orbit change of radiation reaction and waveform are calculated by Eqs.(63) and (64).

For checking the validation of our code, we compare our results with the post-Newtonian expansion at weak field without spin. They match very well. An an example, we give the comparison of the energy fluxes at infinity from $a = 0$ to 1 while $r = 25M$ in Fig. 1. In addition, we compare our results with the published data of Fujita and Tagoshi [16] when spin is zero, and find they have good agreement too. After this check, we think our numerical results are reliable.

IV. NUMERICAL RESULTS

In this section, we present the numerical results of our calculations. We focus on how spin influences the gravitational waves.

A. Orbital frequency with spin

In Figs.(2) and (3), we show the shifts: $\Delta\Omega_\phi = \Omega_\phi(S \neq 0) - \Omega_\phi(S = 0)$ as a function of the orbital radius r while $a = 0$ and 0.996 respectively. It is clear that when the direction of spin inverses, the orbital frequency is bigger than

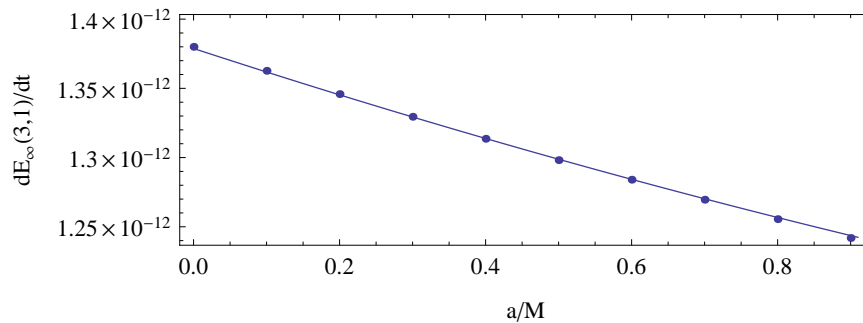


FIG. 1. Comparison of the energy flux at infinity for orbit at $r = 25M$ as a function of Kerr parameter a for $l = 3$, $m = 1$. Agreement between the numerical and post-Newtonian fluxes is good.

TABLE I. The orbitally angular frequency shift $\Delta\Omega_\phi$ due to the spin for $a = 0$.

S	-10^{-3}	-10^{-4}	-10^{-5}	10^{-5}	10^{-4}	10^{-3}
$r = 10$	1.50008×10^{-6}	1.50001×10^{-7}	1.50000×10^{-8}	-1.50000×10^{-8}	-1.49999×10^{-7}	-1.49992×10^{-6}
$r = 8$	2.92991×10^{-6}	2.92971×10^{-7}	2.92969×10^{-8}	-2.92969×10^{-8}	-2.92966×10^{-7}	-2.92946×10^{-6}
$r = 6$	6.94527×10^{-6}	6.94453×10^{-7}	6.94445×10^{-8}	-6.94444×10^{-8}	-6.94436×10^{-7}	-6.94362×10^{-6}
$r = 4$	2.34426×10^{-5}	2.34380×10^{-6}	2.34376×10^{-7}	-2.34374×10^{-7}	-2.34370×10^{-6}	-2.34324×10^{-5}
$r = 2$	1.87616×10^{-4}	1.87512×10^{-5}	1.87501×10^{-6}	-1.87499×10^{-6}	-1.87488×10^{-5}	-1.87384×10^{-4}

the nonspinning particle. This is because the total angular momentum is composed of the orbital angular momentum and the spin angular momentum of the particle. If spin has the same direction as the total angular momentum, the orbital angular momentum would decrease, whereas the orbital angular momentum must be added to counteract the spin. In addition, as the rotating angular momentum of the black hole increase, both the orbital angular momentum and the influence of spin on the angular momentum decrease. Usually, the absolute value of $\Delta\Omega_\phi$ increases as the orbital radius r decreases. But near the horizon of the Kerr black hole, perhaps there are some exceptions, for example, in Fig.(3), the $S = 10^{-2}$ one.

And in Tables I and II, we list the detailed values of the frequency shifts at several orbital radiuses. It can be found that the influence of physical spin on the orbital frequency, as well as the frequency of gravitational waves, is small and can be negligible. But during the long evolution time of EMRIs $\sim M^2/\mu$, the accumulatively phasic error without considering the spin of the inspiraling object is considerable. The phasic shift due to spin should be calculated by an integration $\Delta\phi = \int_{t(r_{\text{out}})}^{t(r_{\text{in}})} \Delta\Omega_\phi(r(t))dt$. This integration is difficult to calculate out, but we can do a simple estimate. For $\mu/M \sim 10^{-5}$, the average frequency shift is about 10^{-7} (see Table II) of the extreme Kerr black hole. During highly relativistic regime, inspiraling about $\sim M/\mu$ cycles, and the typical frequency $\Omega_\phi \sim 10^{-2}$, the dephasing of spinning and nonspinning particles is

$$\Delta\phi \sim \Delta\Omega_\phi \frac{M}{\mu} \frac{2\pi}{\Omega_\phi} \sim 2\pi. \quad (85)$$

We can find that though the wave-frequency shift of spin is very small, during the evolution period in a highly relativistic regime, it can produce considerable (about a 1 cycle) phase error. From the above estimate, we think the role of spin in the EMRIs should be calculated to obtain the template of gravitational waveform accurately.

TABLE II. The orbitally angular frequency shift $\Delta\Omega_\phi$ due to the spin for $a = 0.996$.

S	-10^{-3}	-10^{-4}	-10^{-5}	10^{-5}	10^{-4}	10^{-3}
$r = 10$	9.65822×10^{-7}	9.65768×10^{-8}	9.65762×10^{-9}	-9.65761×10^{-9}	-9.65756×10^{-8}	-9.65702×10^{-7}
$r = 8$	1.74151×10^{-6}	1.74137×10^{-7}	1.74136×10^{-8}	-1.74135×10^{-8}	-1.74134×10^{-7}	-1.74120×10^{-6}
$r = 6$	3.61475×10^{-6}	3.61431×10^{-7}	3.61426×10^{-8}	-3.61425×10^{-8}	-3.61421×10^{-7}	-3.61377×10^{-6}
$r = 4$	9.30694×10^{-6}	9.30480×10^{-7}	9.30459×10^{-8}	-9.30454×10^{-8}	-9.30433×10^{-7}	-9.30219×10^{-6}
$r = 2$	3.03510×10^{-5}	3.03302×10^{-6}	3.03281×10^{-7}	-3.03276×10^{-7}	-3.03255×10^{-6}	-3.03047×10^{-5}
$r = 1.2$	2.55531×10^{-5}	2.55116×10^{-6}	2.55075×10^{-7}	-2.55065×10^{-7}	-2.55024×10^{-6}	-2.54610×10^{-5}

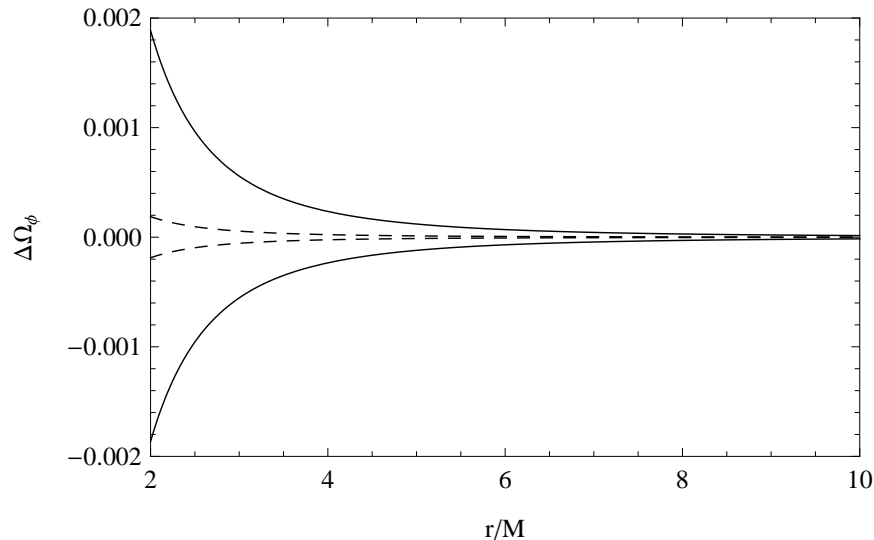


FIG. 2. The shift of the orbital frequencies between spin and nonspinning particles as a function of radius r while $a = 0$. From the top down, the spin magnitude S is -10^{-2} , -10^{-3} , 10^{-3} , 10^{-2} respectively.

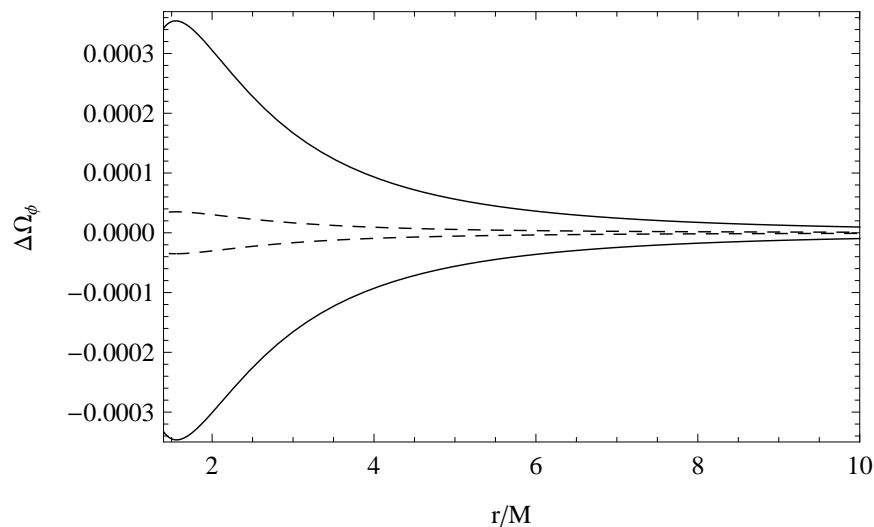


FIG. 3. The shift of the orbital frequencies between spin and nonspinning particles as a function of radius r while $a = 0.996$. From the top down, the spin magnitude S is -10^{-2} , -10^{-3} , 10^{-3} , 10^{-2} respectively.

B. Energy fluxes vs spin

Now we show the results of energy fluxes of the spinning particles. In Figs.(4, 5 and 6), we list the energy fluxes of a particle orbiting the black hole with $a = 0$, 0.996 and -0.996 respectively at infinity and the horizon. All the orbital radiuses are $r = 10$. We can find the energy fluxes to infinity and the horizon being a function of the spin S looks like quadratic functions. Usually, the dE_∞/dt decreases as the spin magnitude increases. This result coincides with the known relation between the energy fluxes and black hole rotation. In other words, the spin of particle together with the black hole's rotating angular momentum, if they are "positive" (the same direction with orbital angular momentum), will reduce the energy flux down to infinity, and if "negative", will add dE_∞/dt . At the same time the fluxes down to the horizon are no longer monotonic varying with spin; this is exhibited clearly in Figs.4-6 except for the $dE_h(2,1)/dt$ for $a = 0$. This phenomenon is also analogous with the case of the dE_h/dt vs the black hole's angular momentum a . An interesting phenomenon is that when $a \leq 0$, the function of dE_h/dt of S is a concave function, but a convex function when $a > 0$.

