Self Gravitating Systems, Galactic Structures and Galactic Dynamics

Contents

0.1 Topics

- General theory of self gravitating fluids
- Self gravitating systems: analytic solutions
- Self gravitating systems: Numerical Solutions
- Self gravitating systems: acoustic induced geometries
- Near-integrable dynamics and galactic structures
- Geometric approach to the integrability of Hamiltonian systems
- Stochasticity in galactic dynamics
- Influence of the expanding universe on galactic dynamics

0.2 Participants

0.2.1 ICRANet participants

- Donato Bini (IAC CNR, Rome, Italy)
- Dino Boccaletti (University La Sapienza, Rome, Italy)
- Christian Cherubini (University Campus Bio-Medico, Rome, Italy)
- Simonetta Filippi, Project Leader (University Campus Bio-Medico, Rome, Italy)
- Andrea Geralico (University La Sapienza, Rome, Italy)
- Giuseppe Pucacco (University of Rome "Tor Vergata", Rome, Italy)
- Kjell Rosquist (University of Stockholm, Sweden)
- Remo Ruffini (University La Sapienza, Rome, Italy)
- Alonso Sepulveda (Universidad de Antioquia, Medellin Colombia)
- Jorge Zuluaga (Universidad de Antioquia , Medellin Colombia)

0.2.2 Past Collaborators

- Giovanni Busarello(Astron. Obs. Capodimonte, Naples, Italy)
- Piero Cipriani (ICRA, Pescara, Italy)
- Fernando Pacheco (Icra, Rome, Italy)
- Giovanni Sebastiani(IAC,CNR, Rome, Italy)

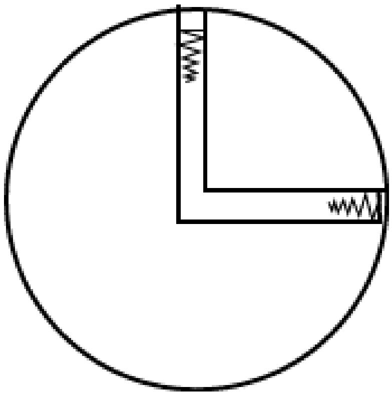

Figure 0.1: Newton's argument for an oblate earth.

0.3 Brief description

0.3.1 Self Gravitating Systems and Galactic Structures: historical Review of the Problem.

The investigation of the gravitational equilibrium of self-gravitating masses started with Newton in the third book of his Principia (1687). In this book Newton developed for the first time the idea of an oblate form of the earth due to rotation. In fact, using an argument based on the picture of a hole drilled up to the center of the earth from a point of the equator, and another one drilled from the pole, both meeting at the center and both filled with water in equilibrium in a rotating earth, Newton proposed the relationship among the ellipticity and the ratio centrifugal force/gravitational force in the form for slow rotation (Figure 1).

In this theory the earth is a spheroidal object and it is assumed that the distribution of mass is homogeneous. Newton concluded that the ellipticity is 1/230; however, the actual ellipticity is 1/294, a smaller result than Newton's predicted value; this discrepancy is interpreted in terms of the inhomogeneity of the earth. This work initiated the study of the rotation and configuration of the celestial bodies. A further step was given by Maclaurin (1742) who generalized the theory on the case when rotation cannot be considered slow but the density is homogeneous. The main result of this research is summarized in Maclaurin's formula giving the connection among angular velocity, eccentricity and density (Figure 2).

In Maclaurin's work is asserted that the direction of the composition of gravitational and centrifugal forces is perpendicular to the surface of the configuration. From Maclarin's formula the existence of two types of oblong configu-

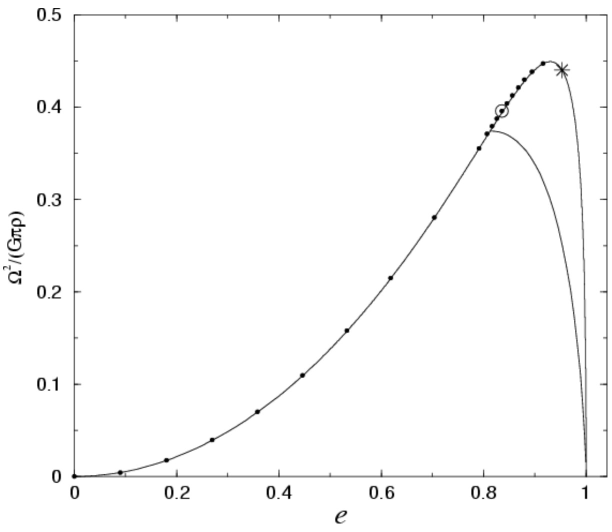

Figure 0.2: Bifurcation diagram of the Maclaurin's spheroids into Jacobi sequences

rations can be deduced, one of them of small ellipticity ($e \rightarrow 0$, in the limit $\Omega^2 \rightarrow 0$), the other one being highly flattened ($e \rightarrow 1$, in the limit $\Omega^2 \rightarrow 0$). These figures are known as Maclaurin's spheroids. In 1834 Jacobi recognized that ellipsoids with three unequal axes, studied before by Lagrange (1811), can be indeed equilibrium configurations. In fact, the existence of such figures can be established by a direct extension of Newton's original arguments. In the triaxial case we may imagine three canals drilled along the direction of the principal axis of the ellipsoid and filled with water. From this argument it is possible to find the relationship among the size of the axes, the angular velocity and the density. As a consequence, the following inequality among the axis can be calculated $1/a_3^2 > 1/a_1^2 + 1/a_2^2$. According to Mayer (1842), the Jacobian sequence bifurcates from the Maclaurin spheroids at the point where the eccentricity is $e = 0.81267$, a result that can be deduced from the Jacobi's formula. In fact, if $a_1 = a_2$, it can be demonstrated that $e = 0.81267$ and $\Omega^2/\pi G\rho = 0.37423$. As the maximum value of $\Omega^2/\pi G\rho$ along the Maclaurin sequence is 0.4493, it follows that for $\Omega^2/\pi G\rho < 0.37423$ there are three equilibrium figures: two Maclaurin spheroids and one Jacobi ellipsoid; for $0.4493 > \Omega^2/\pi G\rho > 0.3742$ only the Maclaurin figures are possible; for $\Omega^2/\pi G\rho > 0.4493$ there are not equilibrium figures. These results were found by Liouville again (1846) using angular momentum instead of the angular velocity as the variable. Liouville demonstrated that increasing angular velocity from zero the Jacobi configurations are possible only for angular momenta up a critical value, that on the point of bifurcation. Later, on 1857, Dedekind, considering configurations with a linear profile of velocity as seen from an inertial frame, proved explicitly a theorem regarding the existence of a relation among this configuration and another one with the same form but uniformly rotating. Thus, a rotating ellipsoid without vorticity has the same dynamics as the same ellipsoid with a linear profile of velocity and without rotation. The next step was given by Riemann (1860), who showed that for linear profiles of velocity, the more general type of motion compatible with an ellipsoidal figure of equilibrium consists in a superposition of a uniform rotation and an internal motion with constant vorticity ζ in the rotating frame. Exactly, he showed that the possible motions are given by: (1) uniform rotation without vorticity; this case leads to sequences of Maclaurin and Jacobi, (2) directions of Ω and ζ coinciding with a principal axis of the ellipsoid; these configurations are known as Riemann-S ellipsoids and spheroids. This case leads as special cases to Jacobi and Dedekind sequences, and (3) the directions of Ω and ζ lie in a principal plane of the ellipsoid. This case leads to three other classes of ellipsoids, namely I, II, III. The subsequent important discovery, due to Poincare (1885), was that, along the Jacobian sequence, there is a point of bifurcation similar to that found in the Maclaurin sequence and that a new sequence of pear-shaped configurations branches off from the Jacobi sequence, corresponding to neutral modes of oscillation of the third harmonics. As a further conclusion there are neutral modes belonging to fourth, fifth and higher harmonics. The fission theory of the origin of the double stars comes from this considerations as conjectured by Poincare and Darwin (1906). In 1924 Cartan established that the Jacobi ellipsoids become unstable at the first point of bifurcation and behave in a different mode with respect to the Maclaurin sequence that is stable on both sides of the bifurcation point. When the density of the configuration is inhomogeneous it is necessary to provide an equation of state. In this case the barotropes are particularly relevant for the construction of the theory. In 1889 an important theorem was proved by Hamy: a mass ellipsoidally stratified cannot have a uniform rotation. Another classical theorem is the one given by Dive (1930): a stratified heterogeneous spheroid, rotating and without differential rotation, cannot be a barotrope. A new approach to the problem of the equilibrium configurations was started in the works by Chandrasekhar and Lebovitz (1961-69) using an integral formulation of the hydrodynamical problem. This approach, known as virial theory allows one to recover the fundamental results of Maclaurin, Jacobi, Dedekind and Riemann and sheds a new light on the problem of stability of these configurations. The tensor virial equations are integral relations, consequences of the equations of stellar hydrodynamics, and they yield necessary conditions that can furnish useful insights for the construction of ellipsoidal models. The virial method developed in Chandrasekhar (1987), shows that only in the case of homogeneous self-gravitating masses having a linear velocity field, the virial equations of second order result equivalent to the complete set of hydrodynamic equations. In the general case of heterogeneous density and non linear velocity field, this equivalence does not exist, and the n-th order virial equations represent necessarly global conditions to be satisfied by any equilibrium configuration.

0.3.2 The ICRANET project on galactic structures.

A series of papers of the ICRANET group (I to IX) have been devoted on the generalization of the theory of ellipsoidal figures of equilibrium, endowed with rotation Ω and vorticity ζ, obtained for the homogeneous case in the classic work of Maclaurin, Jacobi, Dedekind and Riemann, and treated in a unified manner by Chandrasekhar in the virial equation formalism in his book "Ellipsoidal figures of equilibrium" (1987). This series of papers has followed a variety of tentative approaches, consisting of subsequent generalizations of known results: looking at more general density distributions, non-linear velocity fields, selected forms of the pressure tensor, and finally analysing the constraints imposed by the n-th order virial equations. Clearly we have proceeded step by step. The first new step in the theory of ellipsoidal figures in equilibrium was the introduction by Pacheco, Pucacco and Ruffini (paper I, 1986a) of an heterogeneous density distribution and an anisotropic pressure. Using only the second order virial equations, the equilibrium and stability of heterogeneous generalized Riemann ellipsoids was analysed for the case of a linear velocity field with a corresponding uniform vorticity. The stability of second harmonic perturbations of these equilibrium solutions was also analyzed. In paper II, (1986b) by Pacheco, Pucacco and Ruffini additional special solutions of the equations introduced in paper I were considered: some generalized Maclaurin-Dedekind spheroids with anisotropic pressure and their stability properties were analyzed. It was shown how the presence of anisotropic pressure extends the region of stability towards greater values of the eccentricity, which is similar to the homogeneous case considered by Wiegand (1980). In paper III, (1988) by Busarello, Filippi and Ruffini, a second step was made to the generalization of the solutions by introducing a fully general stratified density distribution of the form $\rho = \rho(m^2)$, where ρ is an arbitrary function of the equidensity surfaces. As in the previous papers the pressure is still anisotropic and the velocity field linear. The equilibrium and stability properties of anisotropic and heterogeneous generalized Maclaurin spheroids and Jacobi and Dedekind ellipsoids were studied. The Dedekind theorem, originally proved for homogeneous and isotropic configurations is still valid for this more general case. In Pacheco, Pucacco, Ruffini and Sebastiani (paper IV, 1989) several applications of the previous treatment of the generalized Riemann sequences were studied. Special attention was given to the axial ratios of the equilibrium figures, compatible with given values of anisotropy. A stability analysis of the equilibrium was performed against odd modes of second harmonic perturbations. In Busarello, Filippi and Ruffini, (paper V, 1989), the heterogeneous and anisotropic ellipsoidal Riemann configurations of equilibrium, obtained in the previous paper and characterized by non zero angular velocity of the figure and constant vorticity were used to model a class of elliptical galaxies. Their geometrical and physical properties were discussed in terms of the anisotropy, the uniform figure rotation, and the internal streaming motion. In Busarello, Filippi and Ruffini, (paper VI, 1990), the equilibrium, stability, and some physical properties of a special case of oblate spheroidal configurations which rotate perpendicularly to the symmetry axis were analyzed, still within the framework of the second order virial equations. In Filippi, Ruffini and Sepulveda (paper VII, 1990), the authors made an additional fundamental generalization by introducing a non linear velocity field with a cylindrical structure an a density distribution originally adopted in paper I of the form $\rho = \rho_c(1-m^2)^n$. The generalized anisotropic Riemann sequences coming from second order virial equations were studied. Some of the results obtained in that article have been modified by the consideration of the virial equations of n-th order, specially the claim regarding the validity of the Dedekind theorem, made on the basis of an unfortunate definition of certain coefficients. In Filippi, Ruffini and Sepulveda (paper VIII, 1991), following the theoretical approach of its predecessor, the nonlinear velocity field was extended to cover the most general directions of the vorticity and angular velocity. The more general form for the density $\rho = \rho(m^2)$ was adopted. Equilibrium sequences were determined, and their stability was analyzed against odd and even modes of second harmonic perturbations. In the next paper, discussed below in some detail, (Filippi, Ruffini and Sepulveda, 1996, Paper IX), the authors have considered an heterogeneous, rotating, self-gravitating fluid mass with anisotropic pressure and internal motions that are nonlinear functions of the coordinates in an inertial frame. We present here the complete results for the virial equations of n-th order, and we discuss the constraints for the equilibrium of spherical, spheroidal, and ellipsoidal configurations imposed by the higher order virial equations. In this context, the classical results of Hamy (1887) and Dive (1930) are also confirmed and generalized. In particular, (a) the Dedekind theorem is proved to be invalid in this more general case: the Dedekind figure with $\Omega = 0$ and $Z \neq 0$ cannot be obtained by transposition of the Jacobi figure, endowed $\Omega \neq 0$ and $Z = 0$; (b) the considerations contained in the previous eight papers on the series, concerning spherical or spheroidal configurations, are generalized to recover as special cases; (c) the n-th order virial equations severely constraint all heterogeneous ellipsoidal figures: as shown from tables in figures $3, 4$, and 5, all the heterogeneous ellipsoidal figures cannot exist.

The n-th order virial approach. Paper IX

Let us consider an ideal self-gravitating fluid of density ρ, stratified as concentric ellipsoidal shells and with an anisotropic diagonal pressure P_i. The fluid will be considered from a rotating frame with angular velocity Ω respect to an inertial frame. If u is the velocity of any point of the fluid and v is the gravitational potential, the hydrodynamical equation governing the motion, referred to the rotating frame is given by (Goldstein 1980)

$$\rho\frac{du}{dt} = -\nabla P + \rho\nabla v + \frac{1}{2}\rho\nabla|\Omega \times x|^2 + 2\rho u \times \Omega - \rho\dot{\Omega} \times x\,.$$

In this equation appear, respectively, the contributions of pressure, gravitation, centrifugal force, Coriolis force and a "Faraday" term, corresponding to the time change in the angular velocity. As usual, the gravitational potential satisfies the Poisson equation $\nabla^2 v = -4\pi G\rho$. We want to generalize the form of the virial equations, given for the second, third and fourth order by Chandrasekhar (1987), to the n-th order and for non-linear velocities. In the rotating frame the generalization can be done in complete analogy with the treatment presented by Chandrasekhar, multiplying the hydrodynamical equation by $x_i^{a-1}x_j^b x_k^c$ and integrating over the volume V of an ellipsoid. That means that the boundary of the configuration is defined; on the surface the pressure vanishes. The index i appears a-1 times, the indices j and k appear b and c times, respectively, and $a, b, c \geq 1$. Now, the velocity field considered in Chandrasekhar (1987, pag 69) is a linear function of the coordinates, corresponding to a constant vorticity. In order to generalize the velocity profile we propose a more general form, preserving the ellipsoidal stratification and the ellipsoidal boundary. Specifically, we assume the existence of a constant unit vector n fixed in the rest frame of the ellipsoid such that the velocity field circulates in planes perpendicular to n and having the same direction as Ω. Additionally the continuity equation must be preserved so that the velocity profile can be written as $u = \hat{n} \times (\mathcal{M}\dot{r})\hat{\phi}$. The dimensionless function ϕ describes the characteristic features of the velocity field. Thus, the velocity is linear if ϕ is 1. In this equation $\mathcal{M} = \sum_{i=1}^{3} \hat{e}_i x_i$. Then, the equation of the ellipse has the form $m^2 = r \cdot \mathcal{M} \cdot r$: In the steady state regime we may rewrite the generalized virial equations by introducing generalizations of kinetic energy tensor, angular momentum and moment of inertia respectively. Note first that the virial equations with odd values of $n = a+b+c$ are identically zero if the density, , and the function contain powers of $x_i x_j$ with $i, j = 1, 2$. We now turn our attention to the virial equations with even values of n. The analysis can be performed easily by classifying the powers of the coordinates. In fact, there are only the following possibilities: (1) a=even, b,c= odd; (2) b=even, a,c=odd; (3) c=even, a,b=odd; (4) a,b,c=even. Cases (2) and (3) are equivalent because of the interchangeable positions of j and k in eq (1). Cases (1) and (2) are non-equivalent owing to the privileged position of the index i. So, the steady state virial equations can be classified in three families. In this way the classical homogeneous and linear Maclaurin spheroids, Jacobi, Dedekind and Riemann ellipsoids can be generalized to cover heterogeneous systems with non uniform vorticity and anisotropic pressure, denoted as generalized Maclaurin spheroids, and generalized Jacobi, Dedekind and Riemann ellipsoids.

The generalized S-type Riemann ellipsoids (homogeneous and heterogeneous systems with uniform figure rotation Ω parallel to the vorticity Z, and isotropic or anisotropic pressure) encompass as special cases the generalized Dedekind ellipsoids (homogeneous and heterogeneous systems with $\Omega = 0, Z \neq 0$ and isotropic or anisotropic pressure), the generalized Maclaurin spheroids and and the generalized Jacobi ellipsoids (homogeneous and heterogeneous axysymmetric or ellipsoidal, respectively, having $\Omega \neq 0, Z = 0$, with isotropic or anisotropic

TABLE 1

EQUILIBRIUM CONFIGURATIONS WITH $\mathbf{\Omega} \neq 0$ AND $Z = 0$[a]

Configurations	Density	Shape	Pressure
Spheres	Homogeneous	...	$P_1 < P_3$, B
	Heterogeneous	...	$P_1 < P_3$, B
Generalized Maclaurin spheroids......	Homogeneous	Oblate	isotropic ($P_1 = P_2 = P_3$), B
			anisotropic ($P_1 = P_2 \neq P_3$), B
		Prolate	anisotropic ($P_1 < P_3$), B
	Heterogeneous	..	isotropic, BC
			anisotropic ($P_1 = P_2 \neq P_3$), BC
Generalized Jacobi ellipsoids...........	Homogeneous	Oblate	isotropic $P_1 = P_2 = P_3$), B
			anisotropic ($P_1 \neq P_2 \neq P_3$, B
		Prolate	anisotropic ($P_1 < P_3$; $P_2 < P_3$), B
	Heterogeneous	...	...

[a] Homogeneous and heterogeneous figures of equilibrium having a uniform angular velocity $\mathbf{\Omega}$, with isotropic or anisotropic pressure (the different components P_1, P_2, and P_3 can be barotropic or stratified as the density [denoted by B], or baroclinic, $P_i = P_i(\tilde{x}_1^2 + \tilde{x}_2^2, \tilde{x}_3^2)$ [denoted by BC]).

Figure 0.3: Table 1

TABLE 2

EQUILIBRIUM CONFIGURATIONS WITH $\mathbf{\Omega} = 0$ AND $Z \neq 0$[a]

Configuration	Density	Shape	Pressure	Velocity
Spheres	Homogeneous	...	anisotropic, $P_1 < P_3$, B	$\tilde{\phi}$
			anisotropic, P_1 (B, BC) $< P_3$ (B)	$\tilde{\phi}^*$
	Heterogeneous	...	$P_1 < P_3$, B	$\tilde{\phi}$
			$P_1 < P_3$, B	$\tilde{\phi}^*$
Spheroids................................	Homogeneous	Oblate	isotropic, B	$\tilde{\phi}$
			anisotropic ($P_1 \neq P_3$), B	$\tilde{\phi}$
		Prolate	$P_1 = P_2 < P_3$ (B)	$\tilde{\phi}$
		Oblate	isotropic, B	$\tilde{\phi}^*$
			anisotropic $P_1 = P_2$, (B, BC), P_3 (B)	$\tilde{\phi}^*$
		Prolate	$P_1 = P_2$, (B, BC) $< P_3$ (B)	$\tilde{\phi}^*$
	Heterogeneous		isotropic (BC)	$\tilde{\phi}$
			anisotropic ($P_1 = P_2 \neq P_3$), BC	$\tilde{\phi}$
			isotropic, BC	$\tilde{\phi}^*$
			anisotropic $P_1 = P_2$ (B, BC), P_3 (BC)	$\tilde{\phi}^*$
Generalized Dedekind ellipsoids......	Homogeneous	Oblate	isotropic and anisotropic, B	$\tilde{\phi}$
		Prolate	anisotropic ($P_1 < P_3$), B	$\tilde{\phi}$
		Oblate	isotropic and anisotropic, B	$\tilde{\phi}^*$
		Prolate	anisotropic ($P_1 < P_3$), B	$\tilde{\phi}^*$
	Heterogeneous	...	...	

[a] Homogeneous and heterogeneous figures of equilibrium, isotropic or anisotropic cases are shown, and the different components of the pressure, barotropic (B) or baroclinic (BC) are noted, with differential rotation Z, and a velocity field defined by the functional form $\tilde{\phi} = \tilde{\phi}(m^2)$; $\tilde{\phi}^*$ includes $\tilde{\phi}(m^2, \tilde{x}_3^2)$, $\tilde{\phi}(m^2, \tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2)$, $\tilde{\phi}(\tilde{x}_3^2)$.

Figure 0.4: Table 2

TABLE 3

EQUILIBRIUM CONFIGURATIONS WITH $\Omega \neq \boldsymbol{Z} \neq 0$[a]

Configuration	Density	Shape	Pressure	Velocity	$\boldsymbol{\Omega}, \boldsymbol{Z}$
Spheres	Homogeneous	...	anisotropic $P_1 < P_3$, B	$\tilde{\phi}, \tilde{\phi}^*$	↑↑
			isotropic and anisotropic, B	$\tilde{\phi}, \tilde{\phi}^*$	↑↓
	Heterogeneous	...	anisotropic $P_1 < P_3$, B	$\tilde{\phi}, \tilde{\phi}^*$	↑↑
			isotropic and anisotropic, B	$\tilde{\phi}, \tilde{\phi}^*$	↑↓
Spheroids	Homogeneous	Oblate	isotropic, anisotropic, B	$\tilde{\phi}, \tilde{\phi}^*$	↑↓
		Prolate	anisotropic, $P_1 < P_3$, B	$\tilde{\phi}, \tilde{\phi}^*$	↑↑
	Heterogeneous	Oblate	isotropic (BC) and anisotropic	$\tilde{\phi}$	↑↑
		Prolate		...	↑↓
		Oblate	isotropic (BC) and anisotropic	$\tilde{\phi}^*$	↑↑
		Prolate	$P_1 = P_2$(B, BC), P_3(BC)	...	↑↓
Generalized Riemann ellipsoids......	Homogeneous	Prolate	anisotropic ($P_1 < P_3$, $P_2 < P_3$), B	$\tilde{\phi}, \tilde{\phi}^*$	↑↑
		Oblate	isotropic, B	$\tilde{\phi}, \tilde{\phi}^*$	↑↓
		Oblate	anisotropic, P_1(B), P_2(BC), P_3(B)	$\tilde{\phi}^*$	↑↓
	Heterogeneous	...	...	...	...

[a] Homogeneous and heterogeneous figures of equilibrium in which the direction $\boldsymbol{\Omega}$ and $\boldsymbol{Z}$ are parallel or antiparallel and lie along the rotation axis x_3 ($\boldsymbol{\Omega}$, $\boldsymbol{Z}$ parallel = ↑↑, $\boldsymbol{\Omega}$, $\boldsymbol{Z}$ antiparallel = ↑↓); isotropic and anisotropic cases are noted, and the different components of the pressure, barotropic (B) or baroclinic (BC), are shown. The various forms of the velocity field $\tilde{\phi}, \tilde{\phi}^*$ are considered.

