Generalizations of the Kerr-Newman solution

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2 Brief description

One of the most important metrics in general relativity is the Kerr-Newman solution that describes the gravitational and electromagnetic fields of a rotating charged mass. For astrophysical purposes, however, it is necessary to take into account the effects due to the moment of inertia of the object. To attack this problem, we investigate new exact solutions of Einstein-Maxwell equations which posses an infinite set of gravitational and electromagnetic multipole moments and contain the Kerr-Newman solution as special case.

There are several methods that can be used to find generalizations of the Kerr-Newman metric. In particular, one can use the curvature of the spacetime that can be measured by local observers. We argue that the curvature generated by a gravitational field can be used to calculate the corresponding metric, which determines the trajectories of freely falling test particles. To this end, we present a method to compute the metric from a given curvature tensor. We use Petrov's classification to handle the structure and properties of the curvature tensor, and Cartan's structure equations in an orthonormal tetrad to investigate the differential equations that relate the curvature with the metric. The second structure equation is integrated to obtain the explicit expression for the connection $1-$form from which the components of the orthonormal tetrad are obtained by using the first structure equation. This opens the possibility of using the curvature of astrophysical objects like the Earth to determine the position of freely falling satellites that are used in modern navigation systems.

The Newman-Janis Ansatz was used first to obtain the stationary Kerr metric from the static Schwarzschild metric. Many works have been devoted to investigate the physical significance of this Ansatz, but no definite answer has been given so far. We show that this Ansatz can be applied in general to conformastatic vacuum metrics, and leads to stationary generalizations which, however, do not preserve the conformal symmetry. We investigate also the particular case when the seed solution is given by the Schwarzschild spacetime and show that the resulting rotating configuration does not correspond to a vacuum solution, even in the limiting case of slow rotation. In fact, it describes in general a relativistic fluid with anisotropic pressure and heat flux. This implies that the Newman-Janis Ansatz strongly depends on the choice of representation for the seed solution. We interpret this result as as a further indication of its applicability limitations.

One of the applications of the Kerr-Newman metric is the description of the gravitational field of compact objects like white dwarfs. The equilibrium configurations of uniformly rotating white dwarfs at finite temperatures are investigated, exploiting the Chandrasekhar equation of state for different isothermal cores. The Hartle-Thorne formalism is applied to construct white dwarf configurations in the framework of Newtonian physics. The equations of structure are considered in the slow rotation approximation and all basic parameters of rotating hot white dwarfs are computed to test the so-called moment of inertia, tidal Love number and quadrupole moment (I-Love-Q) relations. It is shown that even within the same equation of state the I-Love-Q relations are not universal for white dwarfs at finite temperatures.

3 Introduction

It is hard to overemphasize the importance of the Kerr geometry not only for general relativity itself, but also for the very fundamentals of physics. It assumes this position as being the most physically relevant rotating generalization of the static Schwarzschild geometry. Its charged counterpart, the Kerr-Newman solution, representing the exterior gravitational and electromagnetic fields of a charged rotating object, is an exact solution of the Einstein-Maxwell equations.

Its line element in Boyer–Lindquist coordinates can be written as

$$\begin{aligned} ds^2 &= \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2 \cos^2\theta}(dt - a\sin^2\theta d\varphi)^2 \\ &\quad - \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}[(r^2 + a^2)d\varphi - adt]^2 \\ &\quad - \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2}dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 \,, \end{aligned} \tag{3.0.1}$$

where M is the total mass of the object, $a = J/M$ is the specific angular momentum, and Q is the electric charge. In this particular coordinate system, the metric functions do not depend on the coordinates t and ϕ, indicating the existence of two Killing vector fields $\xi^I = \partial_t$ and $\xi^{II} = \partial_\varphi$ which represent the properties of stationarity and axial symmetry, respectively.

An important characteristic of this solution is that the source of gravity is surrounded by two horizons situated at a distance

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \tag{3.0.2}$$

from the origin of coordinates. Inside the interior horizon, r_-, a ring singularity is present which, however, cannot be observed by any observer situated outside the exterior horizon. If the condition $M^2 < a^2 + Q^2$ is satisfied, no horizons are present and the Kerr–Newman spacetime represents the exterior field of a naked singularity.

Despite of its fundamental importance in general relativity, and its theoretical and mathematical interest, this solution has not been especially useful for describing astrophysical phenomena, first of all, because observed astrophysical objects do not possess an appreciable net electric charge. Furthermore, the limiting Kerr metric takes into account the mass and the rotation, but does not consider the moment of inertia of the object. For astrophysical applications it is, therefore, necessary to use more general solutions with higher multipole moments which are due not only to the rotation of the body but also to its shape. This means that even in the limiting case of a static spacetime, a solution is needed that takes into account possible deviations from spherically symmetry.

4 The general static vacuum solution

In general relativity, stationary axisymmetric solutions of Einstein's equations [1] play a crucial role for the description of the gravitational field of astrophysical objects. In particular, the black hole solutions and their generalizations that include Maxwell fields are contained within this class.

This type of exact solutions has been the subject of intensive research during the past few decades. In particular, the number of know exact solutions drastically increased after Ernst [2] discovered an elegant representation of the field equations that made it possible to search for their symmetries. These studies lead finally to the development of solution generating techniques [1] which allow us to find new solutions, starting from a given seed solution. In particular, solutions with an arbitrary number of multipole moments for the mass and angular momentum were derived in [3] and used to describe the gravitational field of rotating axially symmetric distributions of mass.

The first analysis of stationary axially symmetric gravitational fields was carried out by Weyl [4] in 1917, soon after the formulation of general relativity. In particular, Weyl discovered that in the static limit the main part of the vacuum field equations reduces to a single linear differential equation. The corresponding general solution can be written in cylindrical coordinates as an infinite sum with arbitrary constant coefficients. A particular choice of the coefficients leads to the subset of asymptotically flat solutions which is the most interesting from a physical point of view. In this section we review the main properties of stationary axisymmetric gravitational fields. In particular, we show explicitly that the main field equations in vacuum can be represented as the equations of a nonlinear sigma model in which the base space is the 4-dimensional spacetime and the target space is a 2-dimensional conformally Euclidean space.

4.1 Line element and field equations

Although there exist in the literature many suitable coordinate systems, stationary axisymmetric gravitational fields are usually described in cylindric coordinates (t, ρ, z, φ). Stationarity implies that t can be chosen as the time coordinate and the metric does not depend on time, i.e. $\partial g_{\mu\nu}/\partial t = 0$. Consequently, the corresponding timelike Killing vector has the components δ^μ_t. A second Killing vector field is associated to the axial symmetry with respect to the axis $\rho = 0$. Then, choosing φ as the azimuthal angle, the metric satisfies the conditions $\partial g_{\mu\nu}/\partial\varphi = 0$, and the components of the corresponding spacelike Killing vector are δ^μ_φ.

Using further the properties of stationarity and axial symmetry, together with the vacuum field equations, for a general metric of the form $g_{\mu\nu} = g_{\mu\nu}(\rho, z)$, it is possible to show that the most general line element for this type of gravitational fields can be written in the Weyl-Lewis-Papapetrou form as [4, 5, 6]

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] , \qquad (4.1.1)$$

where f, ω and γ are functions of ρ and z, only. After some rearrangements which include the introduction of a new function $\Omega = \Omega(\rho, z)$ by means of

$$\rho\partial_\rho\Omega = f^2\partial_z\omega , \qquad \rho\partial_z\Omega = -f^2\partial_\rho\omega , \qquad (4.1.2)$$

the vacuum field equations $R_{\mu\nu} = 0$ can be shown to be equivalent to the following set of partial differential equations

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \partial_z^2 f + \frac{1}{f}[(\partial_\rho\Omega)^2 + (\partial_z\Omega)^2 - (\partial_\rho f)^2 - (\partial_z f)^2] = 0 , \qquad (4.1.3)$$

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Omega) + \partial_z^2\Omega - \frac{2}{f}\left(\partial_\rho f\,\partial_\rho\Omega + \partial_z f\,\partial_z\Omega\right) = 0 , \qquad (4.1.4)$$

$$\partial_\rho\gamma = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right] , \qquad (4.1.5)$$

$$\partial_z\gamma = \frac{\rho}{2f^2}\left(\partial_\rho f\,\partial_z f + \partial_\rho\Omega\,\partial_z\Omega\right) . \qquad (4.1.6)$$

It is clear that the field equations for γ can be integrated by quadratures, once f and Ω are known. For this reason, the equations (4.1.3) and (4.1.4) for f and Ω are usually considered as the main field equations for stationary axisymmetric vacuum gravitational fields. In the following subsections we will focus on the analysis of the main field equations, only. It is interesting to mention that this set of equations can be geometrically interpreted in the context of nonlinear sigma models [7].

Let us consider the special case of static axisymmetric fields. This corresponds to metrics which, apart from being axially symmetric and independent of the time coordinate, are invariant with respect to the transformation $\varphi \to -\varphi$ (i.e. rotations with respect to the axis of symmetry are not allowed). Consequently, the corresponding line element is given by (4.1.1) with $\omega = 0$, and the field equations can be written as

$$\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi + \partial_z^2\psi = 0\,, \quad f = \exp(2\psi)\,, \tag{4.1.7}$$

$$\partial_\rho\gamma = \rho\left[(\partial_\rho\psi)^2 - (\partial_z\psi)^2\right]\,, \quad \partial_z\gamma = 2\rho\partial_\rho\psi\,\partial_z\psi\,. \tag{4.1.8}$$

We see that the main field equation (4.1.7) corresponds to the linear Laplace equation for the metric function ψ.

4.2 Static solution

The general solution of Laplace's equation is known and, if we demand additionally asymptotic flatness, we obtain the Weyl solution which can be written as [4, 1]

$$\psi = \sum_{n=0}^{\infty}\frac{a_n}{(\rho^2+z^2)^{\frac{n+1}{2}}}P_n(\cos\theta)\,, \qquad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}\,, \tag{4.2.1}$$

where a_n $(n = 0, 1, ...)$ are arbitrary constants, and $P_n(\cos\theta)$ represents the Legendre polynomials of degree n. The expression for the metric function γ can be calculated by quadratures by using the set of first order differential equations (4.1.8). Then

$$\gamma = -\sum_{n,m=0}^{\infty}\frac{a_n a_m (n+1)(m+1)}{(n+m+2)(\rho^2+z^2)^{\frac{n+m+2}{2}}}\left(P_nP_m - P_{n+1}P_{m+1}\right)\,. \tag{4.2.2}$$

Since this is the most general static, axisymmetric, asymptotically flat vacuum solution, it must contain all known solution of this class. In particular, one of the most interesting special solutions which is Schwarzschild's spherically symmetric black hole spacetime must be contained in this class. To see this, we must choose the constants a_n in such a way that the infinite sum (4.2.1) converges to the Schwarzschild solution in cylindric coordinates. But, or course, this representation is not the most appropriate to analyze the interesting physical properties of Schwarzchild's metric.

In fact, it turns out that to investigate the properties of solutions with multipole moments it is more convenient to use prolate spheroidal coordinates (t, x, y, φ) in which the line element can be written as

$$ds^2 = f dt^2 - \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right]$$

where

$$x = \frac{r_+ + r_-}{2\sigma}\,, \quad (x^2 \geq 1), \quad y = \frac{r_+ - r_-}{2\sigma}\,, \quad (y^2 \leq 1) \tag{4.2.3}$$

$$r_\pm^2 = \rho^2 + (z \pm \sigma)^2\,, \quad \sigma = const\,, \tag{4.2.4}$$

and the metric functions are f, ω, and γ depend on x and y, only. In this coordinate system, the general static solution which is also asymptotically flat can be expressed as

$$f = \exp(2\psi)\,, \quad \psi = \sum_{n=0}^{\infty}(-1)^{n+1} q_n P_n(y) Q_n(x)\,, \quad q_n = const$$

where $P_n(y)$ are the Legendre polynomials, and $Q_n(x)$ are the Legendre functions of second kind. In particular,

$$P_0 = 1, \quad P_1 = y, \quad P_2 = \frac{1}{2}(3y^2 - 1)\,, ...$$

$$Q_0 = \frac{1}{2}\ln\frac{x+1}{x-1}\,, \quad Q_1 = \frac{1}{2}x\ln\frac{x+1}{x-1} - 1\,,$$

$$Q_2 = \frac{1}{2}(3x^2-1)\ln\frac{x+1}{x-1} - \frac{3}{2}x\,, ...$$

The corresponding function γ can be calculated by quadratures and its general expression has been explicitly derived in [8]. The most important special cases contained in this general solution are the Schwarzschild metric

$$\psi = -q_0 P_0(y) Q_0(x) \, , \quad \gamma = \frac{1}{2} \ln \frac{x^2-1}{x^2-y^2} \, ,$$

and the Erez-Rosen metric [9]

$$\psi = -q_0 P_0(y) Q_0(x) - q_2 P_2(y) Q_2(x) \, , \quad \gamma = \frac{1}{2} \ln \frac{x^2-1}{x^2-y^2} + ... $$

In the last case, the constant parameter q_2 turns out to determine the quadrupole moment. In general, the constants q_n represent an infinite set of parameters that determines an infinite set of mass multipole moments.

5 Stationary generalization

The solution generating techniques [12] can be applied, in particular, to any static seed solution in order to obtain the corresponding stationary generalization. One of the most powerful techniques is the inverse method (ISM) developed by Belinski and Zakharov [13]. We used a particular case of the ISM, which is known as the Hoenselaers–Kinnersley-Xanthopoulos (HKX) transformation to derive the stationary generalization of the general static solution in prolate spheroidal coordinates.

5.1 Ernst representation

In the general stationary case $(\omega \neq 0)$ with line element

$$
\begin{aligned}
& ds^2 = f(dt - \omega d\varphi)^2 \\
- & \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right]
\end{aligned}
$$

it is useful to introduce the the Ernst potentials

$$
E = f + i\Omega\,, \quad \xi = \frac{1-E}{1+E}\,,
$$

where the function Ω is now determined by the equations

$$
\sigma(x^2-1)\Omega_x = f^2\omega_y\,, \quad \sigma(1-y^2)\Omega_y = -f^2\omega_x\,.
$$

Then, the main field equations can be represented in a compact and symmetric form:

$$
(\xi\xi^* - 1)\left\{[(x^2-1)\xi_x]_x + [(1-y^2)\xi_y]_y\right\} = 2\xi^*[(x^2-1)\xi_x^2 + (1-y^2)\xi_y^2]\,.
$$

This equation is invariant with respect to the transformation $x \leftrightarrow y$. Then, since the particular solution

$$\xi = \frac{1}{x} \to \Omega = 0 \to \omega = 0 \to \gamma = \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2}$$

represents the Schwarzschild spacetime, the choice $\xi^{-1} = y$ is also an exact solution. Furthermore, if we take the linear combination $\xi^{-1} = c_1 x + c_2 y$ and introduce it into the field equation, we obtain the new solution

$$\xi^{-1} = \frac{\sigma}{M}x + i\frac{a}{M}y\,,\ \sigma = \sqrt{M^2 - a^2}\,,$$

which corresponds to the Kerr metric in prolate spheroidal coordinates.

In the case of the Einstein-Maxwell theory, the main field equations can be expressed as

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\xi = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\xi\,,$$

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\mathcal{F} = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\mathcal{F}$$

where ∇ represents the gradient operator in prolate spheroidal coordinates. Moreover, the gravitational potential ξ and the electromagnetic $\mathcal{F}$ Ernst potential are defined as

$$\xi = \frac{1-f-i\Omega}{1+f+i\Omega}\,, \quad \mathcal{F} = 2\frac{\Phi}{1+f+i\Omega}\,.$$

The potential Φ can be shown to be determined uniquely by the electromagnetic potentials A_t and A_φ One can show that if ξ_0 is a vacuum solution, then the new potential

$$\xi = \xi_0\sqrt{1-e^2}$$

represents a solution of the Einstein-Maxwell equations with effective electric charge e. This transformation is known in the literature as the Harrison transformation [10]. Accordingly, the Kerr–Newman solution in this representation acquires the simple form

$$\xi = \frac{\sqrt{1-e^2}}{\frac{\sigma}{M}x + i\frac{a}{M}y}\,, \qquad e = \frac{Q}{M}\,, \qquad \sigma = \sqrt{M^2 - a^2 - Q^2}\,.$$

In this way, it is very easy to generalize any vacuum solution to include the case of electric charge. More general transformations of this type can be used in order to generate solutions with any desired set of gravitational and electromagnetic multipole moments [11].

5.2 Representation as a nonlinear sigma model

Consider two (pseudo)-Riemannian manifolds (M, γ) and (N, G) of dimension m and n, respectively. Let M be coordinatized by x^a, and N by X^μ, so that the metrics on M and N can be, in general, smooth functions of the corresponding coordinates, i.e., $\gamma = \gamma(x)$ and $G = G(X)$. A harmonic map is a smooth map $X : M \to N$, or in coordinates $X : x \longmapsto X$ so that X becomes a function of x, and the X's satisfy the motion equations following from the action [14]

$$S = \int d^m x \sqrt{|\gamma|}\, \gamma^{ab}(x)\, \partial_a X^\mu\, \partial_b X^\nu\, G_{\mu\nu}(X)\,, \tag{5.2.1}$$

which sometimes is called the "energy" of the harmonic map X. The straightforward variation of S with respect to X^μ leads to the motion equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b \left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu \right) + \Gamma^\mu_{\ \nu\lambda}\, \gamma^{ab}\, \partial_a X^\nu\, \partial_b X^\lambda = 0\,, \tag{5.2.2}$$

where $\Gamma^\mu_{\ \nu\lambda}$ are the Christoffel symbols associated to the metric $G_{\mu\nu}$ of the target space N. If $G_{\mu\nu}$ is a flat metric, one can choose Cartesian-like coordinates such that $G_{\mu\nu} = \eta_{\mu\nu} = diag(\pm 1, ..., \pm 1)$, the motion equations become linear, and the corresponding sigma model is linear. This is exactly the case of a bosonic string on a flat background in which the base space is the 2-dimensional string world-sheet. In this case the action (5.2.1) is usually referred to as the Polyakov action [16].

Consider now the case in which the base space M is a stationary axisymmetric spacetime. Then, γ^{ab}, $a, b = 0, ..., 3$, can be chosen as the Weyl-Lewis-Papapetrou metric (4.1.1), i.e.

$$\gamma_{ab} = \begin{pmatrix} f & 0 & 0 & -f\omega \\ 0 & -f^{-1}e^{2k} & 0 & 0 \\ 0 & 0 & -f^{-1}e^{2k} & 0 \\ -f\omega & 0 & 0 & f\omega^2 - \rho^2 f^{-1} \end{pmatrix} . \tag{5.2.3}$$

Let the target space N be 2-dimensional with metric $G_{\mu\nu} = (1/2)f^{-2}\delta_{\mu\nu}$, $\mu, \nu = 1, 2$, and let the coordinates on N be $X^\mu = (f, \Omega)$. Then, it is straightforward to show that the action (5.2.1) becomes

$$S = \int \mathcal{L}\, dt d\varphi d\rho dz\,, \qquad \mathcal{L} = \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right]\,, \tag{5.2.4}$$

and the corresponding motion equations (5.2.2) are identical to the main field equations (4.1.3) and (4.1.4).

Notice that the field equations can also be obtained from (5.2.4) by a direct variation with respect to f and Ω. This interesting result was obtained originally by Ernst [2], and is the starting point of what today is known as the Ernst representation of the field equations.

The above result shows that stationary axisymmetric gravitational fields can be described as a $(4 \to 2)-$nonlinear harmonic map, where the base space is the spacetime of the gravitational field and the target space corresponds to a 2-dimensional conformally Euclidean space. A further analysis of the target space shows that it can be interpreted as the quotient space $SL(2, R)/SO(2)$ [15], and the Lagrangian (5.2.4) can be written explicitly [17] in terms of the generators of the Lie group $SL(2, R)$. Harmonic maps in which the target space is a quotient space are usually known as nonlinear sigma models [14].

The form of the Lagrangian (5.2.4) with two gravitational field variables, f and Ω, depending on two coordinates, ρ and z, suggests a representation as a harmonic map with a 2-dimensional base space. In string theory, this is an important fact that allows one to use the conformal invariance of the base space metric to find an adequate representation for the set of classical solutions. This, in turn, facilitates the application of the canonical quantization procedure. Unfortunately, this is not possible for the Lagrangian (5.2.4). Indeed, if we consider γ^{ab} as a 2-dimensional metric that depends on the parameters ρ and z, the diagonal form of the Lagrangian (5.2.4) implies that $\sqrt{|\gamma|}\gamma^{ab} = \delta^{ab}$. Clearly, this choice is not compatible with the factor ρ in front of the Lagrangian. Therefore, the reduced gravitational Lagrangian (5.2.4) cannot be interpreted as corresponding to a $(2 \to n)$-harmonic map. Nevertheless, we will show in the next section that a modification of the definition of harmonic maps allows us to "absorb" the unpleasant factor ρ in the metric of the target space, and to use all the advantages of a 2-dimensional base space.

Notice that the representation of stationary fields as a nonlinear sigma model becomes degenerate in the limiting case of static fields. Indeed, the underlying geometric structure of the $SL(2,R)/SO(2)$ nonlinear sigma models requires that the target space be 2-dimensional, a condition which is not satisfied by static fields. We will see below that by using a dimensional extension of generalized sigma models, it will be possible to treat the special static case, without affecting the underlying geometric structure.

The analysis performed in this section for stationary axisymmetric fields can be generalized to include any gravitational field containing two commuting Killing vector fields [1]. This is due to the fact that for this class of gravitational fields it is always possible to find the corresponding Ernst representation in which the Lagrangian contains only two gravitational variables which depend on only two spacetime coordinates.

5.3 Representation as a generalized harmonic map

Consider two (pseudo-)Riemannian manifolds (M,γ) and (N,G) of dimension m and n, respectively. Let x^a and X^μ be coordinates on M and N, respectively. This coordinatization implies that in general the metrics γ and G become functions of the corresponding coordinates. Let us assume that not only γ but also G can explicitly depend on the coordinates x^a, i.e. let $\gamma=\gamma(x)$ and $G=G(X,x)$. This simple assumption is the main aspect of our generalization which, as we will see, lead to new and nontrivial results.

A smooth map $X:M\to N$ will be called an $(m\to n)-$generalized harmonic map if it satisfies the Euler-Lagrange equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right)+\Gamma^\mu_{\ \nu\lambda}\,\gamma^{ab}\,\partial_a X^\nu\partial_b X^\lambda+G^{\mu\lambda}\gamma^{ab}\,\partial_a X^\nu\,\partial_b G_{\lambda\nu}=0\,, \tag{5.3.1}$$

which follow from the variation of the generalized action

$$S=\int d^m x\sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a\,X^\mu\partial_b X^\nu G_{\mu\nu}(X,x)\,, \tag{5.3.2}$$

with respect to the fields X^μ. Here the Christoffel symbols, determined by the metric $G_{\mu\nu}$, are calculated in the standard manner, without considering the explicit dependence on x. Notice that the new ingredient in this generalized definition of harmonic maps, i.e., the term $G_{\mu\nu}(X,x)$ in the Lagrangian density implies that we are taking into account the "interaction" between the base space M and the target space N. This interaction leads to an extra term in the motion equations, as can be seen in (5.3.1). It turns out that this interaction is the result of the effective presence of the gravitational field.

Notice that the limiting case of generalized linear harmonic maps is much more complicated than in the standard case. Indeed, for the motion equations (5.3.1) to become linear it is necessary that the conditions

$$\gamma^{ab}(\Gamma^{\mu}_{\ \nu\lambda}\,\partial_b X^\lambda + G^{\mu\lambda}\,\partial_b G_{\lambda\nu})\partial_a X^\nu = 0\,, \tag{5.3.3}$$

be satisfied. One could search for a solution in which each term vanishes separately. The choice of a (pseudo-)Euclidean target metric $G_{\mu\nu} = \eta_{\mu\nu}$, which would imply $\Gamma^{\mu}_{\ \nu\lambda} = 0$, is not allowed, because it would contradict the assumption $\partial_b G_{\mu\nu} \neq 0$. Nevertheless, a flat background metric in curvilinear coordinates could be chosen such that the assumption $G^{\mu\lambda}\partial_b G_{\mu\nu} = 0$ is fulfilled, but in this case $\Gamma^{\mu}_{\ \nu\lambda} \neq 0$ and (5.3.3) cannot be satisfied. In the general case of a curved target metric, conditions (5.3.3) represent a system of m first order nonlinear partial differential equations for $G_{\mu\nu}$. Solutions to this system would represent linear generalized harmonic maps. The complexity of this system suggests that this special type of maps is not common.

As we mentioned before, the generalized action (5.3.2) includes an interaction between the base space N and the target space M, reflected on the fact that $G_{\mu\nu}$ depends explicitly on the coordinates of the base space. Clearly, this interaction must affect the conservation laws of the physical systems we attempt to describe by means of generalized harmonic maps. To see this explicitly we calculate the covariant derivative of the generalized Lagrangian density

$$\mathcal{L} = \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X,x)\,, \tag{5.3.4}$$

and replace in the result the corresponding motion equations (5.3.1). Then, the final result can be written as

$$\nabla_b \widetilde{T}_a^{\ b} = -\frac{\partial \mathcal{L}}{\partial x^a} \tag{5.3.5}$$

where $\widetilde{T}_a{}^b$ represents the canonical energy-momentum tensor

$$\widetilde{T}_a{}^b = \frac{\partial\mathcal{L}}{\partial(\partial_b X^\mu)}(\partial_a X^\mu) - \delta_a^b\mathcal{L} = 2\sqrt{\gamma}G_{\mu\nu}\left(\gamma^{bc}\partial_a X^\mu\,\partial_c X^\nu - \frac{1}{2}\delta_a^b\gamma^{cd}\partial_c X^\mu\,\partial_d X^\nu\right). \tag{5.3.6}$$

The standard conservation law is recovered only when the Lagrangian does not depend explicitly on the coordinates of the base space. Even if we choose a flat base space $\gamma_{ab} = \eta_{ab}$, the explicit dependence of the metric of the target space $G_{\mu\nu}(X, x)$ on x generates a term that violates the standard conservation law. This term is due to the interaction between the base space and the target space which, consequently, is one of the main characteristics of the generalized harmonic maps introduced in this work.

An alternative and more general definition of the energy-momentum tensor is by means of the variation of the Lagrangian density with respect to the metric of the base space, i.e.

$$T_{ab} = \frac{\delta\mathcal{L}}{\delta\gamma^{ab}}\,. \tag{5.3.7}$$

A straightforward computation shows that for the action under consideration here we have that $\widetilde{T}_{ab} = 2T_{ab}$ so that the generalized conservation law (5.3.5) can be written as

$$\nabla_b T_a{}^b + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial x^a} = 0\,. \tag{5.3.8}$$

For a given metric on the base space, this represents in general a system of m differential equations for the "fields" X^μ which must be satisfied "on-shell".

If the base space is 2-dimensional, we can use a reparametrization of x to choose a conformally flat metric, and the invariance of the Lagrangian density under arbitrary Weyl transformations to show that the energy-momentum tensor is traceless, $T_a{}^a = 0$.

In Section 5.1 we described stationary, axially symmetric, gravitational fields as a $(4 \to 2)-$nonlinear sigma model. There it was pointed out the convenience of having a 2-dimensional base space in analogy with string theory. Now we will show that this can be done by using the generalized harmonic maps defined above.

Consider a $(2 \to 2)-$generalized harmonic map. Let $x^a = (\rho, z)$ be the coordinates on the base space M, and $X^\mu = (f, \Omega)$ the coordinates on the target space N. In the base space we choose a flat metric and in the target space a conformally flat metric, i.e.

$$\gamma_{ab} = \delta_{ab} \qquad \text{and} \qquad G_{\mu\nu} = \frac{\rho}{2f^2}\delta_{\mu\nu} \qquad (a, b = 1, 2;\ \mu, \nu = 1, 2). \tag{5.3.9}$$

A straightforward computation shows that the generalized Lagrangian (5.3.4) coincides with the Lagrangian (5.2.4) for stationary axisymetric fields, and that the equations of motion (5.3.1) generate the main field equations (4.1.3) and (4.1.4).

For the sake of completeness we calculate the components of the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$. Then

$$T_{\rho\rho} = -T_{zz} = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right], \tag{5.3.10}$$

$$T_{\rho z} = \frac{\rho}{2f^2}\left(\partial_\rho f\, \partial_z f + \partial_\rho\Omega\, \partial_z\Omega\right). \tag{5.3.11}$$

This tensor is traceless due to the fact that the base space is 2-dimensional. It satisfies the generalized conservation law (5.3.8) on-shell:

$$\frac{dT_{\rho\rho}}{d\rho} + \frac{dT_{\rho z}}{dz} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\rho} = 0\,, \tag{5.3.12}$$

$$\frac{dT_{\rho z}}{d\rho} - \frac{dT_{\rho\rho}}{dz} = 0\,. \tag{5.3.13}$$

Incidentally, the last equation coincides with the integrability condition for the metric function k, which is identically satisfied by virtue of the main field equations. In fact, as can be seen from Eqs.(4.1.5,4.1.6) and (5.3.10,5.3.11), the components of the energy-momentum tensor satisfy the relationships $T_{\rho\rho} = \partial_\rho k$ and $T_{\rho z} = \partial_z k$, so that the conservation law (5.3.13) becomes an identity. Although we have eliminated from the starting Lagrangian (5.2.4) the variable k by applying a Legendre transformation on the Einstein-Hilbert Lagrangian (see [17] for details) for this type of gravitational fields, the formalism of generalized harmonic maps seems to retain the information about k at the level of the generalized conservation law.

The above results show that stationary axisymmetric spacetimes can be represented as a $(2 \to 2)-$generalized harmonic map with metrics given as in (5.3.9). It is also possible to interpret the generalized harmonic map given above as a generalized string model. Although the metric of the base space M is Euclidean, we can apply a Wick rotation $\tau = i\rho$ to obtain a Minkowski-like structure on M. Then, M represents the world-sheet of a bosonic string in which τ is measures the time and z is the parameter along the string. The string is "embedded" in the target space N whose metric is conformally flat and explicitly depends on the time parameter τ. We will see in the next section that this embedding becomes more plausible when the target space is subject to a dimensional extension. In the present example, it is necessary to apply a Wick rotation in order to interpret the base space as a string world-sheet. This is due to the fact that both coordinates ρ and z are spatial coordinates. However, this can be avoided by considering other classes of gravitational fields with timelike Killing vector fields; examples will be given below.

The most studied solutions belonging to the class of stationary axisymmetric fields are the asymptotically flat solutions. Asymptotic flatness imposes conditions on the metric functions which in the cylindrical coordinates used here can be formulated in the form

$$\lim_{x^a\to\infty} f = 1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \omega = c_1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \Omega = O\left(\frac{1}{x^a}\right) \tag{5.3.14}$$

where c_1 is an arbitrary real constant which can be set to zero by appropriately choosing the angular coordinate φ. If we choose the domain of the spatial coordinates as $\rho \in [0,\infty)$ and $z \in (-\infty,+\infty)$, from the asymptotic flatness conditions it follows that the coordinates of the target space N satisfy the boundary conditions

$$\dot{X}^\mu(\rho,-\infty) = 0 = \dot{X}^\mu(\rho,\infty) , \quad X'^\mu(\rho,-\infty) = 0 = X'^\mu(\rho,\infty) \tag{5.3.15}$$

where the dot stands for a derivative with respect to ρ and the prime represents derivation with respect to z. These relationships are known in string theory [16] as the Dirichlet and Neumann boundary conditions for open strings, respectively, with the extreme points situated at infinity. We thus conclude that if we assume ρ as a "time" parameter for stationary axisymmetric gravitational fields, an asymptotically flat solution corresponds to an open string with endpoints attached to $D-$branes situated at plus and minus infinity in the $z-$direction.

5.4 Dimensional extension

In order to further analyze the analogy between gravitational fields and bosonic string models, we perform an arbitrary dimensional extension of the target space N, and study the conditions under which this dimensional extension does not affect the field equations of the gravitational field. Consider an $(m \to D)-$generalized harmonic map. As before we denote by $\{x^a\}$ the coordinates on M. Let $\{X^\mu, X^\alpha\}$ with $\mu = 1,2$ and $\alpha = 3,4,...,D$ be the coordinates on N. The metric structure on M is again $\gamma = \gamma(x)$, whereas the metric on N can in general depend on all coordinates of M and N, i.e. $G = G(X^\mu, X^\alpha, x^a)$. The general structure of the corresponding field equations is as given in (5.3.1). They can be divided into one set of equations for X^μ and one set of equations for X^α. According to the results of the last section, the class of gravitational fields under consideration can be represented as a $(2 \to 2)-$generalized harmonic map so that we can assume that the main gravitational variables are contained in the coordinates X^μ of the target space. Then, the gravitational sector of the target space will be contained in the components $G_{\mu\nu}$ ($\mu, \nu = 1,2$) of the metric, whereas the components $G_{\alpha\beta}$ ($\alpha, \beta = 3,4,...,D$) represent the sector of the dimensional extension.

Clearly, the set of differential equations for X^μ also contains the variables X^α and its derivatives $\partial_a X^\alpha$. For the gravitational field equations to remain unaffected by this dimensional extension we demand the vanishing of all the terms containing X^α and its derivatives in the equations for X^μ. It is easy to show that this can be achieved by imposing the conditions

$$G_{\mu\alpha} = 0\,, \quad \frac{\partial G_{\mu\nu}}{\partial X^\alpha} = 0\,, \quad \frac{\partial G_{\alpha\beta}}{\partial X^\mu} = 0\,. \tag{5.4.1}$$

That is to say that the gravitational sector must remain completely invariant under a dimensional extension, and the additional sector cannot depend on the gravitational variables, i.e., $G_{\alpha\beta} = G_{\alpha\beta}(X^\gamma, x^a)$, $\gamma = 3,4,...,D$. Furthermore, the variables X^α must satisfy the differential equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\alpha\right) + \Gamma^\alpha_{\ \beta\gamma}\,\gamma^{ab}\,\partial_a X^\beta \partial_b X^\gamma + G^{\alpha\beta}\gamma^{ab}\,\partial_a X^\gamma\,\partial_b G_{\beta\gamma} = 0\,. \tag{5.4.2}$$

This shows that any given $(2 \to 2)-$generalized map can be extended, without affecting the field equations, to a $(2 \to D)-$generalized harmonic map.

It is worth mentioning that the fact that the target space N becomes split in two separate parts implies that the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$ separates into one part belonging to the gravitational sector and a second one following from the dimensional extension, i.e. $T_{ab} = T_{ab}(X^\mu, x) + T_{ab}(X^\alpha, x)$. The generalized conservation law as given in (5.3.8) is satisfied by the sum of both parts.

Consider the example of stationary axisymmetric fields given the metrics (5.3.9). Taking into account the conditions (5.4.1), after a dimensional extension the metric of the target space becomes

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\rho}{2f^2} & 0 & \cdots & 0 \\ 0 & 0 & G_{33}(X^\alpha, x) & \cdots & G_{3D}(X^\alpha, x) \\ . & . & \cdots & \cdots & \cdots \\ 0 & 0 & G_{D3}(X^\alpha, x) & \cdots & G_{DD}(X^\alpha, x) \end{pmatrix} . \tag{5.4.3}$$

Clearly, to avoid that this metric becomes degenerate we must demand that $\det(G_{\alpha\beta}) \neq 0$, a condition that can be satisfied in view of the arbitrariness of the components of the metric. With the extended metric, the Lagrangian density gets an additional term

$$\begin{aligned} \mathcal{L} = \quad & \frac{\rho}{2f^2} \left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2 \right] \\ & + \left(\partial_\rho X^\alpha \partial_\rho X^\beta + \partial_z X^\alpha \partial_z X^\beta \right) G_{\alpha\beta} \, , \end{aligned} \tag{5.4.4}$$

which nevertheless does not affect the field equations for the gravitational variables f and Ω. On the other hand, the new fields must be solutions of the extra field equations

$$\left(\partial_\rho^2 + \partial_z^2 \right) X^\alpha + \Gamma^\alpha{}_{\beta\gamma} \left(\partial_\rho X^\beta \partial_\rho X^\gamma + \partial_z X^\beta \partial_z X^\gamma \right) \tag{5.4.5}$$

$$+ G^{\alpha\gamma} \left(\partial_\rho X^\beta \partial_\rho G_{\beta\gamma} + \partial_z X^\beta \partial_z G_{\beta\gamma} \right) = 0 \, . \tag{5.4.6}$$

An interesting special case of the dimensional extension is the one in which the extended sector is Minkowskian, i.e. for the choice $G_{\alpha\beta} = \eta_{\alpha\beta}$ with additional fields X^α given as arbitrary harmonic functions. This choice opens the possibility of introducing a "time" coordinate as one of the additional dimensions, an issue that could be helpful when dealing with the interpretation of gravitational fields in this new representation.