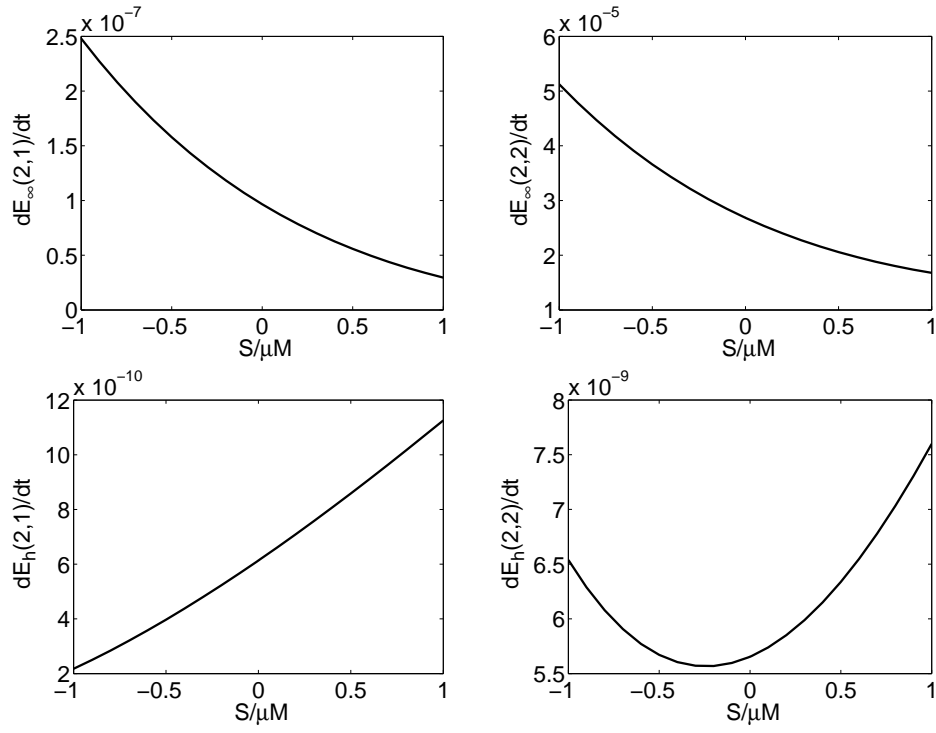


FIG. 4. The energy fluxes of $l = 2$ modes at infinity and the horizon as a function of the spin magnitude S . Where the Kerr parameter $a = 0$ and the orbital radius $r = 10M$.

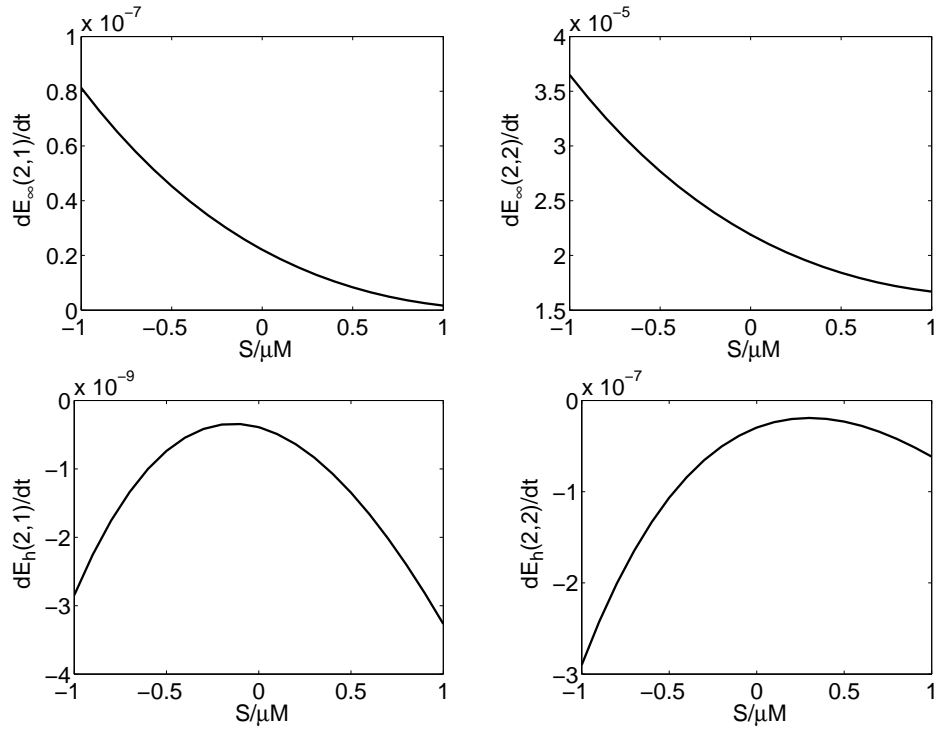


FIG. 5. The energy fluxes of $l = 2$ modes at infinity and the horizon as a function of the spin magnitude S . Where the Kerr parameter $a = 0.996$ and the orbital radius $r = 10M$.

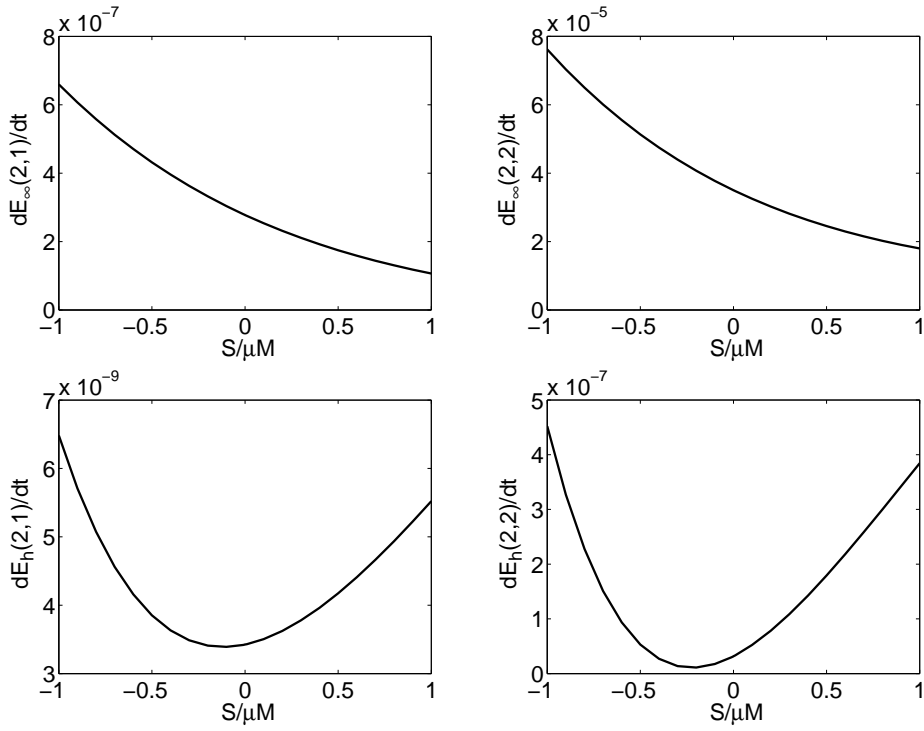


FIG. 6. The energy fluxes of $l = 2$ modes at infinity and the horizon as a function of the spin magnitude S . Where the Kerr parameter $a = -0.996$ and the orbital radius $r = 10M$.

In Fig.7, we give function images of the gravitational wave luminosity vs the spin S , and we produce enough accuracy of the luminosity up through $l = 8$. All of the results we figured out should be enough to clear the effect of spin acting on the energy fluxes. The angular momentum flux, from Eqs.(58, 59), is just the energy flux over Ω_ϕ . It is not necessary to list the results of the angular momentum flux here.

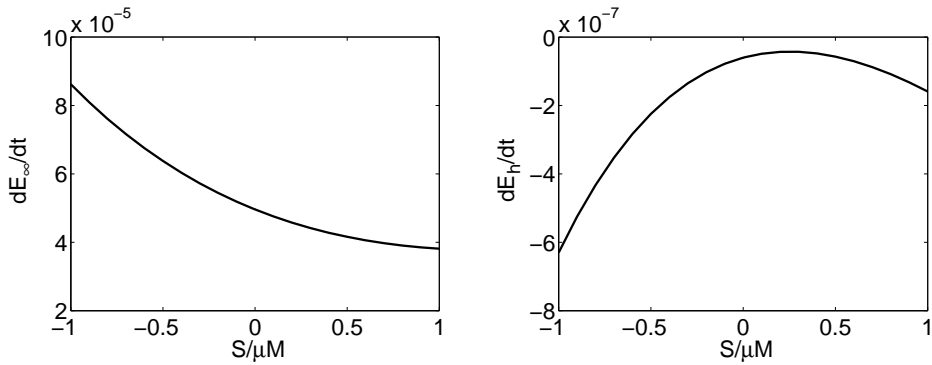


FIG. 7. The gravitational wave luminosity at infinity and the horizon (up through $l = 8$ to produce the accuracy) as a function of the spin magnitude S . Where the Kerr parameter $a = 0.996$ and the orbital radius $r = 10M$.

C. Energy loss vs spin

Because of the gravitational radiation, the energy of a particle decreases, and so does the orbital radius. The total energy loss can be calculated by Eq.(62). Through computing the energy-loss rates for different spin values at a certain radius, we can compare how the spin influences the energy loss of a particle due to gravitational radiation.

TABLE III. The relative differences of energy loss $\Delta\dot{E}/\dot{E}$ vs varied spin values for $a = 0.996$.

S	10^{-3}	10^{-4}	10^{-5}	-10^{-5}	-10^{-4}	-10^{-3}
$r = 10$	-4.31993×10^{-4}	-4.32215×10^{-5}	-4.32303×10^{-6}	4.32140×10^{-6}	4.32255×10^{-5}	4.32475×10^{-4}
$r = 8$	-5.98264×10^{-4}	-5.98605×10^{-5}	-5.98697×10^{-6}	5.98834×10^{-6}	5.98760×10^{-5}	5.99122×10^{-4}
$r = 6$	-9.31789×10^{-4}	-9.32626×10^{-5}	-9.32635×10^{-6}	9.32736×10^{-6}	9.32800×10^{-5}	9.33652×10^{-4}
$r = 4$	-1.84199×10^{-3}	-1.84453×10^{-4}	-1.84480×10^{-5}	1.84482×10^{-5}	1.84510×10^{-4}	1.84764×10^{-3}
$r = 2$	-9.76634×10^{-3}	-9.60818×10^{-4}	-9.59232×10^{-5}	9.58882×10^{-5}	9.57298×10^{-4}	9.41434×10^{-3}
$r = 1.5$	-7.57872×10^{-2}	-7.47114×10^{-3}	-7.46037×10^{-4}	7.45798×10^{-4}	7.44723×10^{-3}	7.33930×10^{-2}

TABLE IV. The particle energy at $r = 10M$, $r = 1.2M$ and energy loss vs varied spin values for $a = 0.996$.

S	$E(r = 10M)$	$E(r = 1.2M)$	ΔE
1.0	0.9504110109413	0.2966990902654	0.6537119206759
0.1	0.9517720321309	0.6831909453624	0.2685810867685
10^{-3}	0.9519178894668	0.7337036132776	0.2182142761892
10^{-5}	0.9519193435776	0.7342411704276	0.2176781731500
0	0.9519193582782	0.7342466042358	0.2176727540423
-10^{-5}	0.9519193729484	0.7342520379715	0.2176673349769
-10^{-3}	0.9519208270598	0.7347903703284	0.2171304567313
-0.1	0.9520657813190	0.7932472876811	0.1588184936379

For the convenience of our comparison, we introduce relative differences of the energy-loss rates,

$$\frac{\Delta\dot{E}(S)}{\dot{E}} = \frac{\dot{E}(S \neq 0) - \dot{E}(S = 0)}{\dot{E}(S = 0)}. \quad (86)$$

In Table III, the values of $\Delta\dot{E}(S)/\dot{E}$ are listed in detail.

It is assumed that LISA will observe gravitational waves of EMRIs at the typical frequency $\sim 10^{-2}$ Hz and the total wave cycle is about $N \sim 10^5$ for one year. Thus, the relative difference of energy luminosity due to spin $\Delta\dot{E}(S)/\dot{E}$ required to establish the accuracy for the cycle $\Delta N \leq 1$ must be $\leq 10^{-5}$ in circular orbit cases [42]. For more complicated orbits, such as zoom-whirl orbits, the requirement for accuracy would be stronger than for this circular one. From Table III, we can clearly find for typical $S = 10^{-5}$ that the relative differences of luminosity reach even exceed the above requirement. The results here argue again that the spin of a small object should be computed for the data analysis of LISA et al., and is consistent with the previous calculation of frequency shift in Sec. IV A.

If we want to calculate the total energy radiation during the whole inspiraling process from a big r to the horizon, we need to integrate \dot{E} step by step. But this calculation requires too much computer time, for example, for nonspinning particles, perhaps a CPU year or more. For spinning particles on circular or equatorial orbits, it would not add much cost [we need to calculate additional three terms in Eq.(79)]. But for general orbits, it needs to numerically work out the Papapetrou Eq.(2-4). By previous experiences [27][29], we know that it adds about half cpu year compared to nonspinning cases. Fortunately, thanks to the theory which circular remains circular under adiabatic radiation reaction, the energy loss of the particle is just the energy difference of the particle at the circular orbit radius r and near the horizon. In Table IV, we list the particle energy at radius $r = 10M$ and $r = 1.2M$ (very near the horizon), and give the energy difference between the two cases. We can find that the particle with positive spin will radiate more energy during the inspiral to the black hole than nonspinning one. At the same time, the particle with negative spin radiates less energy. Remember that the gravitational wave luminosity decreases when spin is positive and increases when spin is negative(see Fig.(7)). This means the inspiraling time of positively spinning particle should be longer, and shorter for negative one.

Especially, for the extreme spinning particle ($S \sim 1$), the radiated energy is much more than nonspinning one (see the first row in Table IV). This phenomenon only occurs when $a \approx 1$ and S is extreme (and the same direction as the rotation of black hole). It is known that the maximum fractional binding energy per unit rest mass is 42% for $a = 1$ and about 30% for $a = 0.998$ [43]. But we point out for extreme spinning particle, the binding energy is much larger than the nonspinning one. For example, for $a = 0.996$ and $S = 1$, the maximum fractional binding energy is about 77%. But we should take care that the physical value of spin is $\ll 1$. Thus, this big binding energy here only has theoretical meaning.

V. CONCLUSION AND DISCUSSION

In this paper, we have calculated the gravitational waves emitted by a spinning particle in circular orbits on equatorial plane around a Kerr black hole. We solved the completely orbital parameters of the spinning particles: orbital frequency, energy and total angular momentum and nonzero four-momenta. We also give the relation between the four-velocity and linear momentum.

Using the Teukolsky formalism of the Kerr black hole perturbation, we analyze the effect of spin on gravitation waves in detail. Our calculations were done in the strong-field area and can apply to extreme spin value. Perhaps this is the first time that the numerical results indicate the spin of a small body cannot be ignored for the future gravitational waveform data analysis .

The main effects of spin presented in this paper are: a positive spin can decrease wave frequency and energy luminosity, but this result is reversed when spin is negative; the relations of energy flux of spin are close to quadratic functions; in the whole inspiral process, the evolution time of positively spinning particle is longer and the total energy radiated is larger; the overall phase difference compared to the nonspinning case achieves the critical value, 1 cycle; the maximum binding energy of extreme spinning particles is much more than for a nonspinning one.

In future, it will be interesting to discuss the spin effect for more complicated orbits such as zoom-whirl orbits with a big eccentricity even with inclination.

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