Figure 0.5: Table 3

pressure). Its interesting to note that Dedekind's theorem, which transforms Dedekind ellipsoids into Jacobi ellipsoids and vice versa, no longer applies in the non linear velocity regime, being limited to the linear case. The density may be inhomogeneous and the pressure may be anisotropic. For generalized S-type ellipsoids we may write $n = (0, 0, 1), \Omega = (0, 0, \Omega), Z = (0, 0, Z)$. With this choice the explicit virial equation are reduced to the infinite set. It is easy to show that in the linear and homogeneous case ($\tilde{\phi} = P_i = P_{ic}(m^2)$), this infinite set reduces just to three equations which coincide with hydrodynamical equations (Chandrasekhar 1987, pp 74-75). The analysis of the possible configurations can be performed for two independent cases in which the form of the velocity profile is decided by $\tilde{\phi} = \tilde{\phi}(m^2)$ and $\tilde{\phi} = \tilde{\phi}^*$, the last one containing combinations of m^2, r^2, x_3^2. The analysis of equilibrium configurations are summarized by the tables in figures $3, 4$, and 5 for Maclaurin, Dedekind and Riemann S configurations. These tables list the existent configurations in this specific sequence. Some other classical results can be recovered from our formulation, as the Dive's theorem (a stratified heterogeneous and rotating spheroid, without differential rotation, cannot be a barotrope) that can be generalized to ellipsoidal, anisotropic configurations. The Hamy theorem (a mass ellipsoidally stratified cannot have a uniform rotation) is confirmed also in the anisotropic case. All the results in Chandrasekhar (1987) for homogeneous configurations are extended to the anisotropic case. Our analysis is restricted to ellipsoidal or spheroidal configurations. Many other, non ellipsoidal, self-gravitating figures can exist as announced by Dive's theorem and must be studied by doing a further generalization of the virial theory to cover integration on non ellipsoidal volumes. In this case the form of the configuration is unknown.

0.3.3 The functional approach

The n-th order virial method is an integral approach to the problem of configuration of self-gravitating systems. A different approach, the functional method, begins directly with the differential hydrodynamical equation and tries to solve an essential question: how may be expressed the functional dependence among velocity potential, the density, the pressure, the gravitational potential, the vorticity, and the form of the configuration? Using a functional method based on the introduction of a velocity potential to solve the Euler, continuity and Poisson equations, a new analytic study of the equilibrium of self-gravitating rotating systems with a polytropic equation of state allows the formulation of the conditions of integrability. For the polytropic index n, and a state equation $P = \alpha\rho^{1+1/n}$, in the incompressible case $\nabla \cdot \vec{v} = 0$, we are able to find the conditions for solving the problem of the equilibrium of polytropic self-gravitating systems that rotate and have a non uniform vorticity. In the paper X, an analysis of the hydrodynamic equation for self-gravitating systems is presented from the point of view of functional analysis. We demonstrate that the basic quantities such as the density, the geometric form of the fluid, the pressure, the velocity profile and the vorticity ζ can be expressed as functionals of the velocity potential Ψ and of a function $g(z)$ of the coordinate z on the fluid rotation axis. By writing the hydrodynamical equations in terms of the velocity potential, it is possible to establish the integrability conditions according to which the pressure and the third component of the vorticity ζ_3 have the functional form $P = P(\rho)$ and $\zeta_3 = \zeta_3(\Psi)$; these conclusions suggest the form $\Psi = \Psi(x, y)$ for the velocity potential. In this way the steady state non linear hydrodynamic equation can be written as a functional equation of Ψ and we may propose some simple arguments to construct analytic solutions. In the special case $n = 1$, an explicit analytic solution can be found. The axisymmetric, linear and non homogeneous configurations can be revisited and we may describe how the properties of the configurations obtained compare to the well-known homogeneous ellipsoids of Maclaurin, Jacobi and Riemann and the discussion can be extended to related works using analytical and numerical tools in Newtonian gravity.

0.3.4 Recent studies

The theory of self-gravitating rotating bodies is well known to be quite complicated even if many simplifications are assumed. The main problem comes from the fact that the equations are in general highly nonlinear and boundary conditions refer to the surface of the configuration, which is not known at the beginning but can be located after numerical studies only. Analytical results can be obtained for constant density, incompressible bodies and linear velocity profiles only, as discussed before. Nowadays numerically two-dimensional (2D) and 3D codes allow to study complicated scenarios in the temporal domain, but there is the need to get initial values for the equations. An analysis

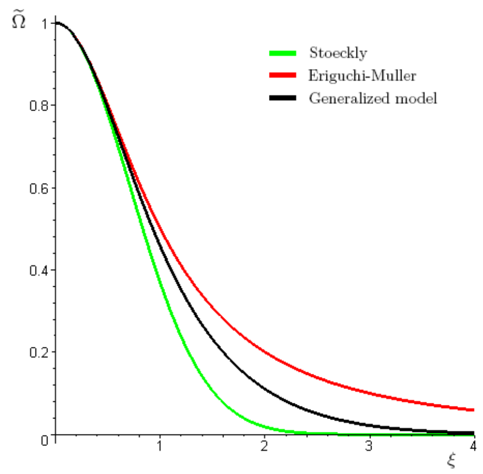

Figure 0.6: Dimensionless angular velocity versus dimensionless equatorial radius for the three models taken into exam.

still in progress (Cherubini, Filippi, Ruffini, Sepulveda and Zuluaga 2007) uses the notion of velocity potential to formulate the hydrodynamical problem and gives the solution for arbitrary values of the polytropic index n based on the computational method of Eruguchi-Muller (1985) that can be implemented for arbitrary profiles of the differential angular velocity. More in detail, the equations are studied with a totally general functional form which interpolates a dimensionless angular velocity profile of gaussian type due to Stoeckly (see Tassoul's monograph) and rational polynomial one (Eriguchi-Muller), i.e.

$$\tilde{\Omega}(\xi) = \frac{e^{-\alpha\xi^2}}{1+\delta^2\xi^2}\,,$$

where ξ is the non dimensional cylindrical radius. While Stoeckly's model is typically used to describe fast non-uniformly rotating configuration close to fission, Eriguchi-Muller one instead is used for j-constant law related to constant specific angular momentum near to the axes of rotation. More in detail in this case for small values of the parameter the rotation law approaches the one of a constant specific angular momentum and the rotation law tend to give rigidly rotating configuration for larger values of the parameter. The plot of our generalized choice for the non dimensional angular velocity is shown in Figure 6. Introducing a computational grid as the one shown in Figure 7, after calibration of the code with well known results of the literature (James and Williams), we have been able to get some new plots of equilibrium configurations (figure 8), with associated mass diagrams (figure 9). The stability of these configurations, a nontrivial point in the theory, is still under exam.

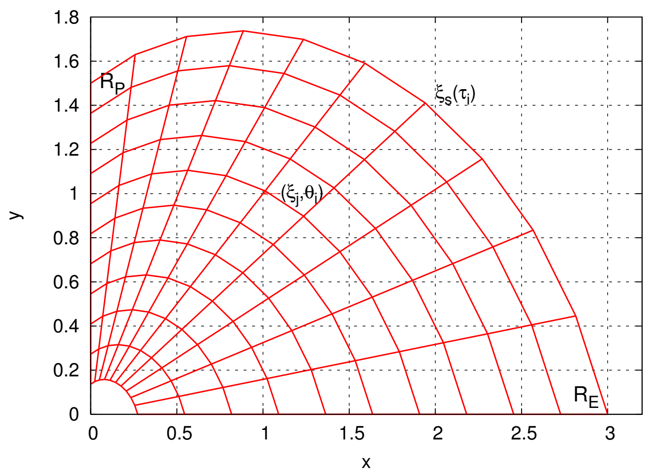

Figure 0.7: Numerical grid used in simulations.

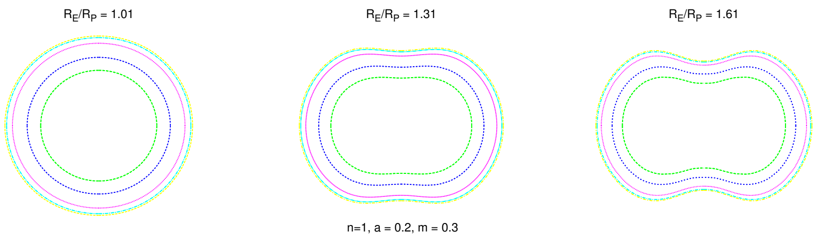

Figure 0.8: Isopicnics for $n = 1$ with $\alpha = 0.2$ and $\delta^2 = 0.3$ for increasing values of the ratio of equatorial and polar radii.

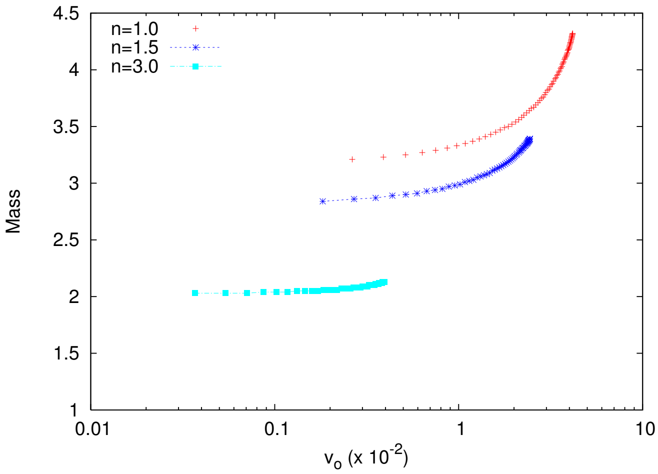

Figure 0.9: Mass Sequence for three values of n in our simulations.

0.4 2010 Results

In recent decades an analogy between General Relativity (GR) and other branches of Physics has been noticed. The central idea is that a number of systems of non relativistic condensed matter physics manifest a mathematical structure similar to the dynamics of fields in a curved manifold. In biology, as an example, it has been hypothesized that stationary scroll wave filaments in cardiac tissue describe a geodesic in a curved space whose metric is represented by the inverse diffusion tensor, with a dynamics close to cosmic strings in a curved universe. In fluid dynamics, in particular, this analogy becomes more striking: given a perfect barotropic and irrotational Newtonian fluid a study for the perturbations of the velocity potential with respect to a background exact solution has been performed. The equations satisfied by the perturbed quantities can be unified in a linear second order hyperbolic equation with non constant coefficients for the velocity potential only, while other quantities like density and pressure can be obtained by differentiation. This wave equation can be rewritten as describing the dynamics of a massless scalar field on a pseudo-Euclidean four dimensional Riemannian manifold. In fact, an induced "effective gravity" in the fluid arises, in which the local speed of sound plays the role of the speed of light in GR. However, while this geometric analogy is quite appealing, finding a relation with experiments is still problematic. All the performed studies in fact, have been essentially linked with superfluid physics experiments, for which viscous contributions can be neglected and perfect fluid approximation is valid, i.e. what Feynman defined as "dry water", although the quantized nature of these systems still poses some formal problems. It is then natural to look for other systems such that the perfect fluid approximation is still valid but quantization complications may be neglected, i.e. a purely classical perfect fluid. Self-gravitating classical fluids and gaseous masses, as described by Euler's equations, appear as the best candidates to satisfy this requirement. For these systems however, one must solve a coupled problem of hydrodynamics and gravitation which is absent in acoustic analogy literature. In all the existing studies on analog models in fact, the contribution of gravitational field was assumed to be externally fixed and practically constant, with no back-reaction. On the other hand in the self-gravitating problem, one must take into account the effect of gravitational back-reaction, present both in the exact background solution as well as in its perturbations. This effect of coupled acoustic disturbances which travel at finite speed and the gravitational field which rearranges itself instantaneously has never been analyzed before using analog geometry models and has been examined by ICRA scientists, focusing in particular on the classical problem of self-gravitating polytropes. Polytropic systems, as discussed before, play an important role in galactic dynamics as well as in the theory of stellar structure and evolution. For these systems the pressure is simply related to the density, while remaining independent of the temperature. Such a choice has specific physical grounds: in the case of a degenerate electron gas, central in the theory of stars, it is well known, as an example, that the pressure and density behave as $\rho \sim p^{\frac{3}{5}}$. Assuming that such a relation exists for other states of the star one has a general relation of polytropic form $p \propto \rho^{1+\frac{1}{n}}$ with a general polytropic index n. Regarding galactic dynamics on the other hand, spherical polytropes are globally stable solutions to the collisionless Boltzmann, or Vlasov, equation of galactic dynamics. While in galactic dynamics n must be larger than $\frac{1}{2}$, in the case of the theory of stellar structure quantity n ranges in $0 \le n < +\infty$. For selfgravitating polytropes the Lane-Emden equation is central: there exist very few analytic solutions and only for selected values of polytropic index and typically for non rotating spherically symmetric configurations or for incompressible fluids only. In more general cases, while uniformly rotating polytropes have analytic solutions in the case $n = 1$ only (assuming truncations of power series) as found by williams[34], solutions for the other non spherical configurations are obtained by numerical techniques only. The acoustic analogy, studied in the past [38] for these spherical cases, has been extended in 2010 on selected axisymmetric systems. Summarizing the perturbative equations for a rotating $n = 1$ polytropic background solution in the Chlebsh formalism we have:

$$\begin{aligned}
\nabla^\mu \nabla_\mu \psi_1 &= \frac{1}{\sqrt{-g}} \left[\nabla \cdot \left(\rho_0 \vec{\xi}_1 \right) - \rho_0 \left(\frac{D^{(0)}}{dt} \right) \left(\frac{\Phi_1}{c^2} \right) \right], \\
\frac{D^{(0)} \vec{\xi}_1}{dt} &= \nabla \psi_1 \times \vec{\omega}_0 - (\vec{\xi}_1 \cdot \nabla) \vec{v}_0 \,, \\
\left[\nabla^2 + k_J^2 \right] \Phi_1 &= -k_J^2 \frac{D^{(0)}}{dt} \psi_1 \,,
\end{aligned} \tag{0.1}$$

which mixes the hydrodynamical (with finite speed) and gravitational (instantaneous) problems through first order time and space partial derivatives of the fields. Here ∇_μ in the first equation denotes the covariant derivative with respect to the acoustic metric $g_{\mu\nu}$, whereas ∇^2 in the third equation is the standard Laplace operator of Euclidean space in three dimensions. Finally, the quantity $k_J = \sqrt{4\pi G \rho_0 / c^2}$ is the magnitude of a generalized Jeans' wave-vector. Here ψ_1 and $\vec{\xi}^1$ are perturbed hydrodynamical potentials (in Clebsch's formulation of perfect fluid hydrodynamics) while Φ_1 is the perturbed gravitational potential inside a self-gravitating configuration. The acoustic metric associated with Williams' solution is given by the line element

$$\begin{aligned}
ds^2 &= \frac{\rho_0}{c} [-(c^2 - v_0^2) dt^2 - 2 v_0 r \sin\theta dt d\phi + dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)] \\
&= \frac{\rho_0}{c} [-c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta (\Omega dt - d\phi)^2] \,,
\end{aligned} \tag{0.2}$$

where $v_0 = \Omega r \sin\theta$, $c = \sqrt{2K\rho_0}$ (the speed of sound) and the background density $\rho_0 = \rho_c \Theta$. In order to explicitly construct the background solution we start from the basic equations governing the hydrostatic equilibrium of a self-gravitating axially symmetric fluid rotating with uniform angular velocity Ω are given by

$$\nabla p = -\rho\nabla\left[\Phi - \frac{1}{2}\Omega^2 r^2 \sin^2\theta\right] ,$$
$$\nabla^2\Phi = 4\pi G\rho , \qquad (0.3)$$

the gravitational potential outside the fluid being governed by the Laplace equation

$$\nabla^2\Phi_{\rm ext} = 0 . \qquad (0.4)$$

The velocity field in this case is in fact given by $\vec{v} = \Omega(-y\partial_x + x\partial_y) = \Omega r \sin\theta \partial_\phi$, implying that the fluid has nonzero vorticity $\vec{\omega} = 2\Omega\partial_z$. Let's assume that the fluid is described by a polytropic equation of state , i.e.

$$p = K\rho^{1+1/n} , \qquad (0.5)$$

with the inverse relation

$$\rho = \left(\frac{p}{K}\right)^{n/(n+1)} , \qquad (0.6)$$

so that sound speed in this case results in

$$c = \left(\frac{\partial\rho}{\partial p}\right)^{-1/2} = K^{1/2}\left(1 + \frac{1}{n}\right)^{1/2} \rho^{1/2n} . \qquad (0.7)$$

Introducing the so called Lane-Emden parametrization , i.e. $\rho = \rho_c\Theta^n$, $r = \alpha\xi$ with $\alpha = (4\pi G)^{-1/2}(n+1)^{1/2}K^{1/2}\rho_c^{(1-n)/2n}$, we denote with ρ_c the density at the center of the fluid configuration. The first of Eqs. (0.3) can be integrated yielding then

$$(n+1)K\rho_c^{1/n}\Theta = -\Phi + \frac{1}{2}\Omega^2 r^2 \sin^2\theta . \qquad (0.8)$$

After a suitably rescaling of the gravitational potential we get the following algebraic relation

$$\Theta = -\chi + \frac{1}{4}\beta\xi^2(1-\mu^2) , \qquad (0.9)$$

where $\mu = \cos\theta$, $\beta = \Omega^2/(2\pi G\rho_c)$ and $\chi = \Phi/[(n+1)K\rho_c^{1/n}]$. The second of Eqs. (0.3)becomes the well known Lane-Emden equation

$$\nabla^2\Theta = -\Theta^n + \beta , \qquad (0.10)$$

and in presence of uniform rotation ($\beta \neq 0$) it admits an exact analytic solutions for $n = 1$ due to the linear nature of the PDE. In the case of axial symmetry and for $n = 1$ it becomes

$$\partial_\xi(\xi^2\partial_\xi\Theta) + \partial_\mu[(1-\mu^2)\partial_\mu\Theta] = \xi^2(-\Theta + \beta) , \qquad (0.11)$$

which should be solved using the conditions $\Theta = 1$ and $\partial_\xi \Theta = 0$ for $\xi = 0$. This is an elliptic system subjected to the above initial conditions but with free boundary, since the surface of the star is not known a priori. On the other hand, on the unknown star's surface both the external and internal gravitational potential as well as their gradients projected on the outgoing and ingoing normal directions respectively must coincide. Only after imposing all conditions on the unknown common boundary one will succeed in determining it, as explicitly shown by Williams[34]. The solution of Eq. (0.11) results in

$$\Theta = \beta + (1-\beta)\frac{\sin\xi}{\xi} + \sum_{l=2}^{\infty} \frac{b_l}{\sqrt{\xi}} J(l+1/2,\xi) P_l(\mu) \ , \tag{0.12}$$

where $J(k,x)$ are Bessel polynomials and $P_l(x)$ are Legendre polynomials, with $b_3 = 0, \ldots, b_{2k+1} = 0, \ldots$, since the polytrope has symmetry also about the equatorial plane $\theta = \pi/2$. The rescaled internal gravitational potential is given by Eq. (0.9) while the external one is given by

$$\chi_{\rm ext} = \nu + \sum_{l=0}^{\infty} \frac{c_l}{\xi^{l+1}} P_l(\mu) \ , \tag{0.13}$$

with $c_1 = 0, c_3 = 0, \ldots, c_{2k+1} = 0, \ldots$. The indeterminate coefficients b_k and c_k evaluated by truncating the infinite series for the potentials at a given l and imposing the matching at the surface, implicitly defined by the equation $\Theta = 0$, i.e.

$$\begin{aligned} 0 &= \int_{-1}^{1} (\chi - \chi_{\rm ext}) P_{2l-2}(\mu) d\mu \ , \\ 0 &= \int_{-1}^{1} \vec{n} \cdot (\nabla\chi - \nabla\chi_{\rm ext}) P_{2l-2}(\mu) d\mu \ , \end{aligned} \tag{0.14}$$

where $\vec{n} = \nabla\Theta$ is the normal to the surface. An algorithm for determining the truncated set of coefficients is presented by Williams and has been implemented again in our study. For instance, for $\beta = 8 \times 10^{-2}$ and taking terms up to the order 8, we find: $\nu \approx -1.36398$; $b_2 \approx -1.03897$; $b_4 \approx 0.38613$; $b_6 \approx -0.79609$; $b_8 \approx -9.34534$; $c_0 \approx -4.15536$; $c_2 \approx 4.44831$; $c_4 \approx -12.31509$; $c_6 \approx 43.35993$; $c_8 \approx -78.90526$. The shape of the boundary of the star is shown in Fig. 0.10.

In this background, for the acoustic metric we have explicitly studied

- the curvature singularities manifested at the stellar surface where the density goes to zero making the acoustic geometry ill-defined
- the light cone structure of the star
- the study of selected classes of null geodesics representing the sound rays of the system (high frequency perturbations)

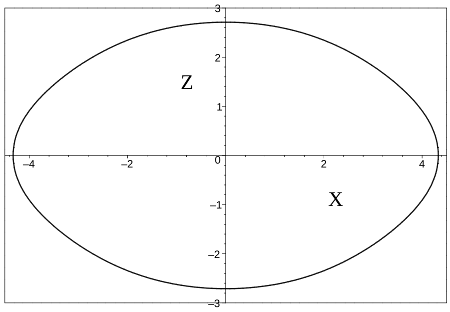

Figure 0.10: The surface of the rotating polytrope is shown for $\beta = 8 \times 10^{-2}$ ($X = \xi \sin\theta$ and $Z = \xi \cos\theta$ are Cartesian-like coordinates). The polar and equatorial radii are given by $\xi_- \approx 2.7117$ and $\xi_+ \approx 4.3302$ respectively, in agreement with both James[35] and Williams[34] results.

At the moment we are implementing these techniques in the case of totally degenerate white dwarfs.

In 2010 we have also approached the perturbations an acoustic perfect fluid black hole (the draining bathtub) in the super-radiant Press-Teukolsky black hole bomb's regime.

The background velocity potential for this system has the mathematical form:

$$\psi(r,\phi) = -ac\log(r/a) + a^2\Omega\phi, \tag{0.15}$$

where a is a length scale while Ω represents the vortex rotation frequency. Linear perturbations $\psi^{(1)} \equiv \Psi$ of the velocity potential satisfy the wave equation

$$\begin{aligned}\Bigg[&-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2a}{cr}\frac{\partial^2}{\partial t\partial r} - \frac{2a^2\Omega}{c^2r^2}\frac{\partial^2}{\partial t\partial\phi} + \left(1-\frac{a^2}{r^2}\right)\frac{\partial^2}{\partial r^2} +\\ &+\frac{2a^3\Omega}{cr^3}\frac{\partial^2}{\partial r\partial\phi} + \frac{c^2r^2 - a^4\Omega^2}{c^2r^4}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\\ &+\frac{r^2+a^2}{r^3}\frac{\partial}{\partial r} - \frac{2a^3\Omega}{cr^4}\frac{\partial}{\partial\phi}\Bigg]\Psi = 0\,,\end{aligned} \tag{0.16}$$

which are a curved space-time scalar field equation: $\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\Psi\right) = 0$. The acoustic metric, which is associated with this equation in fact, has the

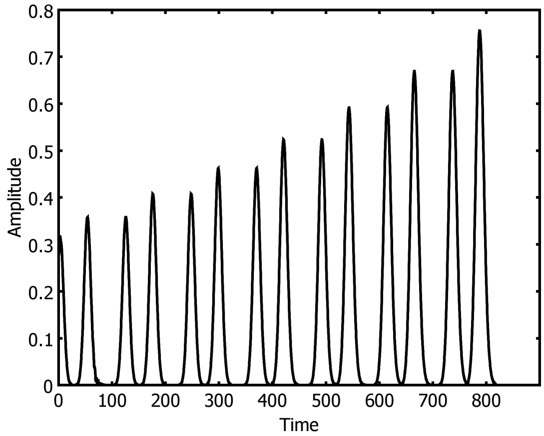

Figure 0.11: Plot of the amplitude $A = |\psi_0|$ in time at a fixed point $r = 25$ (superradiant case with reflecting outer boundary conditions placed at $r = 60$.)

form:

$$\begin{aligned} \mathrm{d}s^2 = & -\left(c^2 - \frac{a^2c^2 + a^4\Omega^2}{r^2}\right)\mathrm{d}t^2 + \frac{2ac}{r}\mathrm{d}t\,\mathrm{d}r \\ & - 2a^2\Omega \mathrm{d}t\,\mathrm{d}\phi + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}z^2. \end{aligned} \tag{0.17}$$

Performing an exponential Fourier decomposition for the variables z and ϕ we have selected as initial data a Gaussian pulse centered at $r = r_0$ and modulated by a monochromatic wave,

$$\psi_1(0, r) = A_0 \exp[-(r - r_0 + ct)^2/b^2 - i\sigma(r - r_0 + ct)/c]|_{t=0}\,. \tag{0.18}$$

If $0 < \sigma < m\Omega$ superradiant scattering should occur. Placing a reflecting cylindrical mirror at a certain distance from the vortex, we have found, trough a constrained evolution numerical scheme, a growing unstable behavior of perturbations which is summarized in Fig.0.11 As expected the wave travels towards the center, gets superradiantly reflected (higher amplitude than incident one), goes towards the border and comes back with the same amplitude and so on. The energy is extracted from the central engine (which looses rotation) and gets accumulated in the outer horizon space. When the level of energy is too big, the mirrored cage gets broken and a blast of acoustic energy travels on the outer world. At the moment we are involved in extending our studies to the case of analog geometries in dielectric media stimulated by lasers, which are a good candidate to probe Hawking effect type effects in condensed matter systems.

0.5 Publications (2010)

- D. Bini, C. Cherubini, S. Filippi and A. Geralico, "Effective geometry of the $n = 1$ uniformly rotating self-gravitating polytrope", Phys Rev D **82**, 044005 (2010).

  Abstract: The effective geometry formalism is used to study the perturbations of a perfect barotropic Newtonian self-gravitating rotating and compressible fluid coupled with gravitational backreaction. The case of a uniformly rotating polytrope with index $n = 1$ is investigated, due to its analytical tractability. Special attention is devoted to the geometrical properties of the underlying background acoustic metric, focusing, in particular, on null geodesics as well as on the analog light cone structure.

- C. Cherubini and S. Filippi, "Boundary Conditions for Scattering Problems from Acoustic Black Holes", JKPS, **56**,1668 (2010).

  Abstract: Analog curved spacetimes emerging from non-relativistic condensed matter systems can be very useful in understanding general relativistic effects. In this article, we analyze different boundary conditions for scattering processes from an acoustic black hole, focusing in particular on the acoustic "black hole bomb" phenomenon in the time domain. More in detail, we develop a superradiant instability, which allows the extraction of energy from the central engine of the system by placing a "mirror"at some distance from the central object.

0.6 Hamiltonian Dynamical Systems and Galactic Dynamics

0.6.1 Near-integrable dynamics and galactic structures.

The study of self-gravitating stellar systems has provided in several occasions important hints to develop powerful tools of analytical mechanics. We may cite the ideas of Jeans (1929) about the relevance of conserved quantities in describing the phase-space structure of large N-body systems and his introduction of the concept of *isolating integral*. Later important contribution are those of Hénon & Heiles (1964), where a paradigmatic example of non-integrable system derived from a simple galactic model was introduced and of Hori (1966), where the theory of Lie transform was introduced in the field of canonical perturbation theory and Hamiltonian normal forms. These and other cues contributed to set up a body of methods and techniques to analyze the near integrable and chaotic regimes of the dynamics of generic non-integrable systems. For a general overview see Boccaletti & Pucacco (Theory of Orbits, Vol. I, 1996; Vol. II, 1999).

On the other side, the payback from analytical mechanics to galactic dynamics has not been as systematic and productive as it could be. The main line of research has been that pursued by Contopoulos (2002) who applied a direct approach to compute approximate forms of effective integrals of motion. The method of Hori (1966), subsequently developed by several other people (Deprit, 1969; Dragt & Finn, 1976; Efthymiopoulos et al. 2004; Finn, 1984; Giorgilli, 2002), has several technical advantages and has gradually become a standard tool in the perturbation theory of Hamiltonian dynamical systems (Boccaletti & Pucacco, 1999). In this respect we have applied the Lie transform method to construct Hamiltonian normal forms of perturbed oscillators and investigate the orbit structure of potentials of interest in galactic dynamics (Belmonte, Boccaletti and Pucacco, 2006, 2007a, 2007b, 2008; Pucacco et al. 2008a, 2008b). The approach allows us to gather several informations concerning the near integrable dynamics below the stochasticity threshold (if any) of the system. Being a completely analytical approach, it has the fundamental value of a complete generality which provides simple recipes to explore the structure of the backbone of phase-space. Exploiting asymptotic properties of the series constructed via the normal form, one can also get quantitative predictions extending the validity of the approach well beyond the radius of convergence of the initial series expansion of the perturbed potential. We have shown how to exploit resonant normal forms to extract information on several aspects of the dynamics of the the logarithmic and the Schwarzschild potential. In particular, using energy and ellipticity as parameters, we have computed the instability thresholds of axial orbits, bifurcation values of low-order boxlets and phase-space fractions pertaining to the families around them. We have also shown how to infer something about the singular limit of the potential.

As in any analytical approach, this method has the virtue of embodying in (more or less) compact formulas simple rules to compute specific properties, giving a general overview of the behavior of the system. In the case in which a non-integrable system has a regular behavior in a large portion of its phase space, a very conservative strategy like the one adopted in our work provides sufficient qualitative and quantitative agreement with other more accurate but less general approaches. In our view, the most relevant limitation of this approach, common to all perturbation methods, comes from the intrinsic structure of the single-resonance normal form. The usual feeling about the problems posed by non-integrable dynamics is in general grounded on trying to cope with the interaction of (several) resonances. Each normal form is instead able to correctly describe only one resonance at the time. However, we remark that the regular dynamics of a non-integrable system can be imagined as a superposition of very weakly interacting resonances. If we are not interested in the thin stochastic layers in the regular regime, each portion of phase space associated with a given resonance has a fairly good alias in the corresponding normal form. An important subject of investigation would therefore be that of including weak interactions in a sort of higher order perturbation theory. For the time being, there are two natural lines of developement of this work: 1. to extend the analysis to cuspy potentials and/or central 'black holes'; 2. to apply this normalization algorithm to three degrees of freedom systems.

Our most recent results concern the investigation of bifurcations of families of periodic orbits ('thin' tubes) in axisymmetric galactic potentials (Pucacco, 2009). We verify that the most relevant bifurcations are due to the (1:1) resonance producing the 'inclined' orbits through two different mechanisms: from the disk orbit and from the 'thin' tube associated to the vertical oscillation. The closest resonances occurring after these are the (4:3) resonance in the oblate case and the (2:1) resonance in the prolate case. The (1:1) resonances are treated in a straightforward way using a 2nd-order truncated normal form. The higher-order resonances are instead cumbersome to investigate, because the normal form has to be truncated to a high degree and the number of terms grows very rapidly. We therefore adopt a further simplification giving analytic formulas for the values of the parameters at which bifurcations ensue and compare them with selected numerical results. Thanks to the asymptotic nature of the series involved, the predictions are reliable well beyond the convergence radius of the original series.

0.6.2 Geometric approach to the integrability of Hamiltonian systems

Integrable systems are still very useful benchmarks to understand the properties of general non-integrable systems, not only for their relevance as starting points for perturbation theory. The topology of invariant surfaces in the phase-space of integrable systems can be highly non-trivial and give rise to complex phenomena (high-order resonances, monodromy, etc.) still not completely understood.

We have started our work by investigating quadratic integrals at fixed and arbitrary energy with a unified geometric approach (Rosquist & Pucacco, 1995; Boccaletti & Pucacco, 1997) solving the Killing tensor equations for 2nd-rank Killing tensors on 2-dim. conformally-Euclidean spaces. In Pucacco & Rosquist (2003) these systems have been shown to be endowed of a bi-Hamiltonian structure and in Pucacco & Rosquist (2004) a class of systems separable at fixed energy has been shown to be non-integrable in Poincarè sense. The case of cubic and quartic integrals of motion, respectively associated to 3rd and 4th-rank Killing tensors, has been investigated in Karlovini & Rosquist (2000) and in Karlovini, Pucacco, Rosquist and Samuelsson (2002). In Pucacco (2004) and Pucacco & Rosquist (2005a) we have obtained new classes of integrable Hamiltonian system with vector potentials and in Pucacco & Rosquist (2005b) we have provided a general treatment of weakly integrable systems.

Recently, Pucacco & Rosquist (2007, 2009), we have presented the theory of separability over 2-dimensional pseudo-Riemannian manifolds ("1+1" separable metrics) by classifying all possible separability structures and providing some non-trivial examples of the additional kinds that appear in the case with indefinite signature of the metric (Pucacco & Rosquist, 2009).

We plan to investigate the existence of higher-order polynomial integrals on general compact surfaces with the topology of the sphere and the torus and to apply the results about pseudo-Riemannian systems to treat integrable time-dependent Hamiltonian systems. We are also working on the general integrability conditions that a system must obey in order to be endowed with one or more integral of motion in a certain polynomial class.

0.6.3 Local dynamics in the expanding universe

The question of whether or not the expansion of the universe may have an influence on the dynamics of local systems such as galaxies has been a topic of debate almost since Einstein presented his theory of general relativity. In this project, we are exploiting a new approach to the problem in which the geometry of the universe appears as a perturbation of the flat metric at the position of the local system.

More specifically, to analyze how the universe influences the dynamics of a local system we take the position of an observer who is comoving with the universe. The local system is assumed to be a test system with respect to the universe. That is, the metric of the universe is to be regarded as a background for the local system. For an arbitrary local system, it has been very difficult to treat this problem exactly. However, there is one case which is known to be amenable to an exact treatment. This is the case of an electromagnetic system for which there is a well-known covariant formulation. The motion is then determined by the Lorentz force equation expressed on a curved background. This system was analyzed by Bonnor for a Bohr hydrogen type system. He came to the conclusion that a circular orbit expands during one revolution, but that the size of the effect is negligible compared to the Hubble expansion. According to our analysis, Bonnor's result should be modified. In fact, the effect turns out be the opposite, nameley that orbits are contracting rather than expanding. The physical reason behind this behavior is that the expansion of the universe leads to a loss of energy. The energy loss is formally analogous to a frictional dissipation. In technical terms, it is the absence of a timelike Killing symmetry which is responsible for the energy loss. Although the relation between energy conservation and time translation symmetry is well-known, that relation has previously received little attention in the cosmological context.

Using the electromagnetic interaction as a model, we have shown that is in fact possible to treat also other types of interaction in a similar way, in particular keeping full covariance. Our plan is to use this result to analyze in detail how orbits in different potentials are affected by the expansion of the universe. Of particular interest in this regard is to apply the method to realistic galactic potentials. Our preliminary work indicates that typical orbits will spiral inwards towards an inner limiting radius. In the coming year we intend to investigate what this behavior means for the galactic rotation curves.

Bibliography

[1] Chandrasekhar, S, *Ellipsoidal Figures of Equilibrium*, 2nd ed. Dover, New York, (1987).

[2] Binney, J.J., Davies, R.L., Illingworth, G.D. , ApJ, 361,78 (1990).

[3] Dive, P. , Rotations Internes des Astres Fluides (Paris:Blanchard)(1930).

[4] Goldstein, H.. *Classical Mechanics*, New York : Addison Wesley,(1980).

[5] Hamy, M. 1887, *Etude sur la figure des Corpes Celestes* (thesis, Ann. Obs. Paris , Memories, 19) (1889).

[6] F. Pacheco, G. Pucacco, and R. Ruffini, Astron. Astrophys. 161, 39 ,(1986) (Paper I).

[7] F. Pacheco, G. Pucacco, and R. Ruffini, *Proc 4th Marcel Grossmann Meeting*, ed R. Ruffini, et al. (Amsterdam : North Holland), 1511 (1986) (Paper II).

[8] G. Busarello, S. Filippi, and R. Ruffini, Astron. Astrophys. 197, 39 , (1988) (Paper III).

[9] F. Pacheco, G. Pucacco, and R. Ruffini, and Sebastiani, G.Astron. Astrophys. 210, 42 ,(1986) (Paper IV).

[10] G. Busarello, S. Filippi, and R. Ruffini, Astron. Astrophys. 213, 80 ,(1989) (Paper V).

[11] G. Busarello, S. Filippi, and R. Ruffini, Astron. Astrophys. 227, 30 ,(1990) (Paper VI).

[12] S. Filippi, R. Ruffini, and A. Sepulveda, Astron. Astrophys. 231, 30 ,(1990) (Paper VII).

[13] S. Filippi, R. Ruffini, and A. Sepulveda, Astron. Astrophys. 246, 59 ,(1991) (Paper VIII).

[14] S. Filippi, R. Ruffini, and A. Sepulveda, Astrophys. J. 460, 762,. (1996) (Paper IX)

[15] J. R. Ipser and R. A. Managan, Astrophys. J. 250, 362 (1981).

[16] P. O. Vandervoort and D. E. Welty, Astrophys. J. 263, 654. (1989.

[17] Y. Eriguchi and I. Hachisu, Astron. Astrophys. 142, 256 (1985).

[18] D. Lai, F. A. Rasio, and S. L. Shapiro, Astrophys. J., Suppl. Ser. 88, 205 (1993).

[19] K. Uryu and Y. Eriguchi, Mon. Not. R. Astron. Soc. 282, 653 (1996).

[20] K. Taniguchi and T. Nakamura, Phys. Rev. Lett. 84, 581 (2000).

[21] Y. Eriguchi, I. Hachisu, and K. Nomoto, Mon. Not. R. Astron. Soc. 266, 179 (1994).

[22] T. Nozawa, N. Stergioulas, E. Gourgoulhon, and Y. Eriguchi, Astron. Astrophys., Suppl. Ser. 132, 431 (1998).

[23] P. Marronetti, G. J. Mathews, and J. R. Wilson, Phys. Rev. D 60, 087301 (1999).

[24] S. Bonazzola, E. Gourgoulhon, and J.-A. Marck, Phys. Rev. Lett. 82, 892 (1999).

[25] J. R. Ipser and R. A. Managan, Astrophys. J. 292, 517 (1985).

[26] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, (1953).

[27] A. Toomre, Astrophys. J. 138, 385 (1963).

[28] J. Binney and S. Tremaine, *Galactic Dynamics*,Princeton University Press, Princeton, NJ, (1987).

[29] J. E. Cazes and J. E. Tohline, Astrophys. J. 532, 1051. (2000).

[30] Weigand, R. , Astron. Astrophys. 82,177, (1980).

[31] S. Filippi, R. Ruffini, and A. Sepulveda, Phys. Rev. 65, 044019 ,(2002) (Paper X)

[32] Eriguchi, Y. and Muller, E., Astron. Astrophys. 146,260 (1985)

[33] Tassoul JL.*Theory of Rotating Stars*, Princeton University Press (1978)

[34] Williams P.S., Astrophysics and Space Science, 143, 349(1988).

[35] James R.A., Astrophys.. J. 140, 552 .(1964).

[36] Stoeckly, R., Astrophys. J., 142, 208 (1965).

[37] Cherubini C. Filippi S., Ruffini R., Sepulveda A. and Zuluaga J. (in preparation 2007).

[38] D. Bini, Cherubini and S. Filippi, *Phys. Rev. D* **78**, 064024 (2008).

[38] Belmonte, C. Boccaletti, D. Pucacco, G. (2006) Stability of axial orbits in galactic potentials, Celestial Mechanics and Dynamical Astronomy, **95**, 101.

[38] Belmonte, C. Boccaletti, D. Pucacco, G. (2007a) Approximate First Integrals for a Model of Galactic Potential with the Method of Lie Transform Normalization, Qualitative Theory of Dynamical Systems, E. Perez-Chavela and J. Xia editors.

[38] Belmonte, C. Boccaletti, D. Pucacco, G. (2007b) On the orbit structure of the logarithmic potential, The Astrophysical Journal, **669**, 202.

[38] Boccaletti, D. & Pucacco, G. (1996) *Theory of Orbits*, Vol. 1: Integrable Systems and Non-Perturbative Methods, Springer-Verlag, Berlin.

[38] Pucacco, G. Boccaletti, D. Belmonte, C. (2008a), Quantitative predictions with detuned normal forms, Celestial Mechanics and Dynamical Astronomy, **102**, 163–176, DOI 10.1007/s10569-008-9141-x.

[38] Pucacco, G. Boccaletti, D. Belmonte, C. (2008b), Periodic orbits in the logarithmic potential, Astronomy & Astrophysics, **489**, 1055–1063.

[38] Belmonte, C. Boccaletti, D. Pucacco, G. (2008), Approximate First Integrals for a Model of Galactic Potential with the Method of Lie Transform Normalization, Qualitative Theory of Dynamical Systems, **7**, 43–71.

[38] Pucacco, G. (2009), *Resonances and bifurcations in axisymmetric scale-free potentials*, Monthly Notices of the Royal Astronomical Society, DOI: 10. 1111 / j.1365-2966.2009.15284.x.

[38] Rosquist, K. & Pucacco, G. (1995) Invariants at fixed and arbitrary energy. A unified geometric approach, Journal of Physics A: Math. Gen., **28**, 3235.

[38] Karlovini, M. Pucacco, G. Rosquist, K. Samuelsson, L. (2002), A unified treatment of quartic invariants at fixed and arbitrary energy, Journal of Mathematical Physics, **43**, 4041–4059

[38] Pucacco, G. & Rosquist, K. (2003) On separable systems in two dimensions, in “Symmetry and Perturbation Theory – SPT2002”, Abenda, S. Gaeta, G. and Walcher, S. editors (World Scientific, Singapore), 196.

[38] Pucacco, G. & Rosquist, K. (2004) Non-integrability of a weakly integrable system, Celestial Mechanics and Dynamical Astronomy, **88**, 185.

[38] Pucacco, G. & Rosquist, K. (2005a) Integrable Hamiltonian systems with vector potentials, J. Math. Phys., **46**, 012701.

[38] Pucacco, G. & Rosquist, K. (2005b) Configurational invariants of Hamiltonian systems, J. Math. Phys., **46**, 052902.

[38] Pucacco, G. & Rosquist, K. (2007) (1+1)-dimensional separation of variables, J. Math. Phys., **48**, 112903.

[38] Pucacco, G. Rosquist, K. (2009), *Nonstandard separability on the Minkowski plane*, Journal of Nonlinear Mathematical Physics, in press.