The dimensional extension finds an interesting application in the case of static axisymmetric gravitational fields. As mentioned in Section 4.1, these fields are obtained from the general stationary fields in the limiting case $\Omega = 0$ (or equivalently, $\omega = 0$). If we consider the representation as an $SL(2,R)/SO(2)$ nonlinear sigma model or as a $(2 \to 2)-$generalized harmonic map, we see immediately that the limit $\Omega = 0$ is not allowed because the target space becomes 1-dimensional and the underlying metric is undefined. To avoid this degeneracy, we first apply a dimensional extension and only then calculate the limiting case $\Omega = 0$. In the most simple case of an extension with $G_{\alpha\beta} = \delta_{\alpha\beta}$, the resulting $(2 \to 2)-$generalized map is described by the metrics $\gamma_{ab} = \delta_{ab}$ and

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{5.4.7}$$

where the additional dimension is coordinatized by an arbitrary harmonic function which does not affect the field equations of the only remaining gravitational variable f. This scheme represents an alternative method for exploring static fields on nondegenerate target spaces. Clearly, this scheme can be applied to the case of gravitational fields possessing two hypersurface orthogonal Killing vector fields.

Our results show that a stationary axisymmetric field can be represented as a string "living" in a D-dimensional target space N. The string world-sheet is parametrized by the coordinates ρ and z. The gravitational sector of the target space depends explicitly on the metric functions f and Ω and on the parameter ρ of the string world-sheet. The sector corresponding to the dimensional extension can be chosen as a $(D-2)-$dimensional Minkowski spacetime with time parameter τ. Then, the string world-sheet is a 2-dimensional flat hypersurface which is "frozen" along the time τ.

5.5 The general solution

If we take as seed metric the general static solution, the application of two HXK transformations generates a stationary solution with an infinite number of gravitoelectric and gravitomagnetic multipole moments. The HKX method is applied at the level of the Ernst potential from which the metric functions can be calculated by using the definition of the Ernst potential E and the field equations for γ. The resulting expressions in the general case are quite cumbersome. We quote here only the special case in which only an arbitrary quadrupole parameter is present. In this case, the result can be written as

$$
\begin{aligned}
f &= \frac{R}{L} e^{-2qP_2Q_2} \,, \\
\omega &= -2a - 2\sigma \frac{\mathcal{M}}{R} e^{2qP_2Q_2} \,, \\
e^{2\gamma} &= \frac{1}{4}\left(1 + \frac{M}{\sigma}\right)^2 \frac{R}{x^2 - y^2} e^{2\hat{\gamma}} \,,
\end{aligned}
\tag{5.5.1}
$$

where

$$
\begin{aligned}
R &= a_+ a_- + b_+ b_- \,, \qquad L = a_+^2 + b_+^2 \,, \\
\mathcal{M} &= \alpha x(1 - y^2)(e^{2q\delta_+} + e^{2q\delta_-}) a_+ + y(x^2 - 1)(1 - \alpha^2 e^{2q(\delta_+ + \delta_-)}) b_+ \,, \\
\hat{\gamma} &= \frac{1}{2}(1 + q)^2 \ln \frac{x^2 - 1}{x^2 - y^2} + 2q(1 - P_2)Q_1 + q^2(1 - P_2)\Big[(1 + P_2)(Q_1^2 - Q_2^2) \\
&\quad + \frac{1}{2}(x^2 - 1)(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2')\Big] \,.
\end{aligned}
\tag{5.5.2}
$$

Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind respectively. Furthermore

$$
\begin{aligned}
a_\pm &= x(1 - \alpha^2 e^{2q(\delta_+ + \delta_-)}) \pm (1 + \alpha^2 e^{2q(\delta_+ + \delta_-)}) \,, \\
b_\pm &= \alpha y(e^{2q\delta_+} + e^{2q\delta_-}) \mp \alpha(e^{2q\delta_+} - e^{2q\delta_-}) \,, \\
\delta_\pm &= \frac{1}{2} \ln \frac{(x \pm y)^2}{x^2 - 1} + \frac{3}{2}(1 - y^2 \mp xy) + \frac{3}{4}[x(1 - y^2) \mp y(x^2 - 1)] \ln \frac{x - 1}{x + 1} \,,
\end{aligned}
$$

the quantity α being a constant

$$
\alpha = \frac{\sigma - M}{a} \,, \qquad \sigma = \sqrt{M^2 - a^2} \,. \tag{5.5.3}
$$

The physical significance of the parameters entering this metric can be clarified by calculating the Geroch-Hansen [18, 19] multipole moments

$$
M_{2k+1} = J_{2k} = 0 \,, \quad k = 0, 1, 2, \ldots \tag{5.5.4}
$$

$$M_0 = M\,, \quad M_2 = -Ma^2 + \frac{2}{15}qM^3\left(1 - \frac{a^2}{M^2}\right)^{3/2}\,, \ldots \tag{5.5.5}$$

$$J_1 = Ma\,, \quad J_3 = -Ma^3 + \frac{4}{15}qM^3a\left(1 - \frac{a^2}{M^2}\right)^{3/2}\,, \ldots. \tag{5.5.6}$$

The vanishing of the odd gravitoelectric (M_n) and even gravitomagnetic (J_n) multipole moments is a consequence of the symmetry with respect to the equatorial plane. From the above expressions we see that M is the total mass of the body, a represents the specific angular momentum, and q is related to the deviation from spherical symmetry. All higher multipole moments can be shown to depend only on the parameters M, a, and q.

We analyzed the geometric and physical properties of the above solution. The special cases contained in the general solution suggest that it can be used to describe the exterior asymptotically flat gravitational field of rotating body with arbitrary quadrupole moment. This is confirmed by the analysis of the motion of particles on the equatorial plane. The quadrupole moment turns out to drastically change the geometric structure of spacetime as well as the motion of particles, especially near the gravitational source.

We investigated in detail the properties of the Quevedo-Mashhoon (QM) spacetime which is a generalization of Kerr spacetime, including an arbitrary quadrupole. Our results show [20] that a deviation from spherical symmetry, corresponding to a non-zero electric quadrupole, completely changes the structure of spacetime. A similar behavior has been found in the case of the Erez-Rosen spacetime. In fact, a naked singularity appears that affects the ergosphere and introduces regions where closed timelike curves are allowed. Whereas in the Kerr spacetime the ergosphere corresponds to the boundary of a simply-connected region of spacetime, in the present case the ergosphere is distorted by the presence of the quadrupole and can even become transformed into non simply-connected regions. All these changes occur near the naked singularity which is situated at $x = 1$, a value that corresponds to the radial distance $r = M + \sqrt{M^2 - a^2}$ in Boyer-Lindquist coordinates. In the limiting case $a/M > 1$, the multipole moments and the metric become complex, indicating that the physical description breaks down. Consequently, the extreme Kerr black hole represents the limit of applicability of the QM spacetime.

Since standard astrophysical objects satisfy the condition $a/M < 1$, we can conclude that the QM metric can be used to describe their exterior gravitational field. Two alternative situations are possible. If the characteristic radius of the body is greater than the critical distance $M + \sqrt{M^2 - a^2}$, i.e. $x > 1$, the exterior solution must be matched with an interior solution in order to describe the entire spacetime. If, however, the characteristic radius of the body is smaller than the critical distance $M + \sqrt{M^2 - a^2}$, the QM metric describes the field of a naked singularity.

6 Kerr solution with higher multipoles: the equatorial plane

6.1 Introduction

One of the most important practical applications of general relativity is the Global Positioning System (GPS), the most advanced navigation system known today. It consists essentially in a set of artificial satellites freely falling in the gravitational field of the Earth. To determine the location of any point on the Earth by using the method of triangulation, it is necessary to know the exact position of several satellites at a given moment of time. This means that the path of each satellite must be determined as exact as possible. In fact, due to the accuracy expected from the GPS, specially for navigation purposes, it is necessary to take into account relativistic effects for the determination of the satellites trajectories and the gravitational field of the Earth. This method is therefore essentially based upon the use of the geodesic equations of motion for each satellite. Moreover, it is necessary to consider the fact that according to special and general relativity clocks inside the satellites run differently than clocks on the Earth surface. Indeed, it is known that not taking relativistic effects into account would lead to an error in the determination of the position which could grow up to 10 kilometers per day.

The curvature seems to be an alternative way to determine the position of any point on the surface of the Earth. Indeed, if we could measure the curvature of the spacetime around the Earth, and from it the corresponding metric, one could imagine that the determination of the position of the satellites could be carried out in a different way. Maybe this method could be more efficient and more accurate. To this end, it is necessary to measure the curvature of spacetime. Several devices have been proposed for this purpose. The five-point curvature detector [24] consists of four mirrors and a light source. By measuring the distances between all the components of the detector, it is possible to determine the curvature. Another method uses a local orthonormal frame which is Fermi-Walker propagated along a geodesic [27]. A gyroscope is directed along each vector of the frame so that the relative acceleration will allow the determination of the curvature components. The gravitational compass [28] is a tetrahedral arrangement of springs with test particles on each vertex. Using the geodesic deviation equation, from the strains in the springs it is possible to infer the components of the curvature. More recently, a generalized geodesic deviation equation was derived which, when applied to a set of test particles, can be used to measure the components of the curvature tensor [29].

It seems therefore to be now well established that the curvature can be measured by using different devices that are within the reach of modern technology. The question arises whether it is possible to obtain the metric from a given curvature tensor. This is the problem we will address in this work. In Sec. 6.2, we study a particular matrix representation of the curvature tensor which allows us to calculate its eigenvalues in a particularly simple way. Petrov's classification is used to represent the curvature matrix in terms of its eigenvalues. In Sec. 6.3, we use Cartan's formalism to derive all the algebraic and differential equations which must be combined and integrated to determine the components of the metric from the components of the curvature. As particular examples, we present the Schwarzschild, Taub-NUT and Kasner metrics with cosmological constant. All the components of the metric are found explicitly in terms of the components of the curvature tensor. It turns out that for a given vacuum solution it is possible to find several generalizations which include the cosmological constant.

6.2 Matrix representation of the curvature tensor

There are several ways to represent and study the properties of the curvature tensor. Here, we will use a method which is based upon the formalism of differential forms and the matrix representation of the curvature tensor. The reason is simple. Imagine an observer in a gravitational field. Locally, the observer can introduce a set of four vectors e_a to perform measurements and experiments. Although it is possible to choose the direction of each vector arbitrarily, the most natural choice would be to construct an orthonormal system, i.e., $e_a \otimes e_b = \eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$. Of course, the observer could also choose a local metric which depends on the point. Nevertheless, the choice of a constant local metric facilitates the process of carrying out measurements in space and time. This choice is also in the spirit of the equivalence principle which states that locally it is always possible to introduce a system in which the laws of special relativity are valid. The set of vectors e_a can be used to introduce a local frame ϑ^a by using the orthonormality condition $e_a \rfloor \vartheta^b = \delta^b_a$, where $\rfloor$ is the internal product. The set of 1-forms ϑ^a determines a local orthonormal tetrad that is the starting point for the construction of the formalism of differential forms which is widely used in general relativity.

There is an additional advantage in choosing a local orthonormal frame. General relativity is a theory constructed upon the assumption of diffeomorphism invariance, i.e, it is invariant with respect to arbitrary changes of coordinates $x^\mu \to x^{\mu'}$ such that $J = \det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) \neq 0$. Once a local orthonormal frame ϑ^a is chosen, the only freedom which remains is the transformation $\vartheta^a \to \vartheta^{a'} = \Lambda^{a'}_{\ a}\vartheta^a$, where $\Lambda^{a'}_{\ a}$ is a Lorentz transformation, satisfying the condition $\Lambda^{a'}_{\ a}\Lambda_{a'\,b} = \eta_{ab}$. This means that the diffeomorphism invariance reduces locally to the Lorentz invariance, which is easier to be handled.

In the local orthonormal frame, the line element can be written as

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab}\vartheta^a \otimes \vartheta^b \ , \tag{6.2.1}$$

with

$$\vartheta^a = e^a_{\ \mu} dx^\mu \ . \tag{6.2.2}$$

The components $e^a_{\ \mu}$ are called tetrad vectors, and can be used to relate tetrad components with coordinate components. For instance, the components of the metric are given in terms of the tetrad vectors by $g_{\mu\nu} = e^a_{\ \mu} e^b_{\ \nu}\eta_{ab}$. The exterior derivative of the local tetrad is given in terms of the connection $1-$form ω_{ab} as [23]

$$d\eta_{ab} = \omega_{ab} + \omega_{ba} \ . \tag{6.2.3}$$

Since the local metric is constant, the above expression vanishes, indicating that the connection $1-$form is antisymmetric. Furthermore, the first structure equation

$$d\vartheta^a = -\omega^a_{\ b} \wedge d\vartheta^b \ , \tag{6.2.4}$$

can be used to calculate all the components of the connection $1-$form. Finally, the curvature $2-$form is defined as

$$\Omega^a_{\ b} = d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b} \tag{6.2.5}$$

in terms of a connection. In this differential form representation, the Ricci and Bianchi identities can be expressed as

$$\Omega^a{}_b \wedge \vartheta^b = 0\,, \qquad d\Omega^a{}_b + \omega^a{}_c \wedge \Omega^c{}_b - \Omega^a{}_c \wedge \omega^c{}_b = 0\,, \tag{6.2.6}$$

respectively.

The curvature 2−form can be decomposed in terms of the canonical basis $\vartheta^a \wedge \vartheta^b$ as

$$\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd} \vartheta^c \wedge \vartheta^d\,, \tag{6.2.7}$$

where $R^a{}_{bcd}$ are the components of the Riemann curvature tensor in the tetrad representation.

It is well known that the curvature tensor can be decomposed in terms of its irreducible parts which are the Weyl tensor [21]

$$W_{abcd} = R_{abcd} + 2\eta_{[a|[c}R_{d]|b]} + \frac{1}{6} R \eta_{a[d}\eta_{c]b}\,, \tag{6.2.8}$$

the trace-free Ricci tensor

$$E_{abcd} = 2\eta_{[b|[c}R_{d]|a]} - \frac{1}{2} R \eta_{a[d}\eta_{c]b}\,, \tag{6.2.9}$$

and the curvature scalar

$$S_{abcd} = -\frac{1}{6} R \eta_{a[d}\eta_{c]b}\,, \tag{6.2.10}$$

where we use the following convention for the components of the Ricci tensor:

$$R_{ab} = \eta^{cd} R_{cabd}\,. \tag{6.2.11}$$

Due to the symmetry properties of the components of the curvature tensor, it is possible to represent it as a (6×6)-matrix by introducing the bivector indices $A, B, ...$ which encode the information of two different tetrad indices, i.e., $ab \to A$. We follow the convention used in [23] which establishes the following correspondence between tetrad and bivector indices

$$01 \to 1\,, \quad 02 \to 2\,, \quad 03 \to 3\,, \quad 23 \to 4\,, \quad 31 \to 5\,, \quad 12 \to 6\,. \tag{6.2.12}$$

This correspondence can be applied to all the irreducible components of the

Riemann tensor given in Eqs.(6.2.8)–(6.2.10). Then, the bivector representation of the Riemann tensor reads

$$R_{AB} = W_{AB} + E_{AB} + S_{AB}\,, \tag{6.2.13}$$

with

$$W_{AB} = \begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \tag{6.2.14}$$

$$E_{AB} = \begin{pmatrix} P & Q \\ Q & -P \end{pmatrix}, \tag{6.2.15}$$

$$S_{AB} = -\tfrac{R}{12}\begin{pmatrix} I_3 & 0 \\ 0 & -I_3 \end{pmatrix}. \tag{6.2.16}$$

Here M, N and P are (3×3) real symmetric matrices, whereas Q is antisymmetric.

We see that the bivector representation of the curvature is in fact given in terms of the (3×3)-matrices M, N, P, Q and the scalar R, suggesting an equivalent representation in terms of only (3×3)-matrices. Indeed, since (6.2.13) represents the irreducible pieces of the curvature with respect to the Lorentz group $SO(3,1)$ and, in turn, this group is isomorphic to the group $SO(3,C)$, it is possible to introduce a local complex basis where the curvature is given as a (3×3)-matrix. This is the $SO(3,C)$-representation of the Riemann tensor [21, 22]:

$$\mathcal{R} = W + E + S\,, \tag{6.2.17}$$

$$W = M + iN\,, \tag{6.2.18}$$

$$E = P + iQ\,, \tag{6.2.19}$$

$$S = \tfrac{1}{12} R\, I_3\,. \tag{6.2.20}$$

In this representation, Einstein's equations can be written as algebraic equations. Consider, for instance, a vacuum spacetime for which $E = 0$ and $S = 0$. Then, the vanishing of the Ricci tensor in terms of the components of the Riemann tensor corresponds to the algebraic condition

$$\mathrm{Tr}(W) = 0\,, \quad W^T = W\,. \tag{6.2.21}$$

In general, from Einstein's equations in the presence of matter

$$R_{ab} - \frac{1}{2}R\eta_{ab} + \Lambda\eta_{ab} = -\kappa T_{ab} \,, \tag{6.2.22}$$

we find that

$$R = 4\Lambda + \kappa T \,, \quad T = \eta^{ab}T_{ab} \,, \tag{6.2.23}$$

and the components of the curvature tensor satisfy the relationships

$$\eta^{cd}R_{cabd} = \kappa T_{ab} + \left(\Lambda + \frac{\kappa}{2}T\right)\eta_{ab} \,. \tag{6.2.24}$$

It is then easy to see that the following curvature tensor

$$S = \frac{1}{12}(4\Lambda + \kappa T)\,\mathrm{diag}(1,1,1) \,, \tag{6.2.25}$$

$$E = \frac{\kappa}{2}\begin{pmatrix} T_{11} - T_{00} + \frac{1}{2}T & T_{12} - iT_{03} & T_{13} + iT_{02} \\ T_{12} - iT_{03} & T_{22} - T_{00} + \frac{1}{2}T & T_{23} - iT_{01} \\ T_{13} - iT_{02} & T_{23} + iT_{01} & T_{33} - T_{00} + \frac{1}{2}T \end{pmatrix} \,, \tag{6.2.26}$$

$$W \ \text{arbitrary } (3\times 3)\text{−matrix with } \ \mathrm{Tr}(W) = 0 \,, \ W^T = W \ , \tag{6.2.27}$$

satisfies Einstein's equations identically. Thus, we see that the matrix W has only ten independent components, the matrix E is hermitian with nine independent components and the scalar piece S has only one component.

The energy-momentum tensor determines completely only the trace-free Ricci tensor and the scalar curvature. The Weyl tensor contains in general ten independent components. However, since the local tetrad ϑ^a is defined modulo transformations of the Lorentz group $SO(3,1)$, we can use the six independent parameters of the Lorentz group to fix six components of the Weyl tensor. Accordingly, we can use the eigenvalues of the matrix W to write the four remaining parameters in the form

$$W^I = \begin{pmatrix} a_1 + ib_1 & & \\ & a_2 + ib_2 & \\ & & -a_1 - a_2 - i(b_1 + b_2) \end{pmatrix} \,. \tag{6.2.28}$$

In fact, this is the most general case of a Weyl tensor, and corresponds to a type I curvature tensor in Petrov's classification. If the eigenvalues of the matrix W are degenerate, then $a_2 = a_1 = a$ and $b_2 = b_1 = b$ and therefore

$$W^D = \begin{pmatrix} a+ib & & \\ & a+ib & \\ & & -2a-2ib \end{pmatrix} , \tag{6.2.29}$$

which represents a type D curvature tensor.

In general, all the eigenvalues can depend on the coordinates x^μ of the spacetime. The real part of the eigenvalues a_1 and a_2 represent the gravitoelectric part of the curvature, whereas the imaginary part b_1 and b_2 correspond to the gravitomagnetic field, i.e., the gravitational field generated by the motion of the source.

6.3 Integration of Cartan's structure equations

Our aim now is to show that for a given curvature tensor it is possible to integrate Cartan's equation in order to compute the components of the metric. To this end, it is necessary to rewrite Cartan's equations so that the dependence on the spacetime coordinates becomes explicit. First, let us introduce the components of the anholonomic connection $\Gamma^a{}_{bc}$ by means of the relationship

$$\omega^a{}_b = \Gamma^a{}_{bc}\vartheta^c , \tag{6.3.1}$$

and the condition $\Gamma_{abc} = -\Gamma_{bac}$. Then, from the definition of the connection $1-$form, we obtain

$$e^a{}_{[\mu,\nu]} = \Gamma^a{}_{bc} e_{[\nu}{}^b e_{\mu]}{}^c , \tag{6.3.2}$$

which represents a differential equation for the components of the tetrad vectors $e^a{}_\mu$. Here, the square brackets denote antisymmetrization. On the other hand, the exterior derivative of the curvature $2-$form yields

$$d\Omega^a{}_b = \frac{1}{2}\left(R^a{}_{bcd,\mu}e_e{}^\mu + 2R^a{}_{bfd}\Gamma^f{}_{ec}\right)\vartheta^e\wedge\vartheta^c\wedge\vartheta^d , \tag{6.3.3}$$

which together with

$$d\Omega^a{}_b = \frac{1}{2}\left(R^f{}_{bed}\Gamma^a{}_{fc}\right)\vartheta^e\wedge\vartheta^c\wedge\vartheta^d , \tag{6.3.4}$$

leads to the following equation

$$R^a{}_{b[cd,|\mu|}e_e{}^\mu = R^a{}_{f[cd}\Gamma^f{}_{|b|e]} - R^a{}_{b[cd}\Gamma^f{}_{|f|e]} - 2R^a{}_{bf[c}\Gamma^f{}_{de]} \ . \qquad (6.3.5)$$

This equation represents an algebraic relationship between the components of the tetrad vectors $e^a{}_\mu$ and the components of the connection $1-$form $\Gamma^a{}_{bc}$.

Finally, the components of the curvature tensor can be expressed in terms of the anholonomic components of the connection as

$$\frac{1}{2}R^a{}_{bcd} = \Gamma^a{}_{b[d,|\mu|}e_c{}^\mu + \Gamma^a{}_{be}\Gamma^e{}_{[cd]} + \Gamma^a{}_{e[c}\Gamma^e{}_{|b|d]} \ , \qquad (6.3.6)$$

which can be considered as a system of partial differential equations for the components of the connection with the components of the curvature and the tetrad vectors as variable coefficients.

To integrate Cartan's equations we proceed as follows. First, we consider the 20 particular independent equations (6.3.6) together with the 18 equations which follow from Eq.(6.3.5). The idea is to obtain from here all the 24 anholonomic components of the connection $\Gamma^a{}_{bc}$. Then, this result is used as input to solve the 24 independent equations which follow from Eq.(6.3.2). This procedure leads to a large number of equations which are complicated to be handled. They have been analyzed with some detail in [11]. Here, we will limit ourselves to quoting the some of the final results obtained previously.

6.4 Type D metrics

Consider a type D curvature tensor with eigenvalue $a + ib$, and suppose that

$$a = a(x^3) \ , \quad b = b(x^3) \ , \qquad (6.4.1)$$

i.e., we assume that the curvature depends on only one spatial coordinate. Furthermore, it is well known that type D spacetimes can have a maximum of four Killing vector fields. Then, we will consider spacetimes with two Killing vector fields which can be taken along the coordinates x^0 and x^1; consequently,

$$g_{\mu\nu,0} = g_{\mu\nu,1} = 0 \ , \quad g_{\mu\nu,0} = \frac{\partial g_{\mu\nu}}{\partial x^0} \ . \qquad (6.4.2)$$

This means that the only relevant spatial direction should be x^3. Therefore, we can use the diffeomorphism invariance of general relativity in order to bring four metric components into any desired form. We then assume that

$$g_{30} = g_{31} = g_{32} = 0 \,, \quad g_{33} = g_{33}(x^3) \,. \tag{6.4.3}$$

In terms of the local tetrad, the above assumption implies that

$$\vartheta^3 = \sqrt{g_{33}} dx^3 = e^3{}_{\dot{3}} dx^3 \,, \tag{6.4.4}$$

where the dot denotes coordinate indices. As a consequence we have that

$$d\vartheta^3 = 0 \,, \tag{6.4.5}$$

which implies that six components of the tetrad vectors vanish, namely,

$$e^0{}_{\dot{3}} = e^1{}_{\dot{3}} = e^2{}_{\dot{3}} = e^3{}_{\dot{0}} = e^3{}_{\dot{1}} = e^3{}_{\dot{2}} \,. \tag{6.4.6}$$

This means that we now have a system of only ten components of $e^a{}_\mu$ that are unknown. On the other hand, the vanishing of the exterior derivative of ϑ^3 implies that

$$\Gamma^3{}_{[ab]} = 0 \,, \tag{6.4.7}$$

which drastically simplifies the set of differential equations for the components of the connection. A detailed analysis of the resulting equations shows that it is convenient to consider particular cases which are obtained for different choices of some components of the connection. In fact, it turns out that the choices

$$\Gamma^1{}_{21} = 0 \,, \quad \Gamma^1{}_{23} \neq 0 \tag{6.4.8}$$

and

$$\Gamma^1{}_{21} \neq 0 \,, \quad \Gamma^1{}_{23} = 0 \tag{6.4.9}$$

lead to completely different solutions which we will analyze in the following subsections.

It is then possible to show that with these simplifying assumptions, we can integrate the set of partial differential equations. Several arbitrary functions arise in the tetrad vectors which can then be absorbed by means of coordinate transformations.

6.4.1 Schwarzschild and Taub-NUT metrics

The particular choice

$$\Gamma^1{}_{21} \neq 0\,, \quad \Gamma^1{}_{23} = 0 \tag{6.4.10}$$

leads to a compatible set of algebraic and differential equations which allow us to calculate all the components of the tetrad vectors. We present the final results without the details of calculations which can be consulted in [22].

Consider, for instance, the following curvature tensor in the $SO(3,C)$ representation:

$$\mathcal{R} = -\frac{M}{r^3}\,\mathrm{diag}(1,1,-2) + \frac{\Lambda}{3}\,\mathrm{diag}(1,1,1)\,, \tag{6.4.11}$$

where $r = x^3$. Then, the integration of all the differential equations yields

$$e^3{}_{\dot 3}\left(\alpha - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1/2}\,, \quad e^2{}_{\dot 2} = r\,, \quad e^1{}_{\dot 2} = rF^1{}_{\dot 2}\,, \tag{6.4.12}$$

$$e^0{}_{\dot m} = C^0{}_{\dot m}\left(\alpha - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{1/2}\,, \quad e^0{}_{\dot 2} = F^0_{\dot 2}\left(\alpha - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{1/2}\,, \tag{6.4.13}$$

where $m = 0,1$, α and $C^0{}_{\dot m}$ are arbitrary real constants and $F^0_{\dot 2}$ and $F^1_{\dot 2}$ are non-zero functions of the coordinate x^2. It is then possible to find a coordinate system in which the above tetrad vector components lead to the line element

$$ds^2 = \left(\alpha - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)dt^2 - \frac{dr^2}{\alpha - \frac{2M}{r} - \frac{\Lambda}{3}r^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{6.4.14}$$

which represents the Schwarzschild-de-Sitter spacetime.

Consider now a curvature tensor with gravitoelectric and gravitomagnetic components:

$$\mathcal{R} = -\frac{M + iP}{(r+iC)^3}\,\mathrm{diag}(1,1,-2) + \frac{\Lambda}{3}\,\mathrm{diag}(1,1,1)\,, \tag{6.4.15}$$

where P and C are arbitrary real constants. It is then possible to show that the result of the integration leads to a line element of the form

$$ds^2 = \Delta_1(dt + 2C\cos\theta d\phi)^2 - \frac{dr^2}{\Delta_1} - (r^2 + C^2)\left(\Delta_2 \sin^2\theta d\phi^2 + \frac{d\theta^2}{\Delta_2}\right) , \tag{6.4.16}$$

with

$$\Delta_1 = (r^2 + C^2)\left[\frac{P}{C}(r^2 - C^2) - 2Mr - \frac{\Lambda}{3}(r^2 + C^2)^2\right]^{-1} , \tag{6.4.17}$$

$$\Delta_2 = \frac{P}{C} + \frac{4}{3}\Lambda C^2 . \tag{6.4.18}$$

Different choices of the parameters P and C lead to different particular solutions of Einstein's equations. For instance, the choice

$$P = l\left(1 - \frac{4}{3}\Lambda l^2\right) , \quad C = l \tag{6.4.19}$$

corresponds to the Taub-NUT metric with cosmological constant [25], where l is the NUT parameter. Furthermore, the choice

$$P = kl\left(1 - \frac{4}{3}\Lambda l^2\right) , \quad C = l , \quad k = -1, 0, +1 \tag{6.4.20}$$

is known as the Cahen-Defrise spacetime [26].

The Taub-NUT metric is obtained for the choice $P = l$ and $C = l$ with $\Lambda = 0$. It is then possible to obtain several different generalizations which include the cosmological constant. In fact, the simplest choice corresponds to $P = C = l$ and the cosmological constant entering only the scalar part of the curvature. Other generalizations are obtained by choosing the free parameter P as a polynomial in Λ, for instance,

$$C = l , \quad P = l\left(c_1 + c_2\Lambda l^2 + c_3\Lambda^2 l^4 + ...\right) , \tag{6.4.21}$$

where c_1, c_2, etc. are dimensionless constants. Another example is obtained for the choice

$$P = l , \quad C = l\left(c_1 + c_2\Lambda l^2 + c_3\Lambda^2 l^4 + ...\right) . \tag{6.4.22}$$

All these examples generalize the Taub-NUT metric to include the cosmological constant. In principle, all of them should represent different physical configurations since they all differ in the behavior of the Weyl tensor. This opens the possibility of analyzing anti-de-Sitter spacetimes which are equivalent from the point of view of the scalar curvature, but different from the point of view of the Weyl curvature.

We conclude that in the particular case analyzed here the method presented above can be used to generate new solutions of Einstein's equations with cosmological constant.

6.4.2 Generalized Kasner metrics

Another particular choice of the connection components given by

$$\Gamma^1{}_{21} = 0\,, \quad \Gamma^1{}_{23} \neq 0 \tag{6.4.23}$$

leads to a set of algebraic and differential equations which can be integrated completely for a curvature tensor with only gravitomagnetic components, i.e.,

$$\mathcal{R} = a(x^3)\,\mathrm{diag}(1,1,-2) + \frac{\Lambda}{3}\,\mathrm{diag}(1,1,1)\,. \tag{6.4.24}$$

Indeed, after applying a series of coordinate transformations, the corresponding line element can be expressed as

$$\begin{aligned} ds^2 = \; & \frac{|a_0|}{\left|3a_0 - \frac{\Lambda}{2}\right|^{2/3}} dt^2 - \frac{(a_0')^2}{2|a_0|\left(3a_0 - \frac{\Lambda}{2}\right)^2} dr^2 \\ & - \frac{1}{\left|3a_0 - \frac{\Lambda}{2}\right|^{2/3}} (dX^2 + dY^2)\,, \end{aligned} \tag{6.4.25}$$

with

$$a_0 = a + \frac{\Lambda}{2} < 0\,, \tag{6.4.26}$$

and the prime represents derivation with respect to $r = x^3$. Here we see that the metric can be calculated immediately from the gravitoelectric component of the curvature $a(r)$. Several particular metrics can be written down. We quote only the metric that follows from the eigenvalue

$$a = -\frac{\gamma}{r^{2\beta}} - \frac{\Lambda}{3}\,, \tag{6.4.27}$$

where γ and β are real constants. For this case, we obtain

$$ds^2 = \left(\gamma - \tfrac{\Lambda}{6} r^{2\beta}\right) r^{-2\beta/3} dt^2 - \tfrac{2}{9}\beta^2 \left(\gamma - \tfrac{\Lambda}{6} r^{2\beta}\right)^{-1} r^{2(\beta-1)} dr^2 - r^{4\beta/3}(dX^2 + dY^2) \,. \tag{6.4.28}$$

In the limiting case $\Lambda = 0$, we obtain for each value of β a particular case of the Kasner metric [1]. In general, the above line element represents a generalization of the Kasner space which includes the cosmological constant. We see that in this particular case we have chosen a curvature eigenvalue which contains the cosmological constant explicitly. This has been done in order to obtain a simple expression for the Kasner metric with Λ. However, one can always change in the function $a(r)$ the term containing the cosmological constant, in order to obtain different solutions. The simplest spacetime would correspond to the one in which the Weyl tensor does not depend on the cosmological constant, i.e.,

$$a = -\frac{\gamma}{r^{2\beta}} \,, \tag{6.4.29}$$

for which we obtain the generalized Kasner metric

$$ds^2 = \frac{\left|-\frac{\gamma}{r^{2\beta}} + \frac{\Lambda}{2}\right|}{\left|-\frac{3\gamma}{r^{2\beta}} + \Lambda\right|^{2/3}} dt^2 - \frac{2\beta^2\gamma^2}{r^{2(2\beta+1)}\left|-\frac{\gamma}{r^{2\beta}} + \frac{\Lambda}{2}\right|\left(-\frac{3\gamma}{r^{2\beta}} + \Lambda\right)^2} dr^2 - \frac{1}{\left|-\frac{3\gamma}{r^{2\beta}} + \Lambda\right|^{2/3}} \left(dX^2 + dY^2\right) \,. \tag{6.4.30}$$

This particular choice seems to be more complicated than the solution (6.4.28); however, from a physical point of view it corresponds to the simplest choice in which the Weyl tensor is not affected by the presence of the cosmological constant.

We see that it is possible to obtain several generalizations of the Kasner metric with cosmological constant and, in principle, each of them should correspond to a different physical configuration.

6.5 Conclusions

In this work, we presented a method to calculate the components of the metric tensor from the components of the Riemann curvature tensor. We use the formalism of differential forms and Cartan's structure equations in order to calculate explicitly the algebraic and differential equations that relate the components of the local tetrad vectors with the components of the connection $1-$form and the curvature $2-$form.

We integrate the differential equations for the case of a type D curvature tensor in Petrov's classification which is characterized by only one complex eigenvalue. We found that for a given curvature eigenvalue, it is possible to obtain different metrics, depending on some assumptions made for the components of the connection $1-$form. For the computation of explicit examples, we assume that the curvature eigenvalue depends on only one spatial coordinate. This simplifies the set of differential equations and allows us to carry out the integration completely. We obtain as concrete examples two classes of spacetimes. The first class contains the Schwarzschild metric, the Taub-NUT metric and several generalizations which include the cosmological constant. The second class contains a family of particular Kasner spacetimes with cosmological constant.

The main result of the present work is that is possible to obtain the metric from the curvature. Furthermore, we found that for any given vacuum spacetime, we can apply the procedure presented in this work to obtain different generalizations which include the cosmological constant. This means that solutions of Einstein's equations with cosmological constant are not unique. The main physical difference between different spacetimes with cosmological constant is reflected in the Weyl tensor which behaves differently for each metric.

The concrete examples of curvature analyzed in this work involve terms with gravitoelectric monopole (mass parameter) and gravitomagnetic monopole (NUT parameter) only. In the case of an astrophysical gravitational source, a more realistic situation involves higher mass and angular momentum multipole moments. It is then easy to see that if we consider the Weyl tensor in the form

$$W = -\sum_{n=1}^{\infty} \frac{m_n}{r^{2n+1}} \,\mathrm{diag}(1, 1, -2)\ , \qquad (6.5.1)$$

the integration of the structure equations can be performed in a way similar to the one used to obtain the Schwarzschild and the Taub-NUT metrics. The explicit metric components can be computed by using the general formula presented here and in [11]. The resulting metric will contain the parameters m_n which correspond to higher mass multipole moments. In this way, one could generate exact solutions with a prescribed set of multipoles. In a realistic situation, for instance in the case of the Earth, one would need only a limited number of moments m_n, whose values can be from the measurement of the curvature components.

To take into account higher gravitomagnetic moments, it will be probably necessary to generalize the method presented here. Indeed, the presence of rotational moments implies that the curvature must depend on at least two spatial coordinates (a radial and an angular coordinate). In addition, it will probably necessary to consider not only type D, but also type I Weyl tensors. In this case, we need to construct a more general method than the one presented here. However, if we fix the angular coordinate and consider, for instance, the equatorial plane of the gravitational source, the curvature will depend only on the radial distance and it will be possible to consider a Weyl tensor of the form

$$W = -\sum_{n=1}^{\infty} \frac{m_n + i j_n}{r^{2n+1}} \, \mathrm{diag}(1, 1, -2) \, , \tag{6.5.2}$$

where the parameters j_n represent the multipoles of the curvature generated by the rotation of the source. Of course, this would be only an approximation of a realistic compact object since the dependence on the angular coordinate is completely neglected. However, since in the case of an object like the Earth, the deviations from spherical symmetry due to rotation are very small, one could expect that this equatorial plane approximation would lead results with a good degree of accuracy.

We conclude that the method presented in this work can be used, in principle, to generate particular metrics, describing the gravitational field of realistic compact objects. It would be interesting to investigate this problem in detail in the case of the Earth, to study the possibility of developing new navigation systems by using as input the curvature of the spacetime around our planet.

7 The Newman-Janis Ansatz

7.1 Introduction

Stationary solutions in general relativity are very important in the context of relativistic astrophysics. If we assume axial symmetry in vacuum, the Kerr solution [30] describes the exterior gravitational field of a rotating stationary configuration. A major open problem in classical general relativity is to find an exact interior solution that could be matched with the exterior Kerr geometry. Soon after the discovery of the Kerr solution, Newman and Janis [31] showed an algorithm for obtaining the Kerr solution from the Schwarzschild solution. The Newman-Janis Ansatz (NJA) can be interpreted as a complex coordinate transformation that acts on the Schwarzschild metric for deriving the Kerr solution. The same method has been used to obtain a Kerr-NUT solution [32] and a solution of the Einstein-Maxwell equations [33], starting from the Schwarzschild and the Reissner-Nordström metric, respectively. The NJA was investigated in general by Talbot [34] who proposed the first explanation for its success. Demianski [25] proved that the Taub-NUT metric with cosmological constant is the most general solution of Einstein's equations with cosmological constant that can be generated by using the NJA. Of course, the reasons why such a method does work can be traced to the behavior of Einstein equations [35, 36]. Moreover, at the level of the curvature tensor it is also possible to apply complex transformations to generate new solutions [22, 37].

The NJA was generalized by Herrera and Jiménez [38] to include the case of static spherically symmetric interior solutions in order to generate stationary interior spacetimes. Several interior Kerr solutions have been obtained by employing this method. For example, in [39] an interior trial solution was obtained which is characterized by a pressure that diverges at the origin of coordinates. In [40], several rotating neutral and charged solutions were obtained, describing the interior field of non-perfect fluids. The case of rotating spacetimes for anisotropic fluids with shear viscosity and heat flux was analyzed in detail in [41, 42], obtaining some particular solutions whose exterior counterpart is unknown, however. In [43], the extension of the NJA was applied to static spacetimes, like the incompressible Schwarzschild interior. The same method can be applied to obtain interior metrics which match a general stationary vacuum spacetime, provided the starting static metric is physically reasonable. Moreover, in [44], the field equations for anisotropic fluids were presented in an Ernst-like form which leads to a precise method for generating interior solutions. It turns out that, when applied to static spherically symmetric interior solutions, the extension of the NJA always destroys the perfect-fluid property [45]; nevertheless, in the case of a pure gravitomagnetic Weyl tensor, the perfect-fluid property can be preserved [46]. Although it has been argued [47] that the NJA would work only in Einstein's theory. Moreover, the application of the NJA to spherically symmetric solutions of alternative gravity theories has been shown to lead to pathologies in the resulting axially symmetric spacetimes [48]. Nevertheless, it has found applications to obtain rotating higher dimensional spacetimes [49], non-commutative black holes [50, 51], loop black holes [52], regular black holes [53, 55] and wormholes [54]. Moreover, a generalization of the NJA was proposed which includes the transformation of a particular gauge field [56].

The main objective of the present work is to apply the NJA to comformastatic spacetimes and, in particular, to the Schwarzschild exterior metric in isotropic coordinates. We follow the original terminology introduced by Synge [57], according to which stationary spacetimes with a conformally flat space of orbits constitute the conformastationary spacetimes, and conformastatic spacetimes comprises the static subset. Several exact solutions belonging to this class have been derived in a series of recent works [58, 59, 60, 61] which are interpreted as describing the gravitational and magnetic fields of static and rotating thin disks. In this work, we start from a general conformastatic metric from which a stationary metric is obtained whose main physical are also analyzed.

7.2 Vacuum conformastatic spacetimes

Consider the following conformastatic line element in spherical coordinates $x^\alpha = (t, r, \theta, \varphi)$:

$$ds^2 = V^2 dt^2 - U^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \,. \tag{7.2.1}$$

In general, the functions U and V can depend on all spatial coordinates. For the sake of simplicity, however, in this work we focus on the simple case in which these functions depend on the radial coordinate r, only. A straightforward computation leads to the following expressions

$$R^1{}_1 = (2rU_rV_r + rUV_{rr} + 2UV_r)/(rU^5V), \tag{7.2.2}$$

$$R^2{}_2 = (4rUVU_{rr} - 4rVU_r^2 - 2rUU_rV_r + rU^2V_{rr} + 4UVU_r)/(rU^6V), \tag{7.2.3}$$

$$R^3{}_3 = R^4{}_4 = (2rUVU_{rr} + 2rVU_r^2 + 2rUV_rU_r + 6UVU_r + U^2V_r)/(rU^6V), \tag{7.2.4}$$

where $R^\alpha{}_\beta$ are the components of the Ricci tensor. In vacuum, it is easy to show that the above system reduces to

$$\nabla^2 U = 0, \quad \nabla \cdot (U^2 \nabla V) = 0, \tag{7.2.5}$$

where ∇ is the usual gradient operator in spherical coordinates. Thus, U is a harmonic function. Having U, the second equation in (7.2.5) gives V. It is easy to prove that the only possible functional dependence $V = V[U]$ is of the form $V = -kU^{-1}$ with k =constant. However, with this kind of relationship between U and V the only possible solution for the complete system $R^\alpha{}_\beta = 0$ is the trivial solution. So, in general, the functions U and V are not related.

Is not easy to find functions U and V satisfying the above system. However, the following solution does exist

$$U = c + \frac{b}{r}, \quad V = \frac{cr - b}{cr + b}, \quad (a, b \quad \text{constants}), \tag{7.2.6}$$

which is equivalent to one of the most important solutions of Einstein's equations, namely, the exterior Schwarzschild solution in isotropic coordinates [1]

$$U = 1 + \frac{m}{2r}, \quad V = \frac{2r - m}{2r + m}\,, \tag{7.2.7}$$

corresponding to $c = 1$ and $b = m/2$.

The NJA is usually applied to obtain new stationary solutions from vacuum static solutions. We will follow the same idea in this work. Indeed, we will assume that we have two functions $U(r)$ and $V(r)$ that satisfy the vacuum conformastatic field equations (7.2.5). We will call this set of functions the seed solution. Our goal is to apply the NJA to a seed solution and to investigate the physical properties of the resulting metrics. In particular, we will be interested in finding explicitly the metric resulting from the Schwarzschild seed solution in order to compare our results with the ones obtained originally by Newman and Janis.

7.3 The Newman-Janis Ansatz

In this section, we apply the NJA to a general conformastatic spacetime given by the line element (7.2.1) with the metric functions U and V satisfying the vacuum field equations (7.2.5). Following the procedure presented originally in [31], we introduce the outgoing Eddington-Finkelstein coordinates (u, r^*) by means of

$$u = t - r^*, \quad dr^* = \frac{U^2}{V} dr \, . \tag{7.3.1}$$

Then, the line element (7.2.1) can be written as

$$ds^2 = V^2 du^2 + 2U^2 V du\, dr - r^2 U^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2). \tag{7.3.2}$$

To apply the NJA to this line element, we introduce the complex null tetrad

$$\begin{aligned} g^{\alpha\beta} &= l^\alpha n^\beta + n^\alpha l^\beta - m^\alpha \bar{m}^\beta - \bar{m}^\alpha m^\beta, \\ l^\alpha &= \delta^\alpha_r, \quad n^\alpha = \frac{1}{U^2 V}\delta^\alpha_u - \frac{1}{2U^4}\delta^\alpha_r, \\ m^\alpha &= \frac{1}{\sqrt{2}U^2 r}\delta^\alpha_\theta - \frac{i}{\sqrt{2}U^2 r \sin\theta}\delta^\alpha_r, \end{aligned} \tag{7.3.3}$$

where $l^\alpha l_\alpha = m^\alpha m_\alpha = n^\alpha n_\alpha = l^\alpha m_\alpha = m^\alpha n_\alpha = 0$ and $l^\alpha n_\alpha = -m^\alpha \bar{m}_\alpha = 1$. Complex conjugation is denoted by a bar over the corresponding quantity. Now, we perform the complex transformation

$$\begin{aligned} r \to \tilde{r} = r + ia\cos\theta, \quad u \to \tilde{u} = u - ia\cos\theta, \\ \{U(r), V(r)\} \to \{M(r,\theta;a), N(r,\theta;a)\} \end{aligned} \tag{7.3.4}$$

where the transformed functions M and N are demanded to be real and to satisfy the condition $\lim_{a\to 0}\{M, N\} = \{U, V\}$. Thus, the null tetrad (7.3.3) transforms into

$$l^\alpha = \delta^\alpha_r, \tag{7.3.5}$$

$$n^\alpha = \frac{1}{M^2 N}\delta^\alpha_u - \frac{1}{2M^4}\delta^\alpha_r, \tag{7.3.6}$$

$$\begin{aligned} m^\alpha &= \frac{(r - ia\cos\theta)}{\sqrt{2}M^2(r^2 + a^2\cos^2\theta)}\delta^\alpha_\theta \\ &+ \frac{a\sin\theta(ir + a\cos\theta)}{\sqrt{2}M^2(r^2 + a^2\cos^2\theta)}(\delta^\alpha_u - \delta^\alpha_r) \\ &+ \frac{(ir + a\cos\theta)}{\sqrt{2}M^2 r\sin\theta(r^2 + a^2\cos^2\theta)}\delta^\alpha_\varphi. \end{aligned}$$

From here, we obtain the transformed inverse metric

$$g^{uu} = -\frac{a^2\sin^2\theta}{M^4(r^2 + a^2\cos^2\theta)}, \quad g^{ur} = \frac{1}{M^2 N} + \frac{a^2\sin^2\theta}{M^4(r^2 + a^2\cos^2\theta)}, \tag{7.3.7}$$

$$g^{u\varphi} = -\frac{a}{M^4(r^2 + a^2\cos^2\theta)}, \quad g^{rr} = -\frac{1}{M^4} - \frac{a^2\sin^2\theta}{M^4(r^2 + a^2\cos^2\theta)}, \tag{7.3.8}$$

$$g^{r\varphi} = \frac{a}{M^4(r^2 + a^2\cos^2\theta)}, \quad g^{\theta\theta} = -\frac{1}{M^4(r^2 + a^2\cos^2\theta)} \tag{7.3.9}$$

$$g^{\varphi\varphi} = -\frac{a}{M^4(r^2 + a^2\cos^2\theta)\sin^2\theta},$$

and the corresponding line element in Eddington-Finkelstein coordinates

$$\begin{aligned} ds^2 &= N^2 du^2 + 2M^2 N du\, dr + 2a(M^2 N - N^2)\sin^2\theta du\, d\varphi \\ &- 2aM^2 N\sin^2\theta dr\, d\varphi - M^4(r^2 + a^2\cos^2\theta)d\theta^2 \\ &- \left[a^2(2M^2 N - N^2)\sin^2\theta + M^4(r^2 + a^2\cos^2\theta)\right]\sin^2\theta d\varphi^2. \end{aligned} \tag{7.3.10}$$

Now, if we choose the transformed functions as

$$M(r,\theta) = U(r) \quad \text{and} \quad N(r,\theta) = \frac{U^2(r^2 + a^2\cos^2\theta)}{\frac{r^2U^2}{V} + a^2\cos^2\theta} \tag{7.3.11}$$

and introduce Boyer-Lindquist (BL) like coordinates by means of the coordinate transformation

$$du = dt - \frac{\frac{r^2U^2}{V} + a^2}{r^2 + a^2}dr, \quad d\varphi = d\phi - \frac{a}{r^2 + a^2}dr\,, \tag{7.3.12}$$

the metric generated by the NJA reduces to

$$\begin{aligned} ds^2 = & \frac{U^4(r^2 + a^2\cos^2\theta)^2}{(a^2\cos^2\theta + \frac{r^2U^2}{V})^2}dt^2 + \frac{2aU^4(r^2 + a^2\cos^2\theta)(\frac{r^2U^2}{V} - r^2)\sin^2\theta}{(a^2\cos^2\theta + \frac{r^2U^2}{V})^2}dt\,d\phi \\ & - \frac{U^4(r^2 + a^2\cos^2\theta)}{a^2 + r^2}dr^2 - U^4(r^2 + a^2\cos^2\theta)d\theta^2 \\ & - U^4(r^2 + a^2\cos^2\theta)\sin^2\theta\left[1 + a^2\sin^2\theta\frac{(\frac{2r^2U^2}{V} - r^2 + a^2\cos^2\theta)}{(a^2\cos^2\theta + \frac{r^2U^2}{V})^2}\right]d\phi^2, \end{aligned} \tag{7.3.13}$$

where, as we mentioned above, $U(r)$ and $V(r)$ are a solution of the Einstein vacuum equations for a conformastatic spacetime. Notice that the above metric can also be obtained as a particular case from Eq.(7.3.13) of Ref. [53] where a general static metric is analyzed. To this end, it is necessary to choose the metric functions as

$$G(r) = V^2(r), \quad H(r) = \frac{r^2}{F(r)}, \quad F(r) = \frac{1}{U^4(r)}, \tag{7.3.14}$$

$$K = \frac{r^2U^2}{V}, \quad \psi = U^4(r^2 + a^2\cos^2\theta)\,. \tag{7.3.15}$$

For our purposes, however, we rewrite the new metric (7.3.13) as

$$\begin{aligned} ds^2 = & \frac{V^2\rho^2}{\Sigma^2}\left[dt^2 - 2a\frac{r^2}{\rho}\left(1 - \frac{U^2}{V}\right)\sin^2\theta dt d\phi + a^2\left(1 - \frac{2\Sigma U^2}{\rho V}\right)\sin^4\theta d\phi^2\right] \\ & - U^4\rho\left[\frac{dr^2}{r^2 + a^2} + d\theta^2 + \sin^2\theta d\phi^2\right]\,, \end{aligned} \tag{7.3.16}$$

with

$$\rho = r^2 + a^2\cos^2\theta\,, \qquad \Sigma = r^2 + a^2\cos^2\theta\frac{V}{U^2}\,. \tag{7.3.17}$$

Notice that in the limit $a \to 0$, the generalized metric reduces to the seed metric (7.2.1). Therefore, the parameter a can be interpreted as responsible for the stationarity of the spacetime and, consequently, should be associated with the rotation of the gravity source.

7.4 Physical interpretation

The generalized metric (7.3.16) is clearly stationary; however, it is not conformastationary. This implies that the NJA does not preserve the conformal invariance in this case. Moreover, if we impose the vacuum field equations for the functions (7.2.5) U and V, one can show that the corresponding Einstein tensor does not vanish; instead, we compute the following structure

$$G_{\alpha\beta} = \begin{pmatrix} G_{tt} & 0 & 0 & G_{t\phi} \\ 0 & G_{rr} & G_{r\theta} & 0 \\ 0 & G_{r\theta} & G_{\theta\theta} & 0 \\ G_{t\phi} & 0 & 0 & G_{\phi\phi} \end{pmatrix} . \tag{7.4.1}$$

It follows that in the case of vacuum comformastatic spacetimes the NJA leads in general to non-vacuum stationary spacetimes.

To investigate the physical interpretation of the spacetimes generated in this manner, we proceed as it is customary in the case of conformastatic and conformastic metrics, namely, we search for the conditions under which the above Einstein tensor can be interpreted as corresponding to a perfect-fluid source possibly endowed with an electromagnetic field [58] in such a way that it could describe the field of a disk-halo configuration.

7.4.1 The linearized limit

For the sake of simplicity, we consider first the approximate linearized limit in which quadratic terms in a can be neglected. Since the parameter a is related to the rotation of the source, we can expect that in the limiting case of slow rotation the resulting solution is related to the Lense-Thirring solution [1]

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - 4ma\frac{\sin^2\theta}{r}dtd\phi - \frac{dr^2}{1 - \frac{2m}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2) . \tag{7.4.2}$$

The non-diagonal term corresponds to the gravitational field generated by the rotation of the source.

The generalized metric (7.3.16) reduces in this case to

$$ds^2 = V^2\left[dt^2 - 2a\left(1-\frac{U^2}{V}\right)\sin^2\theta dt d\phi\right] - U^4\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) . \tag{7.4.3}$$

If the metric functions satisfy the conditions $\lim_{r\to\infty} U = \pm 1$ and $\lim_{r\to\infty} V = 1$, the spacetime is asymptotically flat, implying that the source of gravity is located in a limited region of spacetime. A direct calculation of the Einstein tensor shows that in general it is non-vanishing. For the sake of concreteness, let us consider the Schwarzschild metric (7.2.7) as the seed solution. Then, the only non-vanishing component of the Einstein tensor is

$$G_{t\phi} = 32\,\frac{am^2 r(2r^2+3mr-m^2)\sin^2\theta}{(2\,r+m)^6\,(2\,r-m)} , \tag{7.4.4}$$

Remarkably, all the component G_{tt} vanishes in this limit. This means that there is no energy density to be interpreted as the source of gravity. This makes difficult the interpretation of this approximate solution. In fact, one can try to identify the component $G_{t\phi}$ as due to a particular magnetic distribution in Einstein-Maxwell theory, i.e., satisfying the equations

$$G_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\mu}F_{\beta\nu}g^{\mu\nu} - \frac{1}{4}g_{\alpha\beta}g^{\mu\lambda}g^{\delta\nu}F_{\mu\delta}F_{\lambda\nu}\right) , \tag{7.4.5}$$

where $F_{\alpha\beta}$ is the Faraday tensor which can be expressed in terms of the electromagnetic potential A_α as $F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta\alpha}$. One can prove that starting from a general magnetic potential $A_\phi(r)$ there is no real solution for the components of the Faraday tensor such that the approximate Einstein-Maxwell equations (7.4.5) are satisfied.

If the parameter a, induced by the NJA, would be related only to the stationary rotation of the gravitational source, we would obtain in the linearized limit the Lense-Thirring metric in vacuum or its magnetic generalization. The results presented above show that this is not the case. We can nevertheless force the equivalence for particular cases. Indeed, we see that at the pole $\theta = 0$, the Einstein tensor component (7.4.4) vanishes. However, along the poles the Lense-Thirring metric predicts no gravitational influence due to the rotation and, consequently, this particular value corresponds to the original Schwarzschild metric. The additional particular case for which $2r^2 + 3mr - m^2 = 0$ leads to a radius value located inside the horizon which is, therefore, unphysical because it cannot be detected by an observer located outside the horizon. Thus, we see that it is not possible to recover the Lense-Thirring metric in any particular case.

7.4.2 A relativistic fluid

The non-diagonal structure of the Einstein tensor (7.4.1) for the general stationary metric indicates that it cannot be interpreted as a perfect fluid. Moreover, a straightforward calculation of its trace shows that it is in general different from zero, indicating that the identification with the electromagnetic Maxwell tensor is possible only in very special cases. Let us therefore consider the general case of a relativistic fluid whose energy-momentum tensor is given by

$$T_{\alpha\beta} = (\mu + P)V_\alpha V_\beta - P g_{\alpha\beta} + \mathcal{Q}_\alpha V_\beta + \mathcal{Q}_\beta V_\alpha + \Pi_{\alpha\beta}, \tag{7.4.6}$$

where μ and P represent the energy and pressure, respectively, the worldlines of the fluid are integral curves of the 4-velocity vector field V^α, the heat flux vector is $\mathcal{Q}_\beta$, and $\Pi_{\alpha\beta}$ represents the viscosity tensor. Notice that $\mathcal{Q}_\beta$, and $\Pi_{\alpha\beta}$ are transverse to the worldlines of the fluid in the sense that $\mathcal{Q}_\alpha V^\alpha = \mathcal{Q}^\alpha V_\alpha = 0$, and $\Pi_{\alpha\beta}V^\alpha = 0$. If a particular solution of Einstein's equations is described by the energy-momentum tensor (7.4.6), we may say that the gravitational field is generated by a source in which μ, P, $\mathcal{Q}_\alpha$ and $\Pi_{\alpha\beta}$ are the energy density, the isotropic pressure, the heat flux and the anisotropic tensor of the source. Thus, it is straightforward to see that

$$\mu = T_{\alpha\beta}V^\alpha V^\beta, \tag{7.4.7}$$

$$P = \frac{1}{3}\mathcal{H}^{\alpha\beta}T_{\alpha\beta}, \tag{7.4.8}$$

$$\mathcal{Q}_\alpha = T_{\alpha\beta}V^\beta - \mu V_\alpha, \tag{7.4.9}$$

$$\Pi_{\alpha\beta} = \mathcal{H}_\alpha{}^\mu \mathcal{H}_\beta{}^\nu (T_{\mu\nu} - P\mathcal{H}_{\mu\nu}), \tag{7.4.10}$$

where the projection tensor is defined by $\mathcal{H}_{\alpha\beta} \equiv V_\alpha V_\beta - g_{\alpha\beta}$.

In order to interpret the solution generated by the NJA, we first rewrite the metric (7.3.13) as

$$ds^2 = Adt^2 + 2Bdtd\phi - \frac{\psi}{a^2+r^2}dr^2 - \psi d\theta^2 - Cd\phi^2\,, \tag{7.4.11}$$

where

$$A \equiv \frac{\psi(r^2+a^2\cos^2\theta)}{(K+a^2\cos^2\theta)^2}\,,\quad B \equiv \frac{a\psi\sin^2\theta(K-r^2)}{(K+a^2\cos^2\theta)^2}\,,$$
$$C \equiv \frac{\psi\sin^2\theta\left[(K+a^2)^2 - a^2\sin^2\theta(r^2+a^2)\right]}{(K+a^2\cos^2\theta)^2}, \psi \quad \equiv U^4(r^2+a^2\cos^2\theta)\,,\quad K \equiv \frac{r^2U^2}{V}.$$

It is convenient to introduce a suitable reference frame in terms of an orthonormal tetrad for a local observer in the form

$$V_\alpha = \left\{\sqrt{A},\ 0,\ 0,\ \frac{B}{\sqrt{A}}\right\}, \tag{7.4.12}$$
$$K_\alpha = \left\{0,\ -\frac{\sqrt{\psi}}{\sqrt{a^2+r^2}},\ 0,\ 0\right\}, \tag{7.4.13}$$
$$L_\alpha = \left\{0,\ 0\ -\sqrt{\psi},\ 0,\ 0\right\}, \tag{7.4.14}$$
$$M_\alpha = \left\{0,\ 0\ \ 0,\ -\frac{\sqrt{B^2+AC}}{\sqrt{A}}\right\}. \tag{7.4.15}$$

with the corresponding dual tetrad

$$V^\alpha = \left\{\frac{1}{\sqrt{A}},\ 0,\ 0,\ 0\right\}, \tag{7.4.16}$$
$$K^\alpha = \left\{0,\ \frac{\sqrt{a^2+r^2}}{\sqrt{\psi}},\ 0,\ 0\right\}, \tag{7.4.17}$$
$$L^\alpha = \left\{0,\ 0,\ \frac{1}{\sqrt{\psi}},\ 0\right\}, \tag{7.4.18}$$
$$M^\alpha = \left\{-\frac{B}{\sqrt{A(B^2+AC)}},\ 0\ \ 0,\ \frac{\sqrt{A}}{\sqrt{B^2+AC}}\right\}. \tag{7.4.19}$$

It is easy to see that $V^\alpha V_\alpha = -K^\alpha K_\alpha = -L^\alpha L_\alpha = -M^\alpha M_\alpha = 1$ and that

$V^\alpha K_\alpha = V^\alpha L_\alpha = V^\alpha M_\alpha = K^\alpha L_\alpha = K^\alpha M_\alpha = L^\alpha M_\alpha = 0.$

The idea now is to verify if Einstein's equations, $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, which for the particular Einstein tensor (7.4.1) and the energy-momentum tensor (7.4.6) lead to an algebraic system of equations, can be solved in a consistent manner and without imposing additional conditions on the components of $G_{\alpha\beta}$. To this end, we use the constitutive equations (7.4.7) - (7.4.10) and Eq.(7.4.1). First, we can write the energy density and the pressure of the fluid as

$$\mu = \frac{G_{tt}}{A} \tag{7.4.20}$$

and

$$P = \frac{G_{tt} - AG}{3A}, \tag{7.4.21}$$

respectively, where $G = G_\alpha{}^\alpha$. These quantities must satisfy the corresponding energy conditions to be physically meaningful. Furthermore, the heat function is given by

$$\mathcal{Q}_\alpha = \frac{(AG_{t\phi} - BG_{tt})\delta_\alpha^\phi}{A^{3/2}}, \tag{7.4.22}$$

indicating the heat flux occurs only along the azimuthal direction. Finally, the non-zero components of the anisotropic tensor are

$$\Pi_{rr} = G_{rr} - \frac{\psi}{3(a^2 + r^2)A} G_{tt} + \frac{\psi}{3(a^2 + r^2)} G, \tag{7.4.23}$$

$$\Pi_{\theta\theta} = G_{\theta\theta} - \frac{\psi}{3A} G_{tt} + \frac{\psi}{3} G, \tag{7.4.24}$$

$$\Pi_{r\theta} = G_{r\theta}, \tag{7.4.25}$$

$$\Pi_{\phi\phi} = G_{\phi\phi} + \frac{2B^2 - AC}{3A^2} G_{tt} - \frac{2B}{A} G_{t\phi} + \frac{B^2 + AC}{3A} G, \tag{7.4.26}$$

$$\tag{7.4.27}$$

In terms of the components of the local orthonormal tetrad, the anisotropic tensor can be decomposed as

$$\Pi_{\alpha\beta} = P_r K_\alpha K_\beta + P_\theta L_\alpha L_\beta + P_\phi M_\alpha M_\beta + P_T (L_\alpha K_\beta + L_\beta K_\alpha), \tag{7.4.28}$$

where

$$
\begin{aligned}
P_r &= \frac{a^2 + r^2}{\psi} G_{rr} - \frac{1}{3A} G_{tt} + \frac{1}{3} G, \\
P_\theta &= \frac{1}{\psi} G_{\theta\theta} - \frac{1}{3A} G_{tt} + \frac{1}{3} G, \\
P_\phi &= \frac{A}{B^2 + AC} G_{\phi\phi} + \frac{2B^2 - AC}{3A(B^2 + AC)} G_{tt} - \frac{2B}{B^2 + AC} G_{t\phi} + \frac{1}{3} G, \\
P_T &= \frac{\sqrt{a^2 + r^2}}{\psi},
\end{aligned}
$$

represent the values of the anisotropic pressure in different spatial directions. Notice that $\mathcal{P} \equiv P_r + P_\theta + P_\phi = 0$ and, consequently, the trace $\Pi^\alpha{}_\alpha = 0$. Notice also that to get a fluid without heat flux it is mandatory that $AG_{t\phi} - BG_{tt} = 0$, a condition which in general is not satisfied. Even if we consider the particular metric generated from the Schwarzschild metric in isotropic coordinates, the heat flux cannot be made to vanish.

We see that the NJA generates in this case metrics with properties that are completely different from the properties of the starting seed metric. Consider, for instance, the Schwarzschild metric as seed solution. The NJA generates a non-vacuum solution for a relativistic fluid in which the heat flux is non-trivial and all the components of the anisotropic pressure are different from zero. It is interesting to consider in this case the limit $m \to 0$ with $a \neq 0$. The obtained solution has a non-trivial form, but a straightforward computation shows that its curvature tensor vanishes. This an essential logical test which shows that there is no pressure and heat flux without mass. Nevertheless, the resulting solution has a quite complicated physical interpretation which somehow is not exactly the idea of NJA.

7.5 Conclusions

The NJA was proposed more that half a century ago as a method to obtain the rotating Kerr solution from the static Schwarzschild solution, but it has also been shown to work for the Reissner-Nordström metric from which the Kerr-Newman solution is generated. It has been used extensively to generate stationary perfect-fluid solutions from static ones. Despite its success, the reason why the NJA works is still unknown. In fact, as it is used today, it can be understood as an Ansatz or as a trick. To upgrade the NJA to the status of an algorithm or a genuine method to generate new solutions of Einstein's equations, it must be described as an exact mathematical formalism that allows us to understand why it can be used to generate new solutions. The results obtained in the present work can be interpreted as an indication that the NJA is just a trick that happens to work under very particular circumstances.

In fact, we first applied the NJA to conformastatic vacuum metrics in the hope that we could generate conformastationary vacuum solutions. Our results show that this is not possible. Although the resulting metric has a non-diagonal term which is usually associated with the rotation of the source, it does not preserve the conformal symmetry of the static case. Moreover, the resulting metric does not satisfy Einstein's equations in vacuum. To analyze the physical significance of the metric generated by the NJA, we compare its Einstein tensor with the energy-momentum tensor for a relativistic fluid with anisotropic pressure and heat flux. We proved that in general all the physical quantities determining the fluid can be identified in a consistent manner with the non-zero components of the Einstein tensor. If we consider the particular case of the Schwarzschild metric as seed solution, all the physical quantities of the generated relativistic fluid satisfy the physical condition of vanishing as $m \to 0$, independently of the value of the rotational parameter a.

We thus see that the NJA does not generate the vacuum Kerr metric from the Schwarzschild metric in isotropic coordinates. Even in the limiting case of small a, the resulting linearized solution cannot be identified as the vacuum Lense-Thirring metric. This implies that the NJA depends on the choice of coordinates. We interpret this result as indication that the NJA cannot be considered as an algorithm; instead, it should be interpreted as a trick that happens to work well for particular solutions in spherical coordinates. Recently, in [62], an additional negative fact about the NJA was observed, namely, that in the context of modified gravity theories it leads to pathologies in the resulting metrics and that it should not be used to generate rotating black holes outside general relativity. In the present work, we show that even within general relativity it should be used with caution to construct black hole solutions, because it depends on the coordinates used for the construction.

8 On the gravitational field of Hot White Dwarf Stars

8.1 Introduction

The no-hair theorems for black holes state that in general relativity only a finite number of multipole moments are necessary to describe black holes, namely, mass, charge and angular momentum [87]. It is believed that during the gravitational collapse of an arbitrary mass distribution, higher multipoles disappear as a result of the emission of gravitational waves. One would then expect that other compact objects like neutron stars (NSs) are characterized in general by an infinite number of multipoles. However, some recent intriguing results seem to indicate that compact objects other than black holes are also characterized by a finite number of multipoles.

In fact, the $I-$Love$-Q$ relation states that there exists a connection between the moment of inertia, quadrupole moment and the Love numbers, which in compact objects measure their rigidity and shape response to tidal forces. This relation is valid independently of the equations of state (EoSs) used to describe relativistic compact objects such as NSs and quark stars, if the slow rotation approximation is assumed in the framework of the Hartle-Thorne formalism [110, 98]. Similar approximate relations among multipole moments for NSs have been also investigated in the case of both slow and rapid rotation regimes [113, 106].

The I-Love-Q and I-Love relations, respectively, were investigated for incompressible and realistic stars; it was shown that the EoS-independent behaviour of the I-Love-Q relation can be attributed to its incompressible limit [105, 77]. Moreover, these relations were also calculated by Pani [101] for exotic objects such as thin-shell gravastars at zero temperature, without assessing their validity. The nonvalidity for the thin-shell gravastars was shown for different EoSs [109]. Nonetheless, it was established that in gravastars these relations posses distinct features from the ones of NSs and QSs.

More recently, the validity of the $I-$Love-Q relation was proven also in the case of dark stars by Maselli et al. [99] and white dwarfs (WDs) by Boshkayev et al. [74] at zero temperature. In the case of WDs, the Hartle-Thorne formalism was implemented in Newtonian physics to integrate the field equations together with the condition of hydrostatic equilibrium.

In the present work, we consider an additional important aspect of the internal structure of WDs, namely, their thermodynamic behavior. In particular, we analyze in detail the effects that follow from considering finite temperatures in the EoS. We will use the Hartle-Thorne formalism the validity of which has been well established in the derivation of all physically relevant quantities of rigidly rotating relativistic and classical objects [84, 85, 72]. The parameters describing the structure play a paramount role in the investigation of the stability and the lifespan of WDs [67, 68, 69, 70, 71, 104, see e.g.].

It has been shown that for massive white dwarfs close to the Chandrasekhar mass limit the effects of finite temperatures are negligible. However, for the observed low-mass white dwarfs the effects are crucial [82]. From the astrophysical point of view it is hard to measure the radius of a star with respect to its mass and temperature. Hence, if we know the mass and temperature of a WD, we can theoretically calculate its radius as it will be always different from the cold (degenerate) case [82, 75]. Therefore, we study here the effects that rotation along with temperature cause on the structure of WDs; first, we consider the main physical parameters of WDs and study their dependence on the density of the star for different temperature values. This allows us to investigate in detail the I-Q, I-Love and Love-Q relations, and to demonstrate that they are not universal. The temperature effects are sufficient to break down the universality of the $I-$Love$-Q$ relations. This is shown by integrating numerically the structure equations for slowly rotating WDs with the Chandrasekhar EoS [78, 79, 103] at different temperatures.

8.2 Formalism and stability criteria for rotating hot white dwarfs

A general relativistic analysis of the hydrodynamic equilibrium of WDs has established that relativistic effects lead only to small perturbations of Newtonian gravity [100]; consequently, the essential physical features of WDs can be studied by using Newton's theory. If in addition we use the Hartle-Thorne formalism to analyze perturbatively the structural equations, as proposed recently by Boshkayev et al. [74], it is possible to explore in detail the behavior of all the relevant physical quantities. The main idea consists in solving Newton's equation

$$\nabla^2\Phi = 4\pi G\rho \,, \tag{8.2.1}$$

and the equilibrium condition

$$\frac{dp}{dr} = -\rho\frac{GM}{r^2} \,, \qquad \frac{dM}{dr} = 4\pi r^2\rho \,, \tag{8.2.2}$$

perturbatively by expanding the radial coordinate as $r = R + \xi$, where R is the radial coordinate for a spherical configuration and the function $\xi(R, \theta)$ takes into account the deviations from spherical symmetry due to the rotation of the star. All the relevant quantities such as the total mass M, equatorial radius R_e, moment of inertia I, angular momentum J, quadrupole moment Q, etc. are then Taylor expanded up to the second order in the angular velocity. Within this approximation, due to an appropriate choice of ξ, the density ρ and pressure p can be considered as non affected by the rotation of the star. The structural equations (8.2.1) and (8.2.2) can then be integrated numerically to obtain all the relevant quantities in the desired approximation.

For the analysis of the structural equations it is convenient to introduce the Keplerian angular velocity

$$\Omega_{Kep} = \sqrt{\frac{GM}{R_e^3}} \,, \tag{8.2.3}$$

because it allows us to calculate all the key parameters at the mass-shedding limit, and to determine the stability region inside which rotating configurations can exist [69].

Finally, the inverse β-decay instability determines the critical density which, in turn, defines the onset of instability for a WD to collapse into a neutron star. For the Chandrasekhar EoS we adopt $\rho_{crit} = 1.37 \times 10^{11}$ g/cm^3. The inverse β-decay instability is crucial both for static and rotating configurations. It represents one of the boundaries of the stability region for rotating WDs [69, 72, 74]. According to de Carvalho et al. [82], the occurrence of the inverse β-decay instability is not affected by the presence of temperature, i. e., it is the same as in the degenerate approximation. This is related to the fact that the effects of temperature are negligible in the higher density regime. For the sake of generality, all computations are performed for central densities up to 10^{12} g/cm^3.

8.3 Equations of state for white dwarfs

We will use the simplest EoS for WD matter that correctly describes its main physical properties, namely, the Chandrasekhar EoS [82, 73]. Then, the total pressure is due to the pressure of electrons P_e, because the pressure of positive ions P_N (naked nuclei) is insignificant, whereas the energy density is due to the energy density of nuclei $\mathcal{E}_N$ as the energy density of the degenerate electrons $\mathcal{E}_e$ is negligibly small. Thus, the Chandrasekhar EoS is given by

$$\mathcal{E}_{Ch} = \mathcal{E}_N + \mathcal{E}_e \approx \mathcal{E}_N, \tag{8.3.1}$$

$$P_{Ch} = P_N + P_e \approx P_e. \tag{8.3.2}$$

Hence the energy density of the nuclei is given by

$$\mathcal{E}_N = \frac{A}{Z} M_u c^2 n_e \tag{8.3.3}$$

where A is the average atomic weight, Z is the number of protons, $M_u = 1.6604 \times 10^{-24}$ g is the unified atomic mass, c is the speed of light and n_e is the electron number density. In the following analysis, we will assume the average molecular weight $\mu = A/Z = 2$. In general, the electron number density follows from the Fermi-Dirac statistics, and is determined by

$$n_e = \frac{2}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 dp}{\exp\left[\frac{\tilde{E}(p)-\tilde{\mu}_e(p)}{k_B T}\right] + 1}, \tag{8.3.4}$$

where k_B is the Boltzmann constant, $\tilde{\mu}_e$ is the electron chemical potential without the rest-mass, and $\tilde{E}(p) = \sqrt{c^2p^2 + m_e^2c^4} - m_e c^2$, with p and m_e being the electron momentum and rest-mass, respectively, [82].

The electron number (8.3.4) can be written in an alternative form as

$$n_e = \frac{8\pi\sqrt{2}}{(2\pi\hbar)^3} m^3 c^3 \beta^{3/2} \left[F_{1/2}(\eta,\beta) + \beta F_{3/2}(\eta,\beta)\right], \tag{8.3.5}$$

where

$$F_k(\eta, \beta) = \int_0^\infty \frac{t^k \sqrt{1 + (\beta/2)t}}{1 + e^{t-\eta}} dt \tag{8.3.6}$$

is the relativistic Fermi-Dirac integral, $\eta = \tilde{\mu}_e/(k_B T)$, $t = \tilde{E}(p)/(k_B T)$ and $\beta = k_B T/(m_e c^2)$ are the degeneracy parameters. Consequently, the total electron pressure for $T \neq 0$ K is given by

$$P_e = \frac{2^{3/2}}{3\pi^2\hbar^3} m_e^4 c^5 \beta^{5/2} \left[F_{3/2}(\eta, \beta) + \frac{\beta}{2} F_{5/2}(\eta, \beta) \right]. \tag{8.3.7}$$

When $T = 0$, for a degenerate electron gas we find from Eq. (8.3.4) that

$$n_e = \int_0^{P_e^F} \frac{2}{(2\pi\hbar)^3} d^3 p = \frac{(P_e^F)^3}{3\pi^2\hbar^3} = \frac{(m_e c)^3}{3\pi^2\hbar^3} x_e^3. \tag{8.3.8}$$

Thus, the total electron pressure is

$$\begin{aligned} P_e &= \frac{1}{3} \frac{2}{(2\pi\hbar)^3} \int_0^{P_e^F} \frac{c^2 p^2}{\sqrt{c^2 p^2 + m_e^2 c^4}} 4\pi p^2 dp \\ &= \frac{m_e^4 c^5}{8\pi^2\hbar^3} \left[x_e \sqrt{1 + x_e^2} \left(\frac{2x_e^2}{3} - 1 \right) + \text{arcsinh}(x_e) \right], \end{aligned} \tag{8.3.9}$$

where $x_e = P_e^F/(m_e c)$ is the dimensionless Fermi momentum.

8.4 Results and discussion

For the sake of simplicity, throughout the paper we use a uniform temperature profile for isothermal cores of WDs, i.e. WDs without outer envelop (atmosphere). The atmosphere serves as an isolator and its effect on the structure of WDs can be neglected in this approximation. In order to justify a constant temperature profile within the core, we considered the [107] equilibrium condition for hot relativistic static stars, which is given by $T/u^t = constant$, where T is the local temperature and u^t is the zero-component of the four-velocity. In the case of a static star, $u^t = 1/\sqrt{g_{tt}}$, from which one obtains the known Tolman law: $\sqrt{g_{tt}} T = constant$. So, for the usual spherically symmetric metric: $\exp(\nu/2) T = constant$. In the classical limit $\exp(\nu/2) \approx 1 - \Phi/c^2$,

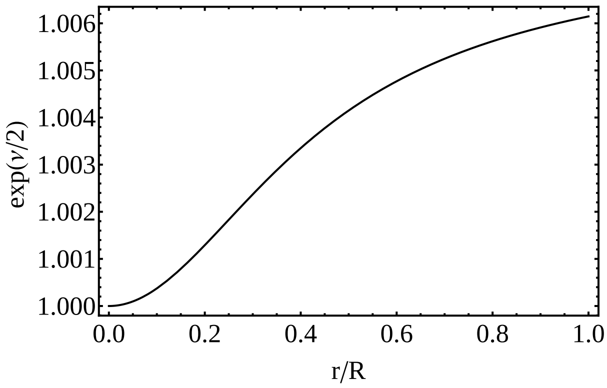

Figure 8.1: $\exp(\nu/2)$ as a function of the radial distance for a zero temperature white dwarf of mass M= 1.44$M_{\odot}$ and radius R=1000 km.

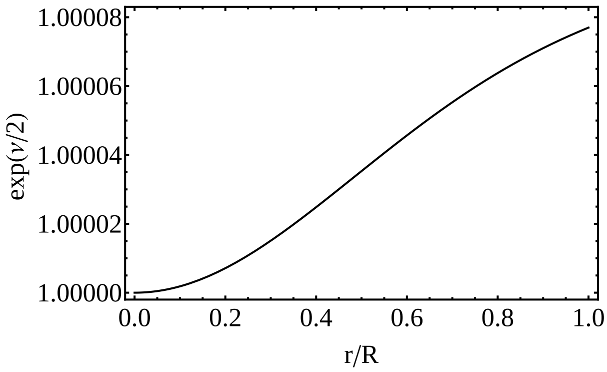

Figure 8.2: $\exp(\nu/2)$ as a function of the radial distance for a zero temperature white dwarf of mass M=0.4$M_{\odot}$ and radius R=10952 km.

where $\Phi = \Phi(r)$ is the internal Newtonian gravitational potential found from Eq. (8.2.1) and c is the speed of light in vacuum. We constructed $\exp(\nu/2)$ as a function of r/R. We then selected a white dwarf with mass 1.44$M_{\odot}$ and radius 1000 km, as an example. As one can see from Fig. 8.1, the function $\exp(\nu/2)$ changes slightly from the center to the surface of a white dwarf core. So, $\exp(\nu/2)$ changes less than 1% from the center to the surface of the isothermal core. This is the main argument to adopt the constant temperature profile.

One can calculate $\exp(\nu/2)$ also for a low mass white dwarf with mass 0.4$M_{\odot}$ and radius 10952 km. In Fig. 8.2, we see that $\exp(\nu/2)$ changes even less than in the previous case. Hence, for the cores of WDs the constant tem-

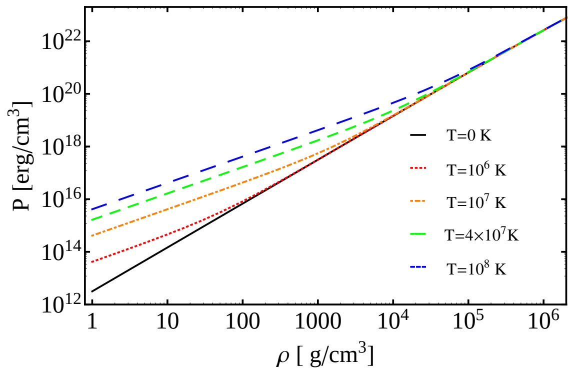

Figure 8.3: Total pressure as a function of the mass density for selected temperatures in the range $T = [0, 10^8]$ K (colour online).

perature profile is a safe assumption. A further generalization of the Tolman condition for slowly rotating stars is given by [66]. Even in that general case the change of function $\exp(\nu/2)$ turned out to be negligible for WDs.

In Fig. 8.3, we plot the total pressure Eq. (8.3.2) as a function of the total density Eq. (8.3.1) for some selected temperatures. We conclude that the effects of temperature are essential only in the range of small densities.

In general, we assume that the slow-rotation approximation can be applied to any realistic star with a Keplerian angular velocity. Indeed, Hartle and Thorne [85] in their pioneering article used this approximation to investigate the effects and deviations produced by rotation starting from massive non-compact stars to neutron stars. The general conditions in the slow-rotation regime are that the velocities of particles at the equatorial plane of the star must be non-relativistic and, of course, that the fractional changes of density, pressure, mass, radius, gravitational potential etc., due to rotation must be smaller than in the static case. However, the most practical condition to check the validity of the slow rotation approximation for WDs would be to compare the mass-radius relations at the mass shedding limit within the slow-rotation approximation with the resuls obtained by using exact numerical computations. Unfortunately, to our knowledge, for white dwarfs this problem has not been considered yet. Some analysis of the validity of the slow approximation for WDs were performed by Boshkayev et al. [see 69, Appendix D, Fig. 9]. Here we employ the Keplerian velocity to set upper bounds for all physical quantities as their realistic values will be between static and maxi-

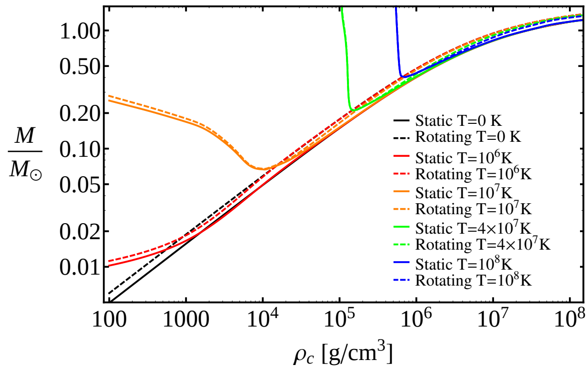

Figure 8.4: Mass versus central density (colour online).

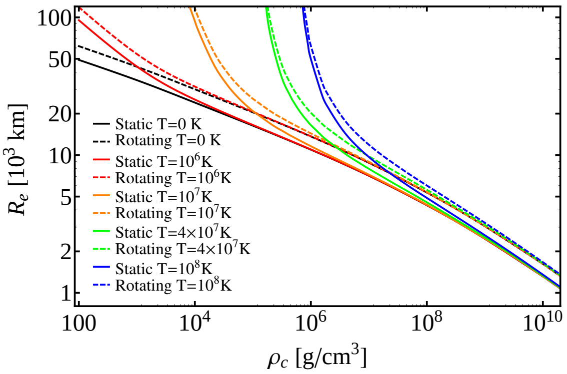

Figure 8.5: Radius versus central density (colour online).

mally rotating configurations. Hence, by solving the structure equations, we construct all necessary relations along the mass shedding sequence with angular velocity Ω_{Kep}.

In Fig. 8.4, the static and rotating mass of a WD is shown as a function of the central density and temperature. Our results show that in general rotating WDs have larger masses than their static counterparts. Due to the choice of the scale the green and blue curves look sudden and sharp, but in reality they are not so abrupt. The curves look sharper with respect to colder white dwarfs, because of the pronounced effects of higher temperatures.

Fig. 8.5 shows the equatorial radius as a function of the central density and temperature for both rotating and static WDs. The plots show that hot WDs

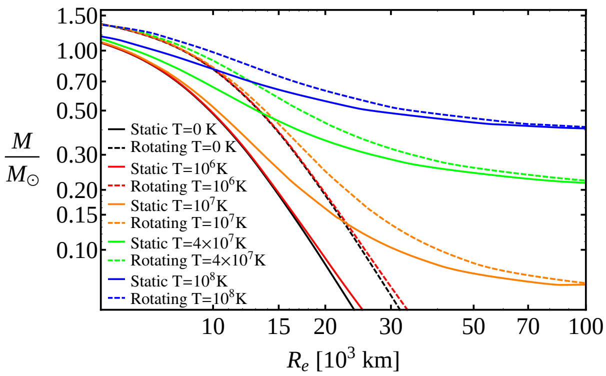

Figure 8.6: Mass versus radius (colour online).

possess larger radius than cold ones. For increasing central densities, WDs become more gravitationally bound and spherical.

Fig. 8.6 shows the mass and equatorial radius relation. Here one can see that the mass-radius relation significantly diverges from the degenerate case especially for lower masses and larger radii, depending on the value of the core temperature. The relationship between the core temperature and observed effective surface temperature is given via the so-called Koester relation [82]. This explains the variety of observed WDs, according to the Sloan Digital Sky Survey Data Release. Indeed, nowadays we have data for more than thirty two thousand WDs and all of them have diverse characteristics [90, 91, 89, 92, 108, 88]. It should be stressed that the scale of the mass is selected for the sake of generality. Indeed, so far observed WDs have masses larger than $0.1M_{\odot}$.

Fig. 8.7 shows the moment of inertia as a function of the central density for both static and rotating, cold and hot WDs. In the static case the moment of inertia of hot WDs is larger than for cold ones, in the entire range of the central density. This was expected as hotter WDs with similar masses will be larger in size than colder ones. However, for rotating WDs the situation is slightly different as hotter (larger in size) WDs cannot rotate faster than colder (smaller in size) ones. This effect becomes more evident starting from the value of the central density 10^6g/cm^3. Consequently, because of the rotation, the moment of inertia of hotter WDs will be smaller than that of colder ones. For the normalized moment of inertia this effect is also valid in the static case as MR^2 goes up faster than the moment of inertia when temperature increases

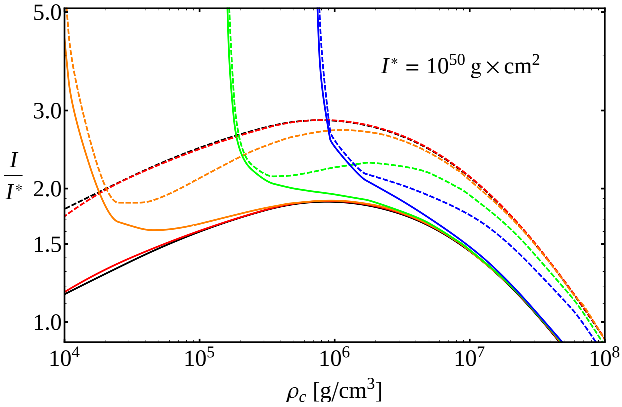

Figure 8.7: Moment of inertia versus central density. The legend is the same as in Fig. 8.6(colour online).

and the EoS becomes softer, for further details see Fig.8.8.

The normalized quadrupole moment is shown as a function of the central density in Fig. 8.9. The effect of the temperature is considerably small in the range of densities higher than 10^{10} g/cm^3. However, as the density diminishes the temperature plays a more important role, leading to a nonlinear behaviour of the analysed quantities. For values of the central density lower than 10^{10} g/cm^3, the quadrupole moment strongly depends on the temperature, but in general it increases in value for less massive stars. Within the approximate interval $\rho_c \in [10^4, 10^6]$ g/cm^3 and for specific values of the temperature, the quadrupole moment drastically decreases, indicating a trend towards spherical symmetry.

The above results show that temperature can play a very important role in the determination of the physical properties of WDs. Moreover, at first glance it seems that the moment of inertia and the quadrupole moment correlate. However, a deeper analysis shows a discrepancy. In Fig.8.10, we plot the dimensionless moment of inertia $\bar{I} = (c^4 I)/(G^2 M^3)$ against the dimensionless quadrupole moment $\bar{Q} = (c^2 Q)/(J^2/M)$, where I is the physical moment of inertia, Q is the physical mass quadrupole moment, M is the static mass and J is the angular momentum of the WD. For the degenerate case, $T = 0$, we corroborate the $\bar{I} - \bar{Q}$ relation established previously by Boshkayev et al. [74]. As the temperature increases towards the range of realistic values the $\bar{I} - \bar{Q}$ relation clearly breaks down. As the temperature increases, the break

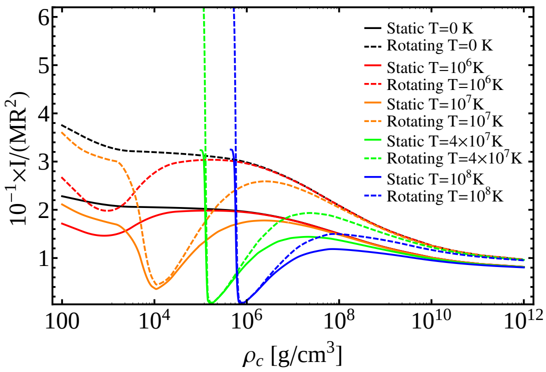

Figure 8.8: Normalized moment of inertia versus central density (colour online).

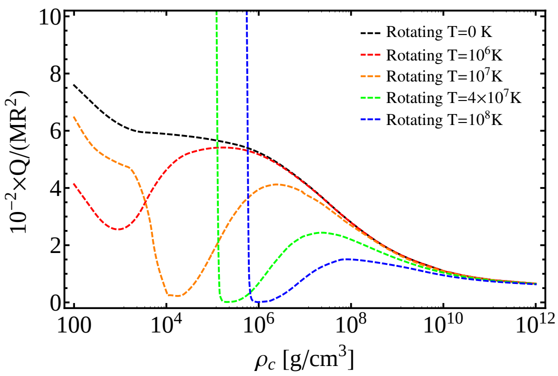

Figure 8.9: Normalized quadrupole moment versus central density (colour online).

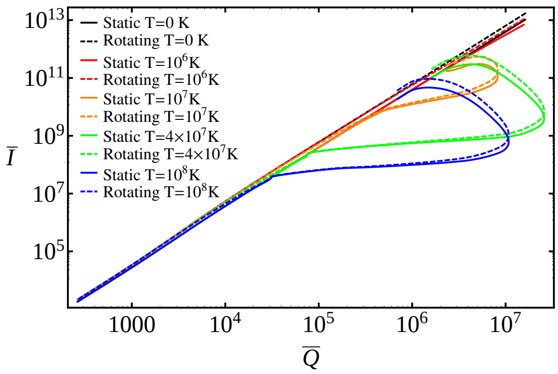

Figure 8.10: Dimensionless moment of inertia versus dimensionless quadrupole moment (colour online).

point moves towards the region of lower quadrupole moment. This proves that the $\bar{I} - \bar{Q}$ is no longer valid in the case of hot WDs.

We investigate the I-Love-Q relations in Figs. 8.11 and 8.12. As the temperature is taken into account, the non-validity of these relations becomes evident. We also see that as the temperature goes up, the breaking occurs at lower values of the dimensionless tidal Love number $\bar{\lambda} = (c^{10}\lambda)/(G^4 M^5)$.

Notice that in this approximation the moment of inertia can be expressed as the sum of a static plus a rotational component, whereas the quadrupole moment has only a rotational component (for the details of this decomposition, see [72]). Therefore, although in Fig. 8.10 we use $\bar{Q}$ as a parameter for the central density also in the static case, it does not mean that there is a static quadrupole moment. Also for this reason, in Fig. 8.12 only the rotational component of the quadrupole moment is plotted. For the sake of clarity, we present in Table 8.1 the numerical values for the I-Love-Q relations in terms of the central density for zero temperature white dwarf stars. Fig. 8.10 and Fig. 8.11 illustrate the behavior of $\bar{I}$ and $\bar{I} + \Delta\bar{I}$ as functions of $\bar{Q}$, where $\bar{Q}$ serves as a parameter for ρ, and $\bar{\lambda}$, respectively. Finally, Fig. 8.12 represents $\bar{Q}$ as a function of $\bar{\lambda}$.

From the above results, we conclude that the I-Love-Q relations proposed by Yagi and Yunes [110] for relativistic objects are not true for hot white dwarf stars, even in the framework of Newtonian gravity. The universality is thus lost for larger values of the moment of inertia, quadrupole moment and tidal Love number. In the region of smaller values of these parameters, which

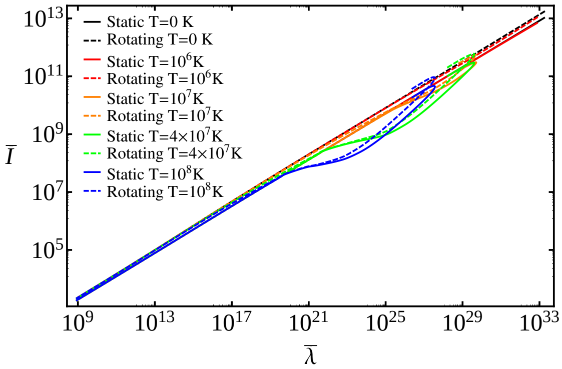

Figure 8.11: Dimensionless moment of inertia versus dimensionless tidal Love number (colour online).

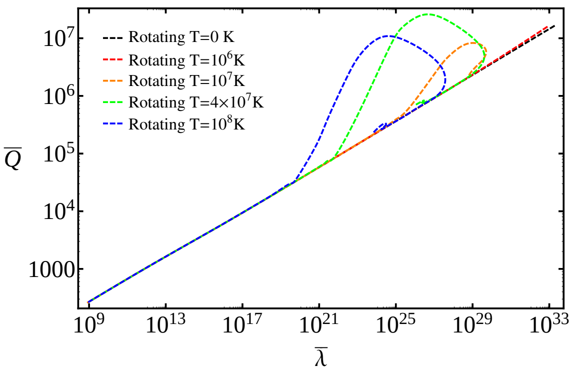

Figure 8.12: Dimensionless quadrupole moment versus dimensionless tidal Love number (colour online).

Table 8.1: I-Love-Q relations for white dwarf stars with $T = 0$ K. Here ρ is the central density, $\bar{I}$ is the dimensionless moment of inertia for static configurations, $\bar{I} + \Delta\bar{I}$ is the dimensionless moment of inertia for rotating configurations, $\bar{Q}$ is the dimensionless quadrupole moment for rotating configurations and $\bar{\lambda}$ is the dimensionless tidal Love number for static configurations.

ρ (g/cm^3)	$\bar{I}$	$\bar{I} + \Delta\bar{I}$	$\bar{Q}$	$\bar{\lambda}$
10^2	1.0×10^{13}	1.7×10^{13}	1.6×10^{7}	1.7×10^{33}
10^3	4.8×10^{11}	7.6×10^{11}	3.5×10^{6}	8.2×10^{29}
10^4	2.3×10^{10}	3.5×10^{10}	7.6×10^{5}	4.0×10^{26}
10^5	1.1×10^{9}	1.7×10^{9}	1.7×10^{5}	2.2×10^{23}
10^6	6.7×10^{7}	1.0×10^{8}	4.2×10^{4}	1.9×10^{20}
10^7	6.2×10^{6}	9.1×10^{6}	1.3×10^{4}	4.9×10^{17}
10^8	9.3×10^{5}	1.3×10^{6}	5.2×10^{3}	4.5×10^{15}
10^9	1.8×10^{5}	2.4×10^{5}	2.4×10^{3}	8.2×10^{13}
10^{10}	3.9×10^{4}	4.9×10^{4}	1.2×10^{3}	1.8×10^{12}
10^{11}	8.5×10^{3}	1.0×10^{4}	5.6×10^{2}	4.0×10^{10}

corresponds to the regime of larger densities when the degeneracy sets in, the behaviour is almost universal as it was shown by Boshkayev et al. [74].

The non-validity of the $I-$Love$-Q$ relations was also found in other studies. For instance, the breakdown of these relations was found by Doneva et al. [83] for rapidly rotating NSs and QSs, although in the slow-rotation approximation and at fixed rotational frequencies, one can still find roughly EoS-independent relations. Similar results have been obtained by Pappas and Apostolatos [102]. The I-Q relations for arbitrarily fast rotating NSs were also considered by Chakrabarti et al. [76], where it was found that the relations can be still universal among various EoSs for constant values of certain dimensionless parameters characterizing the magnitude of the rotation. However, it was demonstrated by Haskell et al. [86] that the universality of the relations is lost in the presence of huge magnetic fields in NSs with rotation period larger than 10 seconds and magnetic fields larger than 10^{12}G. In addition, Yagi et al [112] showed that the universality is also lost for non-compact objects when their opacity was varied. Furthermore, the phase of the proto-NS life, including the effects of both rotation and finite temperatures, was studied by Martinon et al. [97]. It was shown that the I-Love-Q relations are violated in the first second of life, but they are satisfied as soon as the entropy gradients smooth out. Recently, it was found that the I-Q universality is broken when thermal effects become important, independently of the presence of entropy gradients [96].

8.5 Conclusions

We numerically integrated the underlying differential equations in order to determine the structure of slowly and rigidly rotating classical WDs in hydrostatic equilibrium. In particular, using the Chandrasekhar EoS, the relations for the mass, radius, moment of inertia, and quadrupole moment were established as functions of the central density and temperature. All these quantities play a crucial role in describing the equilibrium configurations of uniformly rotating main sequence stars as well as massive stars. In particular, we proved that the temperature affects the behavior of all the physical parameters, especially in the region of realistic temperature values. In addition, we calculated the tidal Love number and investigated the I-Love-Q relations for rotating WDs.

It turned out that the I-Love-Q relations are not universal even within the same EoS when the finite temperature effects are taken into account. This is probably due to the fact that the EoS is not longer barotropic when the thermal effects are included, i.e., the pressure not only depends on the density, but also on the temperature. In a related work by Lau et al. [93], it was found that the universality of the I-Love relation is broken when the elastic properties of crystalline quark matter are accounted for, i.e., the universality is observed only in perfect fluid compact objects (at zero temperature without magnetic fields).

In view of their astrophysical relevance it would be interesting to investigate the validity of the I-Love-Q relations for WDs with different nuclear composition and magnetic field intensity [95, 80, 81, 94, 63, 64, 65]. This will be the issue of future studies.

Bibliography

[1] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, Exact solutions of Einstein's field equations, Cambridge University Press, Cambridge UK, 2003.

[2] F. J. Ernst, *New formulation of the axially symmetric gravitational field problem*, *Phys. Rev.* **167** (1968) 1175; F. J. Ernst, *New Formulation of the axially symmetric gravitational field problem II* Phys. Rev. **168** (1968) 1415.

[3] H. Quevedo and B. Mashhoon, *Exterior gravitational field of a rotating deformed mass*, *Phys. Lett. A* **109** (1985) 13; H. Quevedo, *Class of stationary axisymmetric solutions of Einstein's equations in empty space*, *Phys. Rev. D* **33** (1986) 324; H. Quevedo and B. Mashhoon, *Exterior gravitational field of a charged rotating mass with arbitrary quadrupole moment*, *Phys. Lett. A* **148** (1990) 149; H. Quevedo, Multipole Moments in General Relativity - Static and Stationary Solutions-, *Fort. Phys.* **38** (1990) 733; H.Quevedo and B. Mashhoon *Generalization of Kerr spacetime*, *Phys. Rev. D* **43** (1991) 3902.

[4] H. Weyl, *Zur Gravitationstheorie*, *Ann. Physik (Leipzig)* **54** (1917) 117.

[5] T. Lewis, *Some special solutions of the equations of axially symmetric gravitational fields*, *Proc. Roy. Soc. London* **136** (1932) 176.

[6] A. Papapetrou, *Eine rotationssymmetrische Lösung in de Allgemeinen Relativitätstheorie*, *Ann. Physik (Leipzig)* **12** (1953) 309.

[7] F. J. Hernandez, F. Nettel, and H. Quevedo, *Gravitational fields as generalized string models*, Grav. Cosmol. **15**, 109 (2009).

[8] H. Quevedo, *General Static Axisymmetric Solution of Einstein's Vacuum Field Equations in Prolate Spheroidal Coordinates*, Phys. Rev. D **39**, 2904–2911 (1989).

[9] G. Erez and N. Rosen, Bull. Res. Counc. Israel **8**, 47 (1959).

[10] B. K. Harrison, Phys. Rev. Lett. **41**, 1197 (1978).

[11] H. Quevedo, *Generating Solutions of the Einstein–Maxwell Equations with Prescribed Physical Properties*, Phys. Rev. D **45**, 1174–1177 (1992).

[12] W. Dietz and C. Hoenselaers, *Solutions of Einstein's equations: Techniques and results*, (Springer Verlag, Berlin, 1984).

[13] V. A. Belinski and V. E. Zakharov, Soviet Phys. – JETP, **50**, 1 (1979).

[14] C. W. Misner, *Harmonic maps as models for physical theories*, *Phys. Rev. D* **18** (1978) 4510.

[15] D. Korotkin and H. Nicolai, *Separation of variables and Hamiltonian formulation for the Ernst equation*, *Phys. Rev. Lett.* **74** (1995) 1272.

[16] J. Polchinski, String Theory: An introduction to the bosonic string, Cambridge University Press, Cambridge, UK, 2001.

[17] D. Nuñez, H. Quevedo and A. Sánchez, *Einstein's equations as functional geodesics*, *Rev. Mex. Phys.* **44** (1998) 440; J. Cortez, D. Nuñez, and H. Quevedo, *Gravitational fields and nonlinear sigma models*, *Int. J. Theor. Phys.* **40** (2001) 251.

[18] R. Geroch, *J. Math. Phys.* **11**, 2580 (1970).

[19] R. O. Hansen, *J. Math. Phys.* **15**, 46 (1974).

[20] D. Bini, A. Geralico, O. Luongo, and H. Quevedo, *Generalized Kerr spacetime with an arbitrary quadrupole moment: Geometric properties vs particle motion*, Class. Quantum Grav. **26**, 225006 (2009).

[21] J. Geheniau, R. Debever, Bull. Acad. Roy. Soc. Belgique, **42**, 114 (1956).

[22] H. Quevedo, Gen. Rel. Grav. **24**, 693 (1992).

[23] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation, W. H. Freeman, San Francisco, USA, 1973.

[24] J. L. Synge, Relativity: The general theory, North-Holland, Amsterdam, 1960.

[25] M. Demiański, Phys. Lett. A **42**, 157 (1972).

[26] M. Cahen and L. Defrise, Commun. Math. Phys. **11**, 16 (1968).

[27] F. A. E. Pirani, Acta Phys. Pol. **15**, 389 (1956).

[28] P. Szekeres, J. Math. Phys. **6**, 1387 (1965).

[29] D. Puetzfeld and Y. N. Obukhov, Phys. Rev. D **93**, 044073 (2016).

[30] R. P. Kerr, Phys. Rev. Lett. **11**, 237-238 (1963).

[31] E. T. Newman and A. I. Janis, J. Math. Phys. **6**, 915–917 (1965).

[32] E. T. Newman et al., J. Math. Phys. **6**, 918 (1965).

[33] M. Demianski and E. T. Newman, Bull Acad. Polon. Sci. Set. Math. Astron. Phys. **14**, 653 (1966).

[34] C. G. Talbot, Commun. Math. Phys. **13**, 45 (1969).

[35] M. Schiffer, R. Adler, J. Mark and C. Sheffield, J. Math. Phys. **14** , 52–56 (1973).

[36] R. J. Finkelstein, J. Math. Phys. **16**, 1271–1277 (1975).

[37] H. Quevedo, Gen. Rel. Grav. **24**, 799 (1992).

[38] L. Herrera and J. Jiménez, J. Math. Phys. **23**, 2339 (1982).

[39] S. Drake and R. Turolla, Class. Quantum. Grav. **14**, 1883 (1997).

[40] N. Ibohal, Gen. Rel. Grav. **37**, 19 (2005).

[41] T. Papakostas, J. Phys: Conf.Ser. **8**, 22 (2005).

[42] T. Papakostas, J. Phys: Conf.Ser. **189**, 012027 (2009).

[43] S. Viaggiu, Int. J. Modern. Phys. D **15**, 1441 (2006).

[44] S. Viaggiu, Int. J. Modern. Phys. D **19**, 1783 (2010).

[45] K. Rosquist, Class. Quantum Grav. **16**, 1755 (1999).

[46] C. Lozanovski and L. Wylleman, Class. Quantum. Grav. **28**, 075015 (2011).

[47] R. Ferraro, Gen. Rel. Grav. **46**, 1705 (2014).

[48] D. Hansen and N. Yunes, Phys. Rev. D **88**, 104020 (2013).

[49] G. Lessner, Gen. Rel. Grav. **40**, 2177 (2008).

[50] L. Modesto and P. Nicolini, Phys. Rev. D **82**, 104035 (2010).

[51] Y. Miao, Z. Xue, and S. Zhang, Int. J. Mod. Phys. D **21**, 1250017 (2012).

[52] F. Caravelli and L. Modesto, Class. Quantum. Grav. **27**, 245022 (2010).

[53] M. Azrag-Ainou, Eur. Phys. J. C **74**, 2865 (2014).

[54] M. Azrag-Ainou, Phys. Rev. D **90**, 064041 (2014).

[55] A. Larranaga, A. Cardenas-Avendano, and D. Torres, Phys. Lett. B **743**, 492 (2015).

[56] H. Erbin, Gen. Rel. Grav. **47**, 19 (2015).

[57] J. Synge, Relativity: the general theory, North-Holland Pub. Co. Interscience Publishers, Amsterdam, New York (1960)

[58] A. C. Gutiérrez-Piñeres, G. A. González, and H. Quevedo, Phys. Rev. D **87**, 044010 (2013).

[59] A. C. Gutiérrez-Piñeres, C. S. Lopez-Monsalvo, and H. Quevedo, Gen. Rel. Grav. **47**, 1 (2015).

[60] A. C. Gutiérrez-Piñeres, Gen. Rel. Grav. **47**, 54 (2015).

[61] A. C. Gutiérrez-Piñeres and A. J. Capistrano, Adv. Math. Phys. **15**, 2015 (2015).

[62] D. Hansen and N. Yunes, Phys. Rev. D **88**, 104020 (2013).

[63] Alvear Terrero D., Manreza Paret D., Perez Martinez A., 2017, Astronomische Nachrichten, 338, 1056

[64] Alvear Terrero D., Manreza Paret D., Perez Martinez A., 2018, Int. J. M. P.: Conference Series, 45, 1760025

[65] Alvear Terrero D., Manreza Paret D., Perez Martinez A., 2018, Int. J. M. P. D, 27, 1850016

[66] Belvedere R., Boshkayev K., Rueda J.A., Rufni R., 2014, Nuc. Phys. A, 921, 33

[67] Boshkayev K., Rueda J.A., Ruffini R, 2011, Int. J. M. P. E, 20, 136

[68] Boshkayev K., Izzo L., Rueda J.A., Ruffini R., 2013a, A&A, 555, 9

[69] Boshkayev K., Rueda J.A., Ruffini R., Siutsou I., 2013, ApJ, 762, 117

[70] Boshkayev K., Rueda J.A., Ruffini R., Siutsou I., 2014, J. Kor. Phys. Soc., 65(6), 855

[71] Boshkayev K., Rueda J.A., Ruffini R., Siutsou I., 2015, in Rosquist K. et al. eds. Thirteenth Marcel Grossmann Meeting: On Recent Developments in Theoretical and Experimental General Relativity, Astrophysics and Relativistic Field Theories. World Scientific Publishing Co. Pte. Ltd., p. 2468

[72] Boshkayev K., Quevedo H., Kalymova Zh., Zhami B., 2016, EJP, 37, 065602

[73] Boshkayev K.A., Rueda J.A., Zhami B.A., Kalymova Zh.A., Balgymbekov G.Sh., 2016, IJMPCS, 41, 1660129

[74] Boshkayev K., Quevedo H., Zhami B., 2017, MNRAS, 464, 4349

[75] Carvalho G.A., Marinho R.M., Malheiro M., 2018, Gen. Rel. and Grav., 50, 38.

[76] Chakrabarti S., Delsate T., Gürlebeck N., Steinhoff J., 2014, Phys. Rev. Lett., 112, 2021102

[77] Chan T.K., Chan AtMa P.O., Leung P.T., 2015, Phys. Rev. D., 91, 044017

[78] Chandrasekhar S., 1931, ApJ, 74, 81

[79] Chandrasekhar S., 1939, Introduction to the Study of Stellar Structure. University of Chicago Press, USA

[80] Coelho J.G., Marinho R.M., Malheiro M., Negreiros R., Cáceres D.L., Rueda J.A., Ruffini R., 2014, ApJ, 794, 86

[81] Coelho J.G., Malheiro M., 2014, PASJ, 66, 14

[82] de Carvalho S.M., Rotondo M., Rueda J.A., Ruffini R., Phys. Rev. C, 89, 015801

[83] Doneva D.D., Yazadjiev S.S., Stergioulas N., Kokkotas K.D., 2014, ApJ, 781, L6

[84] Hartle J. B., 1967, ApJ, 150, 1005

[85] Hartle J. B., Thorne K. S., 1968, ApJ, 153, 807

[86] Haskell B., Ciolfi R., Pannarale F., Rezzolla L. 2014 MNRASL, 438, 71

[87] Heusler M., *Black holes uniqueness theorems* (Cambridge University Press, Cambridge, UK, 1996).

[88] Kepler S.O., Kleinman S.J., Nitta A., Koester D., Castanheira B.G., Giovannini O., Costa A.F.M., Althaus L., 2007, MNRAS, 375, 1315

[89] Kepler S.O., Pelisoli I., Koester D., Ourique G., Kleinman S.J , Romero A.D., Nitta A., Eisenstein D.J., Costa J.E.S., Kulebi B., Jordan S., Dufour P., Giommi P., Rebassa-Mansergas A., 2015 MNRAS, 446, 4078

[90] Kepler S.O., Pelisoli I., Koester D., Ourique G., Romero A.D., Reindl N., Kleinman S.J., Eisenstein D.J., Valois A.D.M., Amaral L.A., 2016, MNRAS, 455, 3413

[91] Kepler S.O., Koester D., Ourique G. 2016, Science, 352, 67

[92] Koester D., Kepler S.O., 2015, A&A, 583, 9

[93] Lau S.Y., Leung P.T., Lin L.-M. 2017, Phys. Rev. D, 95, 101302

[94] Lobato R.V., Malheiro M., Coelho J.G., 2016, Int. J. M. P. D, 25, 1641025

[95] Malheiro M., Rueda J.A., Ruffini R., 2012, PASJ, 64, 56

[96] Marques M., Oertel M., Hempel M., Novak J., 2017 Phys. Rev. C., 96 (4), 045806

[97] Martinon G., Maselli A., Gualtieri L., Ferrari V., 2014, Phys. Rev. D., 90, 064026

[98] Maselli A., Cardoso V., Ferrari V., Gualtieri L., Pani P., 2013, Phys. Rev. D, 88, 023007

[99] Maselli A., Pnigouras P., Nielsen N.G., Kouvaris C., Kokkotas K.D. 2017 Phys. Rev. D, 96, 023005

[100] Mathew A., Nandy M.K., 2017 Research in A&A, 17(6), 061, arXiv:1401.0819.

[101] Pani P., 2015, Phys. Rev. D., 92, 124030

[102] Pappas G., Apostolatos T.A., 2014, Phys. Rev. Lett., 112, 121101

[103] Rotondo M., Rueda J.A., Ruffini R., Xue S., 2011, Phys. Rev. D, 84, 084007

[104] Rueda J.A., Boshkayev K., Ruffini R., Loren-Aguilar P., Kulebi B., Aznar-Siguan G. & Garcia-Berro E., 2013, ApJL, 772(2), L24

[105] Sham Y.-H., Chan T.K., Lin L.-M., Leung P.T., 2015, ApJ, 798, 7

[106] Stein L.C., Yagi K., Yunes N., 2014, ApJ, 788, 15

[107] Tolman R.C., 1930, Phys. Rev., 35, 904

[108] Tremblay P.-E., Bergeron P., Gianninas A., 2011, ApJ, 730, 128

[109] Uchikata N., Yoshida Sh., Pani P., 2016, Phys. Rev. D., 94, 064015

[110] Yagi K., Yunes N., 2013, Phys. Rev. D, 88, 023009

[111] Yagi K., Yunes N., 2013, Science, 431, 365

[112] Yagi K., Stein L.C., Pappas G., Yunes N., Apostolatos T.A., 2014, Phys. Rev. D, 90(6), 063010

[113] Yagi K., Kyutoku K., Pappas G., Yunes N., Apostolatos T.A., 2014, Phys. Rev. D, 89, 124013