# Cosmology and Non Linear Relativistic Field Theory

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# 1. Topics

- Efective geometry in non-linear Electrodynamics
- Cosmological effects on non-linear field theory
- Bouncing cosmological models
- Non-linear field theory in flat and curved space-time
- Relativistic Astrophysics

# 2. Participants

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- Azucena Rivasplata (CBPF), co-advisor.

**Visiting professors to ICRA-Br**

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- H. Quevedo (Universidad Nacional Autnoma de Mxico, Mxico)
- K. Rosquist (Stockholm Obser-Dept Astronomy, Sweden)
- S. Joras (Departamento de Fsica Terica (DFT) -Instituto de Fsica (IF) -UFRJ, Brazil)

# 3. ICRA-BR activities

In the years 2009-2010 ICRA-Br developped an intense activity of Conferences, Workshops, Visiting scientists and Schools of advanced topics. We present here a list of the main meetings.

- The XIV Brazilian School of Cosmology and Gravitation was held from August 29 to September 11, 2010. The 32th anniversary of the School was commemorated in this edition. The School started in 1978 under the initiative of the Cosmology and Gravitation Group (CBPF). Its efforts are dedicated to the diffusion of different aspects of Cosmology, Gravitation and Astrophysics. Details can be found in `http://mesonpi.cat.cbpf.br:8080/esccosmologia/site_in/index.html`.

- The I Symposium Mario Novello on Bouncing Models held from 26 to 30 November (2010). This is the first international Meeting dedicated to the idea of the existence of a previous phase of collapsing universe prior to the phase of expansion. This event is organized by Nelson Pinto Neto (ICRA/CBPF), Patrick Peter (IAP,France) and Felipe Tovar (ICRA/CBPF).

- We published in 2010 the Procceedings of the Sobral Second Conference on Cosmology, Relativity and Astrophysics, held in Sobral (Ceara), which aimed at celebrating the most important scientific mission ever realized in the country involving gravitational processes, that is the Eddington Mission. The list of the professors that lectured at this Meeting are:
  - A. Silva
  - A. Rodriguez Ardilla
  - A. Santoro
  - A. Sepulveda
  - E. G. Dal Pino
  - Felix Aharonian
  - F. Mirabel
  - F. Everitt
  - H. Christiansen
  - J. Braga
  - L. Sodr

- L. Urrutia
- L. Herrera
- M. Kaiser
- M. Hamuy
- M. Novello
- N. P. Neto
- O. Aguiar
- P. Laguna
- R. Ruffini
- R. Shellard
- T. Villela
- T. Nash
- Z. Abraham

- During the 2nd semester of 2009 and the first months of 2010, the 4rd edition of the itinerant program of Cosmology (Programa Minimo de Cosmologia, PMC, in Portuguese) was held in the Federal University Amazonas in the city of Manaus (state of Amazonas), (while the first was in Rio Grande do Sul, the second in the state of Rio de Janeiro and the third in the state of Ceara). The main goal of this itinerant School is to communicate basic and advanced knowledge in Cosmology and Relativity throughout Brazil.

  The lecturers are:

  - M. Novello
  - E. Goulart
  - J. Salim
  - L. A. Oliveira
  - S. Bergliaffa
  - S. Jors
  - M. Makler
  - F. Tovar
  - H. Mosquera Cuesta
  - N. P. Neto

- During December 2010 there occurs the fourth ICRA-BR Internal Workshop at CBPF, as it ocurred from 15 to 17 December 2009, where the Third ICRA-BR Internal Workshop took place at CBPF, with oral presentations by researchers and students members of ICRA-BR. They had the opportunity to learn about the research activities that are being developed by the other members of the group. Such an experience was so successful that it was decided to transform it in one of the permanent activities, to be held each year. The list of the participants are:
  - M. Novello
  - A. Nogueira
  - E. Bittencourt
  - E. Goulart
  - J. Salim
  - J. Nogueira Rua
  - M. Borba
  - V. Antunes
  - N. Fux Svaiter
  - S. Joffily
  - G. Menezes
  - M. Alcalde
  - T. Carvalho Aguiar
  - A. Bernui
  - N. P. Neto
  - L. A. Oliveira
  - R. Guida
  - S. Bergliaffa
  - S. Jors
  - M. Makler
  - B. Fraga
  - D. Pantoja
  - D. Celani
  - F. Tovar
  - R. Perez
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  - S. Fernandes Pereira
  - C. Brandt
  - G. Caminha
  - M. Lima
  - P. Ferreira
  - F. Poulis
  - G. Batista Santos
  - M. Carvalho

- In 2009-2010 we developed an english version of the historical-scientific study on the evolution of Cosmology along the XX century performed in 2007-2008 by an ICRA-BR team of researchers . The main topics of this path were included in a poster prepared as a piece of popular science and distributed in technical schools of the Brazilian Ministry of Science and Technology and also in the public schools of the city of Rio de Janeiro. This experience is being extended in 2009-2010 to cover other Brazilian states. To continue this endeavour a review book is being prepared containing reading material and pictures to serve as a guide for the teaching of Cosmology in schools and universities.

- During 2010 there were visits of professor M Novello, Herman Mosquera Cuesta, Erico Goulart and S E P Bergliaffa to ICRANet in Pescara.

- New cooperations with Prof. Timothy C. Beers, Michigan State University (MSU), East Lansing, Michigan (The focus of this cooperation is to establish the pillars for solid scientific cooperation between some Colombian Universities having programs in astronomy and the MSU, so that students from Colombia that are interested in pursuing their PH.D. Astronomy can join the astronomy group led by Prof. Timothy C. Beers at MSU through a fellowship program coordinated by this scientist.) ; Prof. Salvatore Capozziello, Uni-Napoli, Napoli, Italia ( The scientific mainstream of his collaboration is to study the inverse problem in the search for doing cosmology with the observations of Gamma-Ray Bursts (GRBs)); Prof. Luis A. Sanchez, Universidad Nacional de Colombia, Sede Medellin (The purpose of this partnership is to stimulate the new generation of students of the Universidad Nacional de Colombia, Sede Medellin, that are looking for pursuing their Ph.D. in relativistic astrophysics, to join our groups, and through this interaction to be prepared to dispute a scholarship of ICRANet Ph.D. program.)

- Scientific papers

i)The spectrum of scalar fluctuations of a bouncing universe. M. Campista, M. Novello, J. M. Salim. Published in Int.J.Mod.Phys. A 25:3095-3105, 2010.

ii)Geometrizing Relativistic Quantum Mechanics. F. T. Falciano; M. Novello; J.M. Salim. Accepetd for publication in Foundation of Physics (2010)

iii)On a Geometrical Description of Quantum Mechanics. M. Novello, J. M. Salim, F. T. Falciano. Accepted for publication in International Journal of Geometrical Methods of Modern Phys. (2010)

iv) The gravitational mechanism to generate mass. M Novello Arxiv (2010)

v) Mach or Higgs? The mechanisms to generate mass. M. Novello To be published in the Procceedings of the XIV Brazilian School of Cosmology and Gravitation

# 4. Brief description of scientific works

## 4.1. Mach or Higgs? The mechanisms to generate mass

The purpose of this work is to show that the gravitational interaction is able to generate mass for all bodies. The condition for this is the existence of an energy distribution represented by the vacuum or the cosmological constant term $\Lambda\, g_{\mu\nu}$. We review briefly the alternative Higgs mechanism in order to compare both processes.

## 4.2. On a Geometrical Description of Quantum Mechanics

We show that Quantum Mechanics can be interpreted as a modification of the Euclidean nature of 3-d space into a particular affine space, which we call Q-wis. This is proved using the Bohm-de Broglie causal formulation of Quantum Mechanics. In the Q-wis geometry, the length of extended objects changes from point to point. In this formulation, deformation of physical distances are in the core of quantum effects allowing a geometrical formulation of the uncertainty principle.

## 4.3. Bouncing Cosmologies

See Chapter E

## 4.4. Comments on Bouncing

- We introduce analytic solutions for a class of two components bouncing models, where the bounce is triggered by a negative energy density perfect fluid. The equation of state of the two components are constant in time, but otherwise unrelated. By numerically integrating regular equations for scalar cosmological perturbations, we find that the (would-be) growing

mode of the Newtonian potential before the bounce never matches with the growing mode in the expanding stage. For the particular case of a negative energy density component with a stiff equation of state we give a detailed analytic study, which is in complete agreement with the numerical results. We also perform analytic and numerical calculations for long wavelength tensor perturbations, obtaining that, in most cases of interest, the tensor spectral index is independent of the negative energy fluid and given by the spectral index of the growing mode in the contracting stage. We compare our results with previous investigations in the literature.

- We propose a new cosmological paradigm in which our observed expanding phase is originated from an initially large contracting Universe that subsequently experienced a bounce. This category of models, being geodesically complete, is nonsingular and horizon-free and can be made to prevent any relevant scale to ever have been smaller than the Planck length. In this scenario, one can find new ways to solve the standard cosmological puzzles. One can also obtain scale invariant spectra for both scalar and tensor perturbations: this will be the case, for instance, if the contracting Universe is dust-dominated at the time at which large wavelength perturbations get larger than the curvature scale. We present a particular example based on a dust fluid classically contracting model, where a bounce occurs due to quantum effects, in which these features are explicit;

- We study Einstein gravity minimally coupled to a scalar field in a static, spherically symmetric space-time in four dimensions. Black hole solutions are shown to exist for a phantom scalar field whose kinetic energy is negative. These scalar black holes have an infinite horizon area and zero temperature TH and are termed cold black holes (CBHs). The relevant explicit solutions are well-known in the massless case (the so-called anti-Fisher solution), and we have found a particular example of a CBH with a nonzero potential V. All CBHs with V are shown to behave near the horizon quite similarly to those with a massless field. The above solutions can be converted by a conformal transformation to Jordan frames of a general class of scalar-tensor theories of gravity, but CBH horizons in one frame are in many cases converted to singularities in the other, which gives rise to a new type of conformal continuation.

- Cosmological models with two interacting fluids, each satisfying the strong energy condition, are studied in the framework of classical General Relativity. If the interactions are phenomenologically described by a power law in the scale factor, the two initial interacting fluids can be equivalently substituted by two non interacting effective fluids, where one of them may violate the strong energy condition and/or have negative energy density. Analytical solutions of the Friedmann equations of this general setting are obtained and studied. One may have, depending on the scale where

the interaction becomes important, non singular universes with early accelerated phase, or singular models with transition from decelerated to accelerated expansion at large scales. Among the first, there are bouncing models where contraction is stopped by the interaction. In the second case, one obtains dark energy expansion rates without dark energy, like $\lambda$CDM or phantomic accelerated expansions without cosmological constant or phantoms, respectively.

- It is generally believed that one cannot obtain a large universe from quantum cosmological models without an inflationary phase in the classical expanding era because the typical size of the universe after leaving the quantum regime should be around the Planck length, and the standard decelerated classical expansion after that is not sufficient to enlarge the universe in the time available. For instance, in many quantum minisuperspace bouncing models studied in the literature, solutions where the universe leaves the quantum regime in the expanding phase with appropriate size have negligible probability amplitude with respect to solutions leaving this regime around the Planck length. In this paper, I present a general class of moving Gaussian solutions of the Wheeler-DeWitt equation where the velocity of the wave in minisuperspace along the scale factor axis, which is the new large parameter introduced in order to circumvent the above-mentioned problem, induces a large acceleration around the quantum bounce, forcing the universe to leave the quantum regime sufficiently big to increase afterwards to the present size, without needing any classical inflationary phase in between, and with reasonable relative probability amplitudes with respect to models leaving the quantum regime around the Planck scale. Furthermore, linear perturbations around this background model are free of any trans-Planckian problem.

- We show how to obtain the simplest equations for the Mukhanov-Sasaki variables describing quantum linear scalar perturbations in the case of scalar fields without potential term. This was done through the implementation of canonical transformations at the classical level, and unitary transformations at the quantum level, without ever using any classical background equation, and it completes the simplification initiated in investigations by Langlois [D. Langlois, Classical Quantum Gravity 11, 389 (1994).], and Pinho and Pinto-Neto [E. J. C. Pinho and N. Pinto-Neto, Phys. Rev. D 76, 023506 (2007).] for this case. These equations were then used to calculate the spectrum index ns of quantum scalar perturbations of a nonsingular inflationary quantum background model, which starts at infinity past from flat space-time with Planckian size spacelike hypersurfaces, and inflates due to a quantum cosmological effect, until it makes an analytical graceful exit from this inflationary epoch to a decelerated classical stiff matter expansion phase. The result is ns 3, incompatible

with observations.

- We show that minisuperspace quantization of homogeneous and isotropic geometries with phantom scalar fields, when examined in the light of the Bohm-de Broglie interpretation of quantum mechanics, does not eliminate, in general, the classical big rip singularity present in the classical model. For some values of the Hamilton-Jacobi separation constant present in a class of quantum state solutions of the Wheeler-De Witt equation, the big rip can be either completely eliminated or may still constitute a future attractor for all expanding solutions. This is contrary to the conclusion presented in [M. P. Dabrowski, C. Kiefer, and B. Sandhofer, Phys. Rev. D 74, 044022 (2006).], using a different interpretation of the wave function, where the big rip singularity is completely eliminated (smoothed out) through quantization, independently of such a separation constant and for all members of the above mentioned class of solutions. This is an example of the very peculiar situation where different interpretations of the same quantum state of a system are predicting different physical facts, instead of just giving different descriptions of the same observable facts: in fact, there is nothing more observable than the fate of the whole Universe;

- We investigate if theories yielding bouncing cosmological models also generate wormhole solutions. We show that two of them present sensible traversable static wormhole solutions, while for the third possibility such solutions are absent.

## 4.5. Properties of generalized gravitional theories

Although gravity has been shown to be with high accuracy in accordance with General Relativity in a number of situations in which the curvature is small, there is no observational evidence of the behaviour of the gravitational field for very large values of the curvature. In this regard, objects such as black holes and neutron stars are the ideal places to look for deviations from General Relativity in the strong regime [5]. In fact, the Kerr (hence Schwarzschild's) solution is not unique in f(R) theories [6]. Hence, any deviation from these metrics will be unequivocally signaling the need of changes in our description of strong gravity. The task of understanding what kind of deviations can be expected, and their relation to observable quantities is of relevance in view of several developments that offer the prospect of observing properties of black holes in the vicinity of the horizon. Among these, we can cite possibility of obtaining "black hole images" [7], and the precision measurements of the motion of gas and high-eccentricity stars in the immediate vicinity of the SMBH at the center of the Milky Way [8]. As a first step in discussing deviations from General Relativity in strong-field gravity, I am examining some features of static and spherically symmetric (SSS) black hole solutions in theories with a Lagrangian that is a function of the Ricci

scalar. Different aspects of this type of solutions in f(R) theories have been previously in discussed in [9], mostly resorting to exact solutions and/or phase space analysis. I am taking a complementary path which consists in assuming for an arbitrary f(R) that a SSS black hole solution exists, with a horizon at $r = r_0$, defined by the condition $g_{00}(r_0) = 0$.

Another line of research that I am pursuing (with the student Marcio Oliveira Pinheiro) is the study of the propagation of perturbations in theories described by the action

$$\mathcal{S} = \frac{M_P^2}{2} \int d^4x \sqrt{-g} F(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}, ...). \tag{4.1}$$

I showed in [10] that by imposing that the propagation of the perturbations in a nonlinear spin 2 theory be free of shocks [11], the freedom in the dependence of the Lagrangian with the field invariants can be greatly reduced. We are currently applying the same procedure to the action given by Eqn.(4.1).

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## 4.6. Effective Geometry in non-linear Electrodynamics

The idea that gravity may be an emergent phenomenon described by an effective low-energy theory which a is consequence of averaging over (yet unknown) microphysical degrees of freedom dates back at least to the proposal of Sakharov in 1968. Since then, it has been shown that is not difficult to get a low-energy effective metric (although it is considerably harder to get dynamical equations controlling it). In fact, the effective metric arises in systems of very different types ). The basic idea behind this generality has been exposed in [1], where it was shown that given a classical single-field theory described by a Lagrangian that is an arbitrary function of the field and its derivatives, the fluctuations of the linearized theory around a non-trivial background propagate in a curved spacetime. The geometry of this spacetime is encoded by the effective metric, which is unique in the case of a single field, and depends on the background field configuration. This feature of nonlinear theories led to the construction of analog models of gravity, which imitate the kinematical properties of gravitational fields (see [2] for a complete list). In a previous paper [325] E. Goulart and I showed that only two types of effective metrics are possible in certain nonlinear electromagnetic theories. This was achieved by using the dependence of the effective metric on the energy-momentum tensor of the background along with the Segrè classification of the latter. Each of these forms is completely determined by single scalar function, which characterizes the light cone of the nonlinear theory. This light cone was compared with that of Minkowski in two examples. We are currently working in the same problem for nonlinear scalar field theories, which are taken as models for dark energy [4].

M Novello and Erico Goulart published a book (in portuguese) entitled **Eletrodinamica nao-linear (causalidade e efeitos cosmologicos)** in August 2010 that is a consequence of the research we made within this line of research.

In recent years, there has been a growing interest in models that mimic in the laboratory some features of gravitation. The actual realization of these models relies on systems that are very different in nature: ordinary non-viscous fluids, superfluids, flowing and non-flowing dielectrics, non-linear electromagnetism in vacuum, and Bose-Einstein condensates. The basic feature shared by these systems is that the behavior of the fluctuations around a background solution is governed by an effective metric. More precisely, the particles associated to the perturbations do not follow geodesics of the background spacetime but of a Lorentzian geometry described by the effective metric, which depends on the background solution as pointed out some time ago by Unruh and earlier by Plebanski. It is important to notice that only some kinematical aspects of general relativity can be imitated by this method, but not its dynamical features. Although most of these works concerns sound propagation, the most fashionable

results deal with non-linear Electrodynamics. This is related to the possibility of dealing with phenomena that are treatable in actual laboratory experiments. This is one of the main reasons that induce us to analyze carefully a certain number of non-equivalent non-linear electromagnetic configurations. Among these results we can quote the possibility of imitating a non-gravitational Black Hole in laboratory dealing with non-linear electrodynamics effects (see Appendix).

## 4.7. Non-linear field theory in flat and curved space-time

Recent works have shown the important role that Nonlinear Electrodynamics (NLED) can have in two crucial questions of Cosmology, concerning particular moments of its evolution for very large and for low-curvature regimes, that is for very condensed phase and at the period of acceleration. We present here a a toy model of a complete cosmological scenario in which the main factor responsible for the geometry is a nonlinear magnetic field which produces a FRW homogeneous and isotropic geometry. In this scenario we distinguish four distinct phases: a bouncing period, a radiation era, an acceleration era and a re-bouncing. It has already been shown that in NLED a strong magnetic field can overcome the inevitability of a singular region typical of linear Maxwell theory; on the other extreme situation, that is for very weak magnetic field it can accelerate the expansion. The present model goes one step further: after the acceleration phase the universe re-bounces and enter in a collapse era. This behavior is a manifestation of the invariance under the dual map of the scale factor $a(t) \to 1/a(t)$, a consequence of the corresponding inverse symmetry of the electromagnetic field ($F \to 1/F$, where $F \equiv F^{\mu\nu}F_{\mu\nu}$) of the NLED theory presented here. Such sequence collapse-bouncing-expansion-acceleration-re-bouncing-collapse constitutes a basic unitary element for the structure of the universe that can be repeated indefinitely yielding what we call a Cyclic Magnetic Universe (see Appendix).

**Short comment**

In the last years there has been increasing of interest on the cosmological effects induced by Nonlinear Electrodynamics (NLED). The main reason for this is related to the drastic modification NLED provokes in the behavior of the cosmological geometry in respect to two of the most important questions of standard cosmology, that is, the initial singularity and the acceleration of the scale factor. Indeed, NLED provides worthwhile alternatives to solve these two problems in a unified way, that is without invoking different mechanisms for each one of them separately. Such economy of hypotheses is certainly welcome.

The partial analysis of each one of these problems was initiated by our group in ICRA-Br. In this workk we present a new cosmological model, that unifies both descriptions.

The general form for the dynamics of the electromagnetic field, compatible with covariance and gauge conservation principles reduces to $L = L(F)$, where $F \equiv F^{\mu\nu}F_{\mu\nu}$. We do not consider here the other invariant $G \equiv F^{\mu\nu}F^{*}_{\mu\nu}$ constructed with the dual, since its practical importance disappears in cosmological framework once in our scenario the average of the electric field vanishes in a magnetic universe as we shall see in the next sections. Thus, the Lagrangian appears as a regular function that can be developed as positive or negative powers of the invariant $F$. Positive powers dominate the dynamics of the gravitational field in the neighborhood of its moment of extremely high curvatures. Negative powers control the other extreme, that is, in the case of very weak electromagnetic fields. In this case as it was pointed out previously it modifies the evolution of the cosmic geometry for large values of the scale factor, inducing the phenomenon of acceleration of the universe. The arguments presented make it worth considering that only the averaged magnetic field survives in a FRW spatially homogeneous and isotropic geometry. Such configuration of pure averaged magnetic field combined with the dynamic equations of General Relativity received the generic name of Magnetic Universe.

The most remarkable property of a Magnetic Universe configuration is the fact that from the energy conservation law it follows that the dependence on time of the magnetic field $H(t)$ is the same irrespective of the specific form of the Lagrangian. This property allows us to obtain the dependence of the magnetic field on the scale factor $a(t)$, without knowing the particular form of the Lagrangian $L(F)$. Indeed, as we will show later on, from the energy-momentum conservation law it follows that $H = H_0\, a^{-2}$. This dependence is responsible for the property which states that strong magnetic fields dominates the geometry for small values of the scale factor; on the other hand, weak fields determines the evolution of the geometry for latter eras when the radius is big enough to excite these terms.

In order to combine both effects, here we will analyze a toy model. The symmetric behavior of the magnetic field in both extremes – that is for very strong and very weak regimes – allows the appearance of a repetitive configuration of the kind exhibited by an eternal cyclic universe.

Negative power of the field in the Lagrangian of the gravitational field was used in attempting to explain the acceleration of the scale factor of the universe by modification of the dynamics of the gravitational field by adding to the Einstein-Hilbert action a term that depends on negative power of the curvature, that is

$$S = \frac{M_{\rm Pl}^2}{2} \int \sqrt{-g} \left( R - \frac{\alpha^4}{R} \right) d^4x,$$

Although this Lagrangian was shown to be in disagreement with solar system observations, it started a program which introduced polynomial Lagrangian of

the form

$$\sum_n c_n R^n$$

containing positive and negative values of $n$.

This modification introduced an idea that is worth to be generalized: the dynamics should be invariant with respect to the inverse symmetry transformation. In other words,if $\mathbb{X}$ represents the invariant used to construct a Lagrangian for a given field, the Action should be invariant under the map $\mathbb{X} \to 1/\mathbb{X}$. Since the Electrodynamics is the paradigm of field theory, one should start the exam of such a principle into the realm of this theory. In other words we will deal here with a new symmetry between strong and weak electromagnetic field. In a previous work, a model assuming this idea was presented and its cosmological consequences analyzed. In this model, the action for the electromagnetic field was modified by the addition of a new term, namely

$$S = \int \sqrt{-g}(-\frac{F}{4} + \frac{\gamma}{F})d^4x.$$

This action yields an accelerated expansion phase for the evolution of the universe, and correctly describes the electric field of an isolated charge for a sufficiently small value of parameter $\gamma$. The acceleration becomes a consequence of the properties of this dynamics for the situation in which the field is weak.

In another cosmological context, in the strong regime, it has been pointed out in the literature by us, that NLED can produces a bouncing, altering another important issue in Cosmology: the singularity problem. In this article we would like to combine both effects improving the action to discuss the consequences of NLED for both, weak and strong fields.

It is a well-known fact that under certain assumptions, the standard cosmological model unavoidably leads to a singular behavior of the curvature invariants in what has been termed the Big Bang. This is a highly distressing state of affairs, because in the presence of a singularity we are obliged to abandon the rational description of Nature. It is possible that a complete quantum cosmology could describe the state of affairs in a very different and more complete way. For the time being, while such complete quantum theory is not yet known, one should attempt to explore alternatives that are allowed and that provide some sort of phenomenological consequences of a more profound theory.

It is tempting then to investigate how NLED can give origin to an unified scenario that not only accelerates the universe for weak fields (latter cosmological era) but that is also capable of avoiding an initial singularity as a consequence of its properties in the strong regime.

Scenarios that avoid an initial singularity have been intensely studied over the years. As an example of some latest realizations we can mention the pre-big-bang universe and the ekpyrotic universe. While these models are based on deep modifications on conventional physics, that are extremely difficult to be observed, the model we present here relies instead on the electromagnetic field.

The new ingredient that we introduce concerns the dynamics that is rather different from that of Maxwell in distinct regimes. Specifically, the Lagrangian we will work with is given by

$$L_T = \alpha^2 F^2 - \frac{1}{4} F - \frac{\mu^2}{F} + \frac{\beta^2}{F^2}. \tag{4.2}$$

The dimensional constants $\alpha, \beta$ and $\mu$ are to be determined by observation. Thus the complete dynamics of electromagnetic and gravitational fields are governed by Einstein equations plus $L_T$.

We shall see that in Friedmann-Robertson-Walker (FRW) geometry we can distinguish four typical eras which generate a basic unity of the cosmos (BUC) that repeat indefinitely. The whole cosmological scenario is controlled by the energy density $\rho$ and the pressure $p$ of the magnetic field. Each era of the BUC is associated with a specific term of the Lagrangian. As we shall see the conservation of the energy-momentum tensor implies that the field dependence on the scale factor yields that the invariant $F$ is proportional to $a^{-4}$. This dependence is responsible by the different dominance of each term of the Lagrangian in different phases. The first term $\alpha^2 F^2$ dominates in very early epochs allowing a bouncing to avoid the presence of a singularity. Let us call this the ***bouncing era***. The second term is the Maxwell linear action which dominates in the ***radiation era***. The inverse term $\mu^2/F$ dominates in the ***acceleration era***. Finally the last term $\beta^2/F^2$ is responsible for a ***re-bouncing***. Thus each BUC can be described in the following way:

- The bouncing era: There exists a collapsing phase that attains a minimum value for the scale factor $a_B(t)$;
- The radiation era: after the bouncing, $\rho+3p$ changes the sign; the universe stops its acceleration and start expanding with $\ddot{a} < 0$;
- The acceleration era: when the $1/F$ factor dominates the universe enters an accelerated regime;
- The re-bouncing era: when the term $1/F^2$ dominates, the acceleration changes the sign and starts a phase in which $\ddot{a} < 0$ once more; the scale factor attains a maximum and re-bounces

The universe starts a collapsing phase entering a new bouncing era. This unity of four stages, the BUC, constitutes an eternal cyclic configuration that repeats itself indefinitely.

The plan of the work is as follows. First we review the Tolman process of average in order to conciliate the energy distribution of the electromagnetic field with a spatially isotropic geometry, presents the notion of the Magnetic Universe and its generic features concerning the dynamics of electromagnetic field

generated by a Lagrangian $L = L(F)$. Then we present the conditions of bouncing and acceleration of a FRW universe in terms of properties to be satisfied by $L$. Later on we introduce the notion of inverse symmetry of the electromagnetic field in a cosmological context. This principle is used to complete the form of the Lagrangian that guides the combined dynamics of the unique long-range fields yielding a spatially homogeneous and isotropic nonsingular universe. We present then a complete scenario consisting of the four eras: a bouncing, an expansion with negative acceleration, an accelerated phase and a re-bouncing. Finally let us point out that although the total Lagrangian of NLED seems at first sight to induce an energy which is not strictly positive definite, this is not the case in the actual toy model. Furthermore, even in the case of a static spherically symmetric field of a charged particle, the negative contribution to the total energy - as measured in the asymptotic spatial infinity - reduces to a finite constant depending only on the free parameters of the theory. Thus, as it was done by Born and Infeld in their NLED, this constant can be ruled out by the addition of a constant term in the Lagrangian, which do not affect the dynamics of the electromagnetic field and makes the total energy positive definite.

## 4.8. Higgs mechanism without Higgs boson

We analyze the properties of the self-interaction of a spinor field $\Psi$ driven by Heisenberg dynamics. The system has global $\gamma_5-$invariance. It is possible to generalize this symmetry for a local space-time dependence by a minimal coupling of the axial vector constructed with the spinor field to a massless gauge vector field $W_\mu$. As a consequence of this coupling the gauge field acquires a mass when $\Psi$ is in its fundamental state. We can interpret this situation in terms of Higgs mechanism once the mass of the gauge field appears due to the non-linear Heisenberg dynamics that allows an apparent bosonisation which becomes the real vehicle for the generation of the effective mass.

## 4.9. Spinor theory of Gravity

From Einstein Equivalence Principle (EEP) it follows that universality of gravitational processes leads naturally to its identification to a metric tensor $g_{\mu\nu}$. However anyone that accepts this interpretation of the EEP should ask, before adopting the General Relativity approach the following question: giving

the observational fact that any piece of matter/energy provokes a modification of the geometry in which this piece is merged, could one be led to the unique conclusion that this modification is driven by a differential equation containing derivatives up to second order of the metric tensor and by properties of the matter that represents its energy distribution? Should one be obliged to conclude that there is no other logical way to understand this fact? Is there a unique and only way that compels any sort of gravitationally interacting matter to modify space-time geometry through a direct relation between a continuous local modification of the geometry and the corresponding matter-energy content? In other words, are we contrived to accept that geometry is also a physical component of nature, requiring unequivocally a dynamical equation itself? Is this the unique way to implement the Equivalence Principle? General Relativity is a complete realization of EEP that answers **yes** to these questions. These lectures will deal with Pre-Gravity Theory, which provides a distinct and competitive way to implement EEP which answers **no** to all these questions. In Pre-Gravity the gravitational field is represented in terms of two fundamental spinor fields $\Psi_E$ and $\Psi_N$. Its origins goes back to a complementary view of EEP, according to which the geometrical field is an induced quantity that depends on some intimate microscopic sub-structure. This sub-structure does not have by itself a geometric origin but instead it is a matter field. We could say that GR is based on a vision according to which space-time is to be understood as the arena of Physics (in Wheeler's words) and gravity is nothing but the consequence of a direct modification of the intrinsic geometry of such an arena. PG on the other hand, considers that the arena contains only matter and energy and the geometry is nothing but a specific way related to these real quantities or substances interacts among themselves. In this way, in Pre-gravity it has no practical sense to attribute a dynamics to the geometry. Its evolution is just a natural consequence of the dynamics of matter interacting gravitationally, as we shall see. Accepting the idea that the metric tensor is a derived quantity that is, it is not an independent dynamical variable, then we face the question: what should be the intermediate dynamical variables that represents the gravitational phenomenon? In his analysis of similar question, Feynmann argued against the possibility to identify such dynamical entity to different kinds of continuum fields like scalar, spinor and vector. Let us review this analysis. The argument against the scalar field rests on the impossibility of describe the influence of gravity in photon propagation. Accepting that the net effect of a scalar field should produce only conformally flat geometries then it follows that conformal invariance of Maxwell electrodynamics imply the absence of any direct influence of gravity on photon propagation. This was ruled out by the Sobral observation. The impossibility to identify gravity to vector field is related to the purely attractive effect of gravity. For neutrino-like field the Feynmann argument rests on the impossibility of having a $1/r$ static potential. Then he concludes that only a tensorial field $\varphi_{\mu\nu}$ could fulfill this criteria which led that the dynamical quantity of gravitational field has to be identified with the metric tensor.

The Spinor Theory of Gravity provides a distinct answer and circumvent these difficulties.

## 4.10. The Spectrum of Scalar Fluctuations Using Quasi- Maxwellian Formalism

Since the pioneer work of Lifshitz [1] many studies have been devoted to the subject of linear perturba- tions of spatially homogeneous and isotropic universes. There are essentially two approaches by which the per- turbations theory can be tackled. One started by Lif- shitz, is based on small variations of the metric tensor and deals directly with the eld equations of Gen- eral Relativity. The second started by Hawking[2] is based on small variations of the curvature tensor and use the Quasi-Maxwell equa- tions of gravity, based in the Bianchi-Identity.

The first one has the disadvantage that the metric tensor is not a physically sig- nificant quantity and it is plagued by gauge modes. This method has been improved by several authors and a gauge-invariant approach has been developed in a sem- inal paper. The gauge invariant variables are con- structed with linear combinations of gauge dependent perturbations, the dicult is that the physical and geo- metrical meaning of the resulting quantities is obscure[5]. Although there is no consistent quantum theory of grav- ity, perturbations induced by gravitational waves can be quantized in a consistent manner, using the method based on the potential tensor g. This construction has been generalized to in- clude linearized density pertur- bations and the resulting theory has been largely used to model the origin of structures and to describe its evolu- tion in inationary universe models. The other attempt forge to circumvent the dicul- ties introduced by the gauge-dependent variables, is not based on the gauge variables but on the covariant curva- ture variables that are null in the background and has a direct geometrical meaning. This work was extended in ,but still their analysis include contrast density and others gauge-dependent variables. This approach was improved by Ellis and co-workers that substi- tuted the gauge-dependent variables by gauge invariant and covariant new variables with transparent phys- ical or geometrical meaning. Another attempt to overcome the diculties by the presence of remanent gauge-dependent variables, is to eliminate the gauge de- pendent variables used to describe the perturbations and present a closed set of gauge invariant variables that form a consistent dynamical system. This was done in a se- ries of papers. The closed set of gauge inde- pendent variables in- clude only direct observable variables and does not require the knowledge of the components of perturbed metric. The set of variables used constitutes a planar dynamical system. A re-parametrization of these variables allows to establish a gauge-invariant Hamilto- nian treatment for the perturbations and to construct a consistent quantum theory. In this paper we used this method to investigate

perturbations of a pure geomet- ric cosmological model, used has a simple toy model, to describe a dynamical scenario with bounce for the evo- lution of the universe. Our main interest is to obtain the scalar spectrum of the perturbations of the manifold. This perturbation can sow the large-scale structures in a more complete and realistic model.The toy model used to represent the background is asymptotic at in the inn- ity pass and in the innity future.The initial uctu- ation in the past are produced by the quantum uctuations of the vacua. The gauge-invariant variable forge in [7] to described perturbations of a background scalar eld in inationary models cannot be used to propagate these perturbations across the bounce. This is because the variable is not dened at the minimum of the scalar fac- tor. This diculty can be overcame has was done in references [22, 23]. In this paper we decide to used the method of perturbations developed in [16] because be- yond the properties of been gauge independent the vari- ables are well defined across the bounce.

## 4.11. Gravitational Waves in Singular and Bouncing FLRW Universes

We investigate the propagation of gravitational waves in two models belonging to the Friedman-Lemaître-Robertson-Walker (FLRW) class of cosmologies: the singular Einstein-Maxwell Universe (EMU), which has the electromagnetic field described by Maxwell's electrodynamics as the source of its geometry, and the bouncing Nonlinear Electrodynamics Universe (NLEU), which has the electromagnetic field described by a non-linear generalization of Maxwell's electrodynamics as the source of its geometry. We work with an explicitly gauge independent formulation of cosmological perturbations in FLRW models and analyze the qualitative features of the dynamical system that describes the propagation of primordial tensorial perturbations in both geometries. Based on this analysis we show that gravitational waves generated near a singularity or a bounce exhibit qualitatively different behavior.

## 4.12. Cosmic Phenomenology

This research focuses on the connection among models and observations of the Universe, i.e. cosmic phenomenology. Concerning this project, M Makler have been involved in three main research areas: the Large Scale Structure of the Universe, Dark Matter and Dark Energy Unification, and Gravitational Lensing. More recently, we have used the outcome of cosmological N-body simulations to search for signatures of turbulent processes (see Caretta et al. 2008). He is advising a PhD thesis that addresses the abundance of galaxy groups

and clusters and its connections to cosmological models, with a focus on the uncertainties and biases induced by several observational aspects, such as the mass-observable relation.I continued to work on a framework to unify Dark Matter and Dark Energy. An elegant implementation this idea is provided by scalar fields with noncanonical kinetic terms, such as the Extended Born Infeld (EBI) model proposed in Novello et al. (2005). Recently, we have explored the EBI model in the full range of its parameters and compared with SNIa data (C. Furlanetto, MSc thesis, ICRA/CBPF, 2008). More recently he has been deeply involved with cosmological and astrophysical applications of gravitational lensing, more specifically gravitational arcs, computing the arc abundance and arc cross sections, as well as in observational aspects, such as a search for arcs in astronomical images (Estrada et al. 2007). We are also making progress in the theoretical modeling of arc statistics and investigating its uses for cosmology and astrophysics (see, e.g. Caminha et al. 2008). We have developed numerical a code to simulate gravitational arcs, the AddArcs code, which can be used to test and calibrate the semi-analytic modeling and can be used to make predictions about arc statistics and test arcfinding codes. We have continued to devote a substantial part of my time to the participation in the Dark Energy Survey international collaboration, in which the admission of the DES-Brazil consortium was pioneered by ICRA-Brasil. We are actively working both on the science related to DES as well as on the software to be used for the science analysis. Our main focus is on the unique potential for gravitational lensing provided by this project.Our ongoing work on gravitational arcs was recognized with the creation of the DES Strong Lensing Study Group, of which M Makler became one of the co-organizers (together with Elizabeth Buckley-Geer, from Fermilab). Besides investigating the prospects of DES for strong lensing and preparing tools for the scientific exploitation of this survey, we are also working with current data from other surveys, like the Sloan Digital Sky Survey (see, e.g. Estrada et al. 2007) and also the Principal Investigator of a project named SOGRAS to observe 60 galaxy clusters with the SOAR telescope. Up to January 2009 18 clusters were observing and we are in the process of reducing this data.One of our contributions to DES is the development of a whole suite of applications for Strong Lensing studies named SLtools, of which the AddArcs code is one of its subproducts. We are also working on novel models to identify and characterize gravitational arcs in astronomical images.Finally, we are making an effort to develop a high performance computational infrastructure for cosmological and astrophysical applications at ICRA-Brasil. We have built a small computing cluster with resources from ICRA/CBPF and personal grants. This machine has 36 computing nodes and is already available to the community. A larger infrastructure is being purchased with resources from FINEP (about US$ 200.000) a third of which will be dedicated to cosmology and astrophysics.

# 5. Publications

- M. Novello: Cosmologia. A technical book on cosmology for graduate students(in portuguese) published by Brazilian Institute of Physics in 2010
- M. Novello and E. Goulart: Eletrodinamica não linear. A technical book on non linear electrodynamics for graduate students(in portuguese) published by Brazilian Institute of Physics in 2010
- M. Novello: Do big-bang ao Universo eterno. A popular book in portuguese for the main cosmological scenarios envolving non-singular (bouncing) cosmological geometry, published by Edit. Jorge Zahar in 2010
- M. Novello, S.E.Perez Bergliaffa Phys.Rept.463:127-213,2008.
- Cyclic Magnetic Universe. (M. Novello, Aline N. Araujo and J M Salim) in International Journal of Modern Physics A (2009);
- Gravitational waves on singular and bouncing FRW universes: M. Novello, V. Antunes and E, Goulart) in Gravitation and Cosmology (2009) v 15, 191;
- Gaussian coordinate system for Kerr metric: M. Novello and E.H.S. Bittencourt) 2009;
- Spin-2 field theory in terms of Cartan geometry: M. Novello and P.I.Trajtenberg, to appear as a chapter in a book (2009);
- Cosmology; a book to be published in 2010 by Brazilian Institute of Physics (in portuguese)
- Non linear Electrodynamics (causal properties and cosmological effects) with E. Goulart: a book to be published in 2010 by Brazilian Institute of Physics (in portuguese)
- Constructing Dirac linear fermions in terms of non-linear Heisenberg spinors. Europhysics Letters EPL, 80 (2007) 41001t: arXiv:0705.2692[astro-ph];
- Bursts of Gravitational Waves Emitted During Ejection of Jet Superluminal Components in Active Galactic Nuclei Dynamically Dominated by Bardeen-Petterson Effect; Mosquera Cuesta, Herman J.; Caproni, Anderson; Abraham, Zulema in AIP Conference Proceedings, Volume 1168,

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- Neutrino mass spectrum from neutrino spin-flip-driven gravitational waves: Mosquera Cuesta, Herman J.; Lambiase, Gaetano; in International Journal of Modern Physics D, v. 18, p. 435, 2009.

- Luminosity distance vs. proper distance: Effects of nonlinear electrodynamics in cosmology: Mosquera Cuesta, Herman J.; Salim, J. M.; Novello, M.; in Proceedings of Science, v. ISFTG, p. PoS(ISFTG)009, 2009.

- Gravitational wave signal of the short rise fling of galactic run away pulsars: Herman J. Mosquera Cuesta, Carlos A. Bonilla Quintero; in Journal of Cosmology and Astroparticle Physics (0811:006, 2008) — e-Print: arXiv:0711.3046 [astro-ph]

- Lenergie sombre: Un mirage cosmique ? — Luminosity distance vs. proper distance: A cosmological dissimilitude induced by nonlinear electrodynamics: Herman J. Mosquera Cuesta; To be published in Proceedings of the International Conference: Invisible Universe, editor: Jean-Michel Alimi (AIP Conference Proceedings 2010).

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# 6. Mach or Higgs? The mechanisms to generate mass

## 6.1. Introduction

The purpose of these notes is to compare the two known mechanisms to generate mass of the elementary constituents of all bodies, the basic bricks of which will be taken as representations of the Lorentz-Poincaré group and we will analyze them as scalar, spinor, vector and tensor fields. We shall see that in both cases the origin of the mass of any body $\mathbb{A}$ depends on its interaction with its surroundings yielding an overall effect (described either as a scalar field – in the case of the Higgs mechanism – or as the metric tensor of the geometry of space-time - in the case of the gravitational origin) on $\mathbb{A}$ which is represented by a distribution of energy given by the form

$$T_{\mu\nu} = \lambda\, g_{\mu\nu} \tag{6.1}$$

In the literature concerning General Relativity this form of energy-momentum tensor is attributed to the cosmological constant introduced by Einstein in order to be able to construct a model for the geometry of the universe. In the realm of quantum field theory, such distribution is identified to the vacuum. It is true that if one considers the Machian point of view that the inertia of a body $\mathbb{A}$ depends on the energy distribution of all others bodies in the universe, then $\lambda$ is to be interpreted as the cosmological constant [15].

These two mechanisms that contemplate the possibility of determining the mass of any body from elementary principles, are associated to two distinct universal interactions driven by one of the two fields:

- Gravitational field;
- Scalar field.

The idea of using a scalar field to be at the origin of the mass appeared in the domain of high energy physics and it received the name "Higgs mechanism". For the time being there is no evidence of the existence of such scalar in Nature and huge experiments - the LHC experiment – are at this very moment in the verge to be realized in order to prove that such scalar field exists [5], [19].

On the other hand, the relationship of mass with gravity is a very old one and its deep connection has been emphasized in a qualitative way a huge number of

times. We will concentrate our analysis only on a particular process that admits a systematic realization and allows for a quantification.

Although the theory of General Relativity may be understood as completely independent from the Machian idea that inertia of a body $\mathbb{A}$ is related to the global distribution of energy of all particles existing in the universe, we must recognize its historical value in the making the ideology that enabled Einstein to start his journey toward the construction of a theory of gravitation [6].

During the 20th century, the idea of associating the dependence of local characteristics of matter with the global state of the universe came up now and then but without producing any reliable mechanism that could support such proposal [7]. Even the concept of mass – that pervades all gravitational processes – did not find a realization of such dependence on global structure of the universe. On the contrary, the most efficient mechanism and one that has performed an important role in the field of microphysics came from elsewhere, namely the attempt to unify forces of a non-gravitational character, such as long-range electrodynamics with decaying phenomena described by weak interaction. Indeed, the Higgs model produced an efficient scenario for generating mass to the vector bosons [2] that goes in the opposite direction of the proposal of Mach. This mechanism starts with the transformation of a global symmetry into a local one and the corresponding presence of vector gauge fields. Then, a particular form of the dynamics represented by $L_{int}(\varphi)$ of self-interaction of an associated scalar field in its fundamental state represented by an energy-momentum tensor given by $T_{\mu\nu} = L_{int}(\varphi_0)\, g_{\mu\nu}$ appears as the vehicle which provides mass to the gauge fields.

Recently a new mechanism for generation of mass that is a realization of Mach's idea was proposed [17]. The strategy used is to couple the field (scalar, spinor [320], vector [321] and tensor) non minimally to gravity through the presence of terms involving explicitly the curvature of space-time. The distribution of the vacuum energy of the rest-of-the-universe is represented by a cosmological term $\Lambda$. The effect of $\Lambda$ by the intermediary of the dynamics of the metric of space-time in the realm of General Relativity is precisely to give mass to the field. Although this mass depends on the cosmological constant, its value cannot be obtained a priori [11].

## 6.2. The Higgs proposal

Consider a theory of a real scalar field $\varphi$ described by the Lagrangian

$$\mathbb{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi) \tag{6.2}$$

where the potential has the form

$$V = \frac{1}{2}\,\mu^2\,\varphi^2 + \frac{\lambda}{4}\,\varphi^4$$

In the homogeneous case, in order to satisfy the equation of motion, the field must be in an extremum of the potential, which is true for two classes of solution: either

$$\varphi = 0$$

or

$$\varphi_0^2 = -\frac{\mu^2}{\lambda}.$$

In order to be a minimum the constant $\mu^2$ must be negative. This is a problem, once it should imply that the mass of the scalar field is imaginary! However, one can avoid this difficulty in the following manner. Let us redefine the field by introducing a new real variable $\chi$ :

$$\varphi = \varphi_0 + \chi,$$

where $\varphi_0$ is a constant. Substituting this definition on Lagrangian (6.2) it follows

$$\mathbb{L} = \frac{1}{2}\,\partial_\mu\chi\,\partial^\mu\chi + \mu^2\,\chi^2 - \frac{\lambda}{4}\,\chi^4 - \lambda\,\varphi_0\,\chi^3 + \frac{\mu^4}{4\lambda} \tag{6.3}$$

This Lagrangian represents a real scalar field $\chi$ with real positive mass $m^2 = -\,\mu^2$ and extra terms of self-interaction. Note that in the Lagrangian it appears a residual constant term representing a background constant negative energy distribution

$$T_{\mu\nu}(residual) = -\frac{\mu^4}{4\lambda}\,g_{\mu\nu}$$

In the realm of high energy physics it is considered that such term ... " has no physical consequences and can be dropped" [18]. We will come back to this when we analyze its gravitational effects.

Note that now, the potential of field $\chi$ takes the form

$$V = m^2\,\chi^2 + \frac{\lambda}{4}\,\chi^4 + \lambda\,\varphi_0\,\chi^3 + constant$$

Its minimum occurs for $\chi = 0$. The others two extrema that exists for constant values $\chi_0$ are points of maxima. The expansion of the field must be made (for all calculations) around $\chi = 0$ and not around $\varphi = 0$. The reason is that this last is an unstable point and the series will suffer from convergence. Finally, we note that the actual field $\chi$ has a real positive mass.

### 6.2.1. The case of complex field

Let us now turn to the case of a complex field. The Lagrangian for $\phi = \phi_1 + i\,\phi_2$ is given by

$$\mathbb{L} = \partial_\mu\phi^*\,\partial^\mu\phi - V(\phi^*\,\phi) \tag{6.4}$$

where the potential has the form

$$V = \mu^2 \, \phi^* \, \phi + \lambda \, (\phi^* \, \phi)^2$$

It is convenient to write the field as

$$\phi = \frac{1}{\sqrt{2}} \, (\phi_0 + \chi) \, \exp \frac{i}{\phi_0} \, \theta(x)$$

The Lagrangian then becomes

$$\begin{aligned} \mathbb{L} &= \frac{1}{2} \, \partial_\mu \chi \, \partial^\mu \chi + \frac{1}{2} \, \frac{(\phi_0 + \chi)^2}{\phi_0^2} \, \partial_\mu \theta \, \partial^\mu \theta \\ &- \frac{\mu^2}{2} \, (\phi_0 + \chi)^2) - \frac{h}{4} \, (\phi_0 + \chi)^4 \end{aligned} \tag{6.5}$$

The extremum of the potential occurs for $\phi_0 + \chi = 0$. For $\mu^2 > 0$ this extremum is a minimum.

### 6.2.2. From global to local symmetry

The theory of the complex field $\phi$ has a gauge invariance under the constant map

$$\phi' = e^{i\alpha} \, \phi.$$

This means that this transformation occurs in everyplace and does not distinguishes any point of space-time. If the parameter $\alpha$ becomes space-time dependent the symmetry is broken. In order to restore the symmetry, one can use the freedom of the electromagnetic field $A_\mu$ and couple this map with the map

$$A_\mu^{'} = A_\mu - \frac{1}{e} \, \partial_\mu \alpha.$$

This scheme was generalized for more general maps (non-abelian theory) by Yang and Mills in the early 1954 for nonlinear fields, called generically gauge fields. It is immediate to show that by minimal coupling of the scalar field with a gauge field the symmetry is restored. The modification consists in the passage from a global symmetry (valid for transformations that are the same everywhere) to a local symmetry that depends on the space-time location. A global property turns into a local one. It is like going from cosmological framework – that deals with the global structure of space-time — to microphysics.

### 6.2.3. Mass for a vectorial boson

The interaction of the complex field $\phi$ with a vector $W_\mu$ through the substitution of the derivatives of the scalar field $\partial_\mu \, \phi$ by $(\partial_\mu - ieW_\mu) \, \phi$ using the minimum

coupling principle, preserves the gauge invariance when the parameter $\alpha$ becomes a function of space-time $\alpha(x)$. This means that the dynamics is invariant under the map

$$\phi' = \phi \exp i\,\alpha(x)$$

$$W'_\mu = W_\mu + \frac{1}{e}\,\partial_\mu \alpha$$

The Lagrangian, after the above substitution of the field $\phi = \phi_0 + \chi$ turns into

$$\begin{aligned} \mathbb{L} &= -\frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu} + \frac{1}{2}\,e^2\,(\phi_0+\chi)^2\,W_\nu\,W^\nu \\ &+ \frac{1}{2}\,\partial_\mu\chi\,\partial^\mu\chi - V(\chi) \\ &+ \frac{1}{2}\,(\phi_0+\chi)^2\,\partial_\mu\theta\,\partial^\mu\theta \end{aligned} \tag{6.6}$$

Note that this represents the interaction of two real scalar fields $\chi$ and $\theta$ but only the real field $\chi$ interacts with the massive vector boson. Due to the gauge invariance, one can contemplate the possibility of choosing

$$\alpha = -\frac{\theta}{\phi_0}$$

and eliminate $\theta$. The dynamics turn into

$$\begin{aligned} \mathbb{L} &= -\frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu} + \frac{e^2\,\phi_0^2}{2}\,W_\nu\,W^\nu \\ &+ \frac{1}{2}\,\partial_\mu\chi\,\partial^\mu\chi - V(\chi) \\ &+ (e^2\,\varphi_0\,\chi + \frac{e^2}{2}\,\chi^2)\,W_\nu\,W^\nu \end{aligned} \tag{6.7}$$

that represents a massive vector field interacting non-minimally with a real scalar field. Note that one of the degree of freedom of the theory – represented by the scalar field $\theta(x)$ — was eliminated. Indeed, it was transformed into an extra degree of freedom of the massive vector field (that gained one more degree of liberty going from 2 to 3). The total number we had (two for the field $\phi$ and two for the massless field $W_\mu$) is preserved. It only changed the place. The degree of freedom of $\theta$ was conceded to the (now) massive vector boson.

It is not difficult to generalize the above procedure for more than one vector field in such a way that one of them remains massless. This was the procedure for the case of the unified field theory of electro-weak interaction: the intermediary boson gain a mass but the photon remains massless.

### 6.2.4. Mass for a fermion

Let us couple this scalar field with a spinor $\Psi$ through the Lagrangian

$$\mathbb{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi) + \mathbb{L}_D - h\,\varphi\,\bar{\Psi}\,\Psi \tag{6.8}$$

where $\mathbb{L}_D$ is Dirac dynamics for massless free field. Making the same replacement we made previously using $\chi$ instead of $\varphi$ this theory becomes

$$\mathbb{L} = \frac{1}{2}\,\partial_\mu\chi\,\partial^\mu\chi - V(\chi) + \mathbb{L}_D - h\,(\varphi_0 + \chi)\,\bar{\Psi}\,\Psi \tag{6.9}$$

The equation for the spinor field becomes

$$i\gamma^\mu\partial_\mu\,\Psi - h\,\varphi_0\,\Psi - h\,\chi\,\bar{\Psi}\,\Psi = 0 \tag{6.10}$$

which represents a spinor field of mass $h\,\varphi_0 > 0$ interacting with a scalar field $\chi$.

## 6.3. Who gives mass to the scalar field that give mass for the vector and spinor fields?

In the precedent sections we described the Higgs model that produced an efficient scenario for generating mass to the vector bosons in the realm of high-energy physics. At its origin appears a process relating the transformation of a global symmetry into a local one and the corresponding presence of vector gauge fields.

This mechanism appeals to the intervention of a scalar field that appears as the vehicle which provides mass to the gauge vector field $W_\mu$. For the mass to be a real and constant value (a different value for each field) the scalar field $\varphi$ must be in a minimum state of its potential $V$. This fundamental state of the self-interacting scalar field has an energy distribution given by $T_{\mu\nu} = V(\varphi_0)\,g_{\mu\nu}$. A particular form of self-interaction of the scalar field $\varphi$ allows the existence of a constant value $V(\varphi_0)$ that is directly related to the mass of $W_\mu$. This scalar field has its own mass, the origin of which rests unclear. In [17] a new mechanism depending on the gravitational interaction, that can provides mass to the scalar field was presented. In these lectures we shall analyze this mechanism.

Although the concept of mass pervades most of all analysis involving gravitational interaction, it is an uncomfortable situation that still to this day there has been no successful attempt to derive a mechanism by means of which mass is understood a direct consequence of a dynamical process depending on gravity [13].

The main idea concerning inertia in the realm of gravity according to the origins of General Relativity, goes in the opposite direction of the mechanism

that we analyzed in the previous section in the territory of the high-energy physics. Indeed, while the Higgs mechanism explores the reduction of a global symmetry into a local one, the Mach principle suggests a cosmical dependence of local properties, making the origin of the mass of a given body to depend on the structure of the whole universe. In this way, there ought to exist a mechanism by means of which this quantity - the mass – depends on the state of the universe. How to understand such broad concept of mass? Let us describe an example of such mechanism in order to see how this vague idea can achieve a qualitative scheme [14].

### 6.3.1. Mass for scalar field: a trivial case

We start by considering Mach principle as the statement according to which the inertial properties of a body $\mathbb{A}$ are determined by the energy-momentum throughout all space. How could we describe such universal state that takes into account the whole contribution of the rest-of-the-universe onto $\mathbb{A}$ ? There is no simpler way than consider this state as the most homogeneous one and relate it to what Einstein attributed to the cosmological constant or, in modern language, the vacuum of all remaining bodies. This means to describe the energy-momentum distribution of all complementary bodies of $\mathbb{A}$ as

$$T_{\mu\nu} = \lambda\, g_{\mu\nu}$$

Let $\varphi$ be a massless field the dynamics of which is given by the Lagrangian

$$L_\varphi = \frac{1}{2}\,\partial_\alpha\varphi\,\partial^\alpha\varphi$$

In the framework of General Relativity its gravitational interaction is given by the Lagrangian

$$L = \frac{1}{\kappa_0}\,R + \frac{1}{2}\,\partial_\alpha\varphi\,\partial^\alpha\varphi + B(\varphi)\,R - \frac{\lambda}{\kappa_0} \tag{6.11}$$

where for the time being the dependence of $B$ on the scalar field is not fixed. This dynamics represents a scalar field non-minimally coupled to gravity. The cosmological constant is added by the reasons presented above and represents the influence of the rest-of-the-universe on $\varphi$. We shall see that $\lambda$ is the real responsible to provide mass for the scalar field. This means that if we set $\lambda = 0$ the mass of the scalar field should vanish.

Independent variation of $\varphi$ and $g_{\mu\nu}$ yields

$$\Box\varphi - R\,B' = 0 \tag{6.12}$$

$$\alpha_0\,(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) = -T_{\mu\nu} \tag{6.13}$$

where we set $\alpha_0 \equiv 2/\kappa_0$ and $B' \equiv \partial B/\partial\varphi$. The energy-momentum tensor is given by

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}\,\partial_\alpha\varphi\,\partial^\alpha\varphi\,g_{\mu\nu} \\ &+ 2B\,(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) \\ &+ 2\nabla_\mu\nabla_\nu B - 2\Box B\,g_{\mu\nu} + \frac{\lambda}{\kappa_0}\,g_{\mu\nu} \end{aligned} \tag{6.14}$$

Taking the trace of equation (6.55) we obtain

$$(\alpha_0 + 2B)\,R = -\,\partial_\alpha\varphi\,\partial^\alpha\varphi - 6\Box B + \frac{4\lambda}{\kappa_0} \tag{6.15}$$

Inserting this result on the equation (6.46 ) yields

$$\Box\varphi + \mathbb{Z} = 0 \tag{6.16}$$

where

$$\mathbb{Z} \equiv \frac{B'}{\alpha_0 + 2B}\left(\partial_\alpha\varphi\,\partial^\alpha\varphi + 6\Box B - \frac{4\lambda}{\kappa_0}\right)$$

or, equivalently,

$$\mathbb{Z} = \frac{B'}{\alpha_0 + 2B}\left(\partial_\alpha\varphi\,\partial^\alpha\varphi(1 + 6B'') + 6\,B'\,\Box\varphi - \frac{4\lambda}{\kappa_0}\right)$$

Therefore, the scalar field acquires an effective self-interaction through the non-minimal coupling with the gravitational field. At this stage it is worth to select among all possible candidates of $B$ a particular one that makes the factor on the gradient of the field to disappear in the expression of $\mathbb{Z}$ by setting

$$B = a + b\,\varphi - \frac{1}{12}\,\varphi^2$$

where $a$ and $b$ are arbitrary parameters. The quantity $a$ makes only a renormalization of the constant $1/\kappa_0$ and parameter $b$ is responsible for distinguishing different masses for different fields. Making a translation on the field

$$\Phi = -\varphi + 6b$$

it follows

$$\Box\Phi + \mu^2\,\Phi = 0 \tag{6.17}$$

where

$$\mu^2 = \frac{2\lambda}{3}\,\frac{\kappa_{ren}}{\kappa_0}. \tag{6.18}$$

where

$$\kappa_{ren} = \frac{1}{\alpha_0 + 2a + 6b^2}$$

Thus as a result of the above process the scalar field acquires a mass $\mu$ that depends on $\lambda$. If $\lambda$ vanishes then the mass of the field vanishes. The net effect of the non-minimal coupling of gravity with the scalar field corresponds to a specific self-interaction of the scalar field. The mass of the field appears only if we take into account the existence of all remaining bodies in the universe in the state in which all existing matter is on the corresponding vacuum. The values of different masses for different fields is contemplated in the parameter $b$.

### 6.3.2. Mass for scalar field-II

Let us now analyze a more general scenario to provide mass to a scalar field. We start from the Lagrangian that describes a massless field $\varphi$ that is

$$L_\varphi = \frac{1}{2}\,\partial_\alpha\varphi\,\partial^\alpha\varphi$$

The gravitational interaction yields the modified Lagrangian

$$L = \frac{1}{\kappa}\,R + \frac{1}{2}\,W(\varphi)\,\partial_\alpha\varphi\,\partial^\alpha\varphi + B(\varphi)\,R - \frac{1}{\kappa}\,\Lambda \tag{6.19}$$

where for the time being the dependence of $B$ and $W$ on the scalar field is not fixed. We set $\hbar = c = 1$.

This dynamics represents a scalar field coupled non-minimally with gravity. There is no direct interaction between $\varphi$ and the rest-of-the-universe (ROTU), except through the intermediary of gravity described by a cosmological constant $\Lambda$. Thus $\Lambda$ represents the whole influence of the ROTU on $\varphi$.

Independent variation of $\varphi$ and $g_{\mu\nu}$ yields

$$W\,\Box\varphi + \frac{1}{2}\,W'\,\partial_\alpha\varphi\,\partial^\alpha\varphi - B'\,R = 0 \tag{6.20}$$

$$\alpha_0\,(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) = -T_{\mu\nu} \tag{6.21}$$

where $\alpha_0 \equiv 2/\kappa$ and $B' \equiv \partial B/\partial\varphi$. The energy-momentum tensor defined by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\,\frac{\delta(\sqrt{-g}\,L)}{\delta g^{\mu\nu}}$$

is given by

$$\begin{aligned} T_{\mu\nu} &= W\,\partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}\,W\,\partial_\alpha\varphi\,\partial^\alpha\varphi\,g_{\mu\nu} \\ &+ 2B\,(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) \\ &+ 2\nabla_\mu\nabla_\nu B - 2\Box B\,g_{\mu\nu} + \frac{1}{\kappa}\,\Lambda\,g_{\mu\nu} \end{aligned} \tag{6.22}$$

Taking the trace of equation (6.21) we obtain

$$(\alpha_0 + 2B)\, R = -\,\partial_\alpha\varphi\,\partial^\alpha\varphi\,(W + 6\,B'') - 6B'\,\Box\varphi + 4\,\frac{\Lambda}{\kappa} \tag{6.23}$$

where we used that $\Box B = B'\,\Box\varphi + B''\,\partial_\alpha\varphi\,\partial^\alpha\varphi$.

Inserting this result back on the equation (6.20 ) yields

$$\mathbb{M}\,\Box\varphi\; + \mathbb{N}\,\partial_\alpha\varphi\,\partial^\alpha\varphi - \mathbb{Q} = 0 \tag{6.24}$$

where

$$\mathbb{M} \equiv W + \frac{6(B')^2}{\alpha_0 + 2B}$$

$$\mathbb{N} \equiv \frac{1}{2}\,W' + \frac{B'\,(W + 6B'')}{\alpha_0 + 2B}$$

$$\mathbb{Q} = \frac{4\,\Lambda\,B'}{\kappa\,(\alpha_0 + 2B)}$$

Thus, through the non-minimal coupling with the gravitational field the scalar field acquires an effective self-interaction. At this point it is worth to select among all possible candidates of $B$ and $W$ particular ones that makes the factor on the gradient of the field to disappear on the equation of motion by setting $\mathbb{N} = 0$. This condition will give $W$ as a function of $B$ :

$$W = \frac{2q - 6(B')^2}{\alpha_0 + 2B} \tag{6.25}$$

where $q$ is a constant. Inserting this result into the equation (6.24) yields

$$\Box\varphi - \frac{2\,\Lambda}{q\,\kappa}\,B' = 0. \tag{6.26}$$

At this point one is led to set

$$B = -\frac{\beta}{4}\,\varphi^2$$

to obtain

$$\Box\varphi + \mu^2\,\varphi = 0 \tag{6.27}$$

where

$$\mu^2 \equiv \frac{\beta\,\Lambda}{q\,\kappa} \tag{6.28}$$

For the function $W$ we obtain

$$W = \frac{2\,q - 3\,\beta^2\,\varphi^2}{2\,\alpha_0 - \beta\varphi^2}$$

One should set $2q = \alpha_0$ in order to obtain the standard dynamics in case $\beta$ vanishes. Using units were $\hbar = 1 = c$ we write

$$\mathbb{L} = \frac{1}{\kappa} R + \frac{2\,q - 3\,\beta^2\,\varphi^2}{2\,(2\,\alpha_0 - \beta\varphi^2)}\,\partial_\alpha\varphi\,\partial^\alpha\varphi - \frac{1}{4}\,\beta\,\varphi^2\,R - \frac{\Lambda}{\kappa}$$

Thus as a result of the gravitational interaction the scalar field acquires a mass $\mu$ that depends on the constant $\beta$ and on the existence of $\Lambda$ :

$$\mu^2 = \beta\,\Lambda \tag{6.29}$$

If $\Lambda$ vanishes then the mass of the field vanishes. The net effect of the non-minimal coupling of gravity with the scalar field corresponds to a specific self-interaction of the scalar field. The mass of the field appears only if we take into account the existence of all remaining bodies in the universe — represented by the cosmological constant — in the state in which all existing matter is on the corresponding vacuum. The values of different masses for different fields is contemplated in the parameter $\beta$.

### 6.3.3. Renormalization of the mass

The effect of the rest-of-the-universe on a massive scalar field can be analyzed through the same lines as above. Indeed, let us consider the case in which there is a potential $V(\varphi)$

$$L = \frac{1}{\kappa} R + \frac{W}{2}\,\partial_\alpha\varphi\,\partial^\alpha\varphi + B(\varphi)\,R - V(\varphi) - \frac{\Lambda}{\kappa} \tag{6.30}$$

The equation for the scalar field is given by

$$W\,\Box\varphi + \frac{1}{2}\,W^{'}\,\partial_\alpha\,\varphi\,\partial^\alpha\,\varphi - B^{'}\,R + V^{'} = 0 \tag{6.31}$$

Use the equation for the metric to obtain the scalar of curvature in terms of the field and $\Lambda$. It then follows that terms in $\partial_\alpha\,\varphi\,\partial^\alpha\,\varphi$ are absent if we set

$$W = \frac{2q - 6\,(B^{'})^2}{\alpha_0 + 2B}$$

where $q$ is a constant. For the case in which $B = -\beta\,\varphi^2/4$ and for the potential

$$V = \frac{\mu_0}{2}\,\varphi^2$$

and choosing $q = 1/\kappa$ (in order to obtain the standard equation of the scalar field in case $B = 0$) yields

$$\Box\varphi + (\mu_0^2 + \beta\,\Lambda)\,\varphi + \frac{\beta\,\mu_0^2}{4}\,\kappa\,\varphi^3 = 0 \tag{6.32}$$

This dynamics is equivalent to the case in which the scalar field shows an effective potential (in absence of gravity) of the form

$$V_{eff} = (\mu_0^2 + \beta\,\Lambda)\,\frac{\varphi^2}{2} + \frac{\beta\,\mu_0^2\,\kappa}{16}\,\varphi^4$$

Thus the net effect of the gravitational interaction for the dynamics driven by (6.30) is to re-normalize the mass from the bare value $\mu_0$ to the value

$$\mu^2 = \mu_0^2 + \beta\,\Lambda.$$

We can then contemplate the possibility that **all bodies represented by a scalar field could have the same bare mass and as a consequence of gravitational interaction acquires a split into different values characterized by the different values of $\beta$. This result is not exclusive of the scalar field but is valid for any field.**

## 6.4. The case of fermions

Let us now turn our attention to the case of fermions. The massless theory for a spinor field is given by Dirac equation:

$$i\gamma^\mu\partial_\mu\,\Psi = 0 \tag{6.33}$$

This equation is invariant under $\gamma^5$ transformation. In order to have mass for the fermion this symmetry must be broken. Who is the responsible for this?

**Gravity breaks the symmetry**

Electrodynamics appears in gauge theory as a mechanism that preserves a symmetry when one pass from a global transformation to a local one (space-time dependent map). Nothing similar with gravity. On the contrary, in the generation of mass through the mechanism that we are analyzing here, gravity is the responsible to break the symmetry. In the framework of General Relativity the gravitational interaction of the fermion is driven by the Lagrangian

$$\begin{aligned} L &= \frac{i\,\hbar\,c}{2}\bar{\Psi}\gamma^\mu\nabla_\mu\Psi - \frac{i}{2}\nabla_\mu\bar{\Psi}\gamma^\mu\Psi \\ &+ \frac{1}{\kappa}\,R + V(\Phi)\,R - \frac{1}{\kappa}\,\Lambda \\ &+ L_{CT} \end{aligned} \tag{6.34}$$

where the non-minimal coupling of the spinor field with gravity is contained in the term $V(\Phi)$ that depends on the scalar

$$\Phi \equiv \bar{\Psi}\,\Psi$$

which preserves the gauge invariance of the theory under the map $\Psi \to \exp(i\,\theta)\,\Psi$. Note that the dependence on $\Phi$ on the dynamics of $\Psi$ breaks the chiral invariance of the mass-less fermion, a condition that is necessary for a mass to appear.

For the time being the dependence of $V$ on $\Phi$ is not fixed. We have added a counter-term $L_{CT}$ for reasons that will be clear later on. On the other hand, the form of the counter-term should be guessed, from the previous analysis that we made for the scalar case, that is we set

$$L_{CT} = H(\Phi)\,\partial_\mu\Phi\,\partial^\mu\Phi \tag{6.35}$$

This dynamics represents a massless spinor field coupled non-minimally with gravity. The cosmological constant represents the influence of the rest-of-the-universe on $\Psi$.

Independent variation of $\Psi$ and $g_{\mu\nu}$ yields

$$i\gamma^\mu\nabla_\mu\,\Psi + (R\,V' - H'\,\partial_\mu\Phi\,\partial^\mu\Phi - 2H\Box\Phi)\;\Psi = 0 \tag{6.36}$$

$$\alpha_0\,(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) = -T_{\mu\nu} \tag{6.37}$$

where $V' \equiv \partial V/\partial\Phi$. The energy-momentum tensor is given by

$$\begin{aligned} T_{\mu\nu} &= \frac{i}{4}\,\bar{\Psi}\gamma_{(\mu}\nabla_{\nu)}\Psi - \frac{i}{4}\nabla_{(\mu}\bar{\Psi}\gamma_{\nu)}\Psi \\ &+ 2V(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) + 2\nabla_\mu\nabla_\nu V - 2\Box V\,g_{\mu\nu} \\ &+ 2H\,\partial_\mu\Phi\,\partial_\nu\Phi - H\,\partial_\lambda\Phi\,\partial^\lambda\Phi\,g_{\mu\nu} + \frac{\alpha_0}{2}\,\Lambda\,g_{\mu\nu} \end{aligned} \tag{6.38}$$

Taking the trace of equation (6.37) we obtain after some algebraic manipulation:

$$\begin{aligned} (\alpha_0 + 2V + V')\,R &= H'\,\Phi\,\partial_\alpha\Phi\,\partial^\alpha\Phi \\ &+ 2H\,\Phi\,\Box\Phi - 6\Box V + 2\,\alpha_0\,\Lambda \end{aligned} \tag{6.39}$$

Inserting this result back on the equation (6.36) yields

$$i\gamma^\mu\nabla_\mu\,\Psi + \big(\mathbb{X}\,\partial_\lambda\Phi\,\partial^\lambda\Phi + \mathbb{Y}\,\Box\Phi\,\big)\;\Psi + \mathbb{Z}\,\Psi = 0 \tag{6.40}$$

where

$$\mathbb{Z} \equiv \frac{2\,\alpha_0\,\Lambda\,V'}{Q}$$

$$\mathbb{X} = \frac{V'\,(\Phi\,H' - 2H - 6V'')}{Q} - H'$$

$$\mathbb{Y} = \frac{V'\,(2H\,\Phi - 6V')}{Q} - 2H$$

where $Q \equiv \alpha_0 + 2V + \Phi\, V'$.

At this stage it is worth selecting among all possible candidates of $V$ and $H$ particular ones that makes the factor on the gradient and on $\Box$ of the field to disappear from equation (6.40). The simplest way is to set $\mathbb{X} = \mathbb{Y} = 0$ which imply only one condition, that is

$$H = \frac{-\,3(V')^2}{\alpha_0 + 2V} \tag{6.41}$$

The non-minimal term $V$ is such that $\mathbb{Z}$ reduces to a constant, that is

$$V = \frac{\alpha_0}{2}\,\left[(1+\sigma\,\Phi)^{-2} - 1\right] \tag{6.42}$$

Then it follows immediately that

$$H = -3\alpha_0\,\sigma^2\,(1+\sigma\,\Phi)^{-4} \tag{6.43}$$

where $\sigma$ is a constant.

The equation for the spinor becomes

$$i\gamma^\mu \nabla_\mu\,\Psi - m\Psi = 0 \tag{6.44}$$

where

$$m = \frac{4\,\sigma\,\Lambda}{\kappa\,c^2}. \tag{6.45}$$

Thus as a result of the above process the spinor field acquires a mass $m$ that depends crucially on the existence of $\Lambda$. If $\Lambda$ vanishes then the mass of the field vanishes. The non-minimal coupling of gravity with the spinor field corresponds to a specific self-interaction. The mass of the field appears only if we take into account the existence of all remaining bodies in the universe — represented by the cosmological constant. The values of different masses for different fields are contemplated in the parameter $\sigma$.

The various steps of our mechanism can be synthesized as follows:

- The dynamics of a massles spinor field $\Psi$ is described by the Lagrangian

$$L_D = \frac{i}{2}\bar{\Psi}\gamma^\mu\nabla_\mu\Psi - \frac{i}{2}\nabla_\mu\bar{\Psi}\gamma^\mu\Psi;$$

- Gravity is described in General Relativity by the scalar of curvature

$$L_E = R;$$

- The field interacts with gravity in a non-minimal way described by the term

$$L_{int} = V(\Phi)\,R$$

  where $\Phi = \bar{\Psi}\,\Psi$;

- The action of the rest-of-the-universe on the spinor field, through the gravitational intermediary, is contained in the form of an additional constant term on the Lagrangian noted as $\Lambda$;
- A counter-term depending on the invariant $\Phi$ is introduced to kill extra terms coming from gravitational interaction;
- As a result of this process, after specifying $V$ and $H$ the field acquires a mass being described as

$$i\gamma^{\mu}\nabla_{\mu}\Psi - m\Psi = 0$$

  where $m$ is given by equation (6.45) and is zero only if the cosmological constant vanishes.

This procedure allows us to state that the mechanism proposed here is to be understood as a realization of Mach principle according to which the inertia of a body depends on the background of the rest-of-the-universe. This strategy can be applied in a more general context in support of the idea that (local) properties of microphysics may depend on the (global) properties of the universe. We will analyze this in the next session (see also [320]).

Thus, collecting all these terms we obtain the final form of the Lagrangian

$$\begin{aligned} L &= \frac{i}{2}\bar{\Psi}\gamma^{\mu}\nabla_{\mu}\Psi - \frac{i}{2}\nabla_{\mu}\bar{\Psi}\gamma^{\mu}\Psi \\ &+ \frac{1}{\kappa}\left(1+\sigma\,\Phi\right)^{-2} R - \frac{1}{\kappa}\Lambda \\ &- \frac{6}{\kappa}\sigma^{2}\left(1+\sigma\,\Phi\right)^{-4}\partial_{\mu}\Phi\,\partial^{\mu}\Phi \end{aligned} \tag{6.46}$$

**Some comments**

- In the case $\sigma = 0$ the Lagrangian reduces to a massless fermion satisfying Dirac's dynamics plus the gravitational field described by General Relativity;
- The dimensionality of $\sigma$ is $L^3$;
- The ratio $m/\sigma = 4\,\Lambda/\kappa\,c^2$ which has the meaning of a density of mass is an universal constant. How to interpret such universality?

## 6.5. The case of vector fields

We start with a scenario in which there are only three ingredients: a massless vector field, the gravitational field and an homogeneous distribution of energy - that is identified with the vacuum. The theory is specified by the Lagrangian

$$L = -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{\kappa}\,R - \frac{\Lambda}{\kappa} \tag{6.47}$$

The corresponding equations of motion are

$$F^{\mu\nu}{}_{;\nu} = 0$$

and

$$\alpha_0 \left(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}\right) = -T_{\mu\nu}$$

where $F_{\mu\nu} = \nabla_\nu W_\mu - \nabla_\mu W_\nu$ and $\alpha_0 \equiv 2/\kappa$.

In this theory, the vacuum $\Lambda$ is invisible for $W_\mu$. The energy distribution represented by $\Lambda$ interacts with the vector field only indirectly once it modifies the geometry of space-time. In the Higgs mechanism this vacuum is associated to a fundamental state of a scalar field $\varphi$ and it is transformed in a mass term for $W_\mu$. The role of $\Lambda$ is displayed by the value of the potential $V(\varphi)$ in its homogeneous state. We will now show that there is no needs to introduce any extra scalar field by using the universal character of gravitational interaction to generate mass for $W_\mu$.

The point of departure is the recognition that gravity may be the real responsible for breaking the gauge symmetry. For this, we modify the above Lagrangian to include a non-minimal coupling of the field $W_\mu$ to gravity in order to explicitly break such invariance. There are only two possible ways for this [16]. The total Lagrangian must be of the form

$$\begin{aligned} \mathbb{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{\kappa} R \\ &+ \frac{\gamma}{6} R\, \Phi + \gamma\, R_{\mu\nu} W^\mu W^\nu \\ &- \frac{\Lambda}{\kappa} \end{aligned} \tag{6.48}$$

where we define

$$\Phi \equiv W_\mu W^\mu.$$

The first two terms of $\mathbb{L}$ represents the free part of the vector and the gravitational fields. The second line represents the non-minimal coupling interaction of the vector field with gravity. The parameter $\gamma$ is dimensionless. The vacuum – represented by $\Lambda$ – is added by the reasons presented above and it must be understood as the definition of the expression "the influence of the rest-of-the-universe on $W_\mu$". We will not make any further hypothesis on this [17].

In the present proposed mechanism, $\Lambda$ is the real responsible to provide mass for the vector field. This means that if we set $\Lambda = 0$ the mass of $W_\mu$ will vanish.

Independent variation of $W_\mu$ and $g_{\mu\nu}$ yields

$$F^{\mu\nu}{}_{;\nu} + \frac{\gamma}{3} R\, W^\mu + 2\gamma\, R^{\mu\nu} W_\nu = 0 \tag{6.49}$$

$$\alpha_0\,(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) = -T_{\mu\nu} \tag{6.50}$$

The energy-momentum tensor defined by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\,\frac{\delta(\sqrt{-g}\,L)}{\delta g^{\mu\nu}}$$

is given by

$$\begin{aligned} T_{\mu\nu} &= E_{\mu\nu} \\ &+ \frac{\gamma}{3}\,\nabla_\mu\nabla_\nu\Phi - \frac{\gamma}{3}\,\Box\Phi\,g_{\mu\nu} + \frac{\gamma}{3}\,\Phi\,(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) \\ &+ \frac{\gamma}{3}\,RW_\mu\,W_\nu + 2\gamma R_\mu{}^\lambda\,W_\lambda\,W_\nu + 2\gamma R_\nu{}^\lambda\,W_\lambda\,W_\mu \\ &- \gamma\,R_{\alpha\beta}\,W^\alpha\,W^\beta\,g_{\mu\nu} - \gamma\,\nabla_\alpha\,\nabla_\beta\,(W^\alpha\,W^\beta)\,g_{\mu\nu} \\ &+ \gamma\,\nabla_\nu\,\nabla_\beta(W_\mu\,W^\beta) + \gamma\,\nabla_\mu\,\nabla_\beta(W_\nu\,W^\beta) \\ &+ \gamma\,\Box(W_\mu W_\nu) + \frac{1}{\kappa}\,\Lambda\,g_{\mu\nu} \end{aligned} \tag{6.51}$$

where

$$E_{\mu\nu} = F_{\mu\alpha}\,F^\alpha{}_\nu + \frac{1}{4}F_{\alpha\beta}\,F^{\alpha\beta}\,g_{\mu\nu}$$

Taking the trace of equation (6.50) we obtain

$$R = 2\,\Lambda - \kappa\,\gamma\,\nabla_\alpha\,\nabla_\beta\,(W^\alpha\,W^\beta) \tag{6.52}$$

Then, using this result back into equation (6.49) it follows

$$\begin{aligned} F^{\mu\nu}{}_{;\nu} &+ \frac{2\,\gamma\,\Lambda}{3}\,W^\mu \\ &- \frac{\kappa\,\gamma^2}{3}\,\nabla_\alpha\,\nabla_\beta\,(W^\alpha\,W^\beta)\,W^\mu \\ &+ 2\,\gamma\,R^\mu{}_\nu\,W^\nu = 0 \end{aligned} \tag{6.53}$$

The non-minimal coupling with gravity yields an effective self-interaction of the vector field and a term that represents its direct interaction with the curvature of space-time. Besides, as a result of this process the vector field acquires a mass $\mu$ that depends on the constant $\gamma$ and on the existence of $\lambda$. The term

$$2\,\gamma\,R^\mu{}_\nu\,W^\nu$$

gives a contribution (through the dynamics of the metric equation (6.50) of $\gamma\,\Lambda$ yielding for the mass the formula

$$\mu^2 = \frac{5}{3}\,\gamma\,\Lambda \tag{6.54}$$

Note that the Newton's constant does not appear in our formula for the mass. The net effect of the non-minimal coupling of gravity with $W^\mu$ corresponds to a specific self-interaction of the vector field. The mass of the field appears only if we take into account the existence of the rest-of-the-universe — represented by $\Lambda$ — in the state in which this environment is on the corresponding vacuum. If $\Lambda$ vanishes then the mass of the field vanishes.The values of different masses for different fields are contemplated in the parameter $\gamma$.

**Quantum perturbations**

How this process that we have been examining here to give mass to all kind of bodies should be modified in a quantum version? We note, first of all, that the gravitational field is to be treated at a classical level, once there is neither theoretical nor observational evidence that exists a quantum version of gravitational interaction. Thus, any modification of the present scheme means to introduce quantum aspects of the vector field. This will not change the whole scheme of generation of mass described above.Indeed, in the semi-classical approach in which the matter field is quantized but the metric is not, the modification of the equation of general relativity becomes

$$\alpha_0 \left(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}\right) = - < T_{\mu\nu} > \tag{6.55}$$

where the field is in a given specific state. Throughout all the process of gravitational interaction the system does not change its state, allowing the same classical treatment as above.

## 6.6. The case of spin-two field

As in the previous cases we start with a scenario in which there are only three ingredients: a linear tensor field, the gravitational field and an homogeneous distribution of energy identified with the vacuum. We note that there are two possible equivalent ways to describe a spin-two field that is:

- Einstein frame
- Fierz frame

according we use a symmetric second order tensor $\varphi_{\mu\nu}$ or the third-order tensor tensor $F_{\alpha\beta\lambda}$. Although the Fierz representation is not used for most of the works dealing with spin-2 field, it is far better than the Einstein frame when dealing in a curved space-time[23]. Thus, let us review briefly the basic properties of the Fierz frame[1]. We start by defining a three-index tensor $F_{\alpha\beta\mu}$ which is anti-symmetric in the first pair of indices and obeys the cyclic identity:

$$F_{\alpha\mu\nu} + F_{\mu\alpha\nu} = 0, \tag{6.56}$$

[1]We use the notation $A_{(\alpha}B_{\beta)} = A_\alpha B_\beta + A_\beta B_\alpha$, $A_{[\alpha}B_{\beta]} = A_\alpha B_\beta - A_\beta B_\alpha$.

$$F_{\alpha\mu\nu} + F_{\mu\nu\alpha} + F_{\nu\alpha\mu} = 0. \tag{6.57}$$

This expression implies that the dual of $F_{\alpha\mu\nu}$ is trace-free:

$$\overset{*}{F}{}^{\alpha\mu}{}_{\mu} = 0, \tag{6.58}$$

where the asterisk represents the dual operator, defined in terms of $\eta_{\alpha\beta\mu\nu}$ by

$$\overset{*}{F}{}^{\alpha\mu}{}_{\lambda} \equiv \frac{1}{2}\,\eta^{\alpha\mu}{}_{\nu\sigma}\,F^{\nu\sigma}{}_{\lambda}.$$

The tensor $F_{\alpha\mu\nu}$ has 20 independent components. The necessary and sufficient condition for $F_{\alpha\mu\nu}$ to represent an unique spin-2 field (described by 10 components) is [2]

$$\overset{*}{F}{}^{\alpha(\mu\nu)}{}_{,\alpha} = 0, \tag{6.59}$$

which can be rewritten as

$$\begin{aligned} &F_{\alpha\beta}{}^{\lambda}{}_{,\mu} + F_{\beta\mu}{}^{\lambda}{}_{,\alpha} + F_{\mu\alpha}{}^{\lambda}{}_{,\beta} - \frac{1}{2}\delta^{\lambda}_{\alpha}(F_{\mu,\beta} - F_{\beta,\mu}) + \\ &-\frac{1}{2}\delta^{\lambda}_{\mu}(F_{\beta,\alpha} - F_{\alpha,\beta}) - \frac{1}{2}\delta^{\lambda}_{\beta}(F_{\alpha,\mu} - F_{\mu,\alpha}) = 0. \end{aligned} \tag{6.60}$$

A direct consequence of the above equation is the identity:

$$F^{\alpha\beta\mu}{}_{,\mu} = 0\ . \tag{6.61}$$

We call a tensor that satisfies the conditions given in the Eqns.(6.56), (6.57) and (6.59) a Fierz tensor. If $F_{\alpha\mu\nu}$ is a Fierz tensor, it represents an unique spin-2 field. Condition (6.59) yields a connection between the Einstein frame (EF) and the Fierz frame (FF): it implies that there exists a symmetric second-order tensor $\varphi_{\mu\nu}$ that acts as a potential for the field. We write

$$\begin{aligned} 2\,F_{\alpha\mu\nu} &= \varphi_{\nu[\alpha,\mu]} + \left(\varphi_{,\alpha} - \varphi_{\alpha}{}^{\lambda}{}_{,\lambda}\right)\,\eta_{\mu\nu} \\ &- \left(\varphi_{,\mu} - \varphi_{\mu}{}^{\lambda}{}_{,\lambda}\right)\,\eta_{\alpha\nu}. \end{aligned} \tag{6.62}$$

where $\eta_{\mu\nu}$ is the flat spacetime metric tensor, and the factor 2 in the l.h.s. is introduced for convenience.

Taking the trace of equation (6.62) $F_{\alpha} \equiv F_{\alpha\mu\nu}\eta^{\mu\nu}$ it follows that

$$F_{\alpha} = \varphi_{,\alpha} - \varphi_{\alpha}{}^{\lambda}{}_{,\lambda},$$

where . Thus we can write

$$2F_{\alpha\mu\nu} = \varphi_{\nu[\alpha,\mu]} + F_{[\alpha}\,\eta_{\mu]\nu}. \tag{6.63}$$

[2]Note that this condition is analogous to that necessary for the existence of a potential $A_{\mu}$ for the EM field, given by $\overset{*}{A}{}^{\alpha\mu}{}_{,\alpha} = 0$.

Using the properties of the Fierz tensor we obtain the important identity:

$$F^{\alpha}{}_{(\mu\nu),\alpha} \equiv -\,2\,G^{(L)}{}_{\mu\nu}, \tag{6.64}$$

where $G^{(L)}{}_{\mu\nu}$ is the linearized Einstein tensor, defined by the perturbation $g_{\mu\nu} = \eta_{\mu\nu} + \varphi_{\mu\nu}$ by

$$2\,G^{(L)}{}_{\mu\nu} \equiv \Box\,\varphi_{\mu\nu} - \varphi^{\epsilon}{}_{(\mu,\nu)\,,\epsilon} + \varphi_{,\mu\nu} - \eta_{\mu\nu}\,\left(\Box\varphi - \varphi^{\alpha\beta}{}_{,\alpha\beta}\right). \tag{6.65}$$

The divergence of $F^{\alpha}{}_{(\mu\nu),\alpha}$ yields Bianci identity:

$$F^{\alpha(\mu\nu)}{}_{,\alpha\mu} \equiv 0. \tag{6.66}$$

Indeed,

$$F^{\alpha\mu\nu}{}_{,\alpha\mu} + F^{\alpha\nu\mu}{}_{,\mu\alpha} = 0. \tag{6.67}$$

The first term vanishes identically due to the symmetric properties of the field and the second term vanishes due to equation (E.96). Using Eqn.(6.64) the identity which states that the linearized Einstein tensor $G^{(L)}{}_{\mu\nu}$ is divergence-free is recovered.

We shall build now dynamical equations for the free Fierz tensor in flat space-time. Our considerations will be restricted here to linear dynamics. The most general theory can be constructed from a combination of the three invariants involving the field. These are represented by $A$, $B$ and $W$:

$$A \equiv F_{\alpha\mu\nu}\,F^{\alpha\mu\nu}, \qquad B \equiv F_{\mu}\,F^{\mu},$$

$$W \equiv F_{\alpha\beta\lambda}\,\overset{*}{F}{}^{\alpha\beta\lambda} = \frac{1}{2}\,F_{\alpha\beta\lambda}\,F^{\mu\nu\lambda}\,\eta^{\alpha\beta}{}_{\mu\nu}.$$

$W$ is a topological invariant so we shall use only the invariants $A$ and $B$. The EOM for the massless spin-2 field in the ER is given by

$$G^{(L)}{}_{\mu\nu} = 0. \tag{6.68}$$

As we have seen above, in terms of the field $F^{\lambda\mu\nu}$ this equation can be written as

$$F^{\lambda(\mu\nu)}{}_{,\lambda} = 0. \tag{6.69}$$

The corresponding action takes the form

$$S = \frac{1}{k}\int \mathrm{d}^4x\,(A - B). \tag{6.70}$$

Then,

$$\delta S = \int F^{\alpha\,(\mu\nu)}{}_{,\alpha}\,\delta\varphi_{\mu\nu}\,d^4x. \tag{6.71}$$

we obtain

$$\delta S = -2\int G^{(L)}{}_{\mu\nu}\,\delta\varphi^{\mu\nu}\,d^4x, \tag{6.72}$$

where $G^{(L)}{}_{\mu\nu}$ is given in Eqn.(6.65).

Let us consider now the massive case. If we include a mass for the spin 2 field in the Fierz frame, the Lagrangian takes the form

$$\mathcal{L} = A - B + \frac{m^2}{2}\left(\varphi_{\mu\nu}\,\varphi^{\mu\nu} - \varphi^2\right), \tag{6.73}$$

and the EOM that follow are

$$F^{\alpha}{}_{(\mu\nu),\alpha} - m^2\,\left(\varphi_{\mu\nu} - \varphi\,\eta_{\mu\nu}\right) = 0, \tag{6.74}$$

or equivalently,

$$G^{(L)}{}_{\mu\nu} + \frac{m^2}{2}\,\left(\varphi_{\mu\nu} - \varphi\,\eta_{\mu\nu}\right) = 0.$$

The trace of this equation gives

$$F^{\alpha}{}_{,\alpha} + \frac{3}{2}\,m^2\,\varphi = 0, \tag{6.75}$$

while the divergence of Eqn.(6.74) yields

$$F_{\mu} = 0. \tag{6.76}$$

This result together with the trace equation gives $\varphi = 0$.

In terms of the potential, Eqn.(6.76) is equivalent to

$$\varphi_{,\mu} - \varphi^{\epsilon}{}_{\mu,\epsilon} = 0. \tag{6.77}$$

It follows that we must have

$$\varphi^{\mu\nu}{}_{,\nu} = 0.$$

Thus we have shown that the original ten degrees of freedom (DOF) of $F_{\alpha\beta\mu}$ have been reduced to five (which is the correct number for a massive spin-2 field) by means of the five constraints

$$\varphi^{\mu\nu}{}_{,\nu} = 0, \qquad \varphi = 0. \tag{6.78}$$

### Equation of spin-2 in curved background

The passage of the spin-2 field equation from Minkowski spacetime to arbitrary curved riemannian manifold presents ambiguities due to the presence of second order derivatives of the rank two symmetric tensor $\varphi_{\mu\nu}$ that is used in the so called Einstein-frame (see for instance Aragone-Deser). These ambiguities disappear when we pass to the Fierz frame representation that deals with the three index tensor $F_{\alpha\mu\nu}$ as it was shown in Novello-Neves.

There results a unique form of minimal coupling, free of ambiguities. Let us define from $\varphi_{\mu\nu}$ two auxiliary fields ${G^{(I)}}_{\mu\nu}$ and ${G^{(II)}}_{\mu\nu}$ through the expressions:

$$
\begin{aligned}
&2\,{G^{(I)}}_{\mu\nu} \equiv \\
&\Box\,\varphi_{\mu\nu} - \varphi_{\epsilon(\mu;\nu)}{}^{;\epsilon} + \varphi_{;\mu\nu} - \eta_{\mu\nu}\,\left(\Box\varphi - \varphi^{\alpha\beta}{}_{;\alpha\beta}\right), \qquad (6.79)\\
&2\,{G^{(II)}}_{\mu\nu} \equiv \\
&\Box\,\varphi_{\mu\nu} - \varphi_{\epsilon(\mu}{}^{;\epsilon}{}_{;\nu)} + \varphi_{;\mu\nu} - \eta_{\mu\nu}\,\left(\Box\varphi - \varphi^{\alpha\beta}{}_{;\alpha\beta}\right). \qquad (6.80)
\end{aligned}
$$

These objects differ only in the order of the second derivative in the second term on the r.h.s. of the above equations. The equation of motion free of ambiguities concerns the tensor field

$$
\widehat{G}_{\mu\nu} \equiv \frac{1}{2}\,\left({G^{(I)}}_{\mu\nu} + {G^{(II)}}_{\mu\nu}\right) \qquad (6.81)
$$

and is given by

$$
\widehat{G}_{\mu\nu} + \frac{1}{2}\,m^2\,(\varphi_{\mu\nu} - \varphi g_{\mu\nu}) = 0. \qquad (6.82)
$$

which is precisely the usual equations for massive spin-2 field.

**Generating mass for the spin-2 field**

We follow the same strategy as in the previous case and take the dynamics of the spin-2 field as given by

$$
\begin{aligned}
\mathbb{L} \;&=\; F_{\alpha\mu\nu}\,F^{\alpha\mu\nu} - F_\alpha\,F^\alpha + \frac{1}{\kappa}\,R \\
&+\; a\,R_{\alpha\mu\beta\nu}\,\varphi^{\alpha\beta}\,\varphi^{\mu\nu} \\
&-\; \frac{\Lambda}{\kappa} \qquad (6.83)
\end{aligned}
$$

The equations of motion are given by:

$$
F^\alpha{}_{(\mu\nu);\alpha} + 2a\,R_{\alpha\mu\beta\nu}\,\varphi^{\alpha\beta} = 0, \qquad (6.84)
$$

$$
\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} + \frac{\Lambda}{2}\,g_{\mu\nu}\right) + T_{\mu\nu} + a\,Y_{\mu\nu} = 0 \qquad (6.85)
$$

where the quantity $Y_{\mu\nu}$ is given by the variation of the non minimal coupling term:

$$
\delta\int\sqrt{-g}\,R_{\alpha\mu\beta\nu}\,\varphi^{\alpha\beta}\,\varphi^{\mu\nu} = \int\sqrt{-g}\,Y_{\mu\nu}\,\delta g_{\mu\nu} \qquad (6.86)
$$

where $Y_{\mu\nu}$ is given in terms of $S_{\alpha\mu\beta\nu}$ defined as

$$
S_{\alpha\mu\beta\nu} \equiv \varphi_{\alpha\beta}\,\varphi_{\mu\nu} - \varphi_{\alpha\nu}\,\varphi_{\beta\mu}
$$

which has the symmetries:

$$S_{\alpha\mu\beta\nu} = -\, S_{\alpha\mu\nu\beta} = -\, S_{\mu\alpha\beta\nu} = S_{\beta\nu\alpha\mu}.$$

A direct calculation yields

$$\begin{aligned} Y^{\mu\nu} &\equiv S^{\lambda\mu\nu\epsilon}{}_{;\epsilon;\lambda} - \frac{1}{2}\, R_{\alpha\sigma\beta\lambda}\, \varphi^{\alpha\beta}\, \varphi^{\sigma\lambda}\, g^{\mu\nu} \\ &+ \frac{3}{2}\, R_{\alpha\sigma\beta}{}^{(\mu}\, \varphi^{\nu)\sigma}\, \varphi^{\alpha\beta} \\ &- \frac{1}{2}\, R_{\alpha\sigma\beta\lambda}\, \varphi^{\alpha\beta}\, \varphi^{\sigma\lambda}\, g_{\mu\nu} \end{aligned} \tag{6.87}$$

Let us remind that the Riemann curvature can be written in terms of its irreducible quantities involving the Weyl conformal tensor $W_{\alpha\sigma\beta\lambda}$ and the contracted Ricci tensor by the formula:

$$\begin{aligned} R_{\alpha\mu\beta\nu} &= W_{\alpha\mu\beta\nu} \\ &+ \frac{1}{2}\, (R_{\alpha\beta}\, g_{\mu\nu} + R_{\mu\nu}\, g_{\alpha\beta} - R_{\alpha\nu}\, g_{\beta\mu} - R_{\beta\mu}\, g_{\alpha\nu}) \\ &- \frac{1}{6}\, R\, g_{\alpha\mu\beta\nu}. \end{aligned} \tag{6.88}$$

Then

$$\begin{aligned} R_{\alpha\mu\beta\nu}\, \varphi^{\alpha\beta}\, \varphi^{\mu\nu} &= W_{\alpha\mu\beta\nu}\, \varphi^{\alpha\beta}\, \varphi^{\mu\nu} + (R_{\alpha\beta} \\ &- \frac{1}{6}\, R\, g_{\alpha\beta})\, (\varphi\, \varphi^{\alpha\beta} - \varphi^{\alpha}{}_{\lambda}\, \varphi^{\lambda\beta}). \end{aligned} \tag{6.89}$$

We can then re-write the equation of the spin-2 field as

$$F^{\alpha}{}_{(\mu\nu);\alpha} - \frac{a\,\Lambda}{3}\, (\varphi_{\mu\nu} - \varphi\, g_{\mu\nu}) + 2a\, W_{\alpha\mu\beta\nu}\, \varphi^{\alpha\beta} + Q_{\mu\nu} = 0, \tag{6.90}$$

where $Q_{\mu\nu}$ contain non-linear terms of interaction of the spin-2 field with gravity.

## 6.7. Generalized Mach's principle

In this section we present an extension of Mach principle in similar lines as it has been suggested by Dirac, Hoyle and others. This generalization aims to produce a mechanism that transforms the vague idea according to which local properties may depend on the universe's global characteristics into an efficient process. We will apply the strategy that we used in the precedent sections to generate mass in order to elaborate such generalization.

**The cosmological influence on the microphysical world: the case of chiral-invariant Heisenberg-Nambu-Jona-Lasinio dynamics**

There have been many discussions in the scientific literature in the last decades related to the cosmic dependence of the fundamental interactions. The most popular one was the suggestion of Dirac – the so called Large Number Hypothesis – that was converted by Dicke and Brans into a new theory of gravitation, named the scalar-tensor theory. We will do not analyze any of these here. On the contrary, we will concentrate on a specific self-interaction of an elementary field and show that its correspondent dynamics is a consequence of a dynamical cosmological process. That is, to show that dynamics of elementary fields in the realm of microphysics, may depend on the global structure of the universe.

The first question we have to face concerns the choice of the elementary process. There is no better way than start our analysis with the fundamental theory proposed by Nambu and Jona-Lasinio concerning a dynamical model of elementary particles [21]. Since the original paper until to-day hundreds of papers devoted to the NJL model were published [20]. For our purpose here it is enough to analyze the nonlinear equation of motion that they used in their original paper as the basis of their theory which is given by

$$i\gamma^{\mu}\nabla_{\mu}\Psi - 2s(A + i\,B\,\gamma^{5})\Psi = 0$$

This equation, as remarked by these authors, was proposed earlier by Heisenberg [1] although in a quite different context. We will not enter in the analysis of the theory that follows from this dynamics. Our question here is just this: is it possible to produce a model such that HNJL (Heisenberg-Nambu-Jona-Lasinio) equation for spinor field becomes a consequence of the gravitational interaction of a free massless Dirac field with the rest-of-the-universe? We shall see that the answer is yes.

We used Mach's principle as the statement according to which the inertial properties of a body $\mathbb{A}$ are determined by the energy-momentum throughout all space. We follow here a similar procedure and will understand the Extended Mach Principle as the idea which states that the influence of the rest-of-the-universe on microphysics can be described through the action of the energy-momentum distribution identified with the cosmic form

$$T^{U}_{\mu\nu} = \Lambda\, g_{\mu\nu}$$

### Non minimal coupling with gravity

In the framework of General Relativity we set the dynamics of a fermion field $\Psi$ coupled non-minimally with gravity to be given by the Lagrangian (we are using units were $\hbar = c = 1$)

$$L = L_D + \frac{1}{\kappa}\,R + V(X)\,R - \frac{1}{\kappa}\,\Lambda + L_{CT} \tag{6.91}$$

where

$$L_D \equiv \frac{i}{2}\bar{\Psi}\gamma^{\mu}\nabla_{\mu}\Psi - \frac{i}{2}\nabla_{\mu}\bar{\Psi}\gamma^{\mu}\Psi \tag{6.92}$$

The non-minimal coupling of the spinor field with gravity is contained in the term $V(X)$ and depends on the scalar $X$ defined by

$$X = A^2 + B^2$$

where $A = \bar{\Psi}\,\Psi$ and $B = i\bar{\Psi}\,\gamma^5\,\Psi$. We note that we can write, in an equivalent way,

$$X = J_\mu\,J^\mu$$

where $J^\mu = \bar{\Psi}\gamma^\mu\Psi$. This quantity $X$ is chiral invariant, once it is invariant under the map

$$\Psi' = \gamma^5\,\Psi.$$

Indeed, from this $\gamma^5$ transformation, it follows

$$A' = -\,A,\;\; B' = -\,B;\;\; then, X' = X.$$

The case in which the theory breaks chiral invariance and the interacting term $V$ depends only on the invariant $A$ – is the road to the appearance of a mass as we saw in the previous sections [320]. Here we start from the beginning with a chiral invariant theory. For the time being the dependence of $V$ on $X$ is not fixed. We have added $L_{CT}$ to counter-balance the terms of the form $\partial_\lambda X\,\partial^\lambda X$ and $\Box X$ that appear due to the gravitational interaction. The most general form of this counter-term is

$$L_{CT} = H(X)\,\partial_\mu X\,\partial^\mu X \tag{6.93}$$

We shall see that $H$ depends on $V$ and if we set $V = 0$ then $H$ vanishes. This dynamics represents a massless spinor field coupled non-minimally with gravity. The cosmological constant represents the influence of the rest-of-the-universe on $\Psi$.

Independent variation of $\Psi$ and $g_{\mu\nu}$ yields

$$i\gamma^\mu\nabla_\mu\,\Psi + \Omega\,(A + i\,B\,\gamma^5)\Psi = 0 \tag{6.94}$$

where

$$\Omega \equiv 2R\,V' - 2H'\,\partial_\mu X\,\partial^\mu X - 4H\Box X$$

$$\alpha_0\,(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) = -T_{\mu\nu} \tag{6.95}$$

where we set $\alpha_0 \equiv 2/\kappa$ and $V' \equiv \partial V/\partial X$. The energy-momentum tensor is given by

$$\begin{aligned} T_{\mu\nu} &= \frac{i}{4}\,\bar{\Psi}\gamma_{(\mu}\nabla_{\nu)}\Psi - \frac{i}{4}\nabla_{(\mu}\bar{\Psi}\gamma_{\mu)}\Psi \\ &+ 2V(R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}) + 2\nabla_\mu\nabla_\nu V - 2\Box V\,g_{\mu\nu} \\ &+ 2H\,\partial_\mu X\,\partial_\nu X - H\,\partial_\lambda X\,\partial^\lambda X\,g_{\mu\nu} + \frac{\alpha_0}{2}\,\Lambda\,g_{\mu\nu} \end{aligned} \tag{6.96}$$

Taking the trace of equation (6.95), after some simplification and using

$$\Box V = V' \Box X + V'' \partial_\mu X \, \partial^\mu X \tag{6.97}$$

it follows

$$\begin{aligned} (\alpha_0 + 2V + 2\,V'\,X)\,R &= (4HX - 6V')\Box X \\ &+ (2H'\,X - 6V'' - 2H)\,\partial_\alpha X\,\partial^\alpha X \\ &+ 2\,\alpha_0\,\Lambda \end{aligned} \tag{6.98}$$

Then

$$\begin{aligned} \Omega &= (\mathbb{M}\,\Box X + \mathbb{N}\,\partial_\mu X\,\partial^\mu X) \\ &+ \frac{4\,\alpha_0\,\Lambda\,V'}{\alpha_0 + 2V + 2\,V'\,X} \end{aligned} \tag{6.99}$$

where

$$\mathbb{M} = \frac{2V'(4HX - 6V')}{\alpha_0 + 2V + 2\,V'\,X} - 4\,H$$

$$\mathbb{N} = \frac{2V'\,(2\,X\,H' - 6V'' - 2\,H)}{\alpha_0 + 2V + 2\,V'\,X} - 2\,H'$$

Defining $\Delta \equiv \alpha_0 + 2V + 2\,V'\,X$ we re-write $\mathbb{M}$ and $\mathbb{N}$ as

$$\mathbb{M} = -\frac{4}{\Delta}\,\left(3\,V'^2 + H\,(\alpha_0 + 2V)\right)$$

$$\mathbb{N} = -\frac{2}{\Delta}\,\left(3\,V'^2 + H\,(\alpha_0 + 2V)\right)'$$

Inserting this result on the equation (6.94) yields

$$i\gamma^\mu \nabla_\mu\,\Psi + \left(\mathbb{M}\,\Box X + \mathbb{N}\,\partial_\lambda X\,\partial^\lambda X\right)\,\Psi + \mathbb{Z}\,(A + i\,B\,\gamma^5)\Psi = 0 \tag{6.100}$$

where

$$\mathbb{Z} = \frac{4\,\alpha_0\,\Lambda\,V'}{\Delta}$$

At this stage it is worth to select among all possible candidates of $V$ and $H$ particular ones that makes the factor on the gradient and on $\Box$ of the field to disappear from equation (6.100).

The simplest way is to set $\mathbb{M} = \mathbb{N} = 0$, which is satisfied if

$$H = -\frac{3\,V'^2}{\alpha_0 + 2V}$$

Imposing that $\mathbb{Z}$ must reduce to a constant we obtain

$$V = \frac{1}{\kappa}\,\left[\frac{1}{1 + \beta\,X} - 1\right]. \tag{6.101}$$

As a consequence of this,

$$H = -\frac{3\,\beta^2}{2\kappa}\,\frac{1}{(1+\beta\,X)^3} \tag{6.102}$$

where $\beta$ is a constant. Using equations 6.100) and 6.101) the equation for the spinor becomes

$$i\gamma^\mu \nabla_\mu\,\Psi - 2s(A + i\,B\,\gamma^5)\Psi = 0 \tag{6.103}$$

where

$$s = \frac{2\,\beta\,\Lambda}{\kappa(\hbar\,c)}. \tag{6.104}$$

Thus as a result of the gravitational interaction the spinor field satisfies Heisenberg-Nambu-Jona-Lasinio equation of motion. This is possible due to the influence of the rest-of-the-Universe on $\Psi$. If $\Lambda$ vanishes then the constant of the self-interaction of $\Psi$ vanishes.

The final form of the Lagrangian is provided by

$$L = L_D + \frac{1}{\kappa\,(1+\beta X)}\,R - \frac{1}{\kappa}\,\Lambda - \frac{3\beta^2}{2\kappa}\,\frac{1}{(1+\beta X)^3}\,\partial_\mu X\,\partial^\mu X \tag{6.105}$$

In this section we analyzed the influence of all the material content of the universe on a fermionic field when this content is in two possible states: in one case its energy distribution is zero; in another case it is in a vacuum state represented by the homogeneous distribution $T_{\mu\nu} = \Lambda g_{\mu\nu}$. Note that when $\Lambda$ vanishes, the dynamics of the field is independent of the global properties of the universe and it reduces to the massless Dirac equation

$$i\gamma^\mu \nabla_\mu\,\Psi = 0$$

In the second case, the rest-of-the-universe induces on field $\Psi$ the Heisenberg-Nambu-Jona-Lasinio non-linear dynamics

$$i\gamma^\mu \nabla_\mu\,\Psi - 2s\,(A + iB\gamma^5)\,\Psi = 0.$$

Such scenario shows a mechanism by means of which the rules of the micro-physical world depends on the global structure of the universe. It is not hard to envisage others situations in which the above mechanism can be further applied.

## 6.8. Appendix: Vacuum state in non-linear theories

Although the cosmological constant was postulated from first principles, quantum field theory gave a simple interpretation of $\Lambda$ by its association to the fundamental vacuum state. It is possible to describe its origin even classically as a consequence of certain special states of matter. For instance, non linear theories

produce classically a vacuum, defined by its distribution of energy-momentum tensor provided by expression (6.1). Let us review very briefly how this occurs in a specific example. We start by the standard definition of the symmetric energy-momentum tensor as variation of the Lagrangian induced by variation of the metric tensor, that is

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta L \sqrt{-g}}{\delta g^{\mu\nu}} \tag{6.106}$$

In order to present a specific example, let us concentrate on the case of electromagnetic field in which the Lagrangian depends only on the invariant $F$ defined by

$$F \equiv F_{\mu\nu} F^{\mu\nu}$$

Then, the expression of the energy-momentum tensor is given by

$$T_{\mu\nu} = -4\, L_F\, F_{\mu}{}^{\alpha}\, F_{\alpha\nu} - L\, g_{\mu\nu}. \tag{6.107}$$

where $L_F = \partial L / \partial F$ represents the derivative of the Lagrangian with respect to the invariant $F$. The corresponding equation of motion of the field is provided by

$$(L_F\, F^{\mu\nu})_{;\,\nu} = 0. \tag{6.108}$$

where the symbol ; represents covariant derivative. This equation admits a particular solution when $L_F$ vanishes for non-null constant value $F_0$ . When the system is in this state, the corresponding expression of the energy-momentum tensor reduces to

$$T_{\mu\nu} = \Lambda\, g_{\mu\nu}$$

where

$$\Lambda = L_0.$$

The consequences of this state in Cosmology due to non linear theories of Electrodynamics was revisited recently (see [12]). A by-product is the emergence of effective geometries that mimics gravitational processes like, for instance, non-gravitational black holes or analogue expanding universes in laboratory.

## Acknowledgements

The major part of this work was done when visiting the International Center for Relativistic Astrophysics (ICRANet) at Pescara (Italy). I would extend this acknowledgement to its efficient administration. I would also thank my colleagues of ICRA-Brasil and particularly Dr J.M. Salim with whom I exchanged many discussions. This work was partially supported by *Conselho Nacional de Desenvolvimento Científico e Tecnológico* (CNPq), FINEP and *Fundação de Amparo à Pesquisa do Estado de Rio de Janeiro* (FAPERJ) of Brazil.

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[11] Let us remark that it is possible to understand Mach principle in a broad sense. Indeed, for the method of obtaining mass using the gravity mechanism, the notion of Totality (or rest-of-the-universe) admits two alternative interpretations. This is related to the fact that when we deal with the vacuum represented by the distribution of energy by $T_{\mu\nu} = \lambda\, g_{\mu\nu}$ it is completely irrelevant – for the gravitational mechanism of providing mass — if parameter $\lambda$ has a classical global origin (the Universe) identified with the cosmological constant introduced by Einstein; or a local quantum one (the environment) identified with the vacuum of quantum fields.

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[14] At this point one should note that the expression rest-of-the-universe may not have a strict cosmological meaning but instead is a short term to represent the whole background of matter that really affects $\mathbb{A}$.

[15] However this is not mandatory. The term rest-of-the-universe concerns the environment of $\mathbb{A}$, that is the whole domain of influence on $\mathbb{A}$ of the remaining bodies in the universe.

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# 7. On a Geometrical Description of Quantum Mechanics

We show that Quantum Mechanics can be interpreted as a modification of the Euclidean nature of 3-d space into a particular affine space, which we call Q-wis. This is proved using the Bohm-de Broglie causal formulation of Quantum Mechanics. In the Q-wis geometry, the length of extended objects changes from point to point. In this formulation, deformation of physical distances are in the core of quantum effects allowing a geometrical formulation of the uncertainty principle.

## 7.1. Introduction

The early years of quantum mechanics were marked by intense debates and controversies related to the meaning of the new behaviour of matter. While one group was convinced that was unavoidable to abandon the classical picture, the other group tried incessantly to save its main roots and conceptual pillars. To be able to reproduce the atypical quantum effects, the latter group was forced to introduce new ingredients such as de Broglie's pilot wave [1]-[6] or Mandelung's hydrodynamical picture [7].

However, the lack of physical explanations for these ad hoc modifications weakened these pictures. At the same time, the former group leaded by Schrödinger, Bohr and Heisenberg was increasingly gaining new adepts until its climax in the 1927 Solvay's conference when this picture was finally accepted as the orthodox interpretation of quantum mechanics - the Copenhagen interpretation [8]-[10].

Notwithstanding, a marginal group of physicists continued to develop other approaches [11]-[13] to describe quantum mechanics that are more adequate to connect to a classical picture[1]. One of the most prominent amongst these alternative interpretations is the causal interpretation of quantum mechanics also known as Bohm-de Broglie interpretation [14]-[17].

The development of quantum cosmological scenario brought to light some difficulties intrinsic to the Copenhagen interpretation. More specifically, the measurement process in a quantum closed universe seems inevitably inconsistent [341]-[20]. Fortunately, there are some alternative interpretations that are consistently applied simultaneously to cosmology and to the micro-world. As two examples we mention the many-worlds interpretation [341, 21, 22] and the consistent histories formulation [23]-[26].

In the present work we will focus only on the Bohm-de Broglie interpretation since it is amongst the well-defined interpretation that can be applied to any kind of system, and up to date it is completely equivalent to the Copenhagen interpretation when applied to the micro-world.

We will show that it is possible to interpret all quantum phenomena as a modification of the geometrical properties of the physical space. Hence, we will deal with a generalization of the Euclidean geometry. Several papers in the literature have advocated a possible connection between non-Euclidean geometry and quantum effects [2]-[35]. Despite the different approaches, a common feature of all these works is the proposal of a new geometry that was first introduced by Weyl [27]-[28]. The so-called Weyl geometry is a modification of a Riemannian space-time to accommodate the conformal map as a pure gauge transformation. As it is well known, this can be achieved only if not just the metric but all fields are also conformally transformed. In addition, one has to redefine the covariant derivative to maintain the Riemannian structure of the space-time.

[1]Since we are not concerned with relativistic phenomena, the term classical physics should be understood as pre-relativistic physics unless otherwise specified.

Instead of working with this Weyl geometry, we shall define a different space where the connection cannot be specified solely in terms of the metric and which we shall call Q-wis [2]. There are at least two main differences. The Q-wis is not conformally invariant nor there is any kind of geometrical gauge degree of freedom. Furthermore, the covariant derivative is strictly defined only with the connection so that the covariant derivative of the metric does not vanish. Contrarily to the Weyl geometry, the affine degree of freedom of the Q-wis space raises physical implication that allows us to re-interpret the quantum effects. In fact, we shall propose a physical description of quantum effects through a limitation of the classical standards, or in others words, through a limitation of the Euclidean standards used to measure physical distances.

The appendix is reserved to develop in more detail the properties of a Q-wis space but it is worth to mention its main difference from an Euclidean space that is related to the notion of a standard ruler.

A Q-wis is a geometrical space endowed with an euclidian metric. Furthermore, this space also posses an extra degree of freedom, which allows the length of a vector to change from point to point. This means that a ruler of length $l$ if parallel transported will change by an amount

$$\delta\, l = l\, f_{,a}\, \mathrm{d}x^a \quad . \tag{7.1}$$

A Q-wis is distinguished precisely by the fact that the length of the ruler transported along a closed curve does not change. Hence, if the change of the ruler's length is $\mathrm{d}l$, for a closed path in Q-wis we have

$$\oint \mathrm{d}l = 0 \qquad , \tag{7.2}$$

which guarantees the uniqueness of any local measurement. The allowance of an intrinsic modification of the standard rulers is the main geometrical hypothesis of the present work. We shall argue how this geometrical modification can be in the origin of quantum effects. For the sake of clarity we will deal with the simplest system possible, namely an isolated point-like particle possibly subjected to an external potential.

The outline of the article is as follows. In the next section we briefly review the main points of quantum mechanics and in section *Non-Euclidean geometry* we describe how to connect the Q-wis space to the quantum theory. We show that quantum mechanics can be derived from a geometrical variational principle. Then, in the conclusions we present our final remarks. The appendix is reserved to describe the main properties of the Q-wis geometry.

[2] This name is an acronym for quantum weyl integrable space that is motivated from the fact that it is similar to a Weyl integrable space and at the same time describes quantum phenomena. Its mathematical properties are analysed in the appendix.

## 7.2. Quantum Mechanics

Quantum mechanics is a modification of the classical laws of physics to incorporate the uncontrolled disturbance caused by the macroscopic apparatus necessary to realize any kind of measurement. This statement, known as Bohr's complementary principle, contains the main idea of the Copenhagen interpretation of quantum mechanics. The quantization program continues with the correspondence principle promoting the classical variables into operators and the Poisson brackets into commutation relations.

In this non-relativistic scenario, the Schrödinger equation establishes the dynamics for the wave function describing the system. Note that as in Newtonian mechanics time is only a external parameter and the 3-d space is assumed to be endowed with the Euclidean geometry.

Using the polar form for the wave function, $\Psi = A\, e^{iS/\hbar}$, the Schrödinger equation can be decomposed in two equations for the real functions $A\,(x)$ and $S\,(x)$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S.\nabla S + V - \frac{\hbar^2}{2m}\frac{\nabla^2 A}{A} = 0 \quad , \tag{7.3}$$

$$\frac{\partial A^2}{\partial t} + \nabla\left(A^2\frac{\nabla S}{m}\right) = 0 \qquad . \tag{7.4}$$

Solving these two equations is completely analogous to solving the Schrödinger equation. The probabilistic interpretation of quantum mechanics associate $\|\Psi\|^2 = A^2$ with the probability distribution function on configuration space. Hence, eq. (D.7) has exactly the form of a continuity equation with $A^2\nabla S/m$ playing the role of current density.

### 7.2.1. Bohm-de Broglie interpretation

The causal interpretation, which is an ontological hidden variable formulation of quantum mechanics, propose that the wave function does not contain all the information about the system.

An isolated system describing a free particle (or a particle subjected to a potential $V$) is defined simultaneously by a wave function and a point-like particle. In this case, the wave function still satisfies the Schrödinger equation but it should also works as a guiding wave modifying the particle's trajectory.

Note that eq.(7.3) is a Hamilton-Jacobi like equation with an extra term that it is often called quantum potential

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A} \quad , \tag{7.5}$$

while, as already mentioned, eq.(D.7) is a continuity-like equation. The Bohm-de Broglie interpretation takes these analogies seriously and postulate an extra

equation associating the velocity of the point-like particle with the gradient of the phase of the wave function. Hence,

$$\dot{x} = \frac{1}{m}\nabla S \qquad . \tag{7.6}$$

Integrating eq. (7.6) yields the quantum bohmian trajectories. The unknown or hidden variables are the initial positions necessary to fix the constant of integration of the above equation.

For the case of a spinless particle, the quantum potential is the sole responsible for all novelties of quantum effects such as non-locality or tunnelling processes. As a matter of fact, the Bohm-de Broglie interpretation has the theoretical advantage of having a well formulated classical limit. Classical behaviour is obtained as soon as the quantum potential, which has dimensions of energy, becomes negligible compared to other energy scales of the system.

In what follows, we will show that it is possible to reinterpret quantum mechanics as a manifestation of non-Euclidean structure of the 3-dimensional space. Hence, we propose a geometrical interpretation to describe quantum effects.

## 7.3. Non-Euclidean geometry

Since ancient times, Euclidean geometry was considered as the most adequate mathematical formulation to describe the physical space. However, its validity can only be established a posteriori as long as its construction yields useful notions to connect physical quantities such as the Euclidean distance between two given points.

Special relativity modified the notion of 3-dimensional Euclidean space to incorporate time in a four-dimensional continuum (Minkowski spacetime). Later on, General Relativity generalized the absolute Minkowski spacetime to describe gravitational phenomena. General Relativity considers the spacetime manifold as a dynamical field that can be deformed and stretched but in such a way that it always preserves its Riemannian structure. It is worth noting that both the Euclidean and Minkowskian spaces are nothing more than special cases of Riemannian spaces.

Nonetheless, Riemannian manifold are not the most general type of geometrical spaces. In the same way as above, Riemannian geometries can be understood as a special subclass of a more general structure where the connection it is not uniquely determined by the metric. Geometries where the connection is not just the Christoffel's symbol are known as affine space. As to the matter of which geometry is actually realized in Nature, it has to be determined by physical experiments.

Instead of imposing a priori that quantum mechanics has to be constructed over an Euclidean background as it is traditionally done, we shall argue that quantum effects can be interpreted as a manifestation of a non-Euclidean structure derived from a variational principle. The validity of the specific geometrical

structure proposed can be checked a posteriori comparing it to the usual non-relativistic quantum mechanics.

Thus, consider a point-like particle with velocity $v = \nabla S/m$ and subjected to a potential $V$. We shall follow Einstein's idea to derive the geometrical structure of space from a variational principle by considering the connection as an independent variable and hence by using Palatini's variational procedure. The validity of an action principle can only be justified a posteriori by deriving the correct dynamical equation of motion but normally its formulation already specifies the kinematical properties of the theory. In particular, we consider an action that includes geometry and the particle's lagrangian with the peculiarity that the particle is non-minimally coupled to geometry through a scalar field $\Omega$ which we shall show to be related to the affine structure of 3-d space. Thus we define the action by

$$I = \int \mathrm{d}t\mathrm{d}^3x\sqrt{g}\left[\lambda^2\,\Omega^2\mathcal{R} - \Omega^2\mathcal{L}_m\right] \quad , \tag{7.7}$$

where the connection of the 3-d space $\Gamma^i_{jk}$, the Hamilton's principal function $S$ and the scalar function $\Omega$ should be understood as independent variables. Each term in equation (7.7) is defined as follows: we are considering the line element in Cartesian coordinates given by

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{7.8}$$

with

$$g = \det g_{ij} \quad . \tag{7.9}$$

The Ricci curvature tensor is defined in term of the connection through

$$\mathcal{R}_{ij} = \Gamma^m_{mi\,,j} - \Gamma^m_{ij\,,m} + \Gamma^l_{mi}\Gamma^m_{jl} - \Gamma^l_{ij}\Gamma^m_{lm} \tag{7.10}$$

and its trace defines the curvature scalar $\mathcal{R} \equiv g^{ij}\mathcal{R}_{ij}$ which has dimensions of inverse length squared, $[\mathcal{R}] = L^{-2}$. The constant $\lambda^2$ has dimension of energy times length squared, $[\lambda^2] = E.L^2$, and the particle's Lagrangian is defined by the Hamilton's function through

$$\mathcal{L}_m = \frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S.\nabla S + V \quad , \tag{7.11}$$

where $\frac{\partial S}{\partial t}$ is related to the particle's total energy.

From equation (7.7), variation of the action $I$ with respect to the independent variables gives respectively (see appendix for details)

$$\delta\Gamma^i_{jk}: \qquad g_{ij;k} = -4\left(\ln\Omega\right)_{,k}\, g_{ij} \qquad , \tag{7.12}$$

where ";" denotes covariant derivative and a common "," simple spatial derivative. Equation (8.15) characterizes the affine properties of the physical space.

Hence, the variational principle naturally defines a Q-wis space. Variation with respect to $\Omega$ gives

$$\delta\Omega : \qquad \lambda^2\mathcal{R} = \frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S.\nabla S + V \quad . \tag{7.13}$$

The right-hand side of this equation has dimension of energy while the curvature scalar has dimension of $[\mathcal{R}] = L^{-2}$. Furthermore, apart from the particle's energy, the only extra parameter of the system is the particle's mass $m$. Thus, there is only one-way to combine the unknown constant $\lambda^2$, which has dimension of $[\lambda^2] = E.L^2$, with the particle's mass such as to form a physical quantity. Multiplying them, we find a quantity that has dimension of angular momentum squared $[m.\lambda^2] = \hbar^2$.

In terms of the scalar function $\Omega$, the curvature scalar is given by (see appendix)

$$\mathcal{R} = 8\frac{\nabla^2\Omega}{\Omega} \qquad . \tag{7.14}$$

Hence, setting $\lambda^2 = \hbar^2/16m$, equation (7.13) becomes

$$\delta\Omega : \qquad \frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S.\nabla S + V - \frac{\hbar^2}{2m}\frac{\nabla^2\Omega}{\Omega} = 0 \quad , \tag{7.15}$$

Finally, varying the Hamilton's principal function $S$ we find

$$\delta S : \qquad \frac{\partial\Omega^2}{\partial t} + \nabla\left(\Omega^2\frac{\nabla S}{m}\right) = 0 \qquad . \tag{7.16}$$

Equations (7.15) and (7.16) are identical to equations (7.3) and (D.7) if we identify $\Omega = A$. Thus, the "action" of a point-like particle non-minimally coupled to geometry is given by

$$I = \int \mathrm{d}t\mathrm{d}^3x\sqrt{g}\,\Omega^2\left[\frac{\hbar^2}{16\,m}\mathcal{R} - \left(\frac{\partial S}{\partial t} - +\frac{1}{2m}\nabla S.\nabla S + V\right)\right]$$

exactly reproduce the Schrödinger equation and thus the quantum behaviour. The straightest way to compare this geometrical approach to the common quantum theories is to relate it to the Bohm-de Broglie interpretation[3]. Note that this formulation has the advantage of giving a physical explanation of the appearance of the quantum potential, eq. (7.5). In a Q-wis, this term is simply its curvature scalar. The inverse square root of the curvature scalar defines a typical length $L_w$ (Weyl length) that can be used to evaluate the strength of quantum effects

$$L_w \equiv \frac{1}{\sqrt{\mathcal{R}_w}} \quad . \tag{7.17}$$

[3]Up to date, all interpretation of quantum mechanics are on equal footing. Thus, establishing the connection with the causal interpretation automatically links this geometrical interpretation with all others.

As we have already mentioned, the classical limit of Bohm-de Broglie interpretation is achieved when the quantum potential is negligible compared to other energy scales of the system. In the scope of this geometrical approach, the classical behaviour is recovered when the length defined by the Q-wis curvature scalar is small compared to the typical length scale of the system. Once the Q-wis curvature becomes non-negligible the system goes into a quantum regime.

### 7.3.1. Geometrical uncertainty principle

As long as we accept that quantum mechanics is a manifestation of a non-Euclidean geometry, we are faced with the need of reinterpreting geometrically all theoretical issues related to quantum effects. As a first step, we associate the uncertainty principle to a break down of the classical notion of a standard ruler.

In Euclidian space, there is a clear notion of distance between two points. Generalizing to curved spaces, it is still possible to define distance as the smallest length between two given points calculated along geodesics in 3-d space. This is a consistent definition since the 3-d space has a true metric in the mathematical sense that its eigenvalues are all positives. However, this definition does not encompass the classical notion of a standard ruler since standard rulers are based on euclidian space. This means that if quantum mechanics can be interpreted as a modification of euclidian space it shall have a limit of validity for the notion of euclidian distance. In other words, it should not be possible to perform a classical measurement of distances smaller than a given value that of course should be related to the curvature scale of the space, i.e. the Q-wis curvature length. Thus, we propose that any measurement can only measure distances bigger than the Weyl length

$$\Delta L \geq L_w = \frac{1}{\sqrt{\mathcal{R}_w}} \quad . \tag{7.18}$$

The quantum regime is extreme when the Q-wis curvature term dominates. Thus, from equations (7.14) and (7.15) we have

$$\mathcal{R}_w = 2\left(\frac{2\Delta p}{\hbar}\right)^2 - \frac{16m}{\hbar^2}\left(E - V\right) \leq 2\left(\frac{2\Delta p}{\hbar}\right)^2 \tag{7.19}$$

and finally combining equations (7.18) and (7.19) we obtain

$$\Delta L.\Delta p \geq \frac{\hbar}{2\sqrt{2}} \qquad . \tag{7.20}$$

We should emphasize that now the Heisenberg's uncertainty relation has a pure geometrical meaning. Our argument closely resembles Bohr's complementary principle inasmuch as the impossibility of applying the classical definitions

of measurements. However, we strongly diverge with respect to the fundamental origin of the physical limitation.

Bohr's complementary principle is based on the uncontrolled interference of a classical apparatus of measurement. On the other hand, we argue that the notion of a classical standard ruler breaks down because its meaning is intrinsically dependent on the validity of Euclidian geometry. Once it becomes necessary to include the Q-wis curvature, we are no longer able to perform a classical measurement of distance.

There is another way to interpret the uncertainty principle. First, recall that in scattering processes one can defines the classical electron radius as $r_e \equiv e^2/mc^2$ since it has dimension of length and it is the classical radius for which the electrostatic self-energy is equal to the electron mass. Furthermore, the Thomson cross section is approximately the area defined with this classical radius, $\sigma_T \approx 4\pi r_e^2$.

Now, we shall construct a similar notion for our quantum system. For a given particle of mass $m$ and energy $E$ there is only one combination with the free parameter of the theory $\lambda$ which has dimension of length, i.e. $\lambda/\sqrt{E}$. We take this value as a definition of the classical size of the particle, namely

$$l_{part} \equiv \frac{\lambda}{\sqrt{E}} = \sqrt{\frac{\hbar^2}{16mE}} \qquad . \tag{7.21}$$

One might worry the appearance of a $\hbar$ in the definition of a classical quantity. However, as we shall show below, this length also establishes how far the system is from a quantum regime which naturally should depends on $\hbar$. The appearance of the Planck constant in our definition is completely analogous to the appearance of the speed of light $c$ for the classical electron radius in non-relativistic Thomson cross section. The above classical size $l_{part}$ has the same meaning to quantum processes as the classical electron radius has for the Thomson scattering.

Note that this definition coincides with the particle's Compton wavelength if one uses the relativistic relation $E = m.c^2$ which is its rest mass potential energy. For a non-relativistic particle the contribution to its total energy comes mainly from its rest mass. Thus, even though the particle might have a kinetic energy, for a non-relativistic particle one is still allowed to use the above equation and compare it to the particle's Compton wave-length inasmuch the Compton wavelength specifies the limits of validity of non-relativistic quantum mechanics. As soon as the system attains the relativistic regime not only the above equation is no longer valid but also non-relativistic quantum mechanics. Hence, the above equation has the same range of validity as non-relativistic quantum mechanics.

In connection with the definition of a classical radius from the Thomson cross section we shall conceive a free stationary particle. Moreover, the radius $l_{part}$ defines a volume which we suppose to be at rest so that its energy is related to

the curvature through

$$E = \frac{\hbar^2}{16m}\mathcal{R}_W \quad \Rightarrow \quad l_{part} = \frac{1}{\sqrt{\mathcal{R}_W}} \qquad . \tag{7.22}$$

Notwithstanding this finite size picture, the system describes a point-like particle. Thus, from equation (7.19) we can relate the volume defined by $l_{part}$ with the particle's momentum through

$$l_{part} \,.\Delta p \geq \frac{\hbar}{2\sqrt{2}} \qquad . \tag{7.23}$$

From this point of view, the uncertainty principle indicates that it is impossible to perform a measurement smaller than the classical size of the particle defined by equation (7.21). In other words, it is impossible to perform a classical measurement inside what one normally would call a classical particle. This geometrical uncertainty relation attribute to a point-like particle an effective size due to the Q-wis curvature of the 3-d space.

## 7.4. Conclusions

It is well known that as soon as we consider high velocities or high energies one has to abandon the Euclidean geometry as a good description of the physical space. These brought two completely different modifications where the physical space loses its absolute and universal character. In fact, this is the core of classical relativistic physical theories, namely Special and General Relativity.

In a similar way, one should be allowed to consider that the difficulties that appears while going from classical to quantum mechanics comes from an inappropriate extrapolation of the Euclidean geometry to the micro-world. Hence, the unquestioned hypothesis of the validity of the 3-d Euclidean geometry to all length scales might be intrinsically related to quantum effects.

In the present work, we have shown that there is a close connection between the Bohm-de Broglie interpretation of quantum mechanics and the Q-wis spaces. In fact, we point out that the Bohmian quantum potential can be identified with the curvature scalar of the Q-wis. Moreover, we present a variational principle that reproduces the Bohmian dynamical equations considered up to date as equivalent to Schrödinger's quantum mechanics.

The Palatini-like procedure, in which the connection acts as an independent variable while varying the action, naturally endows the space with the appropriate Q-wis structure. Thus, the Q-wis geometry enters into the theory less arbitrarily than the implicit ad hoc Euclidean hypothesis of quantum mechanics.

The identification of the Q-wis curvature scalar as the ultimate origin of quantum effects leads to a geometrical version of the uncertainty principle. This geometrical description considers the uncertainty principle as a break down of the classical notion of standard rulers. Thus, it arises an identification of quantum

effects to the length variation of the standard rulers.

Acknowledgments
We would like to thank CNPq of Brazil and MN also to FAPERJ for financial support. We would also like to thank the participants of the "Pequeno Seminário" of ICRA-CBPF's Cosmology Group for useful discussions, comments and suggestions. MN would like to thank professor E. Elbaz for comments on this paper.

## 7.5. Appendix: Q-wis Geometry

In this section we shall briefly review the mathematical properties of such 3-d Q-wis space. Contrary to the Riemannian geometry, which is completely specified by a metric tensor, the Q-wis space defines an affine geometry. This means that the covariant derivative which is defined in terms of a connection $\Gamma^m_{ik}$ depends not only on the metric coefficients but also on the gradient of a scalar function $f_{,a}(x)$. For instance, given a vector $X_a$ its covariant derivative is

$$X_{a\,;b} = X_{a\,,b} - \Gamma^m_{ab}\, X_m \qquad . \tag{7.24}$$

The non-metricity of the Q-wis geometry implies that rulers, which are standards of length measurement, changes while we transport it by a small displacement $\mathrm{d}x^i$. This means that a ruler of length $l$ will change by

$$\delta\, l = l\, f_{,a}\, \mathrm{d}x^a \quad . \tag{7.25}$$

Note that even though it changes form point to point, in a Q-wis the length does not change along a closed path

$$\oint \mathrm{d}l = 0 \quad . \tag{7.26}$$

Contrary to a Riemannian space, the covariant derivative of the metric does not vanishes but it is given by

$$g_{ab\,;k} = f_{,k}\, g_{ab} \qquad . \tag{7.27}$$

Using Cartesian coordinates, it follows that the expression for the connection in terms of $f_{,k}$ takes the form

$$\Gamma^k_{ab} = -\frac{1}{2}\left(\delta^k_a\, f_{,b} + \delta^k_b\, f_{,a} - g_{ab}\, f^{,k}\right) \quad . \tag{7.28}$$

As a matter of convenience, we define $f = -4\, ln\, \Omega$. The Ricci tensor eq. (7.10) constructed with the above connection equation (.43) is given by

$$\mathcal{R}_{ij} = 2\frac{\Omega_{,ij}}{\Omega} - 6\frac{\Omega_{,i}\Omega_{,j}}{\Omega^2} + 2g_{ij}\left[\frac{\nabla^2\Omega}{\Omega} + \frac{\vec{\nabla}\Omega.\vec{\nabla}\Omega}{\Omega^2}\right] \tag{7.29}$$

and its trace the scalar of curvature $\mathcal{R} \equiv g^{ij}\mathcal{R}_{ij} = 8\,\frac{\nabla^2\Omega}{\Omega}$. In the present paper we have used a variational principle, which proof is as follows. Consider the action

$$I = \int \mathrm{d}t\,\mathrm{d}^3x\sqrt{g}\;\Omega^2\,\mathcal{R} \tag{7.30}$$

then, variation of the connection yields

$$\delta I = \int \mathrm{d}t\,\mathrm{d}^3x\sqrt{g}\;\Omega^2\,g^{ab}\,\delta R_{ab} = \int \mathrm{d}t\,\mathrm{d}^3x\;Z^{ab}_m\,\delta\Gamma^m_{ab} \tag{7.31}$$

with

$$\begin{aligned} Z^{ab}_m \equiv (\sqrt{g}\,g^{ab}\,\Omega^2)_{;\,m} - \frac{1}{2}\,(\sqrt{g}\,g^{ak}\,\Omega^2)_{;\,k}\,\delta^b_m + \\ -\frac{1}{2}\,(\sqrt{g}\,g^{bk}\,\Omega^2)_{;\,k}\,\delta^a_m \end{aligned} \tag{7.32}$$

Taking its trace we obtain $\left(\sqrt{g}\,g^{ak}\,\Omega^2\right)_{;\,k} = 0$. Substituting this expression in (8.9) we finally obtain the condition for a Q-wis geometry

$$g_{ab\,;\,k} = -\,4\,\frac{\Omega_{,\,k}}{\Omega}\,g_{ab} \quad . \tag{7.33}$$

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# 8. Geometrizing Relativistic Quantum Mechanics

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## Abstract

We propose a new approach to describe quantum mechanics as a manifestation of non-euclidean geometry. In particular, we construct a new geometrical space that we shall call Qwist. A Qwist space has a extra scalar degree of freedom that ultimately will be identified with quantum effects. The geometrical properties of Qwist allow us to formulate a geometrical version of the uncertainty principle. This relativistic uncertainty relation unifies the position-momentum and time-energy uncertainty principles in a unique relation that recover both of them in the non-relativistic limit.

## 8.1. Introduction

Non-relativistic theories were formulated to describe natural phenomena as a collection of events occurring on space using time as their external parameter. For that it seemed reasonable to formalize the physical arena in an abstract language as a flat, homogeneous and isotropic space. In fact, there was no other but one geometrical theory available. Hence, euclidean geometry was immediately identified with physical space.

Apart from its peculiarities, quantum mechanics promptly inherited euclidean geometry from classical mechanics. Even though some of its formulations do not even have a well defined notion of trajectory, quantum mechanics is defined using the flat euclidean metric or at most the flat minkowskian metric when we are dealing with relativistic quantum systems.

One of the novelties of relativistic theories is to describe physical phenomena on a non-euclidean four dimensional manifold. There is a rupture in the identification of the euclidean geometry with the physical space. In fact, General Relativity generalize the spacetime structure allowing it to be any possible type of Riemannian geometry.

The geometrical properties of a Riemannian space is completely characterized by a metric tensor. Its connection, the geometrical object that defines the covariant derivative, is identified with the Christoffel symbol, which is completely determined by the metric tensor. These spacetimes have two important properties, namely, all Riemannian spaces are locally minkowskian and the parallel transport prescription which enables us to compare objects at different locations preserve lengths and angles.

Additionally, there is still a wider class of torsion free geometrical space that was first introduced by Weyl [1] as an attempt to include electromagnetism in the properties of spacetime. In a Weyl space, the covariant derivative is again modified to implement a new gauge symmetry. The main idea was to incorporate electromagnetism in the geometrical degrees of freedom and geometrize both gravitation and electromagnetism, the only known classical long-range interactions.

From a different perspective, London proposed [2] that the geometrical space developed by Weyl could be related to quantum phenomena. London hypothesis was that stationary states of a quantum system such as an hydrogen atom should be associated with special geometrical configurations. In particular, he obtained Bohr's atomic orbits for the hydrogen atom as the only possible integrable orbits on that Weyl space. Therefore, it became admissible that euclidean geometry could fail not only on large scales but also on small scales. The possibility of this breakdown of euclidean geometry on small scales was considered by Riemann even before the development of quantum mechanics [3]

> "There arises from this problem of searching out the simplest facts by which the metric relations of space can be determined, a problem which in nature of things is not quite definite ... These facts are, like all facts, not necessary but of a merely empirical certainty; they are hypothesis; one may therefore inquire into their probability, which is truly very great within the bounds of observation, and thereafter decide concerning the admissibility of protracting them outside the limits of observation, not only toward the immeasurably large, but also toward the immeasurably small."

From London's work up to today, there has been very few but also very interesting analysis relating non-euclidean spaces and quantum mechanics [5]-[15]. In general, all these attempts to reproduce quantum phenomena are based on Weyl spaces. After Dirac's work [4], the properties of a Weyl space have been considerably changed with respect to Weyl's original idea. In appendix .1 we describe in some detail what is nowadays called a Weyl space.

Our aim in this work is to propose a similar but different approach to describe quantum effects. We argue that relativistic quantum mechanics can also be understood as a manifestation of what we shall call a Qwist (quantum weyl integrable spacetime). In particular, we construct a geometrical version of Heisenberg's uncertainty principle, which is related to the variation of a vector's

length in Qwist. This length variability also happens in Weyl spaces but with a completely different physical meaning as shall be clear in what follows.

In the next section we shall describe the formal basis of our approach. In advance, it seems convenient to stress that to define a physical theory one has first of all to define its kinematic properties. Given an action principle together with a set of dynamical fields does not completely specify the theory. In particular, each theory has its own internal symmetries, which can be used to construct an equivalent class of observers [16]. The point we would like to stress is that one has to define from the beginning which are the allowed transformations of the dynamical fields.

In particular, one should not be misled by similarities of some of the equations and confuse Qwist with any other geometrical space. Let us now define the structure of the Qwist space that we shall identify with the physical spacetime.

### 8.1.1. Qwist geometry

A Qwist is a geometrical manifold $\left(g_{\mu\nu}\,;\Gamma^{\alpha}_{\mu\nu}\right)$ endowed with a metric tensor and a symmetric affine connection $\Gamma^{\alpha}_{\mu\nu}=\Gamma^{\alpha}_{(\mu\nu)}$. Its symmetry group is the Manifold Mapping Group $\mathcal{MMG}$, which allow us to perform an arbitrary coordinate transformation[1].The connection of this space is defined with an extra degree of freedom given by a scalar field $\Omega(x)$. Thus, Qwist is neither a Riemannian space nor a Weyl geometry (see appendix .1 for more details).

We shall construct our connection such that the non-metricity condition be given by

$$\nabla_{\alpha}g_{\mu\nu}\equiv-2\left(\ln\Omega\right)_{,\alpha}g_{\mu\nu} \qquad . \tag{8.1}$$

One can use the above equation to solve for the connection giving

$$\Gamma^{\alpha}_{\mu\nu}=\left\{{}^{\alpha}_{\mu\nu}\right\}+\frac{1}{\Omega}\left(\Omega_{,\nu}\,\delta^{\alpha}_{\mu}+\Omega_{,\mu}\,\delta^{\alpha}_{\nu}-\Omega^{,\alpha}g_{\mu\nu}\right) \quad . \tag{8.2}$$

Since the connection is not equal to the Christoffel symbol, it is adequate to distinguish between two kind of covariant derivative. The Qwist covariant derivative is constructed with the connection and we shall denote by

$$\nabla_{\alpha}\xi^{\mu}\equiv\xi^{\mu}_{\ ;\alpha}=\xi^{\mu}_{\ ,\alpha}+\Gamma^{\mu}_{\alpha\lambda}\xi^{\lambda} \quad ,$$

and a Riemannian covariant derivative

$$\xi^{\mu}_{\ //\alpha}=\xi^{\mu}_{\ ,\alpha}+\left\{{}^{\mu}_{\alpha\lambda}\right\}\xi^{\lambda} \quad .$$

The non-metricity condition (8.1) implies that the length of a vector is not preserved if parallel transported

$$\delta l=-l\,\Omega^{-1}\Omega_{,\mu}\mathrm{d}x^{\mu} \quad . \tag{8.3}$$

[1]Note that the symmetry group of a Qwist space does not include conformal transformations, which are characteristic to Weyl spaces.

The above equation has an important physical meaning. It describes how the physical length of a ruler changes from point to point. Note that as long as the extra degree of freedom is a scalar field $\Omega'(x') = \Omega(x)$ there is no gauge freedom in (8.3).

Furthermore, condition (8.1) does not suffers from any kind of second clock effect [17, 18] that could be present if instead of the gradient of a scalar function it were a vector field. Considering (8.3), it is immediate to show that the length of a vector does not change along a closed path

$$\oint \delta l = 0 \qquad . \tag{8.4}$$

Thus, this property guarantees that all local measurements such as distances are well defined and can be uniquely determined.

As usual, the curvature tensor can be written in terms of the connection as

$$\mathcal{R}^{\alpha}_{\ \mu\beta\nu} = \Gamma^{\alpha}_{\ \mu\beta,\nu} - \Gamma^{\alpha}_{\ \mu\nu,\beta} + \Gamma^{\varepsilon}_{\ \mu\beta}\Gamma^{\alpha}_{\ \nu\varepsilon} - \Gamma^{\varepsilon}_{\ \mu\nu}\Gamma^{\alpha}_{\ \varepsilon\beta} \quad ,$$

which can be used to calculate its traces.

Hence, the Ricci tensor is given by

$$\mathcal{R}_{\mu\nu} = \hat{\mathcal{R}}_{\mu\nu} - \frac{2\nabla_{\nu}(\Omega_{,\mu})}{\Omega} + \frac{4\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2} - g_{\mu\nu}\left[\frac{\Box\Omega}{\Omega} + \frac{\Omega_{,\lambda}\Omega^{,\lambda}}{\Omega^2}\right] \tag{8.5}$$

where $\Box \equiv \nabla_{\mu}\nabla^{\mu}$ is the d'Alembertian operator and $\hat{\mathcal{R}}_{\mu\nu}$ is the Riemannian part of the Ricci tensor, i.e. the Ricci tensor constructed solely with the Christoffel symbol. For the curvature scalar $\mathcal{R} \equiv g^{\mu\nu}\mathcal{R}_{\mu\nu}$ we find

$$\mathcal{R} \ = \ \hat{\mathcal{R}} - 6\Box\ln\Omega - 6\left(\ln\Omega\right)_{,\alpha}\left(\ln\Omega\right)^{,\alpha} \ = \ \hat{\mathcal{R}} - 6\frac{\Box\Omega}{\Omega} \tag{8.6}$$

where again $\hat{\mathcal{R}} \equiv g^{\mu\nu}\,\hat{\mathcal{R}}_{\mu\nu}$ is its corresponding Riemannian part.

It is interesting to show that it is possible to derive the geometrical properties of a Qwist space from a palatini-like variational principle by considering the connection as an independent field. For this purpose, consider the functional

$$I = \int \mathrm{d}^4x\sqrt{-g}\ \Omega^2\,\mathcal{R} \quad . \tag{8.7}$$

If we demand this functional to be stationary with respect to variation of the connection we find

$$\delta I = \int \mathrm{d}^4x\sqrt{-g}\,\Omega^2\,g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} = \int \mathrm{d}^4x\ Z^{\mu\nu}_{\lambda}\delta\Gamma^{\lambda}_{\mu\nu} = 0 \tag{8.8}$$

where $Z^{\mu\nu}_{\lambda}$ is given by

$$Z^{\mu\nu}_{\lambda} \ \equiv \ (\sqrt{-g}\,g^{\mu\nu}\Omega^2)_{;\lambda} - \frac{1}{2}(\sqrt{-g}\,g^{\mu\alpha}\Omega^2)_{;\alpha}\,\delta^{\nu}_{\lambda} - \frac{1}{2}(\sqrt{-g}\,g^{\nu\alpha}\Omega^2)_{;\alpha}\,\delta^{\mu}_{\lambda} = 0 \tag{8.9}$$

Taking the trace of the above expression yields

$$\left(\sqrt{-g}\, g^{\mu\alpha}\, \Omega^2\right)_{;\,\alpha} = 0 \quad . \tag{8.10}$$

Substituting (8.10) again in (8.9) we finally obtain the condition that characterize a Qwist geometry

$$g_{\mu\nu\,;\,\alpha} = -2\, \frac{\Omega_{,\,\alpha}}{\Omega}\, g_{\mu\nu} \quad . \tag{8.11}$$

In the following sections we will study the dynamics of a spinless charged particle in a Qwist geometry and relate it to a relativistic quantum system. In particular, we will show that this system yields the correct non-relativistic limit, i.e. the Schrödinger equation for a charged particle.

We shall describe quantum mechanics using the Bohm-de Broglie causal interpretation, which are amongst the well-defined formulations that reproduce the same results of the orthodox interpretation but has the advantage of describing matter as point-like particles.

In addition, we propose a geometrical interpretation of Heisenberg's uncertainty principle. This relativistic geometrical version combines both the position-momentum and time-energy relations in an unique principle, which decouples into the usual Heisenberg's uncertainty principles in the non-relativistic limit.

Finally, we discuss how our results are related to Klein-Gordon's equation from a geometrical point of view and conclude in the last section with some final remarks.

## 8.2. Relativistic Quantum Mechanics

In this section, we will describe a system composed of a relativistic charged point-like particle interacting with an external electromagnetic field and geometry. Therefore, we are considering a charged particle wandering in a non-euclidean spacetime that has an independent degree of freedom.

This is perhaps the simplest relativistic system but it will be interesting enough to present the connection between Qwist and quantum mechanics. We decided to include an external electromagnetic field (non-dynamical) insofar as it does not veil the main properties of Qwist. Notwithstanding, if preferable, it is possible to do the same analysis turning the electromagnetic field off, which is equivalent to consider a neutral particle without expense of the quantum physical content.

Our discussion will be based on a variational principle that at the same time provides the dynamical equation for the particle and naturally endows the spacetime with an affine structure that is typical of a Qwist space. As will be clear in what follows, the particle's dynamics is given by integrating a relativistic Hamilton-Jacobi equation where its momentum is described by the derivative

of Hamilton's principle function $P_\mu = \partial_\mu S$. In addition, apart from a kind of non-minimal coupling, the geometrical sector shall be given by the Ricci scalar.

Thus, consider the following action that should be justified a posteriori

$$I = \frac{1}{\kappa} \int \mathrm{d}^4x \sqrt{-g} \Omega^2 \left( \mathcal{L}_g + \mathcal{L}_m \right) \quad , \tag{8.12}$$

with

$$\mathcal{L}_g \equiv \lambda^2 \mathcal{R} \tag{8.13}$$

$$\mathcal{L}_m \equiv \frac{g^{\mu\nu}}{\hbar^2} \left( \partial_\mu S - \frac{e}{c} A_\mu \right) \left( \partial_\nu S - \frac{e}{c} A_\nu \right) - \mu^2 \ . \tag{8.14}$$

In the above expressions $g$ is the determinant of the metric $g_{\mu\nu}$ and $\mathcal{R}$ is the Ricci scalar . As already mentioned, $S$ is the relativistic version of the Hamilton's principle function of the particle which here is coupled to the gauge field $A^\mu = \left( \varphi, \vec{A} \right)$ describing an external electromagnetic field.

The constants appearing above are the speed of light $c$, the gravitational constant $\kappa \equiv 16\pi G/c^3$, a dimensionless $\lambda$ to be determined, the particle's electric charge $e$ and its inverse Compton wavelength $\mu \equiv mc/\hbar$.

We should strongly stress that there is no gravitational interaction in this system. The gravitational constant $\kappa$ appears as a global factor and it is introduce only to adjust the dimensionality of the action $[I] = \hbar$. Actually, one should not be surprised by the introduction of $\kappa$ inasmuch this is the only way to change the dimensionality of the curvature scalar to dimension of action.

This system has three dynamical variables to be varied, namely, the dimensionless scalar function $\Omega$, the connection $\Gamma^\lambda_{\mu\nu}$ and $S$. Variation with respect to the connection give us the geometrical structure of spacetime. Following the derivation of section 8.1.1, equations(8.7)-(8.11), we have

$$\delta I = \int \mathrm{d}^4x \ Z^{\mu\nu}_\lambda \, \delta\Gamma^\lambda_{\mu\nu} = 0 \qquad \Rightarrow \qquad g_{\mu\nu;\lambda} = -2 \left( \ln \Omega \right)_{,\lambda} g_{\mu\nu} \qquad . \tag{8.15}$$

Varying the Hamilton's function $S$ give us a conservation-like equation,

$$\begin{aligned} \delta I &= -\frac{2}{\kappa\hbar^2} \int \mathrm{d}^4x \sqrt{-g} \, \Omega^2 \partial^\mu \left( \delta S \right) \left( \partial_\mu S - \frac{e}{c} A_\mu \right) = 0 \\ &= \frac{2}{\kappa\hbar^2} \int \mathrm{d}^4x \sqrt{-g} \, g^{\mu\nu} \left[ \Omega^2 \left( \partial_\mu S - \frac{e}{c} A_\mu \right) \right]_{//\nu} \delta S = 0 \\ &\qquad \Rightarrow g^{\mu\nu} \left[ \Omega^2 \left( \partial_\mu S - \frac{e}{c} A_\mu \right) \right]_{//\nu} = 0 \quad , \end{aligned} \tag{8.16}$$

where we have dropped a surface term using the four-dimensional Gauss theorem. Note that we have written this expression using a Riemannian covariant derivative, i.e. defined using the Christoffel symbol only.

Finally, variation with respect to $\Omega$ gives

$$\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\left(\partial^\mu S - \frac{e}{c}A^\mu\right) - m^2c^2 + \lambda^2\hbar^2\mathcal{R} = 0 \quad . \tag{8.17}$$

Since we are supposing that the Riemannian part of the Qwist geometry is Minkowskian, $g_{\mu\nu} = diag\,(1,-1,-1,-1)$, by choosing $\lambda^2 = 1/6$, the above equation becomes

$$\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\left(\partial^\mu S - \frac{e}{c}A^\mu\right) - m^2c^2 - \hbar^2\frac{\Box\Omega}{\Omega} = 0 \ . \tag{8.18}$$

Equation (8.18) generalize the relativistic Hamilton-Jacobi equation with the inclusion of the last term. This extra term which is basically the Weyl curvature scalar respond for all relativistic quantum effects of this system.

The Weyl curvature scalar plays the same role in this relativistic scenario as the bohmian quantum potential for the non-relativistic quantum mechanics [19]-[22]. In particular, the classical regime is attained in the limit $\Box\Omega \to 0$.

Given a field configuration for the gauge field $A_\mu$, (8.16) and (8.18) define a closed system that is well defined with the specification of appropriate initial conditions. Notwithstanding, since quantum effects are now given by modifications of the spacetime structure, it shall be convenient to use a hydrodynamical description and associate the particle's world-line with a time-like congruence.

Following [23]-[26] we shall define our canonical momentum one-form $\tilde{P} = P_\mu\,\widetilde{\theta}^\mu$ and de Broglie's mass respectively as

$$P_\mu \equiv \partial_\mu S - \frac{e}{c}A_\mu \quad , \tag{8.19}$$

$$M \equiv \sqrt{m^2 + \frac{\hbar^2}{6c^2}\mathcal{R}} = m\sqrt{1 + \frac{\hbar^2}{m^2c^2}\frac{\Box\Omega}{\Omega}} \quad . \tag{8.20}$$

By virtue of (8.18)-(8.20) we can define a unitary time-like velocity field by

$$U^\mu = c\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \equiv \frac{1}{M}g^{\mu\nu}P_\nu \quad \Longrightarrow \quad U^\mu U_\mu = c^2 \quad \text{i.e.} \quad \mathrm{d}\lambda^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \quad . \tag{8.21}$$

However, the "euclidean' particle velocity field is define as

$$V^\mu = c\frac{\mathrm{d}x^\mu}{\mathrm{d}s} \equiv \frac{1}{m}g^{\mu\nu}P_\nu \quad \Longrightarrow \quad V^\mu V_\mu = \frac{M^2}{m^2}c^2 \quad . \tag{8.22}$$

Thus, there is an intrinsic time re-parameterization along the particle's trajectory given by

$$\mathrm{d}s = \frac{\mathrm{d}\lambda}{\sqrt{1 + \frac{\hbar^2}{m^2c^2}\frac{\Box\Omega}{\Omega}}} = \frac{m}{M}\mathrm{d}\lambda \quad . \tag{8.23}$$

In addition, within this geometrical interpretation, (8.16) can be viewed as a integrability condition. One can immediately show that for any vector $\xi^\mu$ we have

$$\xi^\mu{}_{;\mu} = \frac{1}{\Omega^4}\left(\Omega^4\xi^\mu\right)_{//\mu} \quad \text{or} \quad \xi_{\mu;\nu}\,g^{\mu\nu} = \frac{1}{\Omega^2}\left(\Omega^2\xi^\mu\right)_{//\mu} \quad ,$$

which connects the covariant derivative of Qwist and Riemann geometries. Hence, the continuity-like equation (8.16) can be written as

$$\frac{1}{\Omega^2}\left(\Omega^2 P^\alpha\right)_{//\alpha} = P_{\alpha\,;\,\beta}\, g^{\alpha\beta} = 0 \qquad , \tag{8.24}$$

which can be viewed as an integrability condition on Qwist for the canonical momentum $P_\mu$.

Note that the original system equations (8.16) and (8.18) are equivalent to (8.21) and (8.24). Notwithstanding the mathematical equivalence, one should bear in mind that its physical interpretation is completely different. The Dynamical equations for the two scalar functions, namely $S$ and $\Omega$ is now substituted by kinematic relations for the time-like congruence $U^\mu$.

As a matter of fact, specifying our Cauchy-surface $\Sigma_c$ as a space-like hypersurface where $P^\mu$ is time-like and orthogonal everywhere, one can show [26]-[28] that the integrability condition given by (8.24) guarantees that $P^\mu$ will always remains time-like. This can be proved as follows.

Consider the variation along the congruence $U^\mu$ of the quantity $\Omega^2 M\delta V$, where $\delta V$ is an infinitesimal 3-volume orthogonal to the congruence. It's straightforward to show that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\Omega^{-2}M\delta V\right) &= U^\alpha\partial_\alpha\left(\Omega^{-2}M\right)\delta V + \Omega^{-2}MU^\alpha\partial_\alpha\left(\delta V\right) \\
&= U^\alpha\partial_\alpha\left(\Omega^{-2}M\right)\delta V + \Omega^{-2}M\delta V\, U^\alpha_{\;;\alpha} \\
&= \left(\Omega^{-2}MU^\alpha\right)_{;\alpha}\delta V = \left(\Omega^{-2}P^\alpha\right)_{;\alpha}\delta V \\
&= P_{\alpha\,;\,\beta}\, g^{\alpha\beta}\,\Omega^{-2}\delta V = 0\ .
\end{aligned} \tag{8.25}$$

Hence, as long as $\delta V$ and $\Omega^{-2}$ are always positive and (8.25) shows that $\Omega^{-2}M\delta V = C^{te}$ then $M^2$ should not change sign. Given that $P^\mu$ is time-like on the Cauchy-surface, i.e. $P^\mu P_\mu(\Sigma_c) = M^2(\Sigma_c) > 0$ then it also has to be time-like everywhere $P_\mu P^\mu(x) > 0$.

Our system is suitable to describe relativistic spinless charged particles. Therefore, one might be concerned how to deal with creation-annihilation processes and if it is possible to consistently define a time-like congruence like $P^\mu$ in the presence of both particles and anti-particles.

We should emphasize that this analyses deals only with non-interacting particle which in a sense avoid these kind of difficulties. However, this formalism is as good to describe particles as it is to describe anti-particles due to its invariance under time reversal accompanied by a change of sign of the electrical charge $e$. A time reversal is equivalent to a change of sign of the Hamilton's function $S \to -S$. Hence it's straightforward to show that the system of equations (8.16) and (8.18), remains unchanged if $e \to -e$. We shall come back later to this issues in section 8.2.3.

Finally, consider the particle's trajectory, which is given by integrating (8.21). One should already expect that the particle should not follow a geodetic trajectory as long as it has a quantum force acting on it. Given the particle's velocity

field we can calculate this quantum force using the kinematic equations. As a matter of convenience, we shall for the moment consider a cartesian coordinates so that

$$\frac{\mathrm{d}U^\mu}{\mathrm{d}\lambda} = \frac{c^2}{M}\partial^\mu M - \frac{U^\alpha \partial_\alpha M}{M} U^\mu + \frac{e}{M} F^\mu{}_\alpha U^\alpha \quad ,$$

where we have define the components of the electromagnetic tensor as $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. We can readily recognize the last term as being the Lorentz's force while the two first ones are intrinsically geometrical terms that we commonly associate with quantum effects.

The geodesic equation written in its covariant form gives

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = W^\mu + \frac{e}{Mc^2} F^\mu{}_\alpha U^\alpha \quad , \tag{8.26}$$

with

$$W^\mu \equiv \partial^\mu \ln\left(\frac{M}{\Omega}\right) - \frac{1}{c^2} U^\alpha \partial_\alpha \ln\left(\frac{M}{\Omega^2}\right) U^\mu \quad . \tag{8.27}$$

Note that the quantum force $W^\mu$ also depends on the velocity field $W^\mu = W^\mu\left(\Omega\,, U^\lambda\right)$.

### 8.2.1. Non-relativistic limit

The above system describes a charged relativistic point-like particle interacting with an external electromagnetic field in a Q-wist geometry. This system can be understood as a relativistic generalization of the Schrödinger picture of quantum mechanics. As we shall now show, it is possible to recover the Schrödinger description by redefining the Hamilton's principal function and taking the usual non-relativistic limit $c \to \infty$.

One of the main difference in the description of a relativistic particle is that its energy contains its inertial rest mass as a potential-like energy. Furthermore, inasmuch the energy is related to the Hamilton's function by $E = -\frac{\partial S}{\partial t}$, we define the non-relativistic version of the Hamilton's function by $S_{nr} \equiv S + mc^2 t$ .

Substituting this ansatz in (8.18) we find

$$\frac{\partial S_{nr}}{\partial t}\left(1 - \frac{eA_0}{mc^2}\right) + \frac{1}{2m}\left(\vec{\nabla} S_{nr} - \frac{e}{c}\vec{A}\right)^2 + eA_0 - \frac{e^2 A_0 A^0}{2mc^2} - \frac{\hbar^2}{2m}\frac{\nabla^2\Omega}{\Omega} - \frac{1}{2mc^2}\left[\left(\frac{\partial S_{nr}}{\partial t}\right)^2 - \frac{\hbar^2}{\Omega}\right.$$

By considering the limit $c \to \infty$ it is licit to neglect $\frac{eA_0}{mc^2}$ with respect to 1 and $\frac{e^2 A_0 A^0}{2mc^2}$ to $eA_0$. Furthermore, the last two terms go away and we identify the spatial part of the Weyl curvature as the non-relativistic bohmian quantum potential $Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\Omega}{\Omega}$ [19]-[22].

Thus, in this limit we have

$$\frac{\partial S_{nr}}{\partial t} + \frac{1}{2m}\left(\vec{\nabla} S_{nr} - \frac{e}{c}\vec{A}\right)^2 + eA_0 + Q = 0 \quad , \tag{8.28}$$

which reproduce the first non-relativistic bohmian equation, namely the Hamilton-Jacobi like equation.

The second non-relativistic equation is derived directly from (8.16),

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left[\Omega^2\left(\frac{\partial S_{nr}}{\partial t} - mc^2 - eA_0\right)\right] - \vec{\nabla}.\left[\Omega^2(\vec{\nabla} S_{nr} - \frac{e}{c}\vec{A})\right] = 0$$

and again neglecting $\frac{eA_0}{mc^2}$ with respect to 1 and the terms containing $1/c^2$

$$\frac{\partial \Omega^2}{\partial t} + \vec{\nabla}.\left[\Omega^2\left(\frac{\vec{\nabla} S_{nr}}{m} - \frac{e\vec{A}}{mc}\right)\right] = 0 \qquad . \tag{8.29}$$

As it's well known, (8.28) and (8.29) are equivalent to the Schrödinger equation [2]. Hence it is in fact legitimate to view (8.16) and (8.18) as relativistic generalizations of the Schröndinger picture of quantum mechanics for a spinless charged point-like particle.

Note that in this limit the geometrical description degenerates to

$$M \longrightarrow m \quad , \quad U^\mu \longrightarrow V^\mu \quad , \quad \mathrm{d}\lambda \longrightarrow \mathrm{d}s$$

The de Broglie's Mass $M$ goes to the particle's rest mass $m$ while both velocity fields coincide as well as its parameterization. In non-relativistic quantum mechanics one can understand the quantum force as a deviation from an euclidean geodesic [32]. If we chose a coordinate system which has a vanishing Christoffel symbol, one can show that

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} = h^{\mu\nu}\frac{\partial}{\partial x^\nu}Q + \frac{e}{m}F^\mu{}_\nu V^\nu \qquad ,$$

where we have defined the projector tensor $h^{\mu\nu}$ along the velocity field $V^\mu$ by $h^{\mu\nu} \equiv g^{\mu\nu} - V^\mu V^\nu$. Taking the non-relativistic limit (see [29]-[31] for more details), the spatial part becomes

$$m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = -\vec{\nabla}Q + e\left(\vec{E} + \vec{v}\times\vec{B}\right) \quad ,$$

where $\vec{v}$ is the particle's 3-velocity. As expected, the particle feels the usual electromagnetic force plus the non-relativistic quantum force given by $-\vec{\nabla}Q$.

[2] The above mentioned pair of equations are precisely the Bohm-de Broglie system of equations that one finds when uses the polar form of the wave-function $\psi = \Omega \exp\left\{\frac{i}{\hbar}S_c\right\}$ in Schrödinger's equation.

### 8.2.2. Relativistic uncertainty principle

The uncertainty principle is in the core of the orthodox interpretation of quantum mechanics. In a recent paper [32], we have shown that it is possible to interpret geometrically the uncertainty relation for the position and momentum by virtue of a characteristic length scale defined by a 3-d curvature scalar.

However, as long as we have considered a non-relativistic theory, we have established a purely spatial relation which in the relativistic context is unsatisfactory by the requirement of covariance.

In this section, we shall generalize this "spatial" uncertainty principle to a four dimensional relation. In a relativistic theory only four dimensional quantity acquires physical meaning. Accordingly, the interdependence between space and time compel us to somehow relate the uncertainty principle to the interval $\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 - \mathrm{d}l^2$. In addition, taking the non-relativistic limit from this relation we shall show that we recover both uncertainty relation for position and momentum as well as for time and energy.

As we have shown (8.6), the Qwist curvature scalar can be decomposed in a Riemannian part plus the contribution of the extra degree of freedom $\Omega$. If we suppose that the Riemannian part is flat, $\hat{\mathcal{R}} = 0$, then

$$\mathcal{R} = -\frac{6}{c^2\Omega}\frac{\partial^2\Omega}{\partial t^2} + 6\frac{\nabla^2\Omega}{\Omega} \quad .$$

Apart from the speed of light $c$, the above equation shows that Qwist curvature scalar which has dimension of inverse length squared is a sum of two terms one with dimension of inverse time squared and the other with dimension of inverse length squared. Thus, we define the Weyl length and Weyl time by

$$L_w \equiv \left|\left|6\frac{\nabla^2\Omega}{\Omega}\right|\right|^{-1/2} \quad ; \qquad T_w \equiv \left|\left|\frac{6}{\Omega}\frac{\partial^2\Omega}{\partial t^2}\right|\right|^{-1/2} \quad . \tag{8.30}$$

In a previous paper [32], we have shown that non-relativistic quantum mechanics can be pictured also as a modification of the 3d euclidean space which we called Qwis. Through this analysis we were able to construct an uncertainty principle for space and momentum based on the characteristic length above defined $L_w$.

If in fact the Qwist curvature scalar $\mathcal{R}$ is related to quantum phenomena then the above quantities (8.30) have to somehow yield a measurement of departure from a classical behavior. Recall equation (8.23)

$$\mathrm{d}s = \frac{\mathrm{d}\lambda}{\sqrt{1 + \frac{\hbar^2}{m^2c^2}\frac{\Box\Omega}{\Omega}}} \quad .$$

In the non-relativistic limit, i.e. $c \to \infty$, we have $\mathrm{d}\lambda \to c.\mathrm{d}\tau$. Furthermore, the time derivative of $\Omega$ can be neglected compared to its spatial derivative.

Thus, we have

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\lambda} &= \sqrt{1-\frac{\mathrm{d}l^2}{\mathrm{d}\lambda^2}} = \sqrt{1-\frac{v^2}{c^2}} \approx 1-\frac{v^2}{2c^2} = 1-\frac{p^2}{2m^2c^2} \\ &= \frac{1}{\sqrt{1+\frac{\hbar^2}{m^2c^2}\frac{\Box\Omega}{\Omega}}} \approx 1+\frac{\hbar^2}{2m^2c^2}\frac{\nabla^2\Omega}{\Omega}\end{aligned}$$

Comparing the above equations we find

$$p^2.\,L_w^2 = \frac{\hbar^2}{6}$$

To interpret the above equation as an uncertainty-like relation we need an extra assumption. Suppose that any length measurement $\Delta L$ can only measure distances bigger than the characteristic Weyl length, i.e.

$$\Delta L \geq L_w = \left|\left|6\frac{\nabla^2\Omega}{\Omega}\right|\right|^{-1/2} \quad . \tag{8.31}$$

The reasonability of this hypothesis lies on the notion of a classical measurement. A length measurement is made with a standard ruler which is supposed to be a stiff object. Thus, the notion of classical standard ruler presuppose the validity of euclidean space (see [32] for a more detailed discussion). With the hypothesis (E.93), the above relation becomes

$$\Delta p^2.\,\Delta L^2 \geq \Delta p^2.\,L_w^2 \quad \Rightarrow \quad \Delta p.\Delta L \geq \frac{\hbar}{\sqrt{6}} \quad . \tag{8.32}$$

In addition, it is also possible to derive the uncertainty relation for time and energy. For the moment we shall turn off the electromagnetic interaction ($A_\mu = 0$). Without the electromagnetic potential, equation (8.18) can be recast as

$$\begin{aligned}E^2 = m^2c^4 + p^2c^2 + \frac{\hbar^2}{\Omega}\frac{\partial^2\Omega}{\partial t^2} - \hbar^2c^2\frac{\nabla^2\Omega}{\Omega} \Rightarrow \\ \Rightarrow \quad E = mc^2\sqrt{1+\frac{1}{c^2}\left(\frac{p^2}{m^2}-\frac{\hbar^2\nabla^2\Omega}{m^2\Omega}\right)+\frac{1}{c^4}\frac{\hbar^2}{m^2\Omega}\frac{\partial^2\Omega}{\partial t^2}}\end{aligned}$$

In section 8.2.1, we have argued that the non-relativistic hamilton's principle function should be related to the relativistic one by $S_{nr} = S + mc^2t$ which can be interpreted as $E_{nr} = E - mc^2$. Therefore, expanding in power of $c^{-2}$, the above equation becomes

$$E_{nr} = \frac{p^2}{2m} - \frac{\hbar^2\nabla^2\Omega}{2m\Omega} + \frac{\hbar^2}{2mc^2\Omega}\frac{\partial^2\Omega}{\partial t^2} - \frac{1}{8m^3c^2}\left(p^2 - \hbar^2\Omega^{-1}\nabla^2\right) + \mathcal{O}(1/c^4) \tag{8.33}$$

In zeroth order,

$$E_{nr} = \frac{p^2}{2m} - \frac{\hbar^2 \nabla^2 \Omega}{2m\Omega} \quad .$$

Note that the non-relativistic energy includes a classical term $p^2/2m$ plus a geometrical term $-\frac{\hbar^2\nabla^2\Omega}{2m\Omega}$. Using this result in the second order term we find

$$0 = \frac{\hbar^2}{2mc^2\Omega}\frac{\partial^2\Omega}{\partial t^2} - \frac{E_{cl}^2}{2mc^2}$$

which gives

$$E_{cl}^2 \, . \, T_w^2 = \frac{\hbar^2}{6} \tag{8.34}$$

A time measurement is by definition the length of a 4-d trajectory in a fix spatial point. However, in Qwist appears an intrinsic time re-parameterization (8.23) which is certainly not include in the euclidean definition of a standard clock. Generalizing the hypothesis that a length measurement has to be greater than the weyl length $L_w$, we shall suppose that a time measurement realized by a standard clock has to be greater than the weyl time $T_w$, i.e. $\Delta t \geq T_w$. Through this hypothesis, the above relation becomes

$$\Delta E_{cl}.\Delta t \;\geq\; \Delta E_{cl}.T_w \quad \Rightarrow \quad \Delta E_{cl}.\Delta t \geq \frac{\hbar}{\sqrt{6}} \quad . \tag{8.35}$$

Note that the uncertainty relation for position and momentum (8.32) is naturally incorporated into the uncertainty relation for time and energy. Actually, it is the entanglement between space and time that allows us to derive the above relation (8.35).

In orthodox quantum mechanics, the impossibility of deriving the uncertainty relation for time and energy is commonly associated with the role played by time as an external parameter, i.e. a lack of a time operator. Our geometrical approach does not deal with operators and in fact treats space and time variables on equal footing in the relativistic sense.

### 8.2.3. Geometrical interpretation of Klein-Gordon's equation

The geometrical approach developed above describes a relativistic particle interacting with a non-euclidean geometry. The deviation from a pure classical behavior comes from the non-vanishing of the Ricci scalar $\mathcal{R}$. Once $\Box\Omega \to 0$, the classical regime is recovered. Hence, the extra degree of freedom of the Qwist spacetime, namely the scalar function $\Omega$ is responsible for the departure from classical behavior.

Interesting enough, our geometrical variational method is formally equivalent to a Klein-Gordon system. Suppose that instead of treating separately the particle's and the geometrical degrees of freedom, we have defined a wave-function

by combining both $\psi = \phi.\exp\left\{\frac{i}{\hbar}S\right\}$. The $\phi$ field has dimension of inverse length as commonly is done in field theory and is related to our dimensionless geometrical degree of freedom by $\Omega = \sqrt{\kappa\hbar}\,\phi$.

With respect to this hybrid object $\psi$, as should have been expected, (8.16) and (8.18) can be combined into one equation, a massive Klein-Gordon equation

$$D_\mu D^\mu \psi - m^2c^2\psi = 0 \quad ,$$

where we have defined the gauge covariant derivative operator $D_\mu \equiv i\hbar\partial_\mu + eA_\mu$. As it is well known, the above equation can be derived using a variational principle with Lagrangian density

$$\mathcal{L} = -\hbar^{-1}\left(\bar{D}_\mu\psi^* D^\mu\psi - m^2c^2\psi^*\psi\right) \quad , \tag{8.36}$$

where $\psi$ and $\psi^*$ should be treated as independent dynamical variables. Modulus a total derivative, (8.36) defines an action

$$I_\psi = \frac{1}{\kappa}\int \mathrm{d}^4x\sqrt{-g}\,\Omega^2\left[\frac{\Box\Omega}{\Omega} - \frac{g^{\mu\nu}}{\hbar^2}\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\left(\partial_\nu S - \frac{e}{c}A_\nu\right) + \frac{m^2c^2}{\hbar^2}\right] \quad . \tag{8.37}$$

Note that the above action is precisely (H.1), i.e. $I = I_\psi$, if we impose beforehand a Weyl affine structure for the spacetime and identify the scalar curvature with the first term involving derivatives of $\Omega$. Thus, in a sense, our approach is more general as long as the affine structure is derived as a palatini-like variational principle.

This Lagrangian density naturally defines a conserved current

$$J^\mu \equiv -\frac{1}{2\kappa\hbar}\psi^*\overleftrightarrow{D}^\mu\psi = \Omega^2\left(\partial^\mu S - eA^\mu\right) = \Omega^2 g^{\mu\nu}P_\nu \quad , \tag{8.38}$$

and a energy-momentum tensor

$$T_{\mu\nu} \equiv \frac{2c}{\hbar}\bar{D}_\mu\psi^*\bar{D}_\nu\psi - \frac{c}{\hbar}\left(\bar{D}_\lambda\psi^*\bar{D}^\lambda\psi + m^2\psi^*\psi\right)g_{\mu\nu} \quad . \tag{8.39}$$

The Klein-Gordon equation can be casted in a Schrödinger-like form by defining a two-component wave-function (see [26, 33] for more detail). In this approach, it is possible to identify positive and negative energy solutions. The energy is defined as the eigenvalue of the Hamiltonian of this Schrödinger picture and it is numerically equal to the spatial integral of the $00-$component of the energy-momentum tensor, i.e. $E = \int \mathrm{d}^3x\, T_{00}$.

It can be shown that the negative energy solution can be mapped into the positive energy solution by a charge conjugation operation. Therefore, as it is well known, we can associate the negative energy solutions to anti-particle states. This charge conjugation is intrinsically related to the invariance of the system by a change of $\psi \to \psi^*$. Hence, charge conjugation can be imitated by a change $S \to -S$ and $e \to -e$.

Notwithstanding, as previously emphasized, our analysis deals only with non-interacting particle. Thus, even thought this formalism is as adequate to particles as to anti-particles, we have not yet established how interacting process like creation-annihilation should be described.

## 8.3. Conclusions

It had been shown in the literature that exist a very interesting connection between non-euclidean geometries and quantum phenomena. The predominant mechanism to describe quantum effects by geometrical degrees of freedom has been based on Weyl space. In the present work, we proposed a new geometrical approach based on a new geometrical space that we called Qwist.

In Qwist, its extra scalar degree of freedom produce a length variability, which is responsible for change in size of extended object. Furthermore, this is a physical and in principle measurable effect. The physical interpretation of this length variability allowed us to formulate a relativistic and geometrical version of the uncertainty principle.

The non-euclidean properties of Qwist provide two characteristics dimensions, namely a Weyl length and a Weyl time defined by

$$L_w \equiv \left|\left|\frac{\nabla^2\Omega}{\Omega}\right|\right|^{-1/2} \quad ; \qquad T_w \equiv \left|\left|\frac{1}{\Omega}\frac{\partial^2\Omega}{\partial t^2}\right|\right|^{-1/2} \quad .$$

These quantities quantify the departure from an euclidean geometry, which can be used to restrict the validity of a classical measurement. Therefore, there should have some restriction on determining the properties of the system if there is a significant departure from euclidean geometry, i.e. there is a significant manifestation of non-euclideanity in Qwist.

To support the idea that action (H.1) correctly describes the dynamics of a relativistic charged "quantum" particle, we have studied its non-relativistic limit and shown that it recovers the usual Schrödinger quantum dynamics. In addition, we reformulated our dynamical variables to connect this relativistic system with the Klein-Gordon equation. In particular, it was necessary to define a new complex field that from our point of view mix geometrical degrees of freedom with the particle's hamilton principle function.

The present formalism is adequate to describe particles as well as anti-particles. However, it not yet clear how interactions between particles and anti-particles shall be included in this scenario. In addition, it is still an open issue the meaning of a many-particles system and its physical interpretation in view of the Qwist geometrical interpretation.

## Acknowledgments

We would like to thank CNPq of Brazil and MN also to FAPERJ for financial support. We would also like to thank the participants of "Pequeno Seminário" of ICRA-CBPF's Cosmology Group for useful discussions, comments and suggestions.

## .1. - Weyl geometry

The Weyl space is construct so as to incorporate the gauge transformation of a vector field ($w_\mu \to w_\mu + \Lambda_{,\mu}$) analogous to the electromagnetic gauge transformation [359]-[38]. This new vector field is associated with the non-zero covariant derivative of the metric tensor. The gauge transformation of the Weyl space is a combine transformation of the metric and of the weyl vector field. In a weyl gauge transformation we have

$$g_{\alpha\beta} \longrightarrow g'_{\alpha\beta} = e^{2\Lambda(x)} g_{\alpha\beta} \tag{.40}$$

$$w_\mu \longrightarrow w'_\mu = w_\mu + \Lambda_{,\mu} \tag{.41}$$

while the connection is constructed so that the covariant derivative of the metric is zero $Dg = 0$ (see ref.'s[359, 35]), or in components

$$D_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - g_{\rho\beta}\Gamma^\rho_{\mu\alpha} - g_{\alpha\rho}\Gamma^\rho_{\mu\beta} - 2w_\mu g_{\alpha\beta} = 0 \quad . \tag{.42}$$

This equation can be solved to give

$$\Gamma^\alpha_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} - \left(\delta^\alpha_\mu\, w_\nu + \delta^\alpha_\nu\, w_\mu - g_{\mu\nu}\, w^\alpha\right) \quad , \tag{.43}$$

where $\left\{ {\alpha \atop \mu\nu} \right\}$ is the Christoffel symbol. Note that the covariant derivative defined in (.42) is the usual covariant derivative constructed with the connection plus an extra term related to the weyl vector field ($w_\mu$). To distinguish it from the conventional covariant derivative, one call it co-covariant derivative and defined it as follows. If a tensor is transformed under a weyl gauge transformation into $A_\mu \to A'_\mu = e^{n\Lambda} A_\mu$ then it is called a tensor of power $n$ and its co-covariant derivative is given by

$$D_\mu \xi_\alpha \equiv \nabla_\mu \xi_\alpha - n\xi_\alpha \Lambda_\mu \quad , \tag{.44}$$

where the $\vec{\nabla}$ defines the covariant derivative constructed with the connection $\nabla_\nu \xi_\mu \equiv \xi_{\mu,\nu} - \Gamma^\alpha_{\nu\mu}\, \xi_\alpha$ . Hence, we have defined the metric $g_{\mu\nu}$ as a tensor of power 2 in the weyl sense.

From (.42) one can easily show that

$$\nabla_\mu g_{\alpha\beta} = 2w_\mu g_{\alpha\beta} \quad . \tag{.45}$$

Thus, it is also straightforward to show that the length of a vector $l \equiv \sqrt{g_{\mu\nu} l^\mu l^\nu}$ parallel transported will change from point to point such that in an infinitesimal displacement is given by

$$\delta l = l w_\mu \mathrm{d}x^\mu \quad . \tag{.46}$$

Note that the length of a vector is not gauge invariant. As a matter of fact, under a weyl gauge transformation we have $l \to l' = e^{\Lambda(x)} l$. Thus, by an adequate gauge choice one can always make $\delta l = 0$ in a point. It suffice to choose $\Lambda = -\int w_\mu \mathrm{d}x^\mu$, then

$$\delta l' = \delta\Lambda\, l' + e^\Lambda \delta l = l' \left(\Lambda_{,\mu} + w_\mu\right) \mathrm{d}x^\mu = 0.$$

However, its variation along a closed curve will not be zero since in general the curvature $K_{\mu\nu} \equiv w_{\mu,\nu} - w_{\nu,\mu} \neq 0$. There is a special case of Weyl space when the Weyl vector $w_\mu$ is the gradient of a function, i.e. $w_\mu = \partial_\mu f$. In this case, the length of a vector will not change along a closed curve. Furthermore, one can choose $\Lambda = -f$ so that $w'_\mu = f_{,\mu} + \Lambda_{,\mu} = 0$, which could motivate us to associate this subclass of Weyl spaces with conformal transformations of the metric tensor.

However, even in this special case when the weyl vector is the gradient of a function, the Weyl space is not equivalent to a conformal transformation of the associated Riemannian space.

A conformal transformation of the metric tensor takes for example $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\Lambda} g_{\mu\nu}$. This transformation maps a Riemannian space $\mathcal{M}$ characterized by $(g_{\mu\nu}, \left\{{\alpha \atop \mu\nu}\right\})$ into another Riemannian space $\widetilde{\mathcal{M}}$ with $(\tilde{g}_{\mu\nu}, \widetilde{\left\{{\alpha \atop \mu\nu}\right\}})$. Thus, we have

$$\widetilde{\left\{{\alpha \atop \mu\nu}\right\}} \equiv \frac{1}{2}\tilde{g}^{\alpha\xi}\left(\tilde{g}_{\xi\nu,\mu} + \tilde{g}_{\mu\xi,\nu} - \tilde{g}_{\mu\nu,\xi}\right) = \left\{{\alpha \atop \mu\nu}\right\} + \left(\delta^\alpha_\nu \Lambda_{,\mu} + \delta^\alpha_\mu \Lambda_{,\nu} - g_{\mu\nu} \Lambda^{,\alpha}\right)$$

which can be inverted

$$\left\{{\alpha \atop \mu\nu}\right\} = \widetilde{\left\{{\alpha \atop \mu\nu}\right\}} - \left(\delta^\alpha_\nu \Lambda_{,\mu} + \delta^\alpha_\mu \Lambda_{,\nu} - g_{\mu\nu} \Lambda^{,\alpha}\right) \tag{.47}$$

becoming very similar to (.43).

However, the new Riemannian manifold $\widetilde{\mathcal{M}}$ still satisfies the metricity condition, namely, $\widetilde{\nabla}_\alpha \tilde{g}_{\mu\nu} = 0$. The covariant derivative of the transformed metric tensor is still zero. The conformal transformation does not change the metricity condition. Note that if we blindly calculate $\nabla_\alpha \tilde{g}_{\mu\nu}$ we find

$$\nabla_\alpha \left(\widetilde{g}_{\mu\nu}\right) = \nabla_\alpha \left(e^{2\Lambda} \tilde{g}_{\mu\nu}\right) = 2\Lambda_{,\alpha} g_{\mu\nu} \tag{.48}$$

which is also very similar to (.45). However, contrary to (.45), (.48) has no physical meaning. It is the covariant derivative in $\mathcal{M}$ applied to the metric tensor of $\widetilde{\mathcal{M}}$.

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# A. Higgs mechanism without Higgs boson: M. Novello -U. Moschella

## A.1. abstract

We analyze the properties of the self-interaction of a spinor field $\Psi$ driven by Heisenberg dynamics. The system has global $\gamma_5-$invariance. It is possible to generalize this symmetry for a local space-time dependence by a minimal coupling of the axial vector constructed with the spinor field to a massless gauge vector field $W_\mu$. As a consequence of this coupling the gauge field acquires a mass when $\Psi$ is in its fundamental state. We can interpret this situation in terms of Higgs mechanism once the mass of the gauge field appears due to the non-linear Heisenberg dynamics that allows an apparent bosonisation which becomes the real vehicle for the generation of the effective mass.

## A.2. Heisenberg spinor field

We will call Heisenberg spinor (or H-field for short) a spinor field that satisfies the non linear equation [1]

$$i\gamma^\mu \partial_\mu \Psi - 2s\left(A + iB\gamma^5\right)\Psi = 0 \tag{A.1}$$

in which the constant $s$ has the dimension of (lenght)$^2$ and the quantities $A$ and $B$ are given in terms of the Heisenberg spinor $\Psi$ as:

$$A \equiv \overline{\Psi}\Psi \tag{A.2}$$

and

$$B \equiv i\overline{\Psi}\gamma^5\Psi \tag{A.3}$$

We use the convention as in [17] and the standard definitions which just for completeness we quote:

$$\bar{\Psi} \equiv \Psi^+\gamma^0.$$

The $\gamma^5$ is hermitian and the others obeys the relation

$$\gamma_\mu^+ = \gamma^0 \gamma_\mu \gamma^0.$$

This dynamics is obtained from the Lagrangian

$$L = L_D - \mathbf{V} = \frac{i}{2}\,\overline{\Psi}\,\gamma^\mu\,\partial_\mu\,\Psi - \frac{i}{2}\,\partial_\mu\overline{\Psi}\,\gamma^\mu\;\Psi - \mathbf{V} \tag{A.4}$$

The self-interacting term comes from a potential that can be described in terms either of a current-current or as a quartic of spinors:

$$\mathbf{V} = sJ_\mu\,J^\mu \tag{A.5}$$

or, equivalently

$$\mathbf{V} = s(A^2 + B^2) \tag{A.6}$$

The proof of this equivalence as well as the basis of most of the properties needed to analyse non-linear spinors comes from the Pauli-Kofink (PK) identities that establishes a set of tensor relations concerning elements of the four-dimensional Clifford $\gamma$-algebra. For any element $Q$ of this algebra the PK relation states the validity of

$$(\bar{\Psi}Q\gamma_\lambda\Psi)\gamma^\lambda\Psi = (\bar{\Psi}Q\Psi)\Psi - (\bar{\Psi}Q\gamma_5\Psi)\gamma_5\Psi. \tag{A.7}$$

for $Q$ equal to $I$, $\gamma^\mu$, $\gamma_5$ and $\gamma^\mu\gamma_5$. As a consequence of this relation we obtain two extremely important consequences:

- The norm of the currents $J_\mu \equiv \overline{\Psi}\,\gamma_\mu\,\Psi$ and $I_\mu \equiv \overline{\Psi}\,\gamma_\mu\,\gamma_5\,\Psi$ have the same strenght but opposite sign.
- The vectors $J_\mu$ and $I_\mu$ are orthogonal.

Indeed, using the PK relation we have

$$(\bar{\Psi}\gamma_\lambda\Psi)\gamma^\lambda\Psi = (\bar{\Psi}\Psi)\Psi - (\bar{\Psi}\gamma_5\Psi)\gamma_5\Psi.$$

Multiplying by $\bar{\Psi}$ and using the definitions above it follows

$$J^\mu J_\mu = A^2 + B^2. \tag{A.8}$$

We also have

$$(\bar{\Psi}\gamma_5\gamma_\lambda\Psi)\gamma^\lambda\Psi = (\bar{\Psi}\gamma_5\Psi)\Psi - (\bar{\Psi}\Psi)\gamma_5\Psi.$$

From which it follows that the norm of $I_\mu$ is

$$I^\mu I_\mu = -A^2 - B^2 \tag{A.9}$$

and that the four-vector currents are orthogonal

$$I_\mu J^\mu = 0. \tag{A.10}$$

From these results it follows that the current $J_\mu$ is a time-like vector; and the axial current is space-like. Thus, Heisenberg potential $\mathbf{V}$ is nothing but the norm of the four-vector current $J^\mu$.

### A.2.1. Symmetry

Let us consider the $\gamma_5-$map

$$\widetilde{\Psi} = (\cos\alpha + i\,\sin\alpha\,\gamma_5)\,\Psi \tag{A.11}$$

It yields for the scalars $A$ and $B$ the corresponding changes:

$$\begin{aligned} \widetilde{A} &= \cos 2\alpha\, A + sin2\alpha\, B \\ \widetilde{B} &= -\sin 2\alpha\, A + cos2\alpha\, B \end{aligned} \tag{A.12}$$

It will be convenient for latter use to define the associated scalar field $\varphi$ defined by

$$\varphi \equiv A + i\,B.$$

This scalar field changes under the above map as a rotation of $2\alpha$ :

$$\widetilde{\varphi} = e^{-2i\alpha}\,\varphi.$$

It follows that the Heisenberg potential is invariant under such map $\widetilde{\mathbf{V}} = \mathbf{V}$. The kinematical part of Lagrangian does not change if the parameter $\alpha$ is constant. Thus, Heisenberg dynamics is invariant under such constant $\gamma_5-$map.

## A.3. Fundamental solution

In linear Dirac dynamics a particular class of solutions (plane waves) is characterized by the eigenstate property

$$\partial_\mu \Psi = i\,k_\mu\,\Psi.$$

In the nonlinear Heisenber dynamics it is possible to find solutions that are defined by the property

$$\partial_\mu \Psi = \left(a\,J_\mu + b\,I_\mu \gamma^5\right)\,\Psi \tag{A.13}$$

where $a$ and $b$ are complex numbers of dimensionality $(lenght)^2$.

It is immediate to prove that if $\Psi$ satisfies this condition it satisfies automatically Heisenberg equation of motion if $a$ and $b$ are such that $2s = i\,(a-b)$.

This is a rather strong condition that deals with simple derivatives instead of the scalar structure obtained by the contraction with $\gamma_\mu$ typical of Dirac or even for the Heisenberg operators that appear in Dirac equation and in equation (A.1). Prior to anything one has to examine the compatibility of such condition which concerns all quantities that can be constructed with such spinors. It is a remarkable result that in order that the fundamental condition eq. (A.13) to be integrable constants $a$ and $b$ must satisfy a unique constraint given by $Re(a) - Re(b) = 0$.

Indeed, a direct calculation gives

$$\partial_\mu J_\nu = (a+\overline{a})J_\mu J_\nu + (b+\overline{b})I_\mu I_\nu \tag{A.14}$$

$$\partial_\mu A = (a+\overline{a})\, A\, J_\mu + (b-\overline{b})\, iB\, I_\mu. \tag{A.15}$$

$$\partial_\mu B = (a+\overline{a})\, B\, J_\mu + (b-\overline{b})\, iAI_\mu. \tag{A.16}$$

$$\partial_\mu I_\nu = (a+\overline{a})J_\mu I_\nu + (b+\overline{b})J_\nu I_\mu. \tag{A.17}$$

Thus,

$$[\partial_\mu, \partial_\nu]\, \Psi = \left(a\, \partial_{[\mu}\, J_{\nu]} + b\, \partial_{[\mu}\, I_{\nu]}\, \gamma^5\right)\, \Psi.$$

Now, the derivative of the currents yields

$$\partial_\mu J_\nu - \partial_\nu J_\mu = (a+\overline{a})[J_\mu,\, J_\nu] + (b+\overline{b})\, [I_\mu,\, I_\nu],$$

and

$$\partial_\mu I_\nu - \partial_\nu I_\mu = (a+\overline{a}-b-\overline{b})[J_\mu\, I_\nu - I_\mu\, J_\nu].$$

Thus the condition of integrability is given by

$$Re(a) = Re(b). \tag{A.18}$$

It is a rather long and tedious work to show that any combination $X$ constructed with $\Psi$ and for all elements of the Clifford algebra, the compatibility condition $[\partial_\mu, \partial_\nu]X = 0$ is automatically fulfilled once this unique condition (A.18) is satisfied.

## A.4. Double Heisenberg dynamics

The invariance of Heisenberg potential under the $\gamma_5-$map allows to consider more general non-linear terms to be added to the dynamics which still mantains the invariance of the theory. Let us consider an extra quadratic term such that the potential becomes

$$\mathbf{W} = s(A^2+B^2) + q\,(A^2+B^2)^2 \tag{A.19}$$

The dynamics that follows is

$$i\gamma^\mu \partial_\mu\, \Psi - 2\left(s + 2\, q\, J^2\right)\, (A + iB\gamma^5)\, \Psi = 0 \tag{A.20}$$

Should this new dynamics still admits a fundamental solution along the same lines as in the previous case discussed in the precedent session? We shall prove now that this is indeed possible in the case in which the norm of the current is constant $J^2 = J_0^2$. Let us set

$$\partial_\mu \Psi = \left(\widehat{a}\, J_\mu + \widehat{b}\, I_\mu \gamma^5\right)\, \Psi \tag{A.21}$$

where

$$\widehat{a} \equiv a + \alpha\, J^2$$

and

$$\widehat{b} \equiv b + \beta\, J^2.$$

Then, equation (A.21) satisfies identically equation (A.20 ) if the parameters are related by

$$\begin{aligned} 2\,s &= i(a-b) \\ 4\,q &= i(\alpha-\beta) \end{aligned} \tag{A.22}$$

Following the same lines as above the integrability condition is given by

$$a + \bar{a} = b + \bar{b} \tag{A.23}$$

$$\alpha + \bar{\alpha} = \beta + \bar{\beta} \tag{A.24}$$

$$J_0^2 = -\,\frac{a+\bar{a}}{\alpha+\bar{\alpha}}. \tag{A.25}$$

## A.4.1. Spontaneous symmetry breaking

Among all possible values of the norm of the current $J_0^2$ there are three that are singled out once they extremize the potential (A.19). The first one occurs when the norm of the current vanishes. In this case both scalars $A$ and $B$ vanish and the field reduces to an eigenstate of $\gamma_5$. This case is of no interest for us, once it represents an unstable state. The other two values are symmetrical under reflection $\varphi \Rightarrow -\,\varphi$. In this case the non-linear equation ( A.20 ) is reduced to particular solutions of the linear massless Dirac field, subjected to the constraint that minimizes the potential given by

$$J_0^2 = -\,\frac{s}{2q} > 0. \tag{A.26}$$

### Equivalent dynamics

Let us now prove a remarkable result that occurs when the field satisfy the fundamental solution. From the equation (A.21) it follows

$$\partial_\mu\, A = -i\, p\, B\, I_\mu. \tag{A.27}$$

$$\partial_\mu\, B = i\, p\, A\, I_\mu. \tag{A.28}$$

where

$$p \equiv b - \bar{b} + (\beta - \bar{\beta})\, J_0^2. \tag{A.29}$$

Then,

$$\partial_\mu A\, \partial^\mu A = p^2\, J_0^2\, A^2 \tag{A.30}$$

$$\partial_\mu B\, \partial^\mu B = p^2\, J_0^2\, B^2 \tag{A.31}$$

That is,

$$\partial_\mu A\, \partial^\mu A + \partial_\mu B\, \partial^\mu B = p^2\, (A^2 + B^2)^2 \tag{A.32}$$

In terms of the associated scalar field $\varphi = A + i\, B$ the double potential is nothing but the "kinematical" term once we can write

$$\partial_\mu \varphi^*\, \partial^\mu \varphi = p^2\, J^2. \tag{A.33}$$

The result displayed by equation ( A.33) can be used to convert Lagrangian

$$L = L_D - \mathbf{W} = L_D - s(A^2 + B^2) - q\, (A^2 + B^2)^2 \tag{A.34}$$

into the form

$$L = L_D - s(A^2 + B^2) - \sigma\, \partial_\mu \varphi^*\, \partial^\mu \varphi \tag{A.35}$$

where $\sigma \equiv p^2/q$. Indeed, the dynamics that follows from Lagrangian ( A.35 ) is given by

$$i\gamma^\mu \partial_\mu\, \Psi - 2s\, (A + iB\gamma^5)\, \Psi + 2\sigma\, (\Box A + i\, \Box B\, \gamma_5)\Psi = 0 \tag{A.36}$$

In the fundamental state we can write

$$(\Box A + i\, \Box B\, \gamma_5)\Psi = -\, \frac{2q}{\sigma}\, (A + i\, B\, \gamma_5)\, J^2\, \Psi \tag{A.37}$$

Thus, this equation ( A.36 ) goes into

$$i\gamma^\mu \partial_\mu\, \Psi - 2s\, (A + iB\gamma^5)\, \Psi - 4q\, J^2\, (A + i\, B\, \gamma_5)\, \Psi = 0 \tag{A.38}$$

The same result follows if one changes the fundamental state condition prior to the variation on Lagrangian ( A.35). This led us to conclude that when the system is in the fundamental state, the double potential can be substituted by the term $\partial_\mu \varphi^*\, \partial_\mu \varphi$.

## A.5. From global to local symmetry

In the case the parameter $\alpha$ of equation ( A.11) is a space-time dependent function, the global symmetry is broken. In order to restore locally the symmetry we follow the standard procedure and couple minimally the spinor-field with an external vector field $W_\mu$. To preserve the symmetry the gauge field must couple to the axial current through the form

$$\begin{aligned} L &= \frac{i}{2}\, \overline{\Psi}\, \gamma^\mu\, (\partial_\mu - ig\, W_\mu\, \gamma_5)\Psi \\ &- \frac{i}{2}\, (\partial_\mu - ig\, W_\mu\, \gamma_5)\overline{\Psi}\, \gamma^\mu\ \Psi - s(A^2 + B^2) \\ &- \sigma\, (\partial_\mu + 2i\, g\, W_\mu)\varphi^*\, (\partial^\mu - 2i\, g\, W^\mu)\varphi - \frac{1}{4}\, F \end{aligned} \tag{A.39}$$

where $F = F_{\mu\nu}\, F^{\mu\nu}$ and $F_{\mu\nu} \equiv W_{\mu,\nu} - W_{\nu,\mu}$. The local symmetry is guaranteed by the invariance of $F_{\mu\nu}$ under the gauge map $\widetilde{W}_\mu = W_\mu - g^{-1}\, \partial_\mu \alpha$.

The fundamental state (A.21) that minimizes the potential when $\Psi = \Psi_s$ is such that the norm of the current is constant $J_0^2 = -\, s/2q > 0$. This corresponds to a spontaneous broken symmetry. We then expand the field about this particular value of the minimum of the potential:

$$\Psi = \Psi_s + \chi. \tag{A.40}$$

Introducing this fluctuation in equation ( A.39 ) it follows that, among others, it appears a term of the form $m^2\, W_\mu W^\mu$. The presence of this term is responsible to provide a mass for the gauge field given by

$$m^2 = 4\, g^2 \sigma\, J_0^2.$$

## A.6. Comments

In the present scheme the responsible for giving mass to the gauge vector is the Heisenberg spinor field $\Psi$. This is made through the intervening of the auxiliary field $\varphi$. This scalar field appears as a kind of bosonisation, which can be interpreted as the actual vehicle for the generation of the effective mass.

In the fundamental state, the associated quantity $\varphi$ acts as a scalar field that satisfies the equation

$$\Box\varphi + p^2\, J_0^2\, \varphi = 0. \tag{A.41}$$

Once $p$ given by (A.29) is an imaginary number, $\varphi$ posses an imaginary mass. This causes no difficulty, once $\varphi$ is nothing but an auxiliary field[1]. This situation is very similar as to what occurs in the Higgs mechanism driven by a boson field: originally the boson appears to have an imaginary mass that allows the existence of a stable equilibrium point. Due to this, it follows that the gauge field interacting with such scalar acquires a real value for its mass: the imaginary mass of the scalar field induces a real mass for the gauge field.

Let us emphasize that the present mechanism is able to give mass only to gauge vector fields that couples with the axial current of the Heisenberg spinor. Thus such scheme does not change the fact that the photon is massless.

[1]Note that a plane wave solution of the double Heisenberg field $\Psi$ can be associated to an effective real positive mass term, in the case the norm of the current is constant.

# B. Gravitational Waves in Singular and Bouncing FLRW Universes: V. Antunes - E. Goulart - M. Novello

## B.1. abstract

We investigate the propagation of gravitational waves in two models belonging to the Friedman-Lemaître-Robertson-Walker (FLRW) class of cosmologies: the singular Einstein-Maxwell Universe (EMU), which has the electromagnetic field described by Maxwell's electrodynamics as the source of its geometry, and the bouncing Nonlinear Electrodynamics Universe (NLEU), which has the electromagnetic field described by a non-linear generalization of Maxwell's electrodynamics as the source of its geometry. We work with an explicitly gauge independent formulation of cosmological perturbations in FLRW models and analyze the qualitative features of the dynamical system that describes the propagation of primordial tensorial perturbations in both geometries. Based on this analysis we show that gravitational waves generated near a singularity or a bounce exhibit qualitatively different behavior.

## B.2. Introduction

In the Standard Cosmology, the model adopted to describe the "radiation era" of our universe consists of a Friedman-Lemaître-Robertson-Walker (FLRW) geometry with the electromagnetic field described by Maxwell's electrodynamics as its source [1]. This is the so called Einstein-Maxwell Universe and is generally considered a good description of the primitive universe at large scales after inflation. However, the fact that this model presents an initial singularity in its past, among other problems, suggests the necessity of formulating an alternative description which does not exhibits such undesired features.

Many attempts to formulate nonsingular nonstationary, or bouncing, FLRW models have been made in the past appealing to a variety of different mechanisms [2, 3, 4, 5, 6, 299, 8, 9, 24]. In most of these attempts, however, a radical revision of the Standard Cosmology is proposed, either by introducing exotic

forms of matter with negative energy density in the model, or by modifications in Einstein's theory of gravitation or the space-time structure.

Recently, a less radical alternative was proposed where the radiation era is modeled by a FLRW geometry with the electromagnetic field described by a non-linear generalization of Maxwell's electrodynamics as its source [11]. Matter is identified with a primordial plasma with negligible bulk viscosity terms in its electric conductivity [11, 391, 94], which is equivalent to taking only the average squared of the magnetic field $b^2$ as non-null [391, 14, 15]. In this model, a negative pressure in the primitive universe avoids the singularity, while the energy density is always positive. Non-linear corrections are important only in the initial stages of evolution of the universe, where the fields are very intense, and thus are insignificant at later eras. This will be called the Nonlinear Electrodynamics Universe, and will be taken here as a representative of the Bouncing Cosmology.

Although singular and bouncing models exhibit very different space-time structures in the primitive universe, they obviously must agree on its predictions about later eras. This poses the problem of how can one decide, on experimental grounds, which of these models is a better description of our Universe. A natural way to look for such experimental criteria would be to compare the way the presence of a bounce affects the evolution of primordial perturbations. In particular, gravitational waves seem to be specially suited to provide information about the space-time structure in early epochs of our universe, since, contrary to what happens with electromagnetic radiation, they are not shielded by any kind of matter.

In this article we compare the propagation of gravitational waves in the Einstein-Maxwell Universe (EMU) and the Nonlinear Electrodynamics Universe (NLEU). We work with an explicitly gauge invariant formulation of cosmological perturbations in models of the FLRW class, which we briefly discuss in section B.3. In section B.4 we consider the dynamical system that describes the evolution of tensorial perturbations in the EMU and NLEU. These perturbations represent gravitational waves propagating in an unperturbed background geometry. We next analyze the qualitative features of this dynamical system in both models. Finally, in section B.5 we compare the results and show that waves generated near a singularity or a bounce exhibit qualitatively different behavior.

## B.3. Preliminaries

We adopt Einstein's theory of gravitation, in which space-time is represented as a four-dimensional semi-Riemannian manifold with a metric tensor $g_{\alpha\beta}$ with signature $(+---)$, where Greek indices run into the set $\{0, 1, 2, 3\}$. Gravity

is described by the field equations

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = -T_{\alpha\beta} \ . \tag{B.1}$$

In the FLRW class of cosmological models, matter is identified with a perfect fluid with energy momentum tensor of the form

$$T_{\alpha\beta} = \rho V_\alpha V_\beta + p h_{\alpha\beta} \ , \tag{B.2}$$

where $\rho$ is the density, $p$ the pressure, $V^\alpha$ the velocity field of the fluid, with $g_{\alpha\beta} V^\alpha V^\beta = +1$, and $h_{\alpha\beta} = g_{\alpha\beta} - V_\alpha V_\beta$ is the spatial metric, from which we can define the projecting operator $h_\alpha^{\ \beta} = \delta_\alpha^{\ \beta} - V_\alpha V^\beta$ which projects into the subspace orthogonal to $V^\alpha$. Projected quantities will be represented with a hat, *e.g.* $\hat{V}_\alpha \equiv h_\alpha^{\ \beta} V_\beta$. We define the kinematical quantities, expansion $\Theta$, shear $\sigma_{\alpha\beta}$ and vorticity $\omega_{\alpha\beta}$, as follows[1]

$$\Theta \equiv \nabla_\alpha V^\alpha \ , \tag{B.3}$$

$$\sigma_{\alpha\beta} \equiv \frac{1}{2} \hat{\nabla}_{(\beta} V_{\alpha)} - \frac{1}{3} \Theta h_{\alpha\beta} \ , \tag{B.4}$$

$$\omega_{\alpha\beta} \equiv \frac{1}{2} \hat{\nabla}_{[\beta} V_{\alpha]} \ , \tag{B.5}$$

where the symbol $\nabla$ denotes covariant differentiation. In terms of these quantities, the gradient of the velocity field can be decomposed in its irreducible components

$$\nabla_\beta V_\alpha = \sigma_{\alpha\beta} + \frac{1}{3} \Theta h_{\alpha\beta} + \omega_{\alpha\beta} + \dot{V}_\alpha V_\beta \ , \tag{B.6}$$

where $\dot{V}^\alpha \equiv V^\beta \nabla_\beta V^\alpha$ is the acceleration. We also define the vorticity vector

$$\omega^\alpha \equiv -\frac{1}{2} \eta^{\alpha\beta\mu\nu} \omega_{\beta\mu} V_\nu \ , \tag{B.7}$$

where $\eta_{\alpha\beta\mu\nu}$ is the Levi-Civita alternating tensor. Taking the derivative $(\nabla_\beta V_\alpha)^{\cdot}$ and using this decomposition, one can derive the kinematical equations that describe the evolution of the expansion, shear and vorticity [16, 17].

The Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ can be decomposed in a "source" part, which depends on matter-energy distribution, and a trace-free part, the Weyl conformal curvature tensor $W_{\alpha\beta\mu\nu}$, defined by the expression

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - M_{\alpha\beta\mu\nu} + \frac{1}{6} g_{\alpha\beta\mu\nu} \ , \tag{B.8}$$

where

$$g_{\alpha\beta\mu\nu} \equiv g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \tag{B.9}$$

[1] We adopt the conventions $Q_{(\alpha\beta)} \equiv Q_{\alpha\beta} + Q_{\beta\alpha}$ and $Q_{[\alpha\beta]} \equiv Q_{\alpha\beta} - Q_{\beta\alpha}$, where $Q_{\alpha\beta}$ is any 2-covariant tensor.

and

$$2M_{\alpha\beta\mu\nu} \equiv R_{\alpha\mu}g_{\beta\nu} + R_{\beta\nu}g_{\alpha\mu} - R_{\alpha\nu}g_{\beta\mu} - R_{\beta\mu}g_{\alpha\nu} \ , \tag{B.10}$$

so that $W_{\alpha\beta\mu\nu}$ is the part of the curvature which is not determined locally by the matter-energy distribution, and thus represents the free gravitational field. By analogy with electrodynamics, we can define the electric end magnetic parts of the Weyl tensor with respect to a particular observer with four-velocity $V^\alpha$, respectively, as

$$E_{\alpha\beta} = -W_{\alpha\mu\beta\nu}V^\mu V^\nu \ , \tag{B.11}$$

$$B_{\alpha\beta} = -W^*_{\alpha\mu\beta\nu}V^\mu V^\nu \ . \tag{B.12}$$

where $*$ denotes the (Hodge) dual[2]. These definitions imply that the tensors $E_{\alpha\beta}$ and $B_{\alpha\beta}$ are symmetric, traceless and belong to the three-dimensional space orthogonal to $V^\alpha$. The combination of Einstein's equations and Bianchi identities allows the field equations of gravitation to be written in a form very similar to the field equations of electrodynamics [225, 19]

$$\nabla_\sigma W^{\alpha\beta\rho\sigma} = J^{\alpha\beta\rho} \ , \tag{B.13}$$

$$J^{\alpha\beta\rho} \equiv -\frac{1}{2}\,\nabla^{[\beta}T^{\alpha]\rho} + \frac{1}{6}\,g^{\rho[\beta}\nabla^{\alpha]}T \ . \tag{B.14}$$

The four independent projections of the divergence of the Weyl tensor

$$h^\sigma{}_\alpha V_\beta V_\mu \nabla_\nu W^{\alpha\beta\mu\nu} \ , \tag{B.15}$$

$$\eta_{\lambda\sigma\alpha\beta}V^\lambda V_\mu \nabla_\nu W^{\alpha\beta\mu\nu} \ , \tag{B.16}$$

$$h_{\mu(\sigma}\eta_{\rho)\lambda\alpha\beta}V^\lambda \nabla_\nu W^{\alpha\beta\mu\nu} \ , \tag{B.17}$$

$$h_{\mu(\rho}h_{\sigma)\alpha}V_\beta \nabla_\nu W^{\alpha\beta\mu\nu} \ , \tag{B.18}$$

leads to field equations, for a perfect fluid energy-momentum tensor, involving the tensors $E_{\alpha\beta}$ and $B_{\alpha\beta}$ and the kinematical quantities $\Theta$, $\sigma_{\alpha\beta}$, and $\omega_{\alpha\beta}$, with the form [150, 21]

[2]That is $Q^*_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}Q^{\mu\nu}$.

$$h^{\alpha\beta}h^{\rho\sigma}\nabla_\sigma E_{\beta\rho} + \eta^\alpha{}_{\beta\mu\nu}V^\beta B^{\nu\rho}\sigma^\mu{}_\rho + 3B^{\alpha\beta}\omega_\beta = \frac{1}{3}\,h^{\alpha\beta}\nabla_\beta\rho\ , \tag{B.19}$$

$$h^{\alpha\beta}h^{\rho\sigma}\nabla_\sigma B_{\beta\rho} - \eta^\alpha{}_{\beta\mu\nu}V^\beta E^{\nu\rho}\sigma^\mu{}_\rho - 3E^{\alpha\beta}\omega_\beta = (\rho+p)\omega^\alpha\ , \tag{B.20}$$

$$\begin{aligned} h^\alpha{}_\mu h^\beta{}_\nu \dot{E}^{\mu\nu} &+ \Theta E^{\alpha\beta} - \frac{1}{2}\,E_\mu{}^{(\alpha}\sigma^{\beta)\mu} - \frac{1}{2}\,E_\mu{}^{(\alpha}\omega^{\beta)\mu} \\ &+ \eta^{\alpha\rho\mu\nu}\eta^{\beta\sigma\lambda\theta}V_\rho V_\lambda E_{\mu\theta}\sigma_{\nu\sigma} + \dot{V}_\mu B_\nu{}^{(\alpha}\eta^{\beta)\lambda\mu\nu}V_\lambda \\ &- \frac{1}{2}\,h_\sigma{}^{(\beta}\eta^{\alpha)\lambda\mu\nu}V_\lambda\nabla_\mu B_\nu{}^\sigma = -\frac{1}{2}\,(\rho+p)\sigma^{\alpha\beta}\ , \end{aligned} \tag{B.21}$$

$$\begin{aligned} h^\alpha{}_\mu h^\beta{}_\nu \dot{B}^{\mu\nu} &+ \Theta B^{\alpha\beta} - \frac{1}{2}\,B_\mu{}^{(\alpha}\sigma^{\beta)\mu} - \frac{1}{2}\,B_\mu{}^{(\alpha}\omega^{\beta)\mu} \\ &+ \eta^{\alpha\rho\mu\nu}\eta^{\beta\sigma\lambda\theta}V_\rho V_\lambda B_{\mu\theta}\sigma_{\nu\sigma} - \dot{V}_\mu E_\nu{}^{(\alpha}\eta^{\beta)\lambda\mu\nu}V_\lambda \\ &+ \frac{1}{2}\,h_\sigma{}^{(\beta}\eta^{\alpha)\lambda\mu\nu}V_\lambda\nabla_\mu E_\nu{}^\sigma = 0\ . \end{aligned} \tag{B.22}$$

These are called quasi-Maxwellian equations of Gravitation in the Jordan-Ehlers-Kundt (JEK) formalism [225].

The class of FLRW universes is kinematically characterized by the null quantities $\dot{V}^\alpha = 0$, $\sigma_{\alpha\beta} = 0$ and $\omega_{\alpha\beta} = 0$. In this case, the space-time can be locally foliated by three-dimensional hyper-surfaces orthogonal to the flow lines of the fluid, each of these spatial sections having constant curvature. This implies that FLRW background geometry can be expressed, in the standard Gaussian system of coordinates, by the FLRW metric:

$$ds^2 = dt^2 - A^2(t)\left\{\frac{dr^2}{1-\epsilon r^2} + r^2\,(d\theta^2 + sin^2\theta\,d\varphi^2)\right\}\ , \tag{B.23}$$

where $A$ is the scale factor and $\epsilon = -1, 0, +1$ correspond to open, flat and closed spatial geometries, respectively. In terms of the conformal time $\tau = \int A^{-1}dt$, the metric (B.23) assume a conformally flat form. Since the Weyl tensor is null on conformally flat space-times, the JEK formalism leads to an explicitly gauge independent formulation of cosmological perturbations on models belonging to the FLRW class [22, 23].

## B.4. Gravitational Waves

We now consider perturbations of purely tensorial quantities only in the formalism we have outlined in the previous section. These are exactly the perturbations of the Weyl tensor which are not associated with density perturbations, and thus represent gravitational waves propagating in the unperturbed background geometry. We set $\delta V_\alpha = 0$, $\delta\dot{V}_\alpha = 0$, $\delta\omega_\alpha = 0$, $\delta\Theta = 0$, $\delta\rho = 0$ and $\delta p = 0$. As

already mentioned, the background geometry is kinematically characterized by the null quantities $\dot{V}_\alpha = 0$, $\sigma_{\alpha\beta} = 0$ and $\omega_{\alpha\beta} = 0$. Since the Weyl tensor is null on conformally flat space-times, we also have $E_{\alpha\beta} = 0$ and $B_{\alpha\beta} = 0$ in the background. In the perturbed geometry, therefore, for all these quantities we can set $E_{\alpha\beta} = \delta E_{\alpha\beta}$, $B_{\alpha\beta} = \delta B_{\alpha\beta}$, and so on. Thus, the perturbed quasi-Maxwellian equations, in this case, assume the form [21]

$$h^{\alpha\beta} h^{\rho\sigma} \nabla_\sigma E_{\beta\rho} = 0 \ , \tag{B.24}$$

$$h^{\alpha\beta} h^{\rho\sigma} \nabla_\sigma B_{\beta\rho} = 0 \ , \tag{B.25}$$

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{E}_{\mu\nu} + \Theta E_{\alpha\beta} - \frac{1}{2} h_{(\alpha}^{\ \mu} h_{\beta)}^{\ \nu} V_\theta \eta_\nu{}^{\theta\lambda\rho} \nabla_\lambda B_{\rho\mu} + \frac{1}{2} (\rho + p) \sigma_{\alpha\beta} = 0 \ , \tag{B.26}$$

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{B}_{\mu\nu} + \Theta B_{\alpha\beta} + \frac{1}{2} h_{(\alpha}^{\ \mu} h_{\beta)}^{\ \nu} V_\theta \eta_\nu{}^{\theta\lambda\rho} \nabla_\lambda E_{\rho\mu} = 0 \ . \tag{B.27}$$

We also have the perturbed auxiliary kinematical equation

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{\sigma}_{\mu\nu} + \frac{2}{3} \ \Theta \sigma_{\alpha\beta} + E_{\alpha\beta} = 0 \ , \tag{B.28}$$

and the perturbed constraint equation

$$\frac{1}{2} h_{(\alpha}^{\ \mu} h_{\beta)}^{\ \nu} V_\theta \eta_\nu{}^{\theta\lambda\rho} \nabla_\lambda \sigma_{\rho\mu} = B_{\alpha\beta} \ . \tag{B.29}$$

We define a basis of tensorial harmonics $\hat{U}_{\alpha\beta}^{\ (k)}$ on each spatial section by [259, 25]

$$\hat{\nabla}^2 \hat{U}_{\alpha\beta}^{\ (k)} = \frac{k^2}{A^2} \hat{U}_{\alpha\beta}^{\ (k)} \ , \tag{B.30}$$

where $\hat{\nabla}^2 \equiv h^{\alpha\beta} \nabla_\alpha \nabla_\beta$ and $k$ is the wavenumber. This basis is symmetric, $\hat{U}_{\alpha\beta}^{\ (k)} = \hat{U}_{\beta\alpha}^{\ (k)}$, and has the following additional properties

$$\left( \hat{U}_{\alpha\beta}^{\ (k)} \right)^{\cdot} = 0 \ , \quad h^{\alpha\beta} \hat{U}_{\alpha\beta}^{\ (k)} = 0 \ , \quad \hat{\nabla}^\alpha \hat{U}_{\alpha\beta}^{\ (k)} = 0 \ . \tag{B.31}$$

We also define the associated tensorial basis [25]

$$P\hat{U}_{\alpha\beta}^{\ (k)} \equiv \frac{1}{2} h_{(\alpha}^{\ \lambda} h_{\beta)}^{\ \mu} V_\theta \eta^{\rho\sigma\theta}{}_\lambda \nabla_\sigma \hat{U}_{\mu\rho}^{\ (k)} \ , \tag{B.32}$$

which satisfies the following relations

$$\left( P\hat{U}_{\alpha\beta}^{\ (k)} \right)^{\cdot} = -\frac{1}{3} \ \Theta \ P\hat{U}_{\alpha\beta}^{\ (k)} \ , \tag{B.33}$$

$$PP\hat{U}_{\alpha\beta}^{\ (k)} = \left( \rho - \frac{1}{3} \ \Theta^2 + \frac{k^2}{A^2} \right) \hat{U}_{\alpha\beta}^{\ (k)} \ . \tag{B.34}$$

In terms of these bases, the perturbed tensorial quantities can be expanded as follows

$$\hat{E}_{\alpha\beta} = \sum E^{(k)}\, \hat{U}^{\ (k)}_{\alpha\beta} \ , \tag{B.35}$$

$$\hat{B}_{\alpha\beta} = \sum B^{(k)}\, P\hat{U}^{\ (k)}_{\alpha\beta} \ , \tag{B.36}$$

$$\hat{\sigma}_{\alpha\beta} = \sum \sigma^{(k)}\, \hat{U}^{\ (k)}_{\alpha\beta} \ . \tag{B.37}$$

Using these expansions and relations (B.31)-(B.34), the constraint equation (B.29) assumes the form

$$\sigma - B = 0 \ , \tag{B.38}$$

where we have dropped the indices $k$ for notational simplicity. Thus, the system of perturbed quasi-Maxwellian equations (B.24)-(B.27), together with the perturbed auxiliary kinematical equation (B.28) and constraint equation (B.29), reduces to the following set of closed dynamical systems for the coefficients of the electric part of the Weyl tensor $E$ and the shear $\sigma$ [25]

$$\dot{E} + \Theta E - \left\{ \frac{1}{2}\,(\rho - p) - \frac{1}{3}\,\Theta^2 + \frac{k^2}{A^2} \right\} \sigma = 0 \ , \tag{B.39}$$

$$\dot{\sigma} + \frac{2}{3}\,\Theta\sigma + E = 0 \ , \tag{B.40}$$

where $A$ is the scale factor. This system completely describes the propagation of gravitational waves in FLRW backgrounds. Making the definitions

$$X \equiv \begin{pmatrix} E \\ \sigma \end{pmatrix} \ , \quad S \equiv \begin{pmatrix} -\Theta & f \\ -1 & -\frac{2}{3}\,\Theta \end{pmatrix} \ , \tag{B.41}$$

where $f \equiv \frac{1}{2}\,(\rho - p) - \frac{1}{3}\,\Theta^2 + \left(\frac{k}{A}\right)^2$, we can write the system (B.39)-(B.40) in matrix form as follows

$$\dot{X} = S(t,\mu)X \tag{B.42}$$

where $\mu$ is proportional to $k^2$. We remark that for non-stationary universes the fundamental matrix $S(t,\mu)$ depends explicitly on time, and the linear system (B.42) is non-autonomous. However, it can be trivially mapped into an autonomous 3-dimensional non-linear system by defining a new parameter $s$ and writing

$$\begin{pmatrix} \dot{X} \\ \dot{t} \end{pmatrix} = \begin{pmatrix} S(t,\mu)X \\ 1 \end{pmatrix} \tag{B.43}$$

where the dot now means derivation with respect to $s$. Let us denote the right hand side of equation (B.43) by $\phi(X,t,\mu)$. One can obtain information about the qualitative behavior of system (B.43) in a neighborhood $U(t_0) \times I(t_0)$ of a point $(X,t) = (0,t_0)$, where $U(t_0)$ is contained in the $t = t_0$ section of the

phase-space and $I(t_0)$ is an interval of the $t$-axis containing $t_0$, by making the expansion of $\phi$ with respect to $E$ and $\sigma$ up to first order

$$\phi(X,t)_{|U(t_0)} \approx \phi_0 + \left(\frac{\partial\phi}{\partial E}\right)_{|(0,t_0)} E + \left(\frac{\partial\phi}{\partial\sigma}\right)_{|(0,t_0)} \sigma \tag{B.44}$$

where $\boldsymbol{\phi}_0 = (0,1)$. This leads to the linearized form of the system (B.43)

$$\begin{pmatrix} \dot{X} \\ \dot{t} \end{pmatrix} \approx \begin{pmatrix} S(t_0,\mu)X \\ 1 \end{pmatrix} \tag{B.45}$$

valid in the region $U(t_0) \times I(t_0)$, with the following solution

$$\begin{pmatrix} X(s) \\ t(s) \end{pmatrix} \approx \begin{pmatrix} e^{sS(t_0)}X_0 \\ s + t_0 \end{pmatrix} \tag{B.46}$$

for $(X_0, t_0)$ in $U(t_0)$. This allows one to determine the integral curves of system (B.45) contained in the "tube" $U(t_0) \times I(t_0)$. However, we are mainly interested in the characterization of the qualitative behavior of those integral curves. From equation (B.46) it follows that this behavior is determined by the eigenvalues of the matrix $S$, and since $\det S > 0$ on both models considered here, one concludes that there are only two possible behaviors, according to the sign of the discriminant $\Delta = (\mathrm{tr}S)^2 - 4\det S$. However, instead of studying the behavior of those integral curves directly, using equation (B.46), we will determine for each fixed value of $t$ in $U(t_0) \times I(t_0)$ the phase diagrams of equation (B.45) in the neighborhood $U(t)$ contained in the corresponding $t =$ const. section. This can be viewed as a time projection of the integral curves, in this approximation. We also define, for convenience, the new "discriminant"

$$\delta \equiv \log\left[\frac{(\mathrm{tr}S)^2}{4\det S}\right] , \tag{B.47}$$

which has exactly the same properties as $\Delta$ in what concerns the determination of the qualitative behavior of the integral curves of system (B.45) and its $t =$ const. phase diagrams, namely trajectories form a focus if $\delta < 0$, or they form a node if $\delta > 0$. Since the sign of $\delta$ is invariant in a sufficiently small region $U(t_0) \times I(t_0)$, we can foliate it by $t =$ const. sections and classify their phase-space structure according to the nature of the phase diagrams. We will say that $U(t_0) \times I(t_0)$ has a focus or a node structure, accordingly. This leads to a way to compare the structure of the phase-space of system (B.43) along the $t$-axis for different models.

We now apply this procedure to study the properties of the dynamical system (B.42) in the Einstein-Maxwell Universe (EMU) and the Nonlinear Electrodynamics Universe (NLEU).

### B.4.1. Gravitational Waves in the Einstein-Maxwell Universe

The source of the background geometry in Einstein-Maxwell Universe (EMU) is the electromagnetic field as described by Maxwell's electrodynamics. Since in FLRW universes the spatial sections are isotropic, only a disordered distribution of electromagnetic radiation can generate a model of this class. This is attained by a process of spatial average of electric and magnetic fields over large scales [26, 27]. The spatial average of an arbitrary quantity $Q$ is defined as follows

$$\langle Q \rangle \equiv \lim_{V \to V_0} \frac{1}{V} \int \sqrt{-h}\, Q\, d^3x \ , \tag{B.48}$$

for an arbitrarily large time-dependent spatial volume $V_0 = \int \sqrt{-h}\, d^3x$, where $h \equiv \det(h_{\alpha\beta})$. Electric and magnetic fields will be denoted by lowercase letters. We set the spatial averages

$$\langle e_i \rangle = 0 \ , \quad \langle b_i \rangle = 0 \ , \quad \langle e_i b_j \rangle = 0 \ , \tag{B.49}$$

$$\langle e_i e_j \rangle = -\frac{1}{3}\, e^2 h_{ij} \ , \tag{B.50}$$

$$\langle b_i b_j \rangle = -\frac{1}{3}\, b^2 h_{ij} \ , \tag{B.51}$$

where the Latin indices run into the set $\{1, 2, 3\}$. As a result, the energy-momentum tensor for electromagnetic radiation assumes the form of the one for a perfect fluid

$$\langle T_{\alpha\beta} \rangle = \rho_\gamma V_\alpha V_\beta + p_\gamma h_{\alpha\beta} \ , \tag{B.52}$$

with density and pressure defined by the relation $\rho_\gamma = 3p_\gamma = (e^2 + b^2)/2$.

Restricting our analysis to the flat case[3] ($\epsilon = 0$), from Einstein's field equations and the kinematical equations mentioned in the preceding section, it follows that the EMU is characterized by the following unperturbed quantities [355, 29]

$$A(t) = A_0 t^{1/2} \ , \tag{B.53}$$

$$\rho_\gamma(t) = 3p_\gamma(t) = \frac{3}{4t^2} \ , \tag{B.54}$$

$$\Theta(t) = \frac{3}{2t} \ , \tag{B.55}$$

where $A_0$ is an arbitrary constant. It is clear that $A \to 0$ as $t \to 0$, and so the EMU has an initial singularity, as expected. From these expressions we see that the discriminant $\delta$, in this case, assumes the form

$$\delta(t, \mu) = \log \left[ \frac{9}{16\mu t} \right] \ , \tag{B.56}$$

[3]Since we are interested in the behavior of gravitational waves in the primitive universe, there is no loss of generality in this choice.

where $\mu \equiv k^2/A_0^2$. The figure shows the behavior of $\delta$ for three particular values of the parameter $\mu$. The solution of the equation $\delta(t,\mu) = 0$ determines the time where a change in the local structure of the phase-space of the system occurs

$$t_c(\mu) = \frac{9}{16\mu} = \frac{9A_0^2}{16(2\pi)^2}\,\lambda^2 \ , \tag{B.57}$$

where we have expressed the parameter $\mu$ in terms of the wavelength $\lambda = 2\pi/k$. Thus, for all finite wavelengths, the structure of the phase-space of the system presents a node/focus transition, *i.e.* for $t < t_c$ it has a node structure, and for $t > t_c$ it has a focus structure (table B.1).

We remark that $\mathrm{tr}\, S < 0$ for all $t$, and the integral curves of the system are asymptotically stable in the EMU. This is nothing but a consequence of the fact that the expansion of the universe causes a damping of gravitational waves. Since the frequency is inversely proportional to $A$, these waves are red-shifted by the expansion of the universe.

### B.4.2. Gravitational Waves in the Nonlinear Electrodynamics Universe

We now consider the cosmological model of the FLRW class which has as the source of its geometry the electromagnetic field as described by a toy-model generalization of Maxwell's electrodynamics. This is defined by the non-linear Lagrangian [11]

$$\mathcal{L} = -\frac{1}{4}\,F + \alpha F^2 + \beta G^2 \ , \tag{B.58}$$

where $F \equiv F_{\alpha\beta}F^{\alpha\beta}$ and $2G \equiv \eta_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}$ are the field invariants. By a similar process of macroscopic spatial average as mentioned in the preceding section, and identifying the matter with a primordial plasma with null average electric field, the energy-momentum tensor assumes again the form of the one for a perfect fluid (E.344), with density and pressure given by the expressions

$$\rho_\gamma = \frac{1}{2}\,b^2(1 - 8\alpha b^2) \ , \tag{B.59}$$

$$p_\gamma = \frac{1}{6}\,b^2(1 - 40\alpha b^2) \ . \tag{B.60}$$

From these expressions one can see that for $\alpha$ in the interval $\frac{1}{40b^2} < \alpha < \frac{1}{8b^2}$ the pressure can become negative for high values of $b$, as expected in the primitive universe, while the density is always positive.

Restricting, as before, our analysis to the flat case ($\epsilon = 0$), the Nonlinear Electrodynamics Universe (NLEU) is characterized by the unperturbed quantities [11]

$$A^2(t) = b_0\left[\frac{2}{3}\,(\,t^2 + 12\alpha\,)\right]^{1/2} \ , \tag{B.61}$$

$$\rho_\gamma(t) = \frac{3}{4}\frac{t^2}{(t^2+12\alpha)^2} \ , \tag{B.62}$$

$$p_\gamma(t) = \frac{1}{4}\frac{t^2-48\alpha}{(t^2+12\alpha)^2} \ , \tag{B.63}$$

$$\Theta(t) = \frac{3}{2}\frac{t}{(t^2+12\alpha)} \ , \tag{B.64}$$

where $b_0 = bA^2 = \text{const}$. In this case we see that at $t = 0$ the universe attains a minimum radius $A^2_{min} = b_0\sqrt{8\alpha}$. Making the choice $A_0^2 = \sqrt{\frac{2}{3}}\, b_0$, where $A_0$ is defined as in equation (B.53), the discriminant $\delta$ assumes the form

$$\delta(t,\mu;\alpha) = \log\left[\frac{9t^2-96\alpha}{16\mu(t^2+12\alpha)^{3/2}}\right] \ , \tag{B.65}$$

where again $\mu \equiv k^2/A_0^2$. We can see that the scale factor $A$ and the discriminant $\delta$ here coincides with those for the EMU as $\alpha \to 0$. This is clearly also true for $t \to \infty$, so that non-linear corrections of Maxwell's electrodynamics are only relevant in the primitive universe, as mentioned before. Since $\delta$ is symmetric in $t$, we restrict our analysis to the expansive phase $t > 0$. The figure (bottom) shows the behavior of $\delta$ for three particular values of $\mu$ and for $\alpha = 0.01$. The equation $\delta(t > 0) = 0$ has two solutions, one solution or no solution at all, depending on the value of the parameter $\mu$. Thus, the structure of the phase-space of the system in the expansive phase presents a focus/node/focus transition, or no transition at all (always a focus), according to $\mu < \mu_c(\alpha)$ or $\mu > \mu_c(\alpha)$, respectively[4]. We denote the critical times associated with the case $\mu < \mu_c(\alpha)$ by $t^+_{c\,1}(\alpha)$ and $t^+_{c\,2}(\alpha)$, where $t^+_{c\,1}(\alpha) < t^+_{c\,2}(\alpha)$. Table B.1 shows the possible structures of the phase-space of the system in the expansive phase for each range of wavelengths.

We remark that $\operatorname{tr} S < 0$ for $t > 0$ and the integral curves of the system are asymptotically stable. Again, gravitational waves are red-shifted by the expansion of the universe.

As we have mentioned before, $\delta$ is symmetric in $t$ and thus the same behavior found for the system in the expansive phase is also found in the contractive, pre-bounce, phase ($t < 0$), except that now, since $\operatorname{tr} S > 0$, the integral curves are unstable and gravitational waves are blue-shifted until the bounce ($t \to 0$).

## B.5. Comparative Analysis

The figure compares the behavior of $\delta$ in the EMU (top) and the NLEU (bottom) for values of $\mu$ in the ranges $\mu < \mu_c(\alpha)$ and $\mu > \mu_c(\alpha)$, respectively. This shows

[4]We ignore the case $\mu = \mu_c(\alpha)$ for which the equation $\delta(t > 0) = 0$ has only one solution, since in the corresponding critical time no change of the local phase-space structure occurs, and thus this is equivalent to the case $\mu > \mu_c(\alpha)$

Table B.1.: Structure of the phase-space of $S$ in the EMU and NLEU ($t > 0$) as a function of the wavelength $\lambda$.

| model | wavelength | time interval | structure |
|---|---|---|---|
| EMU[1] | $0 < \lambda < \infty$ | $0 < t < t_c$ | node |
| | | $t_c < t < \infty$ | focus |
| NLEU[2] | $\lambda_c < \lambda < \infty$ | $0 < t < t^+_{c\,1}$ | focus |
| | | $t^+_{c\,1} < t < t^+_{c\,2}$ | node |
| | | $t^+_{c\,2} < t < \infty$ | focus |
| | $0 < \lambda < \lambda_c$ | $0 < t < \infty$ | focus |

a clear difference between the structure of the phase-space of the system (B.43) in these models near $t = 0$, which corresponds either to the singularity or the bounce.

In the NLEU, for waves with wavelengths in the range $\lambda > \lambda_c(\alpha)$, it is the critical time $t^+_{c\,1}(\alpha)$ that determines how near of $t = 0$ those waves should be generated for the different behavior to emerge, while for waves with wavelengths in the range $\lambda < \lambda_c(\alpha)$, it is the critical time $t_c$ defined in eq. (B.57) that determines this boundary. This has the advantage of being independent of $\alpha$.

In the singular EMU model the phase-space structure of the system exhibit a node/focus transition. This structure is independent of the perturbation wavelength. On the other hand, for the bouncing NLEU model the phase-space structure may exhibit a focus/node/focus transition, or no transition at all, according to the values of the perturbation wavelength. Although only in the first case, *i.e.* $\lambda > \lambda_c$, there is the double regime transition, both cases present a focus structure near the bounce, in contrast to what is observed in the singular case. Table B.1 shows a comparison between the structure of the phase-space of system (B.45) in EMU and NLEU. The figure gives schematic representations of the structure of the phase-space of system (B.45) in EMU (top), NLEU for $\lambda > \lambda_c$ (center) and $\lambda < \lambda_c$ (bottom).

## B.6. Conclusion

In this article we have investigated the propagation of gravitational waves in a singular and a bouncing model for the radiation era of our universe. In the model adopted here as a representative of the Bouncing Cosmology, the source of the geometry is a nonlinear generalization of Maxwell's electrodynamics. The avoidance of the singularity is due to nonlinear corrections of Maxwell's theory.

In both cases, we see that the measurement of tidal effects and the shear anisotropy of a system of test particles on a gravitational antenna would result in $E$-$\sigma$ curves for waves generated shortly after the singularity or the bounce (that is, near $t = 0$) with a non-periodic or a periodic behavior, respectively.

This analysis suggests that there will be regimes associated with sufficiently large wavelengths for which we expect a different behavior for primordial perturbations propagating in a singular or a bouncing background geometry. This qualitative analysis could be used, at least in principle, to characterize the presence of a singularity or a bounce in our universe.

Even though the advanced stage of development of some gravitational wave detectors suggests that the observation of these waves is about to become a reality in the next years, the "cooling down" caused by the expansion of the universe would make it extremely difficult to detect waves of cosmic origins, such as those generated in the primitive universe. In any case, the study of tensor perturbations would still be very significant if somehow we could associate the propagation of gravitational waves with other physical processes (see [30]).

We postpone a careful investigation of the subtle experimental issues related to the analysis presented here to another paper.

Acknowledgments
The authors would like to thank CNPq and FAPERJ for a grant.

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# C. Scalar Perturbations in Scalar Field Quantum Cosmology : F. T. Falciano -N. Pinto-Neto

In this paper it is shown how to obtain the simplest equations for the Mukhanov-Sasaki variables describing quantum linear scalar perturbations in the case of scalar fields without potential term. This was done through the implementation of canonical transformations at the classical level, and unitary transformations at the quantum level, without ever using any classical background equation, and it completes the simplification initiated in investigations by Langlois [2], and Pinho and Pinto-Neto [4] for this case. These equations were then used to calculate the spectrum index $n_s$ of quantum scalar perturbations of a non-singular inflationary quantum background model, which starts at infinity past from flat space-time with Planckian size spacelike hypersurfaces, and inflates due to a quantum cosmological effect, until it makes an analytical graceful exit from this inflationary epoch to a decelerated classical stiff matter expansion phase. The result is $n_s = 3$, incompatible with observations.

## C.1. Introduction

The usual theory of cosmological perturbations, with their simple equations Ref. [1], relies essentially on the assumptions that the background is described by pure classical General Relativity (GR), while the perturbations thereof stem from quantum fluctuations. It is a semiclassical approach, where the background is classical and the perturbations are quantized, and the fact that the background satisfies Einstein's equations is heavily used in the simplification of the equations. In Refs. [3, 4, 5], which assume the validity of the Einstein-Hilbert action, it was shown that such simple equations for quantum linear cosmological perturbations can also be obtained without ever using any equations for the background. This can be accomplished through a series of canonical transformations and redefinitions of the lapse function. These results open the way to also quantise the background, and use these simple equations to evaluate the evolution of the quantum linear perturbations on it. Indeed, such results were applied to quantum bouncing backgrounds, and spectral indices for tensor and scalar perturbations were calculated in Refs. [6, 7].

The matter content used in these papers were assumed to be either a single perfect fluid or a single scalar field. In the case of perfect fluids, the equations were simplified up to their simplest possible form, both for tensor and scalar perturbations. For the case of scalar fields, this simplest form was achieved for tensor perturbations but not for scalar perturbations. One ended in a intermediate stage that needed further simplifications in order to be applied to quantum backgrounds Refs. [4, 2].

Meanwhile, a non-singular inflationary model was found Ref. [9] containing a single scalar field without potential term, which starts at infinity past from flat space-time with Planckian size spacelike hypersurfaces, and inflates, due to a quantum cosmological effect, until it makes an analytical graceful exit from this inflationary epoch to a decelerated classical stiff matter expansion phase. It should be interesting to investigate if this model could generate an almost scale invariant spectrum of scalar perturbations, as observed Ref. [8]. However, without simple equations governing the evolution of the perturbations, the investigation becomes rather cumbersome.

The aim of this paper is twofold: complete the simplification initiated in Refs. [4, 2], and apply it to the background described in Ref. [9]. In fact, after performing some canonical transformations at the classical level, and unitary transformations at the quantum level, we were able to obtain the simple equations for linear scalar perturbations of Ref. [1] for the case of scalar fields without potential, without ever using any classical background equation. These perturbation equations were then used to calculate the spectrum index $n_s$ of the background model of Ref. [9] yielding $n_s = 3$, incompatible with observations [8] ($n_s \approx 1$). Hence, even though the quantum background model has some attractive features, the model should be discarded.

The paper is organized as follows: in the next section, we briefly summarize the results of Ref. [9]. In section III, the simplification of the second order hamiltonian for the scalar perturbations is implemented, and the full quantization of the system, background and perturbations, is performed. The quantum background trajectories are then used to induce a time evolution for the Heisenberg operators describing the perturbations, yielding simple dynamical equations for the quantum perturbations. In Section IV, we calculate the spectral index of scalar perturbations in the background presented in Section II, using the equations obtained in Section III. Section V presents our conclusions.

## C.2. Bohm-de Broglie interpretation of a quantum non-singular inflationary background model

In this section, we first briefly highlight the main characteristics of the Bohm-de Broglie quantisation scheme, restricting our discussion to the homogeneous minisuperspace models which have a finite number of degrees of freedom. We

then apply it to the quantisation of the background geometry with a massless scalar field without potential term.

The Wheeler-DeWitt equation of a minisuperspace model is obtained through the Dirac quantization procedure, where the wave function must be annihilated by the operator version of the Hamiltonian constraint

$$\mathcal{H}(\hat{p}^\mu, \hat{q}_\mu)\Psi(q) = 0 \quad . \tag{C.1}$$

The quantities $\hat{p}^\mu, \hat{q}_\mu$ are the phase space operators related to the homogeneous degrees of freedom of the model. Usually this equation can be written as

$$-\frac{1}{2} f_{\rho\sigma}(q_\mu) \frac{\partial \Psi(q)}{\partial q_\rho \partial q_\sigma} + U(q_\mu)\Psi(q) = 0 \quad , \tag{C.2}$$

where $f_{\rho\sigma}(q_\mu)$ is the minisuperspace DeWitt metric of the model, whose inverse is denoted by $f^{\rho\sigma}(q_\mu)$.

Writing $\Psi$ in polar form, $\Psi = R\exp(iS)$, and substituting it into (C.2), we obtain the following equations:

$$\frac{1}{2} f_{\rho\sigma}(q_\mu) \frac{\partial S}{\partial q_\rho} \frac{\partial S}{\partial q_\sigma} + U(q_\mu) + Q(q_\mu) = 0 \quad , \tag{C.3}$$

$$f_{\rho\sigma}(q_\mu) \frac{\partial}{\partial q_\rho} \left( R^2 \frac{\partial S}{\partial q_\sigma} \right) = 0 \quad , \tag{C.4}$$

where

$$Q(q_\mu) \equiv -\frac{1}{2R} f_{\rho\sigma} \frac{\partial^2 R}{\partial q_\rho \partial q_\sigma} \tag{C.5}$$

is called the quantum potential.

The Bohm -de Broglie interpretation applied to quantum cosmology states that the trajectories $q_\mu(t)$ are real, independently of any observations. Equation (C.3) represents their Hamilton-Jacobi equation, which is the classical one added with a quantum potential term Eq. (C.5) responsible for the quantum effects. This suggests to define

$$p^\rho = \frac{\partial S}{\partial q_\rho}, \tag{C.6}$$

where the momenta are related to the velocities in the usual way:

$$p^\rho = f^{\rho\sigma} \frac{1}{N} \frac{\partial q_\sigma}{\partial t}. \tag{C.7}$$

To obtain the quantum trajectories we have to solve the following system of first order differential equations, called the guidance relations:

$$\frac{\partial S(q_\rho)}{\partial q_\rho} = f^{\rho\sigma} \frac{1}{N} \dot{q}_\sigma \quad . \tag{C.8}$$

Eqs. (C.8) are invariant under time reparametrization. Hence, even at the quantum level, different choices of $N(t)$ yield the same space-time geometry for a given non-classical solution $q_\alpha(t)$. There is no problem of time in the Bohm-de Broglie interpretation of minisuperspace quantum cosmology Ref. [10]. We will return to this point in the next section.

We now apply this interpretation to the situation where $\mathcal{H}$ in Eq. (C.1) is given by

$$H_0^{(0)} = \frac{\sqrt{2V}}{2\ell_{Pl}e^{3\alpha}}\left(-P_\alpha^2 + P_\varphi^2\right) \quad , \tag{C.9}$$

which was worked out in Ref. [9]. The variables are dimensionless with $\varphi$ describing the scalar field degree of freedom and $\alpha$ associated to the scale factor through $\alpha \equiv \log(a)$. The main feature of this model is the possibility to obtain a non-singular inflationary model similar to the pre-big bang model Refs. [21]-[24], with a minimum volume spatial section in the infinity past, or the emergent model Ref. [150] for flat spatial sections, without any graceful exit problem.

We take as solution of the background Wheeler-DeWitt equation, $\hat{H}_0^{(0)}\Psi(a,\varphi) = 0$, a gaussian superposition of WKB solutions. The resulting wave function is (see Ref. [9] for details)

$$\begin{aligned}\Psi(\alpha,\varphi) &= 2\sqrt{\pi|h|}\Bigg[\exp i\left(-\frac{h}{2}\left(\alpha+\varphi\right)^2 + d\left(\alpha+\varphi\right) + \frac{\pi}{4}\right) \\ &\quad + \exp i\left(-\frac{h}{2}\left(\alpha-\varphi\right)^2 + d\left(\alpha-\varphi\right) + \frac{\pi}{4}\right)\Bigg] \, ,\end{aligned} \tag{C.10}$$

where $h$ and $d$ are two positive free parameters associated to the variance and the displacement of the gaussian superposition, respectively.

The norm of the wave-function is given by $R = 4\sqrt{\pi|h|}\cos[\varphi(h\alpha - d)]$, yielding the quantum potential, Eq. (C.5),

$$Q = (h\alpha - d)^2 - h^2\varphi^2 \quad . \tag{C.11}$$

The guidance relations, given by Eq. (C.8) with the choice $N = \frac{\ell_{Pl}}{\sqrt{2V}}e^{3\alpha}$, reduce to

$$\begin{aligned}\dot{\alpha} &= -\frac{\partial S}{\partial \alpha} \quad , \\ \dot{\varphi} &= \frac{\partial S}{\partial \varphi} \quad ,\end{aligned} \tag{C.12}$$

yielding

$$\begin{aligned}\dot{\alpha} &= h\alpha - d \quad , \\ \dot{\varphi} &= -h\varphi \quad ,\end{aligned} \tag{C.13}$$

which can be directly integrated to give

$$a = e^\alpha = e^{d/h}\exp(\alpha_0 e^{ht}) \;\text{ and }\; \varphi = \alpha_0 e^{-ht} \quad , \tag{C.14}$$

where $\alpha_0$ is an integration constant. Recall that the time parameter $t$ is related to cosmic time $\tau$ through $\tau = \int dt e^{3\alpha(t)} \Rightarrow \tau - \tau_0 = \mathrm{Ei}(3\alpha_0 e^{ht})/h$, where $\mathrm{Ei}(x)$ is the exponential-integral function.

These solutions represent ever expanding non-singular models (see figure). For $t << 0$ the Universe expands accelerately from its minimum size $a_0 = e^{d/h}$ (remember that for the physical scale factor one has $a_0^{\mathrm{phys}} = \frac{\ell_{Pl}}{\sqrt{2V}} e^{d/h}$), which occurs in the infinity past $t \to -\infty$. The scalar field is very large in that phase. If $|ht| \leq \alpha_0$ is not very large, one has

$$a \approx e^{\alpha_0 + d/h}[1 + \alpha_0 ht + (1 + 1/\alpha_0)(\alpha_0 ht)^2/2! + (1 + 3/\alpha_0 + 1/\alpha_0^2)(\alpha_0 ht)^3/3!...]. \tag{C.15}$$

Taking $\alpha_0 >> 1$, one can write $a \approx e^{\alpha_0 + d/h} \exp(\alpha_0 ht)$. In that case, from $\tau = \int dt a^3(t)$, one obtains that $a \propto (\tau - \tau_0)^{1/3}$ and $\varphi \propto \ln(\tau - \tau_0)$, as in the classical regime. Figure 1 exhibits the bohmian trajectories and quantum potential for the parameters $h = 3/5$, $d = 2$, and $\alpha_0 = 2$.

## C.3. Simplification of the second order hamiltonian and canonical quantisation

The conventional approach to deal with quantum cosmological perturbations is to consider a semi-classical treatment that quantise only the first order perturbations while the background is treated classically. Once the background dynamics has a classical evolution, one can use these equations to significantly simplify the second order lagrangian before quantising the system Ref. [1]. In this case, the background evolution induces a potential term that modifies the quantum dynamics of the perturbations.

One step further is to consider quantum corrections to the background evolution itself, as in minisuperspace models, Ref. [11]-[14]. In this case, the simplifications in the equations for the linear perturbations using the classical background cannot be implemented. It is worth to remind that the original lagrangian is quite involved, and the use of the background equation is a key step to rewrite the system in a treatable form.

Recent works using technics for hamiltonian's systems Refs. [3, 4, 5] showed that it is also possible to simplify the full hamiltonian system by a series of canonical transformations. Their main results focus in the scalar and tensor perturbations considering the matter content of the Universe described by a perfect fluid. Even though in Ref. [2] and in the Appendix A of Ref. [4] it is shown a long development that significantly simplifies the hamiltonian for a scalar field with a generic potential $U(\varphi)$, there were still some delicate issues to be addressed to consistently quantise the scalar field case.

We will not reproduce the development made in these references but we will continue the development of the above mentioned Appendix. The main point to acquaint from this reference is that their simplification procedure use only

canonical transformations, that guarantees the equivalence between the original and the simplified hamiltonians, independently of the background equations of motion.

In the present work we will focus in the case of a vanishing potential $U(\varphi)$ and show how it is possible to consistently quantise simultaneously both the background and the perturbations. The background system is composed of a free massless scalar field in a spatially flat Friedmann-Lemaître-Robertson-Walker metric (FLRW). Since we are only interested in scalar perturbations, the perturbed metric can be written as

$$\begin{aligned} \mathrm{d}s^2 &= N^2(1+2\phi)\mathrm{d}t^2 - NaB_{|i}\mathrm{d}t\mathrm{d}x^i + \\ &\quad -a^2\left[(1+2\psi)\delta_{ij} - 2E_{|i|j}\right]\mathrm{d}x^i\mathrm{d}x^j \quad . \end{aligned} \tag{C.16}$$

The matter content is defined by a free massless scalar field $\varphi(t,x) = \varphi_0(t) + \delta\varphi(t,x)$, where $\varphi_0$ is the background homogeneous scalar field. Using these definitions in the lagrangian density for the scalar field, namely $\mathcal{L}_m = \frac{1}{2}\varphi_{;\mu}\varphi^{;\mu}$, we find

$$\begin{aligned} \mathcal{L}_m &= \frac{(1-2\phi)}{N^2}\left(\frac{\dot{\varphi}_0^2}{2} + \dot{\varphi}\delta\dot{\varphi}\right) + \frac{\dot{\varphi}_0^2}{N^2}\left(2\phi^2 - \frac{B^{|i}B_{|i}}{2}\right) + \\ &\quad -\frac{\dot{\varphi}_0}{Na}B^{|i}\delta\varphi_{|i} + \frac{\delta\dot{\varphi}^2}{2N^2} - \frac{1}{2a^2}\delta\varphi^{|i}\delta\varphi_{|i} \quad . \end{aligned} \tag{C.17}$$

As our starting point, let us consider the hamiltonian (A39) of Ref. [4] with the scalar field potential $U(\varphi)$ taken to be null,

$$\begin{aligned} H &= NH_0 + \int \mathrm{d}^3x\left(-\frac{\ell_{Pl}^2P_a^2}{2a^2V}\phi + \frac{3P_\varphi^2}{a^4P_aV}\psi + \right. \\ &\quad \left. + \frac{3\ell_{Pl}^2P_\varphi}{2a^4V}v\right)\tilde{\phi}_6 + \Lambda_N P_N + \int \mathrm{d}^3x\Lambda_\phi\pi_\phi \quad , \end{aligned} \tag{C.18}$$

where $\tilde{\phi}_6 = \pi_\psi$, $P_N$ e $\pi_\phi$ are first class constrains, and $v$ is the Mukhanov-Sasaki variable. The quantity $H_0$ is defined as

$$\begin{aligned} H_0 &= -\frac{\ell_{Pl}^2P_a^2}{4aV} + \frac{P_\varphi^2}{2a^3V} + \frac{1}{2a}\int \mathrm{d}^3x\left(\frac{\pi^2}{\sqrt{\gamma}} + \sqrt{\gamma}v^{,i}v_{,i}\right) + \\ &\quad + \left[\frac{15\ell_{Pl}^2P_\varphi^2}{4a^5V^2} - \frac{\ell_{Pl}^4P_a^2}{16a^3V^2} - \frac{27P_\varphi^4}{4a^7V^2P_a^2}\right]\int \mathrm{d}^3x\sqrt{\gamma}v^2\,, \end{aligned} \tag{C.19}$$

where $P_a$, $P_\varphi$ and $\pi$ are the momenta canonically conjugate to $a$, $\varphi_0$ and $v$, respectively, $\ell_{Pl}^2 = \frac{8\pi G}{3}$, and $V$ is the comoving volume of the compact spatial sections, i.e. $V < \infty$. The zero order hamiltonian,

$$H_0^{(0)} \equiv -\frac{\ell_{Pl}^2P_a^2}{4aV} + \frac{P_\varphi^2}{2a^3V} \quad , \tag{C.20}$$

can be used to simplify further the mass-like term for the perturbations, i.e. the function inside brackets multiplying the $v^2$ term. To do so, we rewrite $P_\varphi$ as

$$P_\varphi^2 = 2a^3V\left(H_0^{(0)} + \frac{\ell_{Pl}^2 P_a^2}{4aV}\right) \quad .$$

Redefining the lapse function as

$$\tilde{N} = N\left\{1 + \left[\frac{15\ell_{Pl}^2}{2a^2V} - \frac{27}{aP_a^2}\left(H_0^{(0)} + \frac{\ell_{Pl}^2 P_a^2}{2aV}\right)\right]\int \mathrm{d}^3x\sqrt{\gamma}v^2\right\} ,$$

and keeping only second order terms in $NH_0$, we can rewrite it as

$$\begin{aligned} NH_0 &= \tilde{N}\left[H_0^{(0)} + \frac{1}{2a}\int \mathrm{d}^3x\left(\frac{\pi^2}{\sqrt{\gamma}} + \sqrt{\gamma}v^{,i}v_{,i}\right) + \right. \\ &\quad \left. \frac{\ell_{Pl}^4 P_a^2}{8a^3V^2}\int \mathrm{d}^3x\sqrt{\gamma}v^2\right] + \mathcal{O}(v^4, v^2\pi^2) \quad . \end{aligned} \tag{C.21}$$

Thus, by a simple redefinition of the lapse function, the mass-like term simplifies significantly. Nonetheless, it is still tricky to quantise this term due to the momentum $P_a$. Furthermore, the scale factor is defined on the half-line which requires additional care in specifying the Hilbert space. To deal with these two points, it is convenient to define dimensionless variables $\alpha \equiv \log\left(\sqrt{2V}\ell_{Pl}^{-1}a\right)$ and $\varphi \to \frac{\ell_{Pl}}{\sqrt{2}}\varphi$ which give us the following relations:

$$\begin{aligned} P_\alpha &= -\frac{\ell_{Pl}}{\sqrt{2V}}\frac{e^{3\alpha}}{N}\dot{\alpha} \quad , \quad \frac{\ell_{Pl}^2}{4V}\frac{P_a^2}{a} = \frac{\sqrt{2V}}{\ell_{Pl}}\frac{P_\alpha^2}{2e^{3\alpha}} \quad , \\ \frac{P_\varphi^2}{2a^3V} &\to \frac{\sqrt{2V}}{\ell_{Pl}}\frac{P_\varphi^2}{2e^{3\alpha}} \quad , \quad H_0^{(0)} = \frac{\sqrt{2V}}{2\ell_{Pl}e^{3\alpha}}\left(-P_\alpha^2 + P_\varphi^2\right) \quad . \end{aligned}$$

With these new variables we find,

$$H_0 = H_0^{(0)} + \frac{N\sqrt{2V}}{2\ell_{Pl}e^\alpha}\int \mathrm{d}^3x\sqrt{\gamma}\left(\frac{\pi^2}{\gamma} + v^{,i}v_{,i} + \frac{P_\alpha^2}{e^{4\alpha}}v^2\right).$$

To eliminate the momentum in the mass-like term we perform a canonical transformation generated by

$$\mathcal{F} = \mathcal{I} + \frac{P_\alpha}{2}\int \mathrm{d}^3x\sqrt{\gamma}\,\tilde{v}^2 + e^{\tilde{\alpha}}\int \mathrm{d}^3x\,\pi\tilde{v} \quad , \tag{C.22}$$

which implies

$$\begin{aligned} &\alpha = \tilde{\alpha} + \frac{1}{2}\int \mathrm{d}^3x\sqrt{\gamma}\,\tilde{v}^2 \quad , \quad v = e^{\tilde{\alpha}}\tilde{v} \quad , \\ &\tilde{P}_\alpha = P_\alpha + e^{\tilde{\alpha}}\int \mathrm{d}^3x\,\pi\tilde{v} \quad , \quad \tilde{\pi} = \sqrt{\gamma}\tilde{P}_\alpha\tilde{v} + e^{\tilde{\alpha}}\pi \quad , \\ &e^{3\alpha} = e^{3\tilde{\alpha}}\left(1 + \frac{3}{2}\int \mathrm{d}^3x\sqrt{\gamma}\,\tilde{v}^2\right) + \mathcal{O}\left(\tilde{v}^3\right) \quad . \end{aligned}$$

Once more, redefining the lapse function as

$$\tilde{N} = N\left[1 - \frac{3}{2}\int \mathrm{d}^3x\sqrt{\gamma}\tilde{v}^2\right] \quad ,$$

and omitting the tilde in the new variables, the hamiltonian transforms into

$$\begin{aligned} H &= H_0 + \int \mathrm{d}^3x \left( -\frac{2V}{\ell_{Pl}^2}\frac{P_\alpha^2}{e^{4\alpha}}\phi + \frac{3\sqrt{2V}}{\ell_{Pl}}\frac{P_\varphi^2}{e^{3\alpha}P_\alpha}\psi + \right. \\ &\left. + \frac{3\sqrt{2V}}{\ell_{Pl}}\frac{\sqrt{V}P_\varphi}{e^{4\alpha}}v \right)\pi_\psi + \Lambda_N P_N + \int \mathrm{d}^3x\Lambda_\phi\pi_\phi \end{aligned} \tag{C.23}$$

with,

$$H_0 = \frac{\sqrt{2V}}{2\ell_{Pl}e^{3\alpha}}\left[-P_\alpha^2 + P_\varphi^2 + \int \mathrm{d}^3x\left(\frac{\pi^2}{\sqrt{\gamma}} + \sqrt{\gamma}e^{4\alpha}v^{,i}v_{,i}\right)\right] \ . \tag{C.24}$$

The system described by this hamiltonian can be immediately quantised. The Dirac's quantisation procedure for constrained hamiltonian systems requires that the first class constraints must annihilate the wave-function

$$\begin{aligned} &\frac{\partial}{\partial N}\Psi\left(\alpha,\varphi,v,N,\phi,\psi\right) = 0 \quad , \\ &\frac{\delta}{\delta\psi}\Psi\left(\alpha,\varphi,v,N,\phi,\psi\right) = 0 \quad , \\ &\frac{\delta}{\delta\phi}\Psi\left(\alpha,\varphi,v,N,\phi,\psi\right) = 0 \quad . \end{aligned}$$

Thus, the wave-function must be independent of $N$, $\phi$ and $\psi$, i.e. $\Psi = \Psi\left(\alpha,\varphi,v\right)$ where $v$ encode the perturbed degrees of freedom. Note that, due to the transformation (C.22), $v$ is now the Mukhanov-Sasaki variable divided by $a$. The remaining equation is

$$\hat{H}_0\Psi\left(\alpha,\varphi,v\right) = 0 \quad , \tag{C.25}$$

which has only quadratic terms in the momenta.

A well known feature of the quantization of time reparametrization invariant theories is that the state is not explicitly time dependent, hence one should find among intrinsic degrees of freedom a variable that can play the role of time. In the perfect fluid case, the Wheeler- DeWitt's equation assumes a Schrödinger-like form, due to a linear term in the momenta connected with the fluid degree of freedom. However, the hamiltonian (C.24) does not possess such linear term, rendering ambiguous the choice of an intrinsic time variable. Notwithstanding, we still can define an evolutionary time for the perturbations if we use the Bohm-de Broglie interpretation. The procedure is similar to what is done in a

semiclassical approach, where a time evolution for the quantum perturbations is induced from the classical background trajectory (see, e.g., Ref. [16] for details). Let us summarize it in the following paragraphs.

First of all, take the hamiltonian $NH_0$, with $H_0$ given in Eq. (C.24) satisfying the hamiltonian constraint $H_0 \approx 0$, and let us solve it classically using the Hamilton-Jacobi theory. The respective Hamilton-Jacobi equation reads

$$\begin{aligned} &-\frac{1}{2}\left(\frac{\partial S_T}{\partial \alpha}\right)^2 + \frac{1}{2}\left(\frac{\partial S_T}{\partial \varphi}\right)^2 \\ &+\frac{1}{2}\int \mathrm{d}^3x \left[\frac{1}{\sqrt{\gamma}}\left(\frac{\delta S_T}{\delta v}\right)^2 + \sqrt{\gamma}e^{4\alpha}v^{,i}v_{,i}\right] , \end{aligned} \tag{C.26}$$

where the classical trajectories can be obtained from a solution $S_T$ of Eq. (C.26) through

$$\begin{aligned} \dot{\alpha} &= -P_\alpha = -\frac{\partial S_T}{\partial \alpha} \quad , \\ \dot{\varphi} &= P_\varphi = \frac{\partial S_T}{\partial \varphi} \quad , \\ \dot{v} &= \frac{1}{\sqrt{\gamma}}\pi = \frac{1}{\sqrt{\gamma}}\frac{\delta S_T}{\delta v} \quad , \end{aligned} \tag{C.27}$$

where we have chosen $N = l_{Pl}e^{3\alpha}/\sqrt{2V}$, and hence a time parameter $t$ (a dot means derivative with respect to this parameter), related to conformal time through $dt \propto a^2 d\eta$.

We will now use the fact that the $v$ variable is a small perturbation over the background variables $\alpha$ and $\varphi$, and that its back-reaction in the dynamics of the background is negligible. In this case, one can write $S_T(\alpha, \varphi, v)$ as

$$S_T(\alpha, \varphi, v) = S_0(\alpha, \varphi) + S_2(\alpha, \varphi, v), \tag{C.28}$$

where it is assumed that $S_2(\alpha, \varphi, v)$ cannot be splitted again into a sum involving a function of the background variables alone (which would just impose a redefinition of $S_0$). Noting that, in order to be a solution of the Hamilton-Jacobi equation (C.26), $S_2$ must be at least a second order functional of $v$ (see Ref. [27]), then $S_2 << S_0$ as well as their partial derivatives with respect to the background variables. Hence one obtains for the background that

$$\begin{aligned} \dot{\alpha} &\approx -\frac{\partial S_0}{\partial \alpha} \quad , \\ \dot{\varphi} &\approx \frac{\partial S_0}{\partial \varphi} \quad . \end{aligned} \tag{C.29}$$

Inserting the splitting given in equation (C.28) into equation (C.26), one obtains, order by order:

$$-\frac{1}{2}\left(\frac{\partial S_0}{\partial \alpha}\right)^2 + \frac{1}{2}\left(\frac{\partial S_0}{\partial \varphi}\right)^2 = 0, \tag{C.30}$$

$$-\left(\frac{\partial S_0}{\partial \alpha}\right)\left(\frac{\partial S_2}{\partial \alpha}\right)+\left(\frac{\partial S_0}{\partial \varphi}\right)\left(\frac{\partial S_2}{\partial \varphi}\right)+\frac{1}{2}\int \mathrm{d}^3x\left[\frac{1}{\sqrt{\gamma}}\left(\frac{\delta S_2}{\delta v}\right)^2+\sqrt{\gamma}e^{4\alpha}v^{,i}v_{,i}\right]=0, \tag{C.31}$$

$$-\frac{1}{2}\left(\frac{\partial S_2}{\partial \alpha}\right)^2+\frac{1}{2}\left(\frac{\partial S_2}{\partial \varphi}\right)^2+O(4)=0. \tag{C.32}$$

In Eq. (C.32), the symbol $O(4)$ represents terms coming from high order corrections to the hamiltonian (C.24). As we are interested only on linear perturbations, this equation will not be relevant. The first equation (C.30) is the Hamilton-Jacobi equation of the background which solution yields, together with Eqs. (C.29), the background classical trajectories. Once one obtains the classical trajectories $\alpha(t), \varphi(t)$, the functional $S_2(\alpha, \varphi, v)$ becomes a functional of $v$ and a function of $t$, $S_2(\alpha, \varphi, v) \to S_2(\alpha(t), \varphi(t), v) = \bar{S}_2(t, v)$. Hence equation (C.31), using Eqs. (C.29), can be written as

$$\frac{\partial S_2}{\partial t}+\frac{1}{2}\int \mathrm{d}^3x\left(\frac{1}{\sqrt{\gamma}}\left(\frac{\delta S_2}{\delta v}\right)^2+\sqrt{\gamma}e^{4\alpha(t)}v^{,i}v_{,i}\right)=0. \tag{C.33}$$

Equation (C.33) can now be understood as the Hamilton-Jacobi equation coming from the hamiltonian

$$H_2=\frac{1}{2}\int \mathrm{d}^3x\left(\frac{\pi^2}{\sqrt{\gamma}}+\sqrt{\gamma}e^{4\alpha(t)}v^{,i}v_{,i}\right), \tag{C.34}$$

which is the generator of time $t$ translations (and not anymore constrained to be null).

If one wants to quantize the perturbations, the correspoding Schrödinger equation should be

$$i\frac{\partial \chi}{\partial t}=\hat{H}_2\chi\,, \tag{C.35}$$

where $\chi$ is a wave functional depending on $v$ and $t$, and the dependences of $\hat{H}_2$ on the background variables are understood as a dependence on $t$.

Let us now go one step further and quantize both the background and perturbations. When the background is also quantised, this procedure can also be implemented in the framework of the Bohm-de Broglie interpretation of quantum theory, where there is a definite notion of trajectories as well, the bohmian trajectories. In order to do that, we first note that Eqs. (C.25) and (C.24) imply that

$$(\hat{H}_0^{(0)}+\hat{H}_2)\Psi=0, \tag{C.36}$$

where

$$\hat{H}_0^{(0)} = -\frac{\hat{P}_\alpha^2}{2} + \frac{\hat{P}_\varphi^2}{2} \quad , \tag{C.37}$$

$$\hat{H}_2 = \frac{1}{2}\int \mathrm{d}^3x \left( \frac{\hat{\pi}^2}{\sqrt{\gamma}} + \sqrt{\gamma} e^{4\hat{\alpha}} \hat{v}^{,i} \hat{v}_{,i} \right) \quad . \tag{C.38}$$

We write the wave functional $\Psi$ as $\Psi = \exp(A_T + iS_T) \equiv R_T \exp(iS_T)$, where both $A_T$ and $S_T$ are real functionals. Inserting it in the Wheeler-DeWitt equation (C.36), the two real equations we obtain are

$$-\frac{\partial}{\partial\alpha}\left(R_T^2 \frac{\partial S_T}{\partial\alpha}\right) + \frac{\partial}{\partial\varphi}\left(R_T^2 \frac{\partial S_T}{\partial\varphi}\right) + \int \frac{\mathrm{d}^3x}{\sqrt{\gamma}} \frac{\delta}{\delta v}\left(R_T^2 \frac{\delta S_T}{\delta v}\right) = 0 \quad , \tag{C.39}$$

$$-\frac{1}{2}\left(\frac{\partial S_T}{\partial\alpha}\right)^2 + \frac{1}{2}\left(\frac{\partial S_T}{\partial\varphi}\right)^2 + \frac{1}{2}\int \mathrm{d}^3x \left( \frac{1}{\sqrt{\gamma}} \left(\frac{\delta S_T}{\delta v}\right)^2 + \sqrt{\gamma} e^{4\alpha} v^{,i} v_{,i} \right) + \frac{1}{2R_T}\left(\frac{\partial^2 R_T}{\partial\alpha^2} - \frac{\partial^2 R_T}{\partial\varphi^2}\right) - \frac{1}{2}\int \tag{C.40}$$

These two equations correspond to equations (C.4) and (C.5), respectively.

The bohmian guidance relations are the same as in the classical case,

$$\begin{aligned} \dot{\alpha} &= -P_\alpha = -\frac{\partial S_T}{\partial\alpha} \quad , \\ \dot{\varphi} &= P_\varphi = \frac{\partial S_T}{\partial\varphi} \quad , \\ \dot{v} &= \frac{1}{\sqrt{\gamma}}\pi = \frac{1}{\sqrt{\gamma}}\frac{\delta S_T}{\delta v} \quad , \end{aligned} \tag{C.41}$$

with the difference that the new $S_T$ satisfies a Hamilton-Jacobi equation different from the classical one due to the presence of the quantum potential terms (the two last terms in Eq. (C.40)), which are responsible for the quantum effects.

We have again made the choice $N \propto e^{3\alpha}$. Whether this procedure is unambiguously independent on the choice of the lapse function is a delicate point. Indeed, in a general framework (the full superspace), the bohmian evolution of three-geometries may not even form a four-geometry (a spacetime) in the sense described in Refs. [17, 18, 19, 20], although the theory remains consistent (Refs. [18, 19]), and its geometrical properties depends on the choice of the lapse function. However, in the case of homogeneous spacelike hypersurfaces, a preferred foliation of spacetime is selected, the one where the time direction is perpendicular to the Killing vectors of these hypersurfaces. In this case, once one has chosen this preferred foliation, one can prove that the residual ambiguity in the lapse function (which is now independent of space coordinates) is geometrically irrelevant for the Bohmian trajectories (see Ref. [10]). This is also true when linear perturbations are present, where the hamiltonian constraints

reduce to a single one, and the super-momentum constraint can be solved, as it was shown in Ref. [7]. Again, the lapse function is just a time function. In this case, the bohmian quantum background trajectories can be obtained without geometrical ambiguities [10], and they can be used to induce a time dependence on the perturbation quantum state, as we will see.

Let us assume, as in the classical case, that we can split $A_T(\alpha,\varphi,v) = A_0(\alpha,\varphi) + A_2(\alpha,\varphi,v)$ implying that $R_T(\alpha,\varphi,v) = R_0(\alpha,\varphi)R_2(\alpha,\varphi,v)$, and $S_T(\alpha,\varphi,v) = S_0(\alpha,\varphi) + S_2(\alpha,\varphi,v)$, and that $A_2 << A_0$, $S_2 << S_0$, together with their derivatives with respect to the background variables. The approximate guidance relations are

$$\begin{aligned} \dot{\alpha} &\approx -\frac{\partial S_0}{\partial \alpha} \quad , \\ \dot{\varphi} &\approx \frac{\partial S_0}{\partial \varphi} \quad , \end{aligned} \tag{C.42}$$

and the zeroth order terms of Eqs. (C.39) and (C.40) read

$$-\frac{\partial}{\partial \alpha}\left(R_0^2\frac{\partial S_0}{\partial \alpha}\right) + \frac{\partial}{\partial \varphi}\left(R_0^2\frac{\partial S_0}{\partial \varphi}\right) \approx 0 \quad , \tag{C.43}$$

$$-\frac{1}{2}\left(\frac{\partial S_0}{\partial \alpha}\right)^2 + \frac{1}{2}\left(\frac{\partial S_0}{\partial \varphi}\right)^2 + \frac{1}{2R_0}\left(\frac{\partial^2 R_0}{\partial \alpha^2} - \frac{\partial^2 R_0}{\partial \varphi^2}\right) \approx 0 \, . \tag{C.44}$$

which, again, correspond to Eqs. (C.4) and (C.5) for the background, respectively.

A solution $(S_0, R_0)$ of Eqs. (C.43) and (C.44) yield a bohmian quantum trajectory for the background through Eq. (C.42). If $S_0$ and $R_0$ are obtained from Eq. (C.10), then the bohmian trajectories will be given by Eq. (C.14).

As in the classical case, once one obtains the bohmian quantum trajectories $\alpha(t), \varphi(t)$, the functionals $S_2(\alpha,\varphi,v)$, $A_2(\alpha,\varphi,v)$ become functionals of $v$ and functions of $t$, $S_2(\alpha,\varphi,v) \to S_2(\alpha(t),\varphi(t),v) = \bar{S}_2(t,v)$, $A_2(\alpha,\varphi,v) \to A_2(\alpha(t),\varphi(t),v) = \bar{A}_2(t,v)$.

Defining $\chi(\alpha,\varphi,v) \equiv R_2(\alpha,\varphi,v)\exp(iS_2(\alpha,\varphi,v))$, writing it as

$$\chi(\alpha,\varphi,v) = \int \mathrm{d}\lambda \, G(\lambda,v)F(\lambda,\alpha,\phi) \quad , \tag{C.45}$$

where $F$ satifies

$$\frac{1}{2}\left(\frac{\partial^2 F}{\partial \alpha^2} - \frac{\partial^2 F}{\partial \varphi^2}\right) + \frac{1}{R_0}\left(\frac{\partial R_0}{\partial \alpha}\frac{\partial F}{\partial \alpha} - \frac{\partial R_0}{\partial \varphi}\frac{\partial F}{\partial \varphi}\right) = 0 \, , \tag{C.46}$$

and $G$ is an arbitrary functional of $v$, which also depends on an integration constant $\lambda$, then the next-to-leading-order terms of Eqs. (C.39) and (C.40) read

$$\frac{\partial \bar{R}_2^2}{\partial t} + \int \frac{\mathrm{d}^3 x}{\sqrt{\gamma}}\frac{\delta}{\delta v}\left(\bar{R}_2^2\frac{\delta \bar{S}_2}{\delta v}d^3x\right) = 0 \quad , \tag{C.47}$$

$$\frac{\partial \bar{S}_2}{\partial t} + \frac{1}{2}\int \mathrm{d}^3x \left( \frac{1}{\sqrt{\gamma}} \left( \frac{\delta \bar{S}_2}{\delta v} \right)^2 + \sqrt{\gamma} e^{4\alpha(t)} v^{,i} v_{,i} \right) - \frac{1}{2}\int \frac{\mathrm{d}^3x}{\bar{R}_2\sqrt{\gamma}} \frac{\delta^2 \bar{R}_2}{\delta v^2} = 0 \,, \tag{C.48}$$

where $\bar{R}_2(t,v) \equiv \exp(\bar{A}_2(t,v))$. In order to obtain these equations we used that

$$-\left(\frac{\partial S_0}{\partial \alpha}\right)\left(\frac{\partial S_2}{\partial \alpha}\right) + \left(\frac{\partial S_0}{\partial \varphi}\right)\left(\frac{\partial S_2}{\partial \varphi}\right) = \frac{\partial \bar{S}_2}{\partial t}, \tag{C.49}$$

and the same for $R_2$ and $\bar{R}_2$.

These two equations can be grouped into a single Schrödinger equation

$$i\frac{\partial \bar{\chi}}{\partial t} = \hat{H}_2 \bar{\chi} \,, \tag{C.50}$$

where $\bar{\chi}(t,v) = \chi(\alpha(t), \varphi(t), v)$ is a wave functional depending on $v$ and $t$, and, as before, the dependences of $\hat{H}_2$ on the background variables are understood as a dependence on $t$.

For the specific example of section II, Eq. (C.10), one possible solution of Eq.(C.46) yields for $\chi$ through Eq.(C.45)

$$\chi(\alpha,\varphi,v) = \frac{1}{R(\alpha,\varphi)} \int \mathrm{d}\lambda \, G(\lambda, v) \exp\left\{ \frac{(\alpha+\varphi-d/h)^2}{2\lambda} + \frac{\lambda h^2 (\alpha-\varphi-d/h)^2}{8} \right\} \quad . \tag{C.51}$$

From solution (C.51), we can construct $\bar{\chi}(t,v) \equiv \chi(\alpha(t), \varphi(t), v)$ solution of Eq. (C.50). Note that, as $G$ is an arbitrary functional of $v$ and the real parameter $\lambda$, the functional $\bar{\chi}(t,v)$ constructed from (C.51) via $\bar{\chi}(t,v) \equiv \chi(\alpha(t), \varphi(t), v)$ is also an arbitrary functional of $t$ and $v$ (even though $\chi(\alpha,\varphi,v)$ in (C.51) is not arbitrary in $\alpha$ and $\varphi$).

During our procedure, we have supposed that the evolution of the background is independent of the perturbations. This no back-reaction assumption is based on the fact that terms induced by the linear perturbations in the zeroth order hamiltonian are negligible, which should be the case when one assumes that quantum perturbations are initially in a vacuum quantum state, as it is argued in Ref. [15]. We will come back to this point in the conclusion.

Once one obtains the quantum trajectories for the background variables, they can be used to define a time dependent unitary transformation for the perturbative sector. This unitary transformation takes the vector $|\chi\rangle$ into $|\xi\rangle = U|\chi\rangle$, i.e. $|\chi\rangle = U^{-1}|\xi\rangle$. With respect to this transformation the hamiltonian is taken into $\hat{H}_2 \longrightarrow \hat{H}_{2U}$ with

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\xi\rangle = \hat{H}_{2U}|\xi\rangle = \left( U\hat{H}_2 U^{-1} - iU\frac{\mathrm{d}}{\mathrm{d}t}U^{-1} \right)|\xi\rangle \quad . \tag{C.52}$$

Let us define this unitary transformation by

$$U = e^{iA} e^{-iB} \tag{C.53}$$

with,

$$A = \frac{1}{2}\int \mathrm{d}^3x\sqrt{\gamma}\frac{\dot{a}}{a^3}\hat{v}^2 \qquad , \tag{C.54}$$

$$B = \frac{1}{2}\int \mathrm{d}^3x\,(\hat{\pi}\hat{v} + \hat{v}\hat{\pi})\log(a) \qquad . \tag{C.55}$$

Remember that the time derivative, $\dot{a} = \frac{da}{dt}$, is taken with respect to the parametric time $t$ related to the cosmic time $\tau$ by $\mathrm{d}\tau = N\mathrm{d}t \propto a^3\mathrm{d}t$. In these expressions, the scale factor $a = a(t)$ should be understood as a function of time, instead of an operator, since we suppose that the background quantum equations have already been solved. Thus, $a = a(t)$ should be taken as the bohmian trajectory associated with equations $\hat{H}_0^{(0)}|\phi\rangle = 0$.

Naturally, the $\hat{\pi}$ e $\hat{v}$ operators do not commute with the unitary transformation. Using the following relations

$$e^{iA}\,\hat{v}\,e^{-iA} = \hat{v} \qquad , \qquad e^{iA}\,\hat{\pi}\,e^{-iA} = \hat{\pi} - \frac{\dot{a}}{a^3}\sqrt{\gamma}\,\hat{v}$$
$$e^{-iB}\,\hat{v}\,e^{iB} = a^{-1}\,\hat{v} \qquad , \qquad e^{-iB}\,\hat{\pi}\,e^{iB} = a\hat{\pi} \quad .$$

we can calculate the transformed hamiltonian as

$$\hat{H}_{2U} = \frac{a^2}{2}\int \mathrm{d}^3x\left[\frac{\hat{\pi}^2}{\sqrt{\gamma}} + \sqrt{\gamma}\,\hat{v}^{,i}\hat{v}_{,i} - \left(\frac{\ddot{a}}{a^5} - 2\frac{\dot{a}^2}{a^6}\right)\sqrt{\gamma}\,\hat{v}^2\right] \tag{C.56}$$

Note that the unitary transformation $U$ takes us back to the Mukhanov-Sasaki variable.

Recalling that $\mathrm{d}t = a^{-2}\mathrm{d}\eta$, where $\eta$ is the conformal time, we have $\dot{a} = a^2a'$ and $\ddot{a} = a^4a'' + 2a^3a'^2$, and the hamiltonian can be recast as

$$\hat{H}_{2U} = \frac{a^2}{2}\int \mathrm{d}^3x\left[\frac{\hat{\pi}^2}{\sqrt{\gamma}} + \sqrt{\gamma}\,\hat{v}^{,i}\hat{v}_{,i} - \frac{a''}{a}\sqrt{\gamma}\,\hat{v}^2\right] \quad . \tag{C.57}$$

So far our analysis has been made in the Schrödinger picture but now it is convenient to describe the dynamics using the Heisenberg representation. The equations of motion for the Heisenberg operators are written as

$$\dot{\hat{v}} = -i\left[\hat{v}, \hat{H}_{2U}\right] = a^2\frac{\hat{\pi}}{\sqrt{\gamma}} \qquad ,$$
$$\dot{\hat{\pi}} = -i\left[\hat{\pi}, \hat{H}_{2U}\right] = a^2\sqrt{\gamma}\left(\hat{v}^{,i}{}_{,i} + \frac{a''}{a}\hat{v}\right) \qquad .$$

Combining these two equations and changing to conformal time, we find the following equations for the operator modes of wave number $k$, $v_k$:

$$v_k'' + \left(k^2 - \frac{a''}{a}\right)v_k \quad = 0 \qquad . \tag{C.58}$$

This is the same equation of motion for the perturbations known in the literature in the absence of a scalar field potential Ref. [1]. The crucial point is that we have not used the background equations of motion. Thus we have shown that Eq. (C.58) is well defined, independently of the background dynamics, and it is correct even if we consider quantum background trajectories.

Note, however, that this result was obtained using a specific subclass of wave functionals which satisfies the extra condition Eq. (C.46). What are the physical assumptions behind this choice?

When one approaches the classical limit, where $R_0$ is a slowly varying function of $\alpha$ and $\varphi$, condition (C.46) reduces to

$$\frac{\partial^2 F}{\partial \alpha^2} - \frac{\partial^2 F}{\partial \varphi^2} \approx 0\,. \tag{C.59}$$

If Eq. (C.59) were not satisfied, one would not obtain anymore the usual Schrödinger equation for quantum perturbations in a classical background (which arises when $R_0$ is a slowly varying function of $\alpha$ and $\varphi$), due to extra terms in Eqs. (C.47) and (C.48): there would be corrections originated from some quantum entanglement between the background and the perturbations, even when the background is already classical, which would spoil the usual semiclassical approximation. This could be a viable possibility driven by a different type of wave functional than the one considered here, but it seems that our Universe is not so complicated. In fact, the observation that the simple semiclassical model without this sort of entaglement works well in the real Universe indicates something about the wave functional of the Universe[28][1]. In other words, the validity of the usual semiclassical approximation imposes Eq. (C.59).

When $R_0$ is not slowly varying and quantum effects on the background become important causing the bounce, the two last terms of condition (C.46) cannot be neglected. They would also induce extra terms in Eqs. (C.47) and (C.48), again originated from some quantum entanglement between the background and the perturbations, but now in the background quantum domain, and the final quantum equation (C.58) for the perturbations we obtained would not be valid around the bounce. In this case, there is no observation indicating which class of wave functionals one should take and our choice in this no man's land resides only on assumptions of simplicity: there is no quantum entaglement between the background and the perturbations in the entire history of the Universe. This is the physical hypothesis behind the choice of the specific class os wave functionals satisfying condition (C.46).

In the next section we will apply the above formalism implying Eq. (C.58) to the specific example described in section II.

[1] In these references, it is pointed out how the features of our Universe we take for granted (classicality, separability) impose severe restrictions on the initial wave function of the Universe. In fact, our Universe could have been highly nonclassical, completely entagled, even when it is large, depending on the features of this initial wave solution.

## C.4. Application of the formalism

We will now use Eq. (C.58) to evaluate the spectral index of scalar perturbations in the quantum background described by Eq. (C.14). The potential $V \equiv a''/a$ reads

$$\begin{aligned} V \equiv \frac{a''}{a} &= \frac{1}{a^4}\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] \\ &= \frac{\alpha_0 h^2 \exp(ht)[1-\alpha_0 \exp(ht)]}{a^4}. \end{aligned} \tag{C.60}$$

Defining $u_k \equiv v_k/a$, Eq. (C.58) in terms of the $t$ variable can be written as (from now on we will omit the index $k$),

$$\ddot{u} + k^2 a^4 u = 0 \qquad . \tag{C.61}$$

When $ht << 0$, we can approximate $a \approx \exp(d/h)[1+\alpha_0 \exp(ht)]$, and the general solution reads

$$u = A_+(k) J_\nu(z) - A_-(k) J_{-\nu}(z) \qquad , \tag{C.62}$$

where $J$ is the Bessel function of the first type, $\nu = i2k\exp(2d/h)/h$ and $z = 4\alpha_0^{1/2} k \exp(2d/h + ht/2)/h$. At $t \to -\infty$, when the scale factor becomes constant and spacetime is flat, one can impose vacuum initial conditions

$$v_{\rm ini} = \frac{\mathrm{e}^{ik\eta}}{\sqrt{k}}, , \tag{C.63}$$

which implies that $A_+(k) = 0$, and $A_-(k) \propto k^{-1/2}\exp[i2k\ln(k)\exp(2d/h)/h]$. Hence, $v$ in this region reads

$$v_I = aA_-(k) J_{-\nu}(z) \qquad . \tag{C.64}$$

The solution can also be expanded in powers of $k^2$ according to the formal solution (see Ref. [1])

$$\begin{aligned} \frac{v}{a} &\simeq A_1(k)\left[1 - k^2 \int^t \frac{\mathrm{d}\bar{\eta}}{a^2(\bar{\eta})} \int^{\bar{\eta}} a^2(\bar{\bar{\eta}})\,\mathrm{d}\bar{\bar{\eta}}\right] \\ &+ A_2(k)\left[\int^\eta \frac{\mathrm{d}\bar{\eta}}{a^2} - k^2 \int^\eta \frac{\mathrm{d}\bar{\eta}}{a^2} \int^{\bar{\eta}} a^2 \mathrm{d}\bar{\bar{\eta}} \int^{\bar{\bar{\eta}}} \frac{\mathrm{d}\bar{\bar{\bar{\eta}}}}{a^2}\right] + ..., \end{aligned} \tag{C.65}$$

When the mode is deep inside the potential, $k^2 << V$, we can neglect the $k^2$ terms yielding

$$v_{II} \approx a\left[A_1(k) + A_2(k)\int^\eta \frac{\mathrm{d}\bar{\eta}}{a^2}\right] = a\left[A_1(k) + A_2(k)t\right] \qquad . \tag{C.66}$$

We can now perform the matching of $v_I$ with $v_{II}$ in order to calculate $A_1(k)$ and $A_2(k)$. As we are interested on large scales, $k << 1$, this matching can still be made when $ht << 0$. In this region one has $V \approx \alpha_0 h^2 \exp(ht - 4d/h)$, yielding the matching time

$$ht_M = \ln\left(\frac{k^2 \exp(4d/h)}{\alpha_0 h^2}\right) \qquad . \tag{C.67}$$

Note that the potential crossing condition relating the wave number $k$ and the time $t_M$ of the crossing is logarithmic. In fact, since in this region the scale factor is almost constant, the wave number is also logarithmically related to the conformal time. This dependence is drastically different from the slow roll scenario, where the conformal time of potential crossing is inversely proportional to the wave number, $k \propto 1/\eta_M$.

Performing the matching at this time and taking the leading order term in $k$, one obtains that

$$A_1(k) = k^{-1} A_2(k) \propto k^{-1/2} \exp[i6k \ln(k) \exp(2d/h)/h]. \tag{C.68}$$

Note that solution (C.65) is valid everywhere, hence we can use it in the period when the scale factor evolution becomes classical. During this period, unless for some fine tuning, the mode is also deep inside the potential and one can use Eq. (C.66) to calculate the Bardeen potential $\Phi$ through the classical equation Ref. [1]

$$\Phi = -\frac{(\epsilon + p)^{1/2} z}{k^2} \left(\frac{v}{z}\right)', \tag{C.69}$$

where $z \equiv a^2(\epsilon + p)^{1/2}/\mathcal{H}$. For the case of a scalar field without potential (stiff matter), $z \propto a$, yielding

$$\Phi \propto A_1(k) + \frac{A_2(k)}{k^2 a^4}, \tag{C.70}$$

one constant and one decaying mode, as usual. The transition to radiation dominated and matter dominated phases may alter the amplitudes but not the spectrum. The power spectrum

$$\mathcal{P}_\Phi \equiv \frac{2k^3}{\pi^2} |\Phi|^2 \propto k^{n_{\rm S}-1}, \tag{C.71}$$

yields for the spectral index, from the value of $A_1(k)$ in the constant mode given in Eq. (C.68), the value $n_s = 3$, contrary to observational results Ref. [8]. This power law dependence was checked numerically as can be seen by the figure. Hence, the model cannot describe the primordial era of our Universe.

## C.5. Conclusion

In this paper we were able to obtain the simple equation for linear scalar perturbations of Ref. [1] for the case of a scalar field without potential. The simplification procedure was carried out without ever using any classical background

equation. Instead, by a series of canonical transformations and redefinitions of the lapse function we are able to put the hamiltonian in a form susceptible to quantization.

However, contrary to the perfect fluid case, the scalar field minisuperspace model has no natural way to define a time variable since its hamiltonian constraint does not contain a linear term in the momenta. Nevertheless, if one assumes there is no back-reaction, we have shown how to bypass this problem using the quantum background bohmian trajectories. The quantum background dynamics in the Bohm-de Broglie interpretation naturally provides an evolutionary time to the perturbative sector, similarly to what is done at the semiclassical level through the classical background trajectories [16].

These perturbation equations were then used to calculate the spectrum index $n_s$ of the background model of Ref. [9] yielding $n_s = 3$, incompatible with observations Ref. [8] ($n_s \approx 1$). This result is intimately related to the logarithmically dependence of the wave number to the potential crossing time, see eq. (C.67). As a consequence, the model should be discarded. This is an example of an inflationary model without (almost) scale invariant scalar perturbations.

The no back-reaction hypothesis we have used was justified through the assumption that the perturbations are in a quantum vacuum state initially [15]. One could verify the consistency of such hypothesis by checking whether the perturbations calculated under this assumption never departs the linear regime in the region where the background is influenced by quantum effects. This check was done in other frameworks (see Ref. [7]), where self-consistency was verified. This self-consistency check, however, was not implemented here because the model studied in section IV does not present a scale invariant spectrum for long-wavelength perturbations, and the model should be discarded without the need of calculating the amplitude of perturbations.

We have also assumed that there is no quantum entanglement in such a way that the background disturbs the quantum evolution of the perturbations. This is a restriction on the possible wave functionals of the Universe, which should then satisfy condition (C.46). It should be interesting to investigate situations where entaglement is allowed when the background is in the quantum regime, which would imply modifications of Eq. (C.58) at the bounce. In this case, condition (C.46) reduces to condition (C.59) (no entaglement when the background becomes classical).

Some future investigations should be to apply the formalism to bouncing models obtained in the framework of quantum cosmology with scalar fields without potential described in Ref. [26] in order to evaluate their spectral index. We will also study the possibility to generalize the simplification of the perturbation equations obtained here to the case of scalar fields with an arbitrary potential term.

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# D. Large classical universes emerging from quantum cosmology: Nelson Pinto-Neto

## D.1. abstract

It is generally believed that one cannot obtain a large Universe from quantum cosmological models without an inflationary phase in the classical expanding era because the typical size of the Universe after leaving the quantum regime should be around the Planck length, and the standard decelerated classical expansion after that is not sufficient to enlarge the Universe in the time available. For instance, in many quantum minisuperspace bouncing models studied in the literature, solutions where the Universe leave the quantum regime in the expanding phase with appropriate size have negligible probability amplitude with respect to solutions leaving this regime around the Planck length. In this paper, I present a general class of moving gaussian solutions of the Wheeler-DeWitt equation where the velocity of the wave in minisuperspace along the scale factor axis, which is the new large parameter introduced in order to circumvent the abovementioned problem, induces a large acceleration around the quantum bounce, forcing the Universe to leave the quantum regime sufficiently big to increase afterwards to the present size, without needing any classical inflationary phase in between, and with reasonable relative probability amplitudes with respect to models leaving the quantum regime around the Planck scale. Furthermore, linear perturbations around this background model are free of any transplanckian problem.

## D.2. Introduction

The existence of an initial singularity [1] is one of the major drawbacks of classical cosmology. In spite of the fact that the standard cosmological model, based in classical general relativity sourced by ordinary matter, has been successfully tested until the nucleosynthesis era, the extrapolation of this model to higher energies leads to a breakdown of the geometry in a finite cosmic time. It indicates the failure of this conventional approach at high energies, which should be complemented through the intervention of some new physics (presence of exotic matter, modifications of general relativity through non-minimal couplings, non

linear curvature terms in the lagrangian, quantum effects of the gravitational field , etc), leading to a complete regular cosmological model.

In the framework of quantum cosmology in minisuperspace models, non singular bouncing models have been obtained[1], where the bounce occurs due to quantum effects in the background [6, 7, 111]. Some approaches have used an ontological interpretation of quantum mechanics, the Bohm-de Broglie [9] one, to interpret the results [7, 111] because, contrary to the standard Copenhagen interpretation, this ontological interpretation does not need a classical domain outside the quantized system to generate the physical facts out of potentialities (the facts are there *ab initio*), and hence it can be applied to the Universe as a whole. Of course there are other alternative interpretations which can be used in quantum cosmology, like the many worlds interpretation of quantum mechanics [10], but I will not use them in this paper.

In the Bohm-de Broglie interpretation, quantum Bohmian trajectories, the quantum evolution of the scale factor $a_q(t)$, can be defined through the relation $\dot{a} \propto \partial S/\partial a$, where $S$ is the phase of an exact wave solution $\Psi(a,t)$ of the Wheeler-DeWitt equation. It satisfies a modified Hamilton-Jacobi equation, augmented with a quantum potential term derived from $\Psi(a,t)$, and hence $a_q(t)$ is not the classical trajectory: in the regions where the quantum effects cannot be neglected, the quantum trajectory $a_q(t)$ performs a bounce which connect two asymptotic classical regions where the quantum effects are negligible. One then has in hands a definite function of time for the homogeneous and isotropic background part of the Universe, even at the quantum level, which realizes a soft transition from the contracting phase to the expanding one.

When studying the evolution of quantum cosmological perturbations on these backgrounds, which was done in the series of papers [12, 13, 14, 15] for the case of one perfect fluid with equation of state $p = w\rho$, one arrives at the result that, in order to obtain wavelength spectra and amplitudes compatible with CMB data, one must have [15] $|w| << 1$ and $w^{1/4}L_0 \approx 10^2 l_{\rm pl}$, where $L_0$ is the curvature scale at the bounce and $l_{\rm pl}$ is the Planck length. Hence this analysis shows that the model is self consistent because observational constraints impose that the curvature scale at the bounce must be at least a few orders of magnitude greater than the Planck length, a region where one can trust the Wheeler-DeWitt equation without been spoiled by high order quantum gravity effects. Of course this model should be extended to include radiation. In Ref. [4] it is shown that the requirement $|w| << 1$ is important only at the moment

[1]There are many other frameworks where bounces connecting the present expanding phase with a preceding contracting one may occur [2, 3]. In this case, the Universe is eternal, there is no beginning of time, nor horizons. The new features of these models introduce a new picture, where the usual problems of initial conditions [4] (as, for instance, the almost homogeneous beginning of the expanding phase might be explained through the disipation of nonlinear inhomogeneities when the universe was very large and rarefied in the asymptotic far past of the contracting phase) and the evolution of cosmological perturbations [5] are viewed from a very different perspective.

when the perturbation wavelength becomes greater than the curvature scale: the fluid which dominates at the bounce is irrelevant for the spectral index (but is important for the amplitude, as we will see in future publications).

However this scenario has a problem on the quantum background solution itself: in order for the model describe the big Universe we live in, the scale factor at the bounce $a_0$ must be somewhat large, and the probability one can obtain from the wave function of the model for the occurence of this value is incredibly small (in some cases $\exp(-10^{89})$, as we will see later on). Hence, either there is an inflationary phase after the bounce in order to enlarge the Universe from a small $a_0$ (which may lead to transplanckian problems [11] and non linear inhomogeneities at the bounce because of the growth of linear perturbations in the contracting phase if $a_0$ is small), or one should rely very strongly on some anthropic principle in a situation much worst than in the landscape scenario.

The aim of this paper is to overcome this difficulty by proposing some more general wave solutions of the Wheeler-DeWitt equation which lead to realistic bouncing scenarios with parameters with reasonable probabilities and without any transplanckian problem. In Ref. [15], the wave function at the bounce was chosen to be a static gaussian of the scale factor centered at $a = 0$. As we will see, the ratio between the scale factor at the bounce $a_0$ and the width of the gaussian must be very large in order to yield the big Universe we live in, yielding the very small probability of occurrence of these parameters I mentioned above. However, if one generalizes the wave function to be a moving gaussian on the $a$-axis with velocity $u$, there is a minimum value of this parameter from where one can obtain a large Universe, with reasonable probability of occurrence, and without any transplanckian problems. The parameter $u$ induces a very large acceleration around the bounce, leading to a sufficiently large scale factor when the quantum regime is over.

This paper is organized as follows: in Section II I describe in detail the problem I want to solve. In Section III I present the generalized wave solutions from which this problem can be circumvented. I conclude in Section IV with a discussion of our results, their physical meanings, and prospects for future work.

## D.3. Quantum bounce solutions from static initial gaussians and their problems

The Hamiltonian constraint describing a cosmological model with flat homogeneous and isotropic closed spacelike hypersurfaces with comoving volume $V = 1$, and a perfect fluid satisfying $p = \omega\epsilon$, where $\epsilon$ is the perfect fluid energy density, $p$ is the pressure and $\omega$ is a constant, reads [12]

$$H_0 \equiv \frac{P_T}{a^{3\omega}} - \frac{P_a^2}{4a}, \tag{D.1}$$

where $a$ is the scale factor, $P_a$ its canonical momentum, and the conserved quantity $P_T$, the momentum canonically conjugated to the degree of freedom of the fluid $T$ [16], is associated with the constant appearing in the energy density of the fluid through the relation $\epsilon = P_T/a^{3(1+w)}$. All quantities are in Planck unities. One can verify that the Hamiltonian $H = NH_0$ generates the usual Friedmann equations of the model.

The wave function $\Psi(a,T)$ satisfies the Wheeler-DeWitt equation $H_0\Psi = 0$,

$$i\frac{\partial}{\partial T}\Psi(a,T) = \frac{a^{(3\omega-1)/2}}{4}\frac{\partial}{\partial a}\left[a^{(3\omega-1)/2}\frac{\partial}{\partial a}\right]\Psi(a,T), \tag{D.2}$$

where I have chosen the factor ordering in $a$ in order to yield a covariant Schrödinger equation under field redefinitions. The fluid selects a prefered time variable.

I change variables to

$$\chi = \frac{2}{3}(1-\omega)^{-1}a^{3(1-\omega)/2},$$

obtaining the simple equation

$$i\frac{\partial\Psi(\chi,T)}{\partial T} = \frac{1}{4}\frac{\partial^2\Psi(\chi,T)}{\partial\chi^2}. \tag{D.3}$$

This is just the time reversed Schrödinger equation for a one dimensional free particle constrained to the positive axis. As $a$ and $\chi$ are positive, solutions which have unitary evolution must satisfy the condition

$$\left(\Psi^\star\frac{\partial\Psi}{\partial\chi} - \Psi\frac{\partial\Psi^\star}{\partial\chi}\right)\Bigg|_{\chi=0} = 0 \tag{D.4}$$

(see Ref. [111] for details). I can choose the initial normalized wave function

$$\Psi^{(\mathrm{init})}(\chi) = \left(\frac{8}{T_0\pi}\right)^{1/4}\exp\left(-\frac{\chi^2}{T_0}\right), \tag{D.5}$$

where $T_0$ is an arbitrary constant. The gaussian $\Psi^{(\mathrm{init})}$ satisfies condition (D.4), and it gives the probability density for the value of $\chi$ at $T=0$ with minimum uncertainty.

Using the propagator procedure of Refs. [111], we obtain the wave solution for all times in terms of $a$:

$$\Psi(a,T) = \left[\frac{8T_0}{\pi\left(T^2+T_0^2\right)}\right]^{1/4}\exp\left[\frac{-4T_0a^{3(1-\omega)}}{9(T^2+T_0^2)(1-\omega)^2}\right]\exp\left\{-i\left[\frac{4Ta^{3(1-\omega)}}{9(T^2+T_0^2)(1-\omega)^2} + \frac{1}{2}\arctan\left(\frac{T_0}{T}\right.\right.\right. \tag{D.6}$$

Due to the chosen factor ordering, the probability density $\rho(a,T)$ has a non trivial measure and it is given by $\rho(a,T) = a^{(1-3\omega)/2}R^2$, where $R^2 = |\Psi(a,T)|^2$.

Its continuity equation, one of the equations coming from Eq. (D.2) after substitution of $\Psi = Re^{iS}$ in it, reads

$$\frac{\partial \rho}{\partial T} - \frac{\partial}{\partial a}\left[\frac{a^{(3\omega-1)}}{2}\frac{\partial S}{\partial a}\rho\right] = 0, \tag{D.7}$$

which implies, in the Bohm interpretation [9], the definition of a velocity field

$$\dot{a} = -\frac{a^{(3\omega-1)}}{2}\frac{\partial S}{\partial a}, \tag{D.8}$$

in accordance with the classical relations $\dot{a} = \{a, H\} = -a^{(3\omega-1)}P_a/2$ and $P_a = \partial S/\partial a$.

Note that $S$ satisfies the other equation coming from (D.2),

$$\frac{\partial S}{\partial T} - \frac{a^{(3\omega-1)}}{4}\left(\frac{\partial S}{\partial a}\right)^2 + \frac{a^{(3\omega-1)/2}}{4R}\frac{\partial}{\partial a}\left[a^{(3\omega-1)/2}\frac{\partial R}{\partial a}\right] = 0, \tag{D.9}$$

which is a Hamilton-Jacobi-like equation with an extra quantum term, called the quantum potential, given by

$$Q \equiv -\frac{a^{(3\omega-1)/2}}{4R}\frac{\partial}{\partial a}\left[a^{(3\omega-1)/2}\frac{\partial R}{\partial a}\right]. \tag{D.10}$$

Hence, the trajectory (D.8) will not coincide with the classical trajectory whenever $Q$ is comparable with the other terms present in Eq. (D.9) because $S$ will be different from the classical Hamilton-Jacobi function.

Inserting the phase of (D.6) into Eq. (D.8), I obtain the Bohmian quantum trajectory for the scale factor:

$$a(T) = a_0\left[1 + \left(\frac{T}{T_0}\right)^2\right]^{\frac{1}{3(1-\omega)}}, \tag{D.11}$$

or, in terms of $\chi(T)$,

$$\chi(T) = \chi_0\left[1 + \left(\frac{T}{T_0}\right)^2\right]^{\frac{1}{2}}. \tag{D.12}$$

Note that $\chi_0$ is the value of $\chi$ at $T = 0$, the moment of the bounce, $\chi_0 = \chi_{\text{bounce}}$, and the scale factor at the bounce $a_0$ is connected to $\chi_0$ through

$$a_0 = \left[\frac{3}{2}(1-\omega)\chi_0\right]^{2/[3(1-\omega)]}. \tag{D.13}$$

Solution (E.296) has no singularities and tends to the classical solution when $T \to \pm\infty$. Remember that I am in the gauge $N = a^{3\omega}$, and $T$ is related to conformal time through

$$N\mathrm{d}T = a\mathrm{d}\eta \quad \Longrightarrow \mathrm{d}\eta = [a(T)]^{3\omega-1}\,\mathrm{d}T. \tag{D.14}$$

The solution (E.296) can be obtained from other initial wave functions (see Ref. [111]).

However, the above solution suffers from the following drawback: the curvature scale at the bounce reads $L_{\rm bounce} \equiv T_0 a_0^{3w}$, and the quantity $P_T$ associated in the classical limit $|T| \to \infty$ with the constant appearing in the energy density of the fluid through the relation $\epsilon = P_T/a^{3(1+w)}$, can be obtained in the Bohmian approach from the wave function through the relation $P_T = \partial S/\partial T$. It reads

$$P_T = \frac{\partial S}{\partial T} = \frac{T_0}{2(T^2+T_0^2)} - \frac{\chi(T)^2(T_0^2-T^2)}{(T^2+T_0^2)^2}. \tag{D.15}$$

Inserting the solution (D.12) in Eq. (D.15) and taking the classical limit $|T| \to \infty$, one obtains

$$P_T = \frac{\chi_0^2}{T_0^2}. \tag{D.16}$$

In the case of dust, $P_T$ is the total dust mass of the Universe, yielding $P_T \geq 10^{60}$. If one takes the curvature scale at the bounce some few orders of magnitude larger than the Planck length, say $10^3$, in order to not spoil the Wheeler-DeWitt approach used above due to strong quantum gravitational effects, one has to have $\chi_0 \geq 10^{33}$, with probability less than $\exp(-10^{63})$ to occur (see Eq. (D.5)).

The situation is similar with radiation, where now $P_T = \chi_0^2/T_0^2 \geq 10^{116}$. Note that in this case $\chi_0 = a_0$, $T = \eta$, and the curvature scale at the bounce reads $L_{\rm bounce} \equiv T_0 a_0$. Combining the constraints $a_0/T_0 \geq 10^{58}$ and $a_0 T_0 \geq 10^3$, one arrives at the very low probability $\exp(-10^{89})$ for these parameters to occur.

The source of the problem is the fact that the constant $\chi_0$ appearing in Eq. (D.16) is also the value of $\chi$ at $T = 0$, the $\chi$ at the bounce, and the fact that $P_T$ must be large, induces a large $\chi^2/T_0$ in the gaussian (D.5). One possibility to scape from this drawback is to find a different wave solution to Eq. (D.2) which either modifies Eq. (D.16) or yields Bohmian trajectories where $\chi_0$ is not anymore the value of $\chi$ at $T = 0$, allowing the possibility of having a small initial $\chi$, hence a small $\chi^2/T_0$ in (D.5), and a huge $\chi_0$, perhaps through the presence of an inflationary phase between the bounce and the standard decelerated expansion. I will show in the next section that it is indeed possible to obtain a more general class of wave solutions of Eq. (D.2) where the above mentioned problem is circumvented.

## D.4. New bouncing solutions

I will generalize the initial wave function by inserting a velocity term in Eq. (D.5), which of course must satisfy the boundary condition (D.4), and now reds,

$$\Psi^{(\rm init)}(\chi) = \left(\frac{2}{T_0\pi}\right)^{1/4} [1 \pm \exp(-u^2 T_0/8)]^{-1/2} [\exp(iu\chi/2) \pm \exp(-iu\chi/2)] \exp\left(-\frac{\chi^2}{T_0}\right). \tag{D.17}$$

This initial wave function represents two gaussians travelling from the origin in opposite directions (keeping in mind that only the tail of the gaussian traveling in the negative direction with suport on the positive $a$ axis has physical meaning). The solution for all times read

$$\begin{aligned}
\Psi(\chi,T) &= \left[\frac{2T_0}{\pi\left(T^2+T_0^2\right)}\right]^{1/4}(1\pm\exp(-u^2T_0/8)^{-1/2} \\
&\Bigg( \exp\left[-\frac{T_0(\chi-uT)^2}{(T^2+T_0^2)}\right]\exp\left\{-i\left[\frac{T(\chi-uT)^2}{(T^2+T_0^2)}+2u(\chi-uT/2)+\frac{1}{2}\arctan\left(\frac{T_0}{T}\right)-\frac{\pi}{4}\right]\right. \\
&\pm \exp\left[-\frac{T_0(\chi+uT)^2}{(T^2+T_0^2)}\right]\exp\left\{-i\left[\frac{T(\chi+uT)^2}{(T^2+T_0^2)}-2u(\chi+uT/2)+\frac{1}{2}\arctan\left(\frac{T_0}{T}\right)-\frac{\pi}{4}\right]\right.
\end{aligned}$$

which I write as,

$$\Psi = A(R_-e^{iS_-}\pm R_+e^{iS_+}),$$

where

$$\begin{aligned}
R_\pm &\equiv \exp\left[-\frac{T_0(\chi\pm uT)^2}{(T^2+T_0^2)}\right], \\
S_\pm &\equiv \left[-\frac{T(\chi\pm uT)^2}{(T^2+T_0^2)}\pm 2u(\chi\pm uT/2)-\frac{1}{2}\arctan\left(\frac{T_0}{T}\right)+\frac{\pi}{4}\right], \\
A &\equiv \left[\frac{2T_0}{\pi\left(T^2+T_0^2\right)}\right]^{1/4}(1\pm\exp(-u^2T_0/8)^{-1/2}.
\end{aligned}$$

From these equations one obtains the total amplitude and phase as

$$\begin{aligned}
R &= A\ \sqrt{R_+^2+R_-^2\pm 2R_+R_-\cos(S_+-S_-)} \\
S &= \arctan\left(\frac{R_+\sin(S_+)\pm R_-\sin(S_-)}{R_+\cos(S_+)\pm R_-\cos(S_-)}\right)
\end{aligned}$$

The derivative of $S$ with respect to some variable $x$ reads

$$\frac{\partial S}{\partial x} = \frac{R_+^2\frac{\partial S_+}{\partial x}+R_-^2\frac{\partial S_-}{\partial x}\pm\left(\frac{\partial S_+}{\partial x}+\frac{\partial S_-}{\partial x}\right)R_+R_-\cos\left(S_+-S_-\right)\pm\left(R_-\frac{\partial R_+}{\partial x}-R_+\frac{\partial R_-}{\partial x}\right)\sin\left(S_+-S_-\right.}{R_+^2+R_-^2\pm 2R_+R_-\cos(S_+-S_-)}$$

The guidance relation (D.8) leads to the exact differential equation

$$\dot{\chi}(T) = \frac{T\chi(T)}{T^2+T_0^2}+\frac{uT_0^2}{T^2+T_0^2}\left[\frac{\sinh\left(\theta\frac{T}{T_0}\right)\pm\frac{T}{T_0}\sin\theta}{\cosh\left(\theta\frac{T}{T_0}\right)\pm\cos\theta}\right], \tag{D.19}$$

where

$$\theta \equiv \frac{4uT_0^2\chi(T)}{T^2+T_0^2}. \tag{D.20}$$

From the solution (D.18), the new $P_T$ is now given by

$$P_T = \frac{\partial S}{\partial T} = \frac{T_0}{2(T^2+T_0^2)} + \frac{[u^2T_0^2 - \chi^2(T)](T_0^2 - T^2)}{(T^2+T_0^2)^2} + \frac{\left[4uT_0^2T\chi(T)\sinh\left(\theta\frac{T}{T_0}\right) \pm 2T_0u\chi(T)(T^2-T_0^2)\right.}{(T^2+T_0^2)^2\left[\cosh\left(\theta\frac{T}{T_0}\right) \pm \cos\theta\right]} \tag{D.21}$$

From now on I will work with the plus sign solution given in Eq. (D.18). The minus sign solution yields the same qualitative results.

### D.4.1. Quantum solutions for small $u$

Taking $u << 1$, one has

$$\dot{\chi}(T) = \frac{T\chi(T)}{T^2+T_0^2} + \frac{4\chi(T)u^2TT_0^3}{(T^2+T_0^2)^2}, \tag{D.22}$$

with solution

$$\begin{aligned} \chi(t) &= \frac{\chi_0}{T_0}\sqrt{T^2+T_0^2}\exp\left[\frac{-2u^2T_0^3}{T^2+T_0^2}\right] \\ &= \chi_0\sqrt{x^2+1}\exp\left[\frac{-2u^2T_0}{x^2+1}\right], \end{aligned} \tag{D.23}$$

where in the last step I wrote the solution in terms of $x = T/T_0$.

Solution (D.23) has very nice properties. First of all, one can see that the values of $\chi$ and the curvature scale at the bounce are now given by

$$\chi_{\text{bounce}} = \chi_0\exp(-2u^2T_0), \tag{D.24}$$

and

$$L_{\text{bounce}} = \exp\left[-\frac{4wu^2T_0}{(1-w)}\right]\frac{T_0a_0^{3w}}{\sqrt{1+4u^2T_0}}. \tag{D.25}$$

Inserting solution (D.23) into Eq. (D.21) in the limit $|T| \to \infty$, yields

$$P_T = \frac{\chi_0^2}{T_0^2}. \tag{D.26}$$

Note that now $\chi_0 \neq \chi_{\text{bounce}}$, the value of $\chi$ at $T = 0$. In fact, from Eq. (D.24), one may have $\chi_0 >> \chi_{\text{bounce}}$, depending on the value of $T_0$, because of the huge acceleration one may obtain near after the bounce as compared with the case where $u = 0$: it is a bounce followed by inflation. Hence, one may have reasonable probability amplitudes for the free parameters of the theory which are compatible with a huge $P_T$.

However, one must also check whether the Bohmian trajectory (D.23), with such apropriate choice of parameters, reachs classical evolution ($x >> 1$) before the nucleosynthesis epoch. Let us concentrate on the case of radiation

($w = 1/3$), as it is the most interesting physical situation (one expects the quantum effects and the bounce to occur in a very hot radiation dominated universe). Supose the classical limit is already valid at a conformal time where the energy density of radiation is minimally greater than the energy density before nucleosynthesis, say, the energy density around the freeze-out of neutrons, $\rho_{\rm f} \approx 10^{-88}$. Then, from Eqs. (D.23) and (D.26), and from $\rho_{\rm f} = P_T/a_{\rm f}^4$, with $a_{\rm f} = a_0\eta_{\rm f}/T_0$, one obtains $x_{\rm f} \equiv \eta_{\rm f}/T_0 \approx 10^{22}/(T_0 P_T^{1/4})$. Using that $P_T \geq 10^{116}$, one gets that $x_{\rm f} \leq 10^{-7}/T_0$. Hence, $x_{\rm f} >> 1$ if and only if $T_0 << 10^{-7}$. However, as $u << 1$, then $u^2 T_0 << 1$, and the exponential in (D.23) would be irrelevant, turning solution (D.23) very close to solution (E.296) for $w = 1/3$, taking us back to the previous problem. Concluding, the only way to obtain a huge $P_T$ with parameters with reasonable probability amplitudes in this framework is through choices which will change the usual scale factor evolution during nuclosynthesis, spoiling its observed predictions.

### D.4.2. Quantum solutions for large $u$

If $u >> 1$, for $|T|$ not very small, and noting that the unique possible asymptotic behaviour of a solution $\chi(T)$ of Eq. (E.11.1) is $\chi(T) \propto T$, then the hyperbolic functions in (E.11.1) are very large and much greater than the terms with trigonometric functions, yielding

$$\dot{\chi}(T) = \frac{T\chi(T)}{T^2 + T_0^2} \pm \frac{uT_0^2}{T^2 + T_0^2}, \tag{D.27}$$

with solution

$$\chi(T) = \frac{\chi_0}{T_0}\sqrt{T^2 + T_0^2} \pm uT, \tag{D.28}$$

where the $\pm$ sign corresponds to positive and negative values of $T$, respectively. For $|T| \approx 0$, one has to rely on numerical calculations. However, as shown in figure 1 below, for large $u$ this difference is quite unimportant. Hence, again, $\chi_0$ is very close to the value of $\chi$ at the bounce.

Inserting solution (D.28) into Eq. (D.21) in the limit $|T| \to \infty$, we obtain

$$P_T = \left(\frac{\chi_0}{T_0} + u\right)^2. \tag{D.29}$$

Hence, the huge values of $P_T$ can be obtained from large values of $u$, without any imposition on the parameters $\chi_0$ and $T_0$.

Let us calculate the constraints on the parameter space in order to obtain a sensible model. As discussed in section II, one should have $a_0^2/T_0 \leq 1$ in order to have a reasonable probability amplitude for $a_0$, and the curvature scale at the bounce should not be very close to the Planck length in order to avoid strong quantum gravitational effects. I will also impose that $a_0 > 1$ in order to avoid transplanckian problems (see below), which implies that $a_0/T_0 \leq 1$.

The curvature scale at the bounce reads

$$L_{\rm bounce} = \frac{a_0 T_0 \sqrt{a_0[1+\cos(4ua_0)]}}{\sqrt{a_0[1+\cos(4ua_0)] + uT_0[4ua_0 + \sin(4ua_0)]}} \approx \frac{a_0\sqrt{T_0}}{2u}, \tag{D.30}$$

where in the last approximation I used that $uT_0/a_0 >> 1$, which follows from $u >> 1$, $a_0/T_0 \leq 1$, and I assumed that $4ua_0 \neq (2n+1)\pi$, in order to avoid $L_{\rm bounce} << 1$. Hence, as $a_0 \leq T_0^{1/2}$, then

$$L_{\rm bounce} \leq \frac{T_0}{2u} \tag{D.31}$$

Demanding that $L_{\rm bounce} \geq 10^3$, than

$$T_0 > u10^3 >> 1. \tag{D.32}$$

One must check again whether one recovers the classical radiation dominated evolution before nucleosynthesis. As before, I concentrate on the case $w = 1/3$, which implies that $\chi = a$ and $T = \eta$. I will do this in two steps: first I check whether solution (D.28) is valid before nucleosynthesis, and then whether quantum effects are negligible there.

The approximation leading to solution (D.28) requires that the argument of the hyperbolic functions in Eq. (E.11.1) be large, $\theta T/T_0 >> 1$, which implies that $x = \eta/T_0 >> (4u^2T_0 - 1)^{-1/2} \approx 1/(2uT_0^{1/2})$. Hence, the values of the conformal time for which solution (D.28) is reliable are $\eta >> T_0^{1/2}/(2u)$.

Let us now verify whether solution (D.28) is valid around the freeze-out of neutrons, before nucleosynthesis. This will be true if $\eta_{\rm f} >> T_0^{1/2}/(2u)$. However, $\eta_{\rm f} \approx 10^{22}/P_T^{1/4} = 10^{22}/u^{1/2}$ which, when combined with $\eta_{\rm f} >> T_0^{1/2}/(2u)$, implies that $10^{44} >> T_0/(4u) \geq L_{\rm bounce}$, where I used Eq. (D.31). As the curvature scale around freeze-out of neutrons, $L_{\rm f}$, satisfies $L_{\rm f} \approx 10^{44}$, this condition is just the reasonable constraint that

$$L_{\rm bounce} << L_{\rm f} \approx 10^{44}. \tag{D.33}$$

Hence, if the curvature scale at the bounce is much smaller than the curvature scale around freeze-out of neutrons, than solution (D.28) must be valid at nucleosynthesis period.

Finally, one must check that in the regime where solution (D.28) is valid we are already in the classical limit, even though the above condition $x >> 1/(2uT_0^{1/2})$ may still contain a region were $x << 1$ because $u$ and $T_0$ are large. To prove this, note first that at $\eta >> T_0^{1/2}/(2u)$, the term $u\eta >> T_0^{1/2}/2$ dominates over $a_0\sqrt{x^2+1}$ in Eq. (D.28), either for $x << 1$, because $a_0 \leq T_0^{1/2}$, as for $x >> 1$, because $a_0/T_0 << u$. Hence, the quantum potential given by

$$Q := -\frac{\partial^2 R}{4R\partial\chi^2} =: Q_1 + Q_2^2, \tag{D.34}$$

where

$$\begin{aligned} Q_1 &= -T_0\{[4T_0(\chi^2+u^2T^2)-(T^2+T_0^2)]\cosh(\theta T/T_0)-8uT_0T\chi\sinh(\theta T/T_0) \\ &+ [4T_0(\chi^2-u^2T_0^2)-(T^2-T_0^2)]\cos\theta+8T_0^2\chi u\sin\theta\}\{2(T^2+T_0^2)^2[\cosh(\theta T/T_0)+\cos\theta]\}^{-1}, \end{aligned} \tag{D.35}$$

and

$$Q_2 = T_0\frac{X[\cosh(\theta T/T_0)+\cos\theta]-u[T\sinh(\theta T/T_0)-T_0\sin\theta]}{(T^2+T_0^2)[\cosh(\theta T/T_0)+\cos\theta]}, \tag{D.36}$$

reads, around the Bohmian trajectory $a \approx u\eta$,

$$Q \approx \frac{1}{\cosh(2\theta T/T_0)} << 1, \tag{D.37}$$

while the kinetic term of the Hamilton-Jacobi-like equation (D.9) is given by,

$$\left(\frac{\partial S}{2\partial a}\right)^2 \approx u^2 >> 1. \tag{D.38}$$

Note that near the bounce at $\eta \approx 0$, the kinetic term is almost null, while the quantum potential is finite: quantum effects are dominant only very near the bounce. The transition from quantum to classical regime should be around $\theta T/T_0 \approx 1$, where $Q \approx 1$ and solution (D.28) is not reliable. A little bit later, when $x = \eta/T_0 > (4u^2T_0-1)^{-1/2} \approx 1/(2uT_0^{1/2})$, and knowing from Eq. (D.32) that $T_0 > 10^{61}$ because $u > 10^{58}$, we obtain that the scale factor at the beginning of the classical regime is $a > 10^{31}$ which is the minimum value required for the model to reach the size of the observed Universe without needing any classical inflationary phase afterwards. Note from figure 1 that the presence of the $u$ term in Eq. (D.28) induces a much bigger acceleration at the bounce in comparison with the solution with the same $a_0$ but without the $u$ term. It is this term which is the responsible for the big value $a > 10^{31}$ when the model enters the classical regime.

Concluding, solution (D.28) reachs the standard cosmological model before nucleosynthesis, and can indeed describe the observed Universe in the radiation dominated phase with parameters with reasonable relative probabilities.

In order to avoid the transplanckian problem for the scales of physical interest today, $10^{54} < \lambda_{\rm today}^{\rm physical} < 10^{60}$, one should have $\lambda_{\rm bounce}^{\rm physical}$ corresponding to these scales not smaller than, say, $10^3$. As

$$\lambda_{\rm bounce}^{\rm physical} = \frac{a_0}{a_{\rm today}}\lambda_{\rm today}^{\rm physical}, \tag{D.39}$$

this problem can be avoided if

$$\frac{a_0}{a_{\rm today}} > 10^{-51}, \tag{D.40}$$

which implies that $a_0 > 10^9$.

Note that there is an upper limit for $a_0$ coming from

$$L_{\text{bounce}} \approx \frac{a_0 T_0^{1/2}}{2u} \geq 10^{64} \left( \frac{a_0}{a_{\text{today}}} \right)^2, \tag{D.41}$$

where I used that $a_0^2/T_0 \leq 1$, $P_T \approx u^2 \approx a_{\text{today}}^4 10^{-128}$. The constraint $L_{\text{bounce}} << 10^{44}$ (see Eq.(D.33)) then implies that

$$\frac{a_0}{a_{\text{today}}} << 10^{-10}. \tag{D.42}$$

Hence, there is a large domain of values of $a_0$ where the transplanckian problem can be avoided (see Eqs.(D.40,D.42)).

## D.5. Conclusion

I have shown in this paper how a sufficiently big universe can emerge from a quantum cosmological bounce, without needing any classical inflationary phase afterwards to make it grow to its present size. This is caused by a huge acceleration during the quantum bounce, which may be viewed as a quantum inflation. These results were obtained from a moving gaussian function of the scale factor, which is a solution of the Wheeler-DeWitt equation coming from the canonical quantization of general relativity sourced by relativistic particles. The solution is exact, there is no WKB approximation involved here. Its value at $T = 0$ yields reasonable relative probability amplitudes of having the scale factor at the bounce with the value $a_0$ required to avoid any transplanckian problem, and to allow that the curvature scale at the bounce be some few orders of magnitude greater than the Planck length, a region where one can rely on this simple quantization scheme. In fact, as the maximum value the curvature scale can have is at the bounce itself, one never reachs energy scales where more involved quantum gravity theories, like string theory and loop quantum gravity (see Ref. [17] about issues concerning this approach), must be invoked: the model is self-contained.

There are two internal parameters of the wave functon which must be big in order to obtain a large classical universe from a quantum bounce. The first one is the parameter $T_0$, the square root of the width of the gaussian at the moment of the bounce (see Eq. (D.17)), which must satisfy $T_0 > 10^{61}$ (in Planck unities). This value yields the sufficient large value $a > 10^{31}$ for the scale factor in the beginning of the classical regime, and guarantees that the curvature scale at the bounce be some minimum orders of magnitude greater than the Planck length in order for the Wheeler-DeWitt equation I used be reliable. The other one is the velocity $u$ of the gaussian along the scale factor axis which must satisfy $u > 10^{58}$ in order to yield the amount of radiation we observe in the Universe today, without appealing to some huge production of photons during the bounce.

From these considerations, one can see that the parameters emerging from the quantum era of the Universe are not necessarily Planckian: they depend also on the quantum state of the system, on the internal parameters of the wave function of the Universe. Hence, it is not surprising that one may have quantum gravity effects in large (when compared with the Planck length) Universes [18], which could be dramatically seen in a big-rip [284].

However, one may ask why the internal parameters of the wave function we obtained are so large. Note first that these are not coupling constants, but parameters in the quantum state of the Universe. Hence, to answer this question, one should rely on some deep understanding of quantum cosmology and/or new principles which are not available today. Note, however, that the big value of $T_0$ leads to a widely spread gaussian, and hence almost all scales at the bounce are equally probable. This is a reasonable assumption about the wave function of the Universe one can make: it should not intrinsically select any preferable scale at the bounce without any special reason. Concerning the $u$ variable, its large value implies that the peak of the initial wave packet moves very fast towards large scale factors, which induces a large universe. Perhaps some version of the Anthropic Principle could justify the preference for large classical universes, and as consequence for a large $u$, but I think the important message here is the possibility of obtaining a large universe from a huge acceleration of the scale factor in the far past, whose origin differs fundamentally from those considered in usual inflationary scenarios.

In future publications, we will calculate the evolution of linear quantum perturbations and particle production on these quantum backgrounds, as in Ref. [15], and compare the results with observations.

As a final remark, I would like to repeat a comment we made elsewhere [4]: in contradistinction with models in which time begins, there is no point on asking what is the probability of appearance of some particular eternal model out of nothing. Contrary to usual perspectives, one can as well assume existence to be conceptually prior to non-existence, *i.e.* existence itself may not be deserving explanation. This is the idea underlying our category of models: the Universe always existed and its "appearance" is thus a non question.

## Acknowledgements

I would like to thank CNPq of Brazil for financial support. I very gratefully acknowledge various enlightening conversations with Felipe Tovar Falciano, Patrick Peter, Emanuel Pinho, and specially Andrei Linde, whose critcisms inspired this work. I also would like to thank CAPES (Brazil) and COFECUB (France) for partial financial support.

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# E. Bouncing Cosmologies: M. Novello - S E P Bergliaffa

**Published in Physics Reports**

## E.1. Introduction

The standard cosmological model (SCM) furnishes an accurate description of the evolution of the universe, which spans approximately 13.7 billion years. The main hypothesis on which the model is based are the following:

1. Gravity is described by General Relativity.
2. The universe obeys the Cosmological Principle [109]. As a consequence, all the relevant quantities depend only on global Gaussian time.
3. Above a certain scale, the matter content of the model is described by a continuous distribution of matter/energy, which is described by a perfect fluid.

In spite of its success, the SCM suffers from a series of problems such as the initial singularity, the cosmological horizon, the flatness problem, the baryon asymmetry, and the nature of dark matter and dark energy [1], [2]. Although inflation (which for many is currently a part of the SCM) partially or totally answers some of these, it does not solve the crucial problem of the initial singularity [70] [3]. The existence of an initial singularity is disturbing: a singularity can be naturally considered as a source of lawlessness [142], because the spacetime description breaks down "there", and physical laws presuppose spacetime. Regardless of the fact that several scenarios have been developed to deal with the singularity issue, the breakdown of physical laws continues to be a conundrum

[1] There are even claims that standard cosmology does not predict the value of the present CMBR temperature [215].

[2] Some "open questions" may be added to this list, such as why the Weyl tensor is nearly null, and what the future of the universe is.

[3] Inflation also presents some problems of its own, such as the identification of the inflaton with a definite field of some high-energy theory, the functional form of the potential $V$ in terms of the inflaton [52], and the need of particular initial conditions [181]. See also [302]

after almost a hundred years of the discovery of the FLRW solution [4] (which inevitably displays a past singularity, or in the words of Friedmann [166], a *beginning of the world*).

In this review, we shall concentrate precisely on the issue of the initial singularity [5]. We will see that non-singular universes have been recurrently present in the scientific literature. In spite of the fact that the idea of a cosmological bounce is rather old, the first exact solutions for a bouncing geometry were obtained by Novello and Salim [311], and Melnikov and Orlov [286] in the late 70's [6]. It is legitimate to ask why these solutions did not attract the attention of the community then. In the beginning of the 80's, it was clear that the SCM was in crisis (due to the problems mentioned above, to which we may add the creation of topological defects, and the lack of a process capable of producing the initial spectrum of perturbations, necessary for structure formation). On the other hand, at that time the singularity theorems were taken as the last word about the existence of a singularity in "reasonable" cosmological models. The appearance of the inflationary theory gave an answer to some of the issues in a relatively economical way, and opened the door for an explanation of the origin of the spectrum of primordial fluctuations. Faced with these developments, and taking into account the status of the singularity theorems at that time, the issue of the initial singularity was not pressing anymore, and was temporally abandoned in the hope that quantum gravity would properly address it. At the end of the 90's, the discovery of the acceleration of the universe brought back to the front the idea that $\rho + 3p$ could be negative, which is precisely one of the conditions needed for a cosmological bounce in GR, and contributed to the revival of nonsingular universes. Bouncing models even made it to the headlines in the late 90's and early XXI century, since some models in principle embedded in string theory seemed to suggest that a bouncing geometry could also take care of the problems solved by inflation.

Perhaps the main motivation for nonsingular universes is the avoidance of lawlessness, as mentioned above [7]. Also, since we do not know how to handle

[4] This acronym refers to the authors that presented for the first time the solution of EE that describes a universe with zero pressure (Friedmann [166]) and nonzero pressure (Lemâitre [253]), and to those who studied its general mathematical properties and took it to its current form (Robertson [354] and Walker [413]). For historical details, see [288]

[5] We shall not analyze the existence of future singularities, such as the so-called sudden future singularities [36] or the "Big Rip" [92].

[6] An approximate bouncing solution for a massive minimally coupled scalar field in General Relativity was presented in [384].

[7] It is worth noting that Einstein was well aware of the problem of singularities in GR [337], and he made several attempts to regularize some solutions of his theory, such as the so-called Einstein-Rosen bridge, in the early 30s. Indeed, he wrote "The theory (GR) is based on a separation of the concepts of the gravitational field and matter. While this may be a valid approximation for weak fields, it may presumably be quite inadequate for very high densities of matter. One may not therefore assume the validity of the equations for very high densities and it is just possible that in a unified theory there would be no such singularity" [146].

infinite quantities, we would like to have at our disposal solutions that do not entail divergencies. As will be seen in this review, this can be achieved at a classical level, and also by quantum modifications. On a historical vein, this situation calls for a parallel with the status of the classical theory of the electron by the end of the 19th century. The divergence of the field on the world line of the electron led to a deep analysis of Maxwell's theory, including the acceptance of a cooperative influence of retarded and advanced fields [356] [8], and the related causality issues. However, this divergence is milder than that of some solutions of General Relativity, since it can be removed by the interaction of the electron with the environment. Clearly, this is not an option when the singularity is that of a cosmological model.

Another motivation for the elimination of the initial singularity is related to the Cauchy problem. In the SCM, the structure of spacetime has a natural foliation (if no closed timelike curves are present), from which a global Gaussian coordinate system can be constructed, with $g_{00} = 1$, $g_{0i} = 0$, in such a way that

$$ds^2 = dt^2 - g_{ij}\, dx^i dx^j.$$

The existence of a global coordinate system allows a rigorous setting for the Cauchy problem of initial data. However, it is the gravitational field that diverges on a given spatial hypersurface $t = \text{const.}$ (denoted by $\Sigma$) at the singularity in the SCM. Hence, the Cauchy problem cannot be well formulated on such a surface: we cannot pose on $\Sigma$ the initial values for the field to evolve.

There are more arguments that suggest that the singularity should be absent in an appropriate cosmological model. According to [48], the second law of thermodynamics is to be supplemented with a limit on the entropy of a system of largest linear dimension $R$ and proper energy $E$, given by

$$\frac{S}{E} \leq \frac{2\pi R}{\hbar c}.$$

Currently this bound is known to be satisfied in several physical systems [370]. It was shown in [49] that the bound is violated as the putative singularity is approached in the radiation-dominated FLRW model (taking as $R$ the particle horizon size). The restriction to FLRW models was lifted in [370], where it was shown, independently of the spacetime model, and under the assumptions that (1) causality and the strong energy condition (SEC, see Appendix) hold, (2) for a given energy density, the matter entropy is always bounded from above by the radiation entropy, that the existence of a singularity is inconsistent with the entropy bound: a violation occurs at time scales of the order of Planck's time [9].

From the point of view of quantum mechanics, we could ask if it is possible to repeat in gravitation what was done to eliminate the singularity in the classical theory of the electron. Namely, can the initial singularity be smoothed via

[8] In fact, it can be said that the problem of the singularity of the classical theory of the electron was transcended, if not resolved, by the quantization of the EM field.

[9] For an updated discussion of the several types of entropy bounds in the literature, see [86].

quantum theory of gravity? The absence of the initial singularity in a quantum setting is to be expected on qualitative grounds. There exists only one quantity with dimensions of length that can be constructed from Newton's constant $G$, the velocity of light $c$, and Planck's constant $\hbar$ (namely Planck's length $\ell_{Pl} = \sqrt{G\hbar/c^3}$). This quantity would play in quantum gravity a role analogous to that of the energy of the ground state of the hydrogen atom (which is the only quantity with dimensions of energy that can be built with fundamental constants) [62]. As in the hydrogen atom, $\ell_{Pl}$ would imply some kind of discreteness, and a spectrum bounded from below, hence avoiding the singularity [10]. Also, since it is generally assumed that $\ell_{Pl}$ sets the scale for the quantum gravity effects, geometries in which curvature can become larger than $\ell_{Pl}^{-2}$ or can vary very rapidly on this scale would be highly improbable.

Yet another argument that suggests that quantum effects may tame a singularity is given by the Rayleigh-Jeans spectrum. According to classical physics, the spectral energy distribution of radiation in thermal equilibrium diverges like $\omega^3$ at high frequencies, but when quantum corrections are taken into account, this classical singularity is regularized and the Planck distribution applies [177]. We may expect that QG effects would regularize the initial singularity.

As a consequence of all these arguments indicating that the initial singularity may be absent in realistic descriptions of the universe, many cosmological solutions displaying a bounce were examined in the last decades. In fact, the pattern in scientific cosmologies somehow parallels that of the cosmogonic myths in diverse civilizations, which can be classified in two broad classes. In one of them, the universe emerges in a single instant of creation (as in the Jewish-Christian and the Brazilian Carajás cosmogonies [116]). In the second class, the universe is eternal, consisting of an infinite series of cycles (as in the cosmogonies of the Babylonians and Egyptians) [382].

We have seen that there are reasons to assume that the initial singularity is not a feature of our universe. Quite naturally, the idea of a non-singular universe has been extended to encompass cyclic cosmologies, which display phases of expansion and contraction. The first scientific account of cyclic universes is in the papers of Friedmann [268], Einstein [147], Tolman [396], and Lemaître [254] and his Phoenix universe, all published in the 1930's. A long path has been trodden since those days up to recent realizations of these ideas (as for instance [179], see Sect.E.10.2). We shall see in Ch.E.10 that some cyclic models could potentially solve the problems of the standard cosmological model, with the interesting addition that they do not need to address the issue of the initial conditions.

Another motivation to consider bouncing universes comes from the recognition that a phase of accelerated contraction can solve some of the problems of the SCM in a manner similar to inflation. Let us take for instance the flat-

[10] This expectation has received support from the proof that the spectrum of the volume operator in LQG is discrete, see for instance [267].

ness problem (see also Sect.E.10). Present observations imply that the spatial curvature term, if not negligible, is at least non-dominant wrt the curvature term:

$$r^2 = \frac{|\epsilon|}{a^2 H^2} \lesssim 1,$$

but during a phase of standard, decelerated expansion, $r$ grows with time. Indeed, if $a \sim t^\beta$, then $r \sim t^{1-\beta}$. So we need an impressive fine-tuning at, say, the GUT scale, to get the observed value of $r$ [11]. This problem can be solved by introducing an early phase during which the value of $r$, initially of order 1, decreases so much in time that its subsequent growth during FLRW evolution keeps it still below 1 today. This can be achieved by [179] power-law inflation ($a \sim t^\beta,\ \beta > 1$), pole inflation ($a \sim (-t)^\beta,\ \beta < 0, t \to 0_-$), and accelerated contraction ($0 < \beta < 1, t \to 0_-$) [172]. Thus, an era of accelerated contraction may solve the flatness problem (and the other kinematical issues of the SCM [179]). This property helps in the construction of a scenario for the creation of the initial spectrum of cosmological perturbations in non-singular models (see Sect.E.11).

The main goal of this review is to present some of the many non-singular solutions available in the literature, exhibit the mechanism by which they avoid the singularity, and discuss what observational consequences follow from these solutions and may be taken (hopefully) as an unmistakable evidence of a bounce. We shall not pretend to produce an exhaustive list, but we intend to include at least an explicit form for the time evolution of a representative member of each type of solution [12]. The models examined here will be restricted to those close or identical to the FLRW geometry [13]. Although theories other than GR will be examined, we shall not consider multidimensional theories (exception made for models derived from string theory, see Sect.E.3.3) or theories with torsion.

We shall start in Sect.E.1.1 by stating a working definition of nonsingular universe, and giving a brief account of the criteria that can be used to determine whether a certain model is singular or not. It will suffice for our purposes in this review to define a singularity as the region where a physical property of the matter source or the curvature "blows up" [412]. In fact, since we shall be dealing almost exclusively with geometries of the Friedmann type, the singularity is always associated to the divergence of some functional of the curvature [14].

[11] But notice that the flatness problem may actually not be a problem at all if gravity is not described by GR, see Sect.E.2.2

[12] The issue of singularities in cosmology has been previously dealt with in [168].

[13] Notice however the solutions given in [371]. These are non-singular but do not display the symmetries of the observed universe, although they are very useful as checks of general theorems.

[14] But notice that not all types of singularities have large curvature, and diverging curvature is not the basic mechanism behind singularity theorems. If we consider the problem of singularities in a broad sense, we seem to be "treating a symptom rather than the cause" when addressing exclusively unbounded curvature [66].

Let us remark at this point that there are at least two different types of non-singular universes: (a) bouncing universes (in which the scale factor attains a minimum), and (b) "eternal universes", which are past infinity and ever expanding, and exist forever. Class (a) includes cyclic universes. The focus of this review are those models in class (a), although we shall review a few examples of models in class (b) in Sect.E.8.

### Notation, conventions, etc

Throughout this report, the Einstein's equations (EE) are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu},$$

where $\Lambda$ is the cosmological constant, and $\kappa = 8\pi G/c^4$, which we shall set equal to 1, unless stated otherwise, while the metric of the FLRW model is

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1-\epsilon r^2} + r^2 \left( d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right], \tag{E.1}$$

where $\epsilon = -1, 0, +1$. The 3-dimensional surface of homogeneity $t$ =constant is orthogonal to a fundamental class of observers endowed with a four-velocity vector field $v^\mu = \delta^\mu_0$. In the case of a perfect fluid with energy density $\rho$ and pressure $p$, EE take the form

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0, \tag{E.2}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3}, \tag{E.3}$$

in which $\Lambda$ is the cosmological constant, and the dot denotes the derivative w.r.t. cosmological time. These equations admit a first integral given by the so-called Friedmann equation:

$$\frac{1}{3}\rho = \left(\frac{\dot{a}}{a}\right)^2 + \frac{\epsilon}{a^2} - \frac{\Lambda}{3}. \tag{E.4}$$

The energy-momentum tensor of a theory specified by Lagrangian $\mathcal{L}$ is given by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\,\mathcal{L})}{\delta g^{\mu\nu}}, \tag{E.5}$$

where $g = \det(g_{\mu\nu})$. In the case of a perfect fluid, $T_{\mu\nu}$ takes the form

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu},$$

where $u_\mu$ is the velocity of the fluid.

### E.1.1. Singularities, bounces, and energy conditions

The issue of the initial singularity of the FRLW solution was debated for a long time, since it was not clear if this singular state was an inherent trace of the universe or just a consequence of the high degree of symmetry of the model. This question was discussed firstly in an analytical manner by Lifshitz and collaborators in [50], where geometries that are solutions of EE with a maximum number of allowed functions were analyzed. The results wrongly suggested that the singularity was not unavoidable, but a consequence of the special symmetries of the FLRW solution [15].

From a completely different point of view, Hawking, Penrose, Geroch and others developed theorems that give global conditions under which timelike and null geodesics cannot be extended beyond a certain (singular) point [142]. The goal in this case was not about proving the existence of a region of spacetime in which some functional of the metric is divergent. Instead, the issue of the singularity was considered from a wider perspective, characterizing a spacetime as a whole, by way of its global properties, such as the abrupt termination of some geodesics in the manifold. Let us present a typical example of these theorems [205]:

**Theorem:** The following requirements cannot all be true for a given spacetime $\mathfrak{M}$:

1. There exists a compact spacelike hypersurface (without boundary) $\mathfrak{H}$;
2. The divergence $\theta$ of the unit normals to $\mathfrak{H}$ is positive at every point of $\mathfrak{H}$;
3. $R_{\mu\nu}\, v^\mu\, v^\nu \leq 0$ for every non-spacelike vector $v^\mu$;
4. $\mathfrak{M}$ is geodesically complete in past timelike directions.

Notice that the link of this theorem with physics comes through condition (3) via EE, yielding a statement about the energy-momentum tensor:

$$T_{\mu\nu} v^\mu v^\nu - \frac{T}{2} \geq 0, \tag{E.6}$$

called the strong energy condition (SEC), see the Appendix. Notice also that, although not explicitly mentioned, this theorem assumes the absence of closed timelike curves [142]. With hindsight [16], it can be said that the strength of these theorems is the generality of their assumptions (at the time they were conceived), while their weakness is that they give little information about how the approach to the singularity is described in terms of the dynamics of the theory or about the nature of the singularity. In any case, if we assume that the

[15] For a reappraisal of the work in [50], see for instance [352] and references therein.

[16] From a mathematical point of view, a negative energy could also allow for a bounce. We will not examine this possibility in the present paper.

universe is nonsingular, a positive attitude regarding the singularity theorems is to consider that they show the limits of applicability of "reasonable" hypothesis (such as GR or the energy conditions, see the Appendix) [66].

A local definition of a bounce can also be given, in the GR framework, in terms of the so-called Tolman wormhole [291, 210] (see below). Both in this case and in that of the above mentioned theorems, the non-singular behavior in GR is only possible when the SEC is violated. The assumption of such a condition seemed reasonable in the early seventies, but several situations have been examined in the literature that may be relevant in some epoch of the evolution of the universe, for which SEC is not fulfilled, such as curvature-coupled scalar fields and cosmological inflation [27, 291, 357].

Next we shall examine in some detail how the singularity can be avoided. In the following, we shall use a simple form of the singularity theorems [17]. Let us first introduce some definitions (following [150]). The covariant derivative of the 4-velocity $v_\mu$ of the fluid that generates the geometry can be decomposed as follows

$$v_{\nu;\mu} = \frac{1}{3}\theta\, h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\nu\mu} + v_\mu \dot{v}_\nu, \tag{E.7}$$

where $\theta = v^\mu_{;\mu}$ is the expansion, $h_{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu$, the trace-free symmetric shear tensor is denoted by $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ is the vorticity tensor (see Eqns.(E.363) and (E.364)). Defining $S$ by [18]

$$\frac{\dot{S}}{S} = \frac{\theta}{3}, \tag{E.8}$$

the Raychaudhuri equation [350], which follows from Eqn.(E.7) can be written as [19]

$$3\frac{\ddot{S}}{S} + 2(\sigma^2 - \omega^2) - \dot{v}^\mu_{;\mu} = -\frac{1}{2}(\rho + 3p) + \Lambda, \tag{E.9}$$

where $A_\mu = v^\nu v_{\mu;\nu} \equiv \dot{v}^\mu$ is the acceleration.

**Theorem**[148]: In a universe where $\rho + 3p \geq 0$ is valid, $\Lambda \leq 0$, and $\dot{v}^\mu = \omega^{\mu\nu} = 0$ at all times, at any instant when $H = \frac{1}{3}\theta > 0$, there must have been a time $t_0 < 1/H$ such that $S \to 0$ as $t \to t_0$. A space-time singularity occurs at $t = t_0$, in such a way that $\rho$ and the temperature $T$ diverge.

Several remarks are in order. First, EE were used to obtain Eqn.(E.9). Hence, the consequences of the theorem are only valid in the realm of GR. Second, the singularity implied in the theorem is universal: any past-directed causal curve ends at it with a finite proper length, in line with a coherent definition of a cosmological singularity (if null curves are allowed for causal curves, then affine length has to be used for them instead of proper length which would vanish) [20]. Third, since there is no restriction on the symmetries of the geometry, $\theta$ is in

[17] This will suffice for our goals, more refined formulations can be found in [371].
[18] $S$ corresponds to the scale factor $a$ in the case of the FLRW universe.
[19] This equation was independently obtained by A. Komar [248].
[20] See [371] and [102] for a classification of singularities.

principle a function of all the coordinates, so that the theorem applies not only to Friedmann-Lemaitre-Robertson-Walker (FLRW) models, but also to most of the spatially homogeneous, and to some inhomogeneous models (see examples in [371]). Fourth, as we mentioned before, the condition $\rho + 3p \geq 0$, or more generally, SEC, is violated even at the classical level, for instance by the massive scalar field, and also at the quantum level (as in the Casimir effect [21]). So it would be desirable to have singularity theorems founded on more general energy conditions, but this goal has not been achieved yet (see [371]).

Notice that in the general case, acceleration and/or rotation could in principle avoid the singularity [371], but high pressure cannot prevent the initial singularity in the FLRW model. Rather, it accelerates the collapse. This can be seen as follows. The conservation equations $T^{\mu\nu}_{;\mu} = 0$ give

$$v^\mu \rho_{,\mu} + (\rho + p)\theta = 0,$$

$$(\rho + p)A^\mu = -h^{\mu\nu} p_{,\nu}.$$

Since $p_{,i} = 0$ in the FLRW, there is no acceleration. Furthermore, the pressure contributes to the the active gravitational mass $\rho + 3p$. Finally, not even a large and positive $\Lambda$ can prevent the singularity in the context of the theorem [148].

As mentioned before, a bounce can also be defined locally. The minimal conditions from a local point of view for a bounce to happen in the case of a FLRW universe were analyzed in [291], where a Tolman wormhole was defined as a universe that undergoes a collapse, attains a minimum radius, and subsequently expands. Adopting in what follows the metric Eqn.(E.1), to have a bounce it is necessary that $\dot{a}_b = 0$, and $\ddot{a}_b \geq 0$. For this to be a true minimum of the scale factor (conventionally located at $t = 0$) there must exists a time $\tilde{t}$ such that $\ddot{a} > 0$ for all $t \in (-\tilde{t}, 0) \cup (0, \tilde{t})$. From EE in the FLRW universe (neglecting the cosmological constant term) we get

$$\rho = 3\left(\frac{\dot{a}^2}{a^2} + \frac{\epsilon}{a^2}\right),$$

$$p = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\epsilon}{a^2}\right).$$

From these, the combinations relevant for the energy conditions (see Sect.E.1.3) are:

$$\rho + p = 2\left(-\frac{d^2 \ln a}{dt^2} + \frac{\epsilon}{a^2}\right),$$

$$\rho - p = 2\left(\frac{1}{3a^3}\frac{d^2(a^3)}{dt^2} + 2\frac{\epsilon}{a^2}\right),$$

$$\rho + 3p = -6\,\frac{\ddot{a}}{a}.$$

[21] In fact, it has been shown in [208] that the Casimir effect associated to a massive scalar field coupled to the Ricci scalar in a closed universe can lead to a bounce.

From these conditions and $\dot{a}_b = 0$, and $\ddot{a}_b \geq 0$ it follows that [291]

$$\exists \text{ bounce and } \epsilon = -1 \Rightarrow \text{NEC violated},$$ [22]

$$\exists \text{ bounce and } (\epsilon = 0; \ddot{a}_b > 0) \Rightarrow \text{NEC violated},$$
$$\exists \text{ bounce and } (\epsilon = 1; \ddot{a}_b > a_b^{-1}) \Rightarrow \text{NEC violated},$$

The definition of $\rho$ and $p$ and $\ddot{a} > 0$ imply that:

$$\rho + p < 2\,\frac{\epsilon}{a^2},$$
$$\rho - p > 2\,\frac{\epsilon}{a^2},$$
$$\rho - 3p < 0.$$

It follows that

$$\exists \text{ bounce and } \epsilon \neq 1 \Rightarrow \text{NEC violated},$$
$$\exists \text{ bounce} \Rightarrow \text{SEC violated}.$$

The case that minimizes the violations of the energy conditions can be stated as

$$\exists \text{ bounce and } (\epsilon = +1; \ddot{a}_b \leq a_b^{-1}) \Rightarrow \text{NEC}, \text{WEC}, \text{DEC satisfied}; \text{SEC violated}.$$

This result may be expected since the curvature term with $\epsilon = +1$ acts like a negative energy density in Friedmann's equation. Notice that in this analysis, only Einstein's equations and the point-wise energy conditions were used, without assuming any particular equation of state. In a certain sense, this is the inverse of the theorem stated earlier, which assumed the validity of the SEC [23].

The restriction to a FLRW model was lifted in a subsequent paper [210], and the analysis in a general case was done following standard techniques taken from the ordinary wormhole case [209]. It was found that even in the case of a geometry with no particular symmetries, the SEC must be violated if there is to be a bounce in GR. Consequently, one can conclude that the singularity theorems that assume that SEC is valid cannot be improved. A highlight in these analysis is that only the local geometrical structure of the bounce was needed; no assumptions about asymptotic or topology were required, in contrast with the Hawking-Penrose singularity theorems [372]. Equally important is the fact that, as mentioned above, SEC may not be such a fundamental physical restriction.

To summarize what was discussed up to now, we can say that there is a "window of opportunity" to avoid the initial singularity in FLRW models at a classical level by one or a combination of the following assumptions [24]:

[22] For the energy conditions, see Sect.E.1.3.

[23] An analysis along the same lines but with a more general parametrization for the scale factor was carried out in [340].

[24] We shall not consider here the existence of closed timelike curves as a possible cause of a nonsingular universe.

1. Violating SEC in the realm of GR [25];

2. Working with a new gravitational theory, as for instance those that add scalar degrees of freedom to gravity (Brans-Dicke theory being the paradigmatic example of this type, see Sect.E.3), or by adopting an action built with higher-order invariants (see Sect.E.2).

As will be seen below, other ways to avoid the singularity are:

1. Changing the way gravity couples to matter (from minimal to non-minimal coupling, see for instance the case of the scalar field in Sect.E.3);

2. Using a non-perfect fluid as a source, see Sect.E.5.

Finally, quantum gravitational effects also give the chance of a bounce (see Sect.E.9.2) [26].

### E.1.2. Extrema of $a(t)$ and $\rho(t)$

Let us study the relations imposed by EE between extrema of the scale factor, the energy density, and the energy conditions, in the case of one fluid. Let us recall that the sufficient conditions to have a bounce are [27] $\theta_b = 0$ and $\dot{\theta}_b > 0$, where $\theta = 3\dot{a}/a$ , and the subindex $b$ denotes that the quantities are evaluated at the bounce. It follows from Raychaudhuri's equation for the FLRW model (Eqn.(E.9)) with $\Lambda = 0$,

$$\dot{\theta} + \frac{\theta^2}{3} = -\frac{1}{2}(\rho + 3p), \tag{E.10}$$

that at the bounce we must have $(\rho + 3p)|_b < 0$, independently of the value of $\epsilon$ (as was also shown in the previous section). From the conservation equation,

$$\dot{\rho} = -(\rho + p)\theta,$$

we see that there may be extrema of $\rho$ when $\theta_e = 0$ (as in the case of a putative bounce) and/or when $\rho_e = -p_e$. The second derivative of the energy density is given by

$$\ddot{\rho} = -(\dot{\rho} + \dot{p})\theta - (\rho + p)\dot{\theta}. \tag{E.11}$$

Let us assume first that $\theta_e = 0$ with $\rho_e + p_e \neq 0$, which implies that $\dot{\rho}_e = 0$ and

$$\ddot{\rho}_e = -(\rho_e + p_e)\dot{\theta}_e, \qquad \dot{\theta}_e = -\frac{1}{2}(\rho_e + 3p_e).$$

The different possibilities, according to the sign of $\dot{\theta}_e$, $\rho_e + p_e$, and $\rho + 3p$ are displayed in the following table:

[25] A complete analysis of the behavior of the energy conditions for different types of singularities has been presented in [102].

[26] A definition of a nonsingular space using the so-called principle of quantum hyperbolicity has been given in [66]

[27] We are assuming that $\ddot{a} \neq 0$.

| $\rho_e + 3p_e$ | $\dot{\theta}_e$ | $\rho_e + p_e$ | $\ddot{\rho}_e$ | $\rho_e$ | $a_e$ |
|---|---|---|---|---|---|
| $< 0$ | $> 0$ | $< 0$ | $> 0$ | min. | min. |
| | | $> 0$ | $< 0$ | max. | |
| $> 0$ | $< 0$ | $< 0$ | $< 0$ | max. | max. |
| | | $> 0$ | $> 0$ | min. | |

We see that there are two cases that agree with what may be termed "normal matter" (rows 2 and 4), in the sense that maximum (minimum) compression leads to maximum (minimum) energy density. Notice however that the case in row 2 violates the strong energy condition (see Appendix). The other cases are clearly unusual: minimum density with minimum scale factor (row 1), and the opposite (that is, maximum density with maximum scale factor, row 3)[28]. Notice that it is the null energy condition $\rho + p > 0$ (see Appendix) and not the SEC that is violated at these unusual cases. In fact, if the requirement $\rho + p \geq 0$ is not satisfied, then the equation of energy conservation for a perfect fluid,

$$\dot{\rho} = -\theta(\rho + p), \tag{E.12}$$

says that compression would entail a decreasing energy density, which is a rather unexpected behavior for a fluid [29]. Examples of the four behaviors will be found along this review.

When an EOS $p = \lambda\rho$ plus the condition $\rho > 0$ are imposed[30], we see that the case in row 1 is permitted for $\lambda < -1$, and that in row 2, for $\lambda \in (-1, -1/3)$. The case in row 3 is not allowed for any $\lambda$, while that in row for is permitted for $\lambda > -1/3$.

Notice that all the extrema in $\rho$ in Table E.1.2 are global, since the other possibility (given by $\rho_e + p_e = 0$) leads to an inflection point in $\rho$, assuming that $p = \lambda\rho$.

### E.1.3. Appendix: Energy conditions

We shall give next the general expression of the energy conditions, and also their form for the particular case of the energy-momentum tensor given by

$$T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p). \tag{E.13}$$

- The null energy condition (NEC) states that for any null vector,

$$NEC \Leftrightarrow T_{\mu\nu} k^{\mu} k^{\nu} \geq 0. \tag{E.14}$$

  In terms of Eq.(E.13),

$$NEC \Leftrightarrow \rho + p \geq 0. \tag{E.15}$$

[28] The former is precisely the behavior that allows for a bounce in loop quantum gravity [60] (see Sect.(E.9.2)), while the latter is what is found in the so-called big-rip [284].

[29] Fluids that violate the NEC are called *phantom* or *ghost* fluids, and have been studied in [91].

[30] Notice that some models do not satisfy this conditions, see for instance Eqn.(E.184).

- The weak energy condition (WEC) asserts that

$$WEC \Leftrightarrow T_{\mu\nu} v^\mu v^\nu \geq 0 \tag{E.16}$$

for any timelike vector. In terms of Eqn.(E.13),

$$\rho \geq 0, \text{ and } \rho + p \geq 0. \tag{E.17}$$

- The strong energy condition (SEC) is the assertion that, for any timelike vector,

$$SEC \Leftrightarrow \left( T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) v^\mu v^\nu \geq 0. \tag{E.18}$$

In terms of Eqn.(E.13),

$$\rho + p \geq 0, \text{ and } \rho + 3p \geq 0. \tag{E.19}$$

Each of these three conditions has an averaged counterpart [411]. There is yet another condition:

- The dominant energy condition (DEC) says that for any timelike vector

$$DEC \Leftrightarrow T_{\mu\nu} v^\mu v^\nu \geq 0 \text{ and } T_{\mu\nu} v^\nu \text{ is not spacelike.} \tag{E.20}$$

The different energy conditions are not independent. The following relations are valid:

$$WEC \Rightarrow NEC, \tag{E.21}$$

$$SEC \Rightarrow NEC, \tag{E.22}$$

$$DEC \Rightarrow WEC. \tag{E.23}$$

Notice that if NEC is violated then all the other pointwise energy conditions would be violated [411].

## E.2. Higher-order gravitational theories

Higher-order terms in the action for gravity (such as $R^2, R_{\mu\nu}R^{\mu\nu}$, etc.) typically appear due to quantum effects, either in the case of quantized matter in a fixed gravitational background [123], or in the gravitational effective action as corrections from quantum gravity [132] or string theory [31] [121]. These terms are expected to be important in situations of high curvature, when the scale factor is small [32]. The models that are engineered to work in the intermediate regime, where quantized matter fields evolve on a given classical geometry (the so-called semiclassical approximation) mirror the path taken in the early days of quantum field theory, in which quantum matter was in interaction with a classical electromagnetic background field. In the case of gravity, it is generally agreed that this approach may be valid for distances above $\ell_{Pl}$, although this statement can only be verified by a complete quantum theory of gravitation, not yet available. As we shall see in Ch.E.9, some models go below $\ell_{Pl}$, incorporating effects expected to be present in the complete theory, but for the time being the quest of the "correct theory" at this energy level seems far from being settled.

### E.2.1. Quantized matter on a fixed background

Let us start by considering the corrections coming from quantum matter in a given background. As shown for instance in [403], in the models based on the semiclassical approximation the mean value of the stress-energy tensor $T_{\mu\nu}$ of a set of quantized fields interacting with a classical geometry is plagued with infinities. These divergencies can be removed by a suitable modification of EE that follows from a renormalization procedure. In order to render the mean value of $T_{\mu\nu}$ finite, the cosmological constant $\Lambda$ and Einstein's constant $\kappa$ are renormalized, and a counterterm of the form

$$\triangle L = \sqrt{-g}\,(\alpha\, R^2 + \beta\, R^{\mu\nu}\, R_{\mu\nu}) \tag{E.24}$$

must be introduced in the Lagrangian [33]. The corrections arise from the ultraviolet behavior of the field modes, which only probe the local geometry, hence the appearance of geometric quantities. After the elimination of the divergences and with a convenient choice of $\alpha$ and $\beta$, EE with $< T_{\mu\nu} >$ as a source preserve their form [403]:

$$G_{\mu\nu} + \Lambda^{(ren)}\, g_{\mu\nu} = -\kappa^{(ren)}\ < T^{(ren)}_{\mu\nu} > . \tag{E.25}$$

[31] Since in this case the non-linear terms are always coupled to one or more scalar fields we shall consider it in Sect. E.3.3.

[32] As opposed to Lagrangians that are negative powers of $R$, which are currently being considered as candidates to explain the acceleration of the universe [308].

[33] The relevance of this type of series development was discussed also by Sakharov [364].

Note that such renormalization does not affect the conservation of the energy-momentum tensor, that is

$$< T^{\mu\nu}_{(ren)} >_{;\nu} = 0. \tag{E.26}$$

Notice that there is a residual freedom in the constants introduced by the counterterm [300], so they can be chosen in such a way that they cancel the divergencies without eliminating the quadratic contribution to EE (contrary to what was done in [403]). This more general choice amounts to shifting the constants as $\alpha \to \alpha + \eta$ and $\beta \to \beta + \gamma\,\eta$ [300]. The new equations are

$$G_{\mu\nu} + \eta(\chi_{\mu\nu} + \gamma\, Z_{\mu\nu}) + \Lambda^{(ren)}\, g_{\mu\nu} = -\kappa^{(ren)} \; < T^{(ren)}_{\mu\nu} >, \tag{E.27}$$

where

$$\frac{1}{2}\,\chi_{\mu\nu} \equiv R(R_{\mu\nu} - \frac{1}{4}\,R g_{\mu\nu}) + R_{\;;\mu;\nu} - g_{\mu\nu}\Box R, \tag{E.28}$$

and

$$Z_{\mu\nu} \equiv R_{\;;\mu;\nu} - \Box R_{\mu\nu} - \frac{1}{2}\,(\Box R + R_{\alpha\beta}\, R^{\alpha\beta}) g_{\mu\nu} + 2R^{\alpha\beta}\, R_{\alpha\mu\beta\nu}. \tag{E.29}$$

Cosmological solutions of Eqn.(E.27) in the case of the FLRW metric were studied in [301]. For a flat universe, the equations take the form

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3t_c^2\left\{\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right) - 2\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)^{\cdot}\right\} = \rho, \tag{E.30}$$

$$\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)(\rho + p) = 0, \tag{E.31}$$

where the characteristic time $t_c \equiv 1/sqrtc|\mu^2|$ signals the moment in which the corrections play an important role, and $\mu^{-2} \equiv -2\eta(\gamma + 3)$ (it has dimensions of $L^2$). For the case of radiation ($\rho = \rho_c a_c^4/a^4$), we get

$$H^2 + t_c^2\left\{\left(\frac{\ddot{a}}{a} - H^2\right)^2 - 2H\left(\frac{\dddot{a}}{a} - H^3\right)\right\} = \frac{\rho_c}{3}\left(\frac{a_c}{a}\right)^4, \tag{E.32}$$

where $H = \dot{a}/a$, and $\rho_c = \rho(t_c)$. If we impose the existence of a bounce by the conditions $a_b > 0, \dot{a}_b = 0$, and $\ddot{a}_b > 0$, it follows from this equation that $\mu^{-2} > 0$. It as also shown in [301] $t_c \leq 3.33 \times 10^{-4}$ sec. in order that the theory does not conflict with the three classical tests of GR.

Vacuum solutions of Eqn.(E.27) in the FLRW geometry were studied in [296]. Notice that taking the trace of Eqn.(E.27) in the absence of matter we obtain

$$\ddot{R} + h\dot{R} + \sigma R = 0,$$

where $\sigma = 1/(2\eta(1+\gamma))$, $h = d[\ln(-g)^{1/2}]/dt$. This equation is analogous to that of a damped harmonic oscillator. Depending on the sign of the parameter

$\sigma$, and considering $h > 0$, there may be damped oscillations for $R$ around $R = 0$, or exponentially decaying or growing solutions [296].

Corrections coming from one-loop contributions of conformally-invariant matter fields on a FLRW background were studied in [385] (see also [164]). They allow for nonsingular solutions that are not of the bouncing type since they describe a universe starting from a deSitter state. A thorough analysis of this setting was given in [15], where the back-reaction problem for conformally invariant free quantum fields in FLRW spacetimes with radiation was studied, for both zero [15] and non-zero [16] curvature and/or $\Lambda$. It was found that depending on the values of the regularization parameters, there are some bouncing solutions that approach FLRW at late times.

### E.2.2. Lagrangians depending on the Ricci scalar

On approaching the singularity, powers of the curvature may be expected to play an important dynamical role, hence other possible nonlinear Lagrangians are those belonging to the class defined by

$$S = \int \sqrt{-g}\, f(R)\, d^4x, \tag{E.33}$$

where $f(R)$ is an arbitrary function of the curvature scalar, encompassing polynomials as a particular case [34]. The problem of the singularity using this type of Lagrangians has been repeatedly discussed in the literature (see for instance [88, 31]). The EOM that follows from this action is

$$f^{'}\, R_{\mu\nu} - \frac{1}{2}\, f\, g_{\mu\nu} - \Box f\, g_{\mu\nu} + f^{'}{}_{,\mu;\nu} = 0, \tag{E.34}$$

where $f^{'} \equiv df/dR$. This equation can be expressed in $f$ and its derivatives as

$$f^{'}\, R_{\mu\nu} - \frac{1}{2}\, f\, g_{\mu\nu} + f^{''}\, (R_{,\mu;\nu} - \Box R\, g_{\mu\nu}) + f^{'''}\, (R_{,\mu} R_{,\nu} - R_{,\lambda} R^{,\lambda}\, g_{\mu\nu}) = 0, \tag{E.35}$$

or, using the trace,

$$f^{'}\, \left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}\right) + f^{''}\, \left(R_{,\mu;\nu} - \frac{1}{4}\, g_{\mu\nu} \Box R\right) + f^{'''}\, \left(R_{,\mu} R_{,\nu} - \frac{1}{4} R_{,\lambda} R^{,\lambda}\, g_{\mu\nu}\right) = 0. \tag{E.36}$$

[34] More general cases may include terms proportional to $R_{\mu\nu} R^{\mu\nu}$. In principle a term $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ should also be included in the action, but the existence of a topological invariant yields

$$\delta \int (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2)\, \sqrt{-g}\, d^4x = 0,$$

in such a way that the Riemann-squared term can be omitted.

The particular example given by

$$f(R) = R + \alpha\, R^2 \tag{E.37}$$

was studied by many authors [362, 184, 274]. The equations of motion for the Lagrangian introduced in Eqn.(E.37) in the presence of matter are

$$(1 + 2\alpha R)\, R_{\mu\nu} - \frac{1}{2}(R + \alpha R^2)\, g_{\mu\nu} + 2\alpha(R_{,\mu;\nu} - \Box R\, g_{\mu\nu}) = -T_{\mu\nu}. \tag{E.38}$$

If we restrict to ultra-relativistic matter ($p = \rho/3$) the $0-0$ component of this equation in the case of the FLRW geometry yields

$$\rho = \frac{1}{3}\,\theta^2 + \frac{3\epsilon}{a^2} - 2\alpha\dot{\theta}\left(\dot{\theta} + \frac{2}{3}\,\theta^2\right) + \frac{18\epsilon^2\,\alpha}{a^4} + \frac{4\epsilon\,\alpha}{a^2} + 2\alpha\theta\,\dot{R}, \tag{E.39}$$

where $R = 2\dot{\theta} + 4\,\theta^2/3 + 6\epsilon/a^2$, and $\theta = 3\dot{a}/a$. At the point where the bounce occurs, $\theta_b = 0$ and $\dot{\theta}_b > 0$, and Eqn.(E.39) reduces to

$$\rho_b = -\,2\alpha\dot{\theta}_b^{\,2} + \frac{3\epsilon}{a_b^2}\left(1 + \frac{6\alpha\epsilon}{a_b^2} + \frac{4\alpha}{3}\right). \tag{E.40}$$

Let us take as an example the case in which $\epsilon = 0$. If we want to have a minimum with positive energy density, it follows from Eqn.(E.40) that $\alpha < 0$. As shown in [362], such a choice for the action of the gravitational field admits solutions in the FLRW framework that allow a regular transition from a contracting to an expanding phase. Although negative values of $\alpha$ remove the initial singularity, it was shown in [362, 184] that the solutions with $\alpha < 0$ do not go to the corresponding FLRW solution ($a \propto t^{1/2}$) for large $t$.

A theory that generalizes that defined by Eqn.(E.37), namely

$$f(R) = R + \alpha R^n$$

was studied in [361][35]. It was found that the FLRW solution for $n = 4/3$ and $p = \rho/3$ is regular for all values of $t$, and tends to the radiation solution for large values of $t$. Later, solutions of this theory with dust as a source were found to have similar properties in [195].

Another type of corrections, given by the Lagrangian

$$\mathcal{L} = R + \Lambda + BR^2 + CR^2 \ln|R|, \tag{E.41}$$

were studied in [196] (with $B$ and $C$ constants). The quadratic and logarithmic terms are consequences of vacuum polarization [131]. Although this form of the Lagrangian does not eliminate the singularity in the FLRW solutions, addition of particle creation effects through a viscosity term does (see Ch.E.5) [36].

[35] $f(R)$ theories with negative/positive powers of R were first proposed in [306].

[36] A Bianchi I solution of this theory with and without self-consistent particle production was considered in [197]. It was shown that particle production quickly isotropizes the model.

The stability analysis of the FLRW solution in theories with $\mathcal{L} = f(R)$ was performed in [31], along with necessary and sufficient conditions for the existence of singularities. Eqn.(E.35) in the case of a FLRW geometry in the presence of matter reduces to [238]

$$f^{''}\dot{a}(a^2\,\dddot{a} + a\dot{a}\ddot{a} - 2\dot{a}^3 - 2\dot{a}\epsilon) + \frac{1}{6}f^{'}a^3\ddot{a} + \frac{1}{36}fa^4 + \frac{1}{18}a^4T_{00} = 0. \tag{E.42}$$

The argument of the function $f$ is given by

$$R = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + \epsilon). \tag{E.43}$$

Assuming that near the bounce the scale factor can be developed in a power series as

$$a(t) = a_0 + \frac{1}{2}\,a_1t^2 + \frac{1}{6}\,a_2t^3 + ..., \tag{E.44}$$

a necessary condition for the bounce was given [31]:

$$f_0a_0 + 6a_1f_0' \leq 0, \tag{E.45}$$

where $f_0 = f(R_0)$, and $R_0 = -6a_0^{-2}(a_0a_1 + \epsilon)$, and it was assumed that $T_{00} > 0$. In the quadratic case given by Eqn.(E.37), this condition takes the form

$$6\alpha\epsilon^2 - a_0^2\epsilon - 6\alpha a_1^2a_0^2 < 0. \tag{E.46}$$

When $\epsilon = 0$, the condition $\alpha > 0$ is regained, but there are other possibilities when $\epsilon = 1, -1$ [31]. In the same vein, but without using a series development, conditions for a bounce in $f(R)$ theories were studied in [95] [37]. The basic equations are, that follow from Raychaudhuri's equation and the Gauss-Codazzi equation are

$$\frac{\ddot{a}_b}{a_b} = -\frac{\rho_b}{f_b'} + \frac{f_b}{f_b'},$$

$$R = 6\left(\frac{\ddot{a}_b}{a_b} + \frac{\epsilon}{a_b^2}\right).$$

These equations were used in [95] to analyze a possible bounce in the theories given by $f_1(R) = R^n$, $f_2(R) = R + \alpha R^m$, $f_3(R) = \exp(\lambda R)$. Bounces for $\epsilon = \pm 1$ are possible in the case of $f_1$. This case can describe an "almost-FRLW" phase folowed by an accelerated phase if $n > 1$ and $n$ is odd for $\epsilon = -1$ and $R > 0$. The same happens with $n$ even and $n < 0$ with $R > 0$ or $0 < n < 1$ with $R < 0$, where in the second case $n$ can be only rational. For $f_2$, closed bounces are allowed for every integer value of $m$ (often together with open bounces). For $m$ rational, closed bounces are not allowed in general for $0 < m < 1$. For $m$

[37] Bounce solutions were also shown to exist in orthogonal spatially homogeneous Bianchi cosmologies in $f(R) = R^n$ in [188].

rational with even denominator there is no closed bounce for $(m > 1, \alpha < 0)$ and no bounce at all for negative $m$ and $\alpha$. In the case of $f_3$, one of the following two conditions must be satisfied in order to have a bounce: $\lambda > 0$ and $R_b > \ln(2\rho_b)/\lambda$, or $\lambda < 0$ and $R_b < \ln(2\rho_b)/\lambda$.

Some exact solutions have been recently found in [107] for the theory defined by $f(R) = R^{1+\delta}$. For the vacuum case with $\epsilon = 0$, there is bouncing (entirely due to the dynamics of the theory), for $0 < \delta < 1/4$. There are vacuum solutions for $\delta = 1/2$ and $\epsilon \neq 0$, are given by

$$ds^2 = dt^2 - (\kappa - \kappa t^2 \pm t^4)\left(\frac{dr^2}{1-\epsilon r^2} + r^2 d\Omega^2\right).$$

This solution exhibits a bounce for $\kappa > 0$. Bouncing solutions were also obtained for a perfect fluid with $p = (\gamma - 1)\rho$ in the case $\delta = 1/(3\gamma - 1)$ [38].

We would like to close this section by pointing out that Eqn.E.42 illustrates the fact that the flatness problem is not a priori a problem in theories other than GR (no definite behavior of $|\Omega - 1|$ with time follows from E.42).

### Saturation

An interesting idea was proposed in [238] to limit the curvature by adding terms in the Lagrangian, following the lines that Born and Infeld [71] devised to avoid singularities in electromagnetism. The Born-Infeld Lagrangian, given by

$$\mathcal{L}_{BI} = \beta^2 \left[\sqrt{1 - \frac{\mathscr{H}^2 - \mathscr{E}^2}{\beta^4}} - 1\right] \tag{E.47}$$

is such that the invariant $\mathscr{H}^2 - \mathscr{E}^2$ cannot take values higher than $\beta^4$. The fact that it takes more and more energy to increment the field when it takes values near $\beta^2$ is a phenomenon called saturation [39]. A similar cutoff may be postulated for the curvature tensor when quantum gravitational fluctuations become non-negligible, that is (presumably), when

$$R \approx \ell_{Pl}^{-2} \approx 10^{66}\mathrm{cm}^{-2}.$$

In [238], non-polynomial Lagrangians $f(R)$ were considered such that they reduce to $R$ when $R << \ell_{Pl}^{-2}$, and required that $f(R) \to$ constant for $R \to \infty$. This condition is of course not enough to determine the Lagrangian, but a qualitative guess can be made. A typical Lagrangian that fulfills the above given conditions is

$$f(R) = \frac{R}{1 - \ell_{Pl}^2 R}. \tag{E.48}$$

[38] Cyclic solutions were obtained in the case $\delta = (3\gamma - 4)/(2(7 - 3\gamma))$ for a convenient choice of the integration constants.

[39] This is analogous to the fact that it takes an infinite amount of energy to accelerate a mass moving with $v \approx c$ in special relativity.

An approximate solution of the EOM (E.42) for (E.48) by a development as a power series of $t$ for $\epsilon = 0$ was built in [141], the solution being non-singular though strongly dependent on the non-linearities of the chosen Lagrangian.

The idea of saturation was subsequently explored in [140], where an explicit nonsingular solution given by

$$a(t) = \sigma \left(1 + \frac{\beta^4 t^2}{\sigma^4}\right)^{1/4}, \tag{E.49}$$

was inserted in Eqn.(E.42), where $\sigma$ is a small parameter. This expression tends to the radiation-dominated scale factor for $\beta^4 t^2/\sigma^4 >> 1$. With this $a(t)$ and using that $R = -3\beta^4\sigma^4/a^8$, Eqn.(E.42) can be rewritten as an ordinary linear second-order differential equation for $f(R)$. This equation was integrated for all the values of the 3-curvature. The dependence of the resulting $f(R)$ on the chosen form of $a(t)$ was tested in the case $\epsilon = 0$ with that obtained from $a^8(t) = 1 + 2(1+\alpha)t^2 + t^4$, which has the same asymptotic limit of Eqn.(E.49). The result in this second case is not distinguishable from the first.

A related analysis was carried out in [57], where it was asked that the theory defined by $f(R)$ be asymptotically free (implying that gravity becomes weak at short distances, in such a way that pressure may counteract the gravitational attraction, thus avoiding the singularity), and also ghost-free (so that the bounce is not caused by negative-energy-density matter) [40]. The actions studied in [57] that satisfy these requirements were specified by [41]

$$f(R) = R + \sum_{n=0}^{\infty} c_n R \Box^n R, \tag{E.50}$$

and can be rewritten in terms of a higher-derivative scalar-tensor action:

$$S = \int d^4x\sqrt{-g}\left(\Phi R + \psi \sum_{1}^{\infty} c_i \Box^i \psi - (\psi(\Phi - 1) - c_0\psi^2\right),$$

from which it follows that $\psi = R$ (from the EOM of $\Phi$). After a conformal transformation and linearization it follows that the EOM for the scalar fields are [57]

$$\psi = 3\Box\phi, \qquad \phi = 2\left(\sum_{1}^{\infty} c_i \Box^i \psi + c_0\psi\right)$$

with $\Phi = e^{\phi}$. From these we get

$$\left(1 - 6\sum_{0}^{\infty} c_i \Box^{i+1}\right)\phi \equiv \Gamma(\Box)\phi = 0,$$

[40] For the relation between $f(R)$ theories and ghosts, see [104].

[41] It was shown in [57] that polynomial actions in $R$ do not satisfy these requirements.

and the scalar propagator is

$$G(p^2) \propto \frac{1}{\Gamma(-p^2)}.$$

It is precisely the function $\Gamma$ that controls the absence of ghosts and the asymptotic properties of the theory, which was parameterized in [57] as $\Gamma(-p^2) = e^{\gamma(-p^2)}$, with $\gamma$ analytic. To actually show the existence of bouncing solutions with the properties mentioned above, the scale factor

$$a(t) = a_0 \cosh\left(\sqrt{\frac{\omega}{2}}\, t\right),$$

was imposed in the equation for $G_{00}$ written in terms of $\Gamma$ and its derivatives, and compared with the r.h.s. composed of radiation and cosmological constant, thus yielding the following constraints on $\Gamma$:

$$\Gamma'(\omega) = \frac{2}{3}\Gamma'(0) - \frac{1}{3\omega},$$

$$2\omega\Gamma'(\omega) - 1 \geq 0$$

(the latter coming from demanding that the bounce be caused by the nonlinearities, and not by the radiation energy density). The authors go on to show that the kinetic operator defined by

$$\gamma(\omega) = k_1\omega - k_2\omega^2 + k_4\omega^4,$$

where $k_i$ are constants, satisfies the constraints and has the correct Newtonian limit. So a bouncing solution that is ghost and asymptotically free exists for the theory defined by Eqn.(E.50)[42], although the Lagrangian in the original variable $R$ was not exhibited.

### E.2.3. The limiting curvature hypothesis (LCH)

A different proposal to deal with the singularity problem in the higher-order-curvature scenario is to adopt the limiting curvature hypothesis, introduced by M. Markov [278] as the limiting density hypothesis [43]. The LHC postulates the existence of a maximum value for the curvature, in such a way that

$$R^2 < \ell_{Pl}^{-4}, \qquad R_{\mu\nu}R^{\mu\nu} < \ell_{Pl}^{-8}, \qquad W_{\alpha\beta\gamma\delta}W^{\alpha\beta\gamma\delta} < \ell_{Pl}^{-8},$$

etc, and that any geometry must approach a definite nonsingular solution (typically the de Sitter solution) when the limiting curvature is reached. This automatically guarantees that all curvature invariants are finite [279]. A nonsingular

[42] See also [58].

[43] For boucing solutions that implement this hypothesis through modifications of the EOS, see [359].

higher order theory was constructed in [292] in which every contracting and spatially flat, isotropic universe avoids the big crunch by ending up in a deSitter state enforced by the LCH, for all initial conditions and general matter content [44]. The action used in [292] was the linear action plus a non-linear term $I_2$ with the property that

$$I_2(g_{\mu\nu}) = 0 \Leftrightarrow g_{\mu\nu} = g^{DS}_{\mu\nu}, \tag{E.51}$$

and enforced that $I_2 \to 0$ for large curvatures using an auxiliary field (see below). In a subsequent paper [75], the method was applied to an isotropic, homogeneous universe, both in vacuum and in the presence of matter. The solutions corresponding to $\epsilon = 1$ display a deSitter bounce. In the case in which matter is present, it is shown that its coupling to gravity is asymptotically free. Later, the model was generalized to include a dilaton field [77], in which case it admits flat bouncing solutions. The starting point is the dilaton gravity action with an added non-linear term ($I_2$) times a Lagrange multiplier $\psi$ subject to a potential $V(\psi)$:

$$S = -\frac{1}{2\kappa^2}\left(R - \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{\sqrt{12}}\,\psi\, e^{\gamma\phi} I_2 + V(\psi)\right). \tag{E.52}$$

The potential is to be tailored from the EOM and the constraint equations in such a way that $I_2$, given by

$$I_2 = \sqrt{4R_{\mu\nu}R^{\mu\nu}},$$

goes to zero for large curvatures. Notice that this form of $I_2$ satisfies condition (E.51), so all the curvature invariants are automatically bounded. Restricting to an FLRW metric with $k = 0$, the EOM are

$$\dot{\psi} = -3H\psi + 6H - \frac{1}{H}\left(\frac{1}{2}\chi^2 + V(\psi)\right), \tag{E.53}$$

$$\dot{H} = -V'(\psi), \tag{E.54}$$

$$\dot{\chi} = -3H\chi, \tag{E.55}$$

with $\chi = \dot{\phi}$, and a prime denotes derivative wrt $\psi$. An example was given in [77], where

$$V(\psi) = \frac{\psi^2 - \frac{1}{16}\psi^4}{1 + \frac{1}{32}\psi^4}. \tag{E.56}$$

was chosen. This potential yields the dilaton gravity action at low curvatures, enforces that $I_2$ go to zero at large curvatures, and enables a bounce. By means of a phase space analysis of Eqns.(E.53)-(E.55), it was shown [77] that all the solutions are non-singular, and that some of them display a bounce either with

[44] Note that the LFH furnishes in this case a nonsingular universe without bounce.

or without the dilaton. In particular, the flat bouncing solutions with a non-zero dilaton interpolate between a contracting dilaton-dominated phase and an expanding FLRW epoch, thus avoiding the graceful exit problem of pre-big-bang cosmology (see below).

One obvious drawback of the LCH is that the non-linear terms are not dictated by first principles: they are chosen in such a way as to render the theory finite.

### E.2.4. Appendix: $f(R)$ and scalar-tensor theories

Higher-order Lagrangians can be related to scalar-tensor gravity (see for instance [393]). Let us start with the function $f(R)$ is given by

$$f(R) = R + \alpha R^2. \tag{E.57}$$

The EOM that follow from this Lagrangian is

$$2\alpha R_{;\mu\nu} - (1 - 2\alpha R)R_{\mu\nu} + g_{\mu\nu}\left(\frac{1}{2}\alpha R^2 + \frac{1}{2}R - 2\alpha\Box R\right) = 0, \tag{E.58}$$

the trace of this equation being

$$\Box R - \frac{R}{6\alpha} = 0. \tag{E.59}$$

It was shown in [393] that this theory is equivalent to the one given by the action

$$S = \int \sqrt{-g}\, d^4x\, \left[(1 + 2\alpha\varphi)R - \alpha\varphi^2\right]. \tag{E.60}$$

Varying independently $g_{\mu\nu}$ and $\varphi$ in the action given in Eqn.(E.60), one obtains

$$(1 + 2\alpha)(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) + \frac{\alpha}{2}\,\varphi^2 g_{\mu\nu} - 2\alpha(\varphi_{,\mu;\nu} - \Box\varphi\, g_{\mu\nu}) = 0, \tag{E.61}$$

and

$$2\alpha(R - \varphi) = 0. \tag{E.62}$$

In turn, as shown in [418] the conformal transformation

$$\widetilde{g}_{\mu\nu} = (1 + 2\alpha\phi)\, g_{\mu\nu}, \tag{E.63}$$

takes this theory to Einstein gravity with a massive scalar field.

Except in the case in which $\alpha$ vanishes (which is precisely the case in general relativity) the second equation yields that the scalar field is nothing but the scalar of curvature. Inserting this result into Eqn.(E.61) one arrives precisely at Eqn.(E.58). The equivalence can be generalized to functions $f(R)$ (see [414]) [45]. Based on the equivalence, the singularity problem in fourth order theories was analyzed in [250] for homogeneous cosmological models with a diagonal metric.

[45] It was later proved that all higher order, scalar-tensor and string actions are conformally equivalent to general relativity with additional scalar fields which have particular (different in each case) self-interaction potentials [32].

## E.3. Theories with a scalar field

### E.3.1. Scalar field in the presence of a potential

Violations to some of the energy conditions are produced even at the classical level by some scalar field theories. From the singularity theorems discussed in Ch. 1, we can expect the existence of bouncing solutions in this scenario [46]. We shall see next examples of avoidance of the singularity in scalar field models that violate some of the energy conditions, as well as theories with nonminimal coupling.

A universe filled with radiation and pressureless matter coupled to a classical conformal massless scalar field was studied in [47]. The coupling was provided by the action

$$S = -\frac{1}{2}\int(\psi_{,\alpha}\psi^{,\alpha} + \frac{1}{6}R\psi^2)\sqrt{-g} - \int(\mu + f\psi)d\tau, \tag{E.64}$$

where $\mu$ is the mass of the particle, $f$ is a coupling constant, and

$$-f\int\psi d\tau = -f\int d^4x\left[\sqrt{-g}\psi\int(-g)^{-1/2}\delta^4(x^\mu - x^\mu(\tau))d\tau\right],$$

(this interaction was suggested in [47] as a classical analog of the pion-nucleon coupling). Assuming that we have a FLRW universe filled with a uniform distribution of identical $\mu$ particles, in the continuum approximation, the field equation for $\psi$ takes the form

$$F_{,\eta,\eta} + \epsilon F = -fN, \tag{E.65}$$

where $F = a\psi$, $\eta$ is the conformal time, and $N = na^3$=constant is the number of particles. The calculation of the trace of the total stress-energy tensor from Eq.(E.64) yields

$$T_\alpha^{\ \alpha} = -\mu n,$$

so we get for the trace of EE

$$a'' + \epsilon a = \frac{4\pi}{3}N\mu, \tag{E.66}$$

where the prime means derivative wrt conformal time. Finally the Friedmann equation is given by

$$a'^2 + \epsilon a^2 = \frac{4\pi}{3}(F'^2 + \epsilon F^2 + 2Na\mu + 2NfF + 2B), \tag{E.67}$$

where $B$ is a constant that gives the amount of radiation. The system composed of Eqns.(E.65-E.67) was solved in [47] for all values of $\epsilon$, and it was shown

[46] The role of scalar fields in Cosmology has been examined for instance in [239].

that a bounce is possible for the three cases when some relations between the integration constants are fulfilled. However, physical requirements show that only the $\epsilon = +1$ solution can bounce provided $N^2 f^2 > 2B$. A nice feature of this solution is that it satisfies the weak energy condition.

Another non-singular universe based on a scalar field was presented in [45]. A closed FLRW model was considered, with a conformally coupled scalar field $\phi$ as matter content, which can be thought as a perfect fluid with comoving velocity defined by

$$v^\mu = \frac{\phi^{,\mu}}{(\phi_{,\alpha}\phi^{,\alpha})^{1/2}}.$$

In this case, the energy density and the pressure are given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi^2\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\right] + \frac{\dot{a}}{a}\phi\dot{\phi} + V,$$

$$p = \frac{1}{6}\dot{\phi}^2 + \frac{1}{3}\phi\frac{dV}{d\phi} + \frac{1}{6}\phi^2\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\right] + \frac{1}{3}\frac{\dot{a}}{a}\phi\dot{\phi} - V.$$

EE were written as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{\rho}{6},$$

$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\gamma - \frac{2}{3}\right)\rho = 0,$$

with $p = (\gamma - 1)\rho$. From these equations we get

$$\frac{\ddot{a}}{a} + \left(\frac{3}{2}\gamma - 1\right)\left(\frac{\dot{a}^2 + 1}{a^2}\right) = 0.$$

Introducing the conformal time through $dt = a(\eta)d\eta$, and with the changes of variables $u = a'/a$, and $u = w'/(cw)$, with $c = 3\gamma/2 - 1$, the solution for $a(\eta)$ is [45]

$$a(\eta) = a_0[\cos(c\eta + d)]^{1/c},$$

where $a_0$, and $d$ are integration constants, which were fixed resorting to the limiting curvature hypothesis (see Sect. E.2.3) along with the imposition of a "prematter phase (starting from the limiting values), followed by a radiation-dominated era and a matter-domination period afterwards. The constant $c$ is essentially the parameter of the equation of state of the prematter era, and the only constant which is not completely determined in the model. potential $V$ was then reconstructed in terms of the scale factor (assuming that the EOS changes inthe different eras of the universe) and $\phi$ from $\gamma = 1 + p/\rho$, and the evolution of $\phi$ was obtained by numerical integration.

More general models, given by solutions of the theory

$$S = \int d^4x\sqrt{-g}\{F(\phi)R - \partial_\alpha\phi\partial^\alpha\phi - 2V(\phi)\},$$

in which $\phi$ is nonminimally coupled to gravity through $F$, were studied in [193], where it was shown that there are bouncing solutions, which were later proved to be unstable under linear anisotropic perturbations [3]. A phase-space analysis of the models given by $F(\phi) = \xi\phi^2$ showed the existence of bouncing solutions, under certain restrictions on the constants of the potential $V(\phi) = \alpha\phi^2 + \beta\phi^4 + \Lambda$ [194].

Nonsingular solutions for a scalar field in the presence of a potential were also studied in [11], for theories defined by

$$\mathcal{L} = \frac{1}{2}\omega\dot{\phi}^2 - U(\omega),$$

where $\omega$ is determined by $dU/d\omega = \frac{1}{2}\dot{\phi}^2$. The existence of a bounce was shown for a tailored potential given by

$$U(\omega) = \lambda\left(\omega^{-1} + \frac{1-\alpha}{\alpha}\omega^{\alpha/(1-\alpha)} - \frac{1}{\alpha}\right),$$

where $\lambda$ is a constant with dimensions of energy density, and $\alpha$ is a number parameterising the classes of theories [47]. The bounce exists for $\alpha < 1/3$, and $\epsilon = +1$. Later, this approach was generalized to Bianchi I cosmologies in [167].

So far we have examined a classical scalar field on a given background. A quantum scalar field $\phi(x)$ in a classical geometry was studied in [286, 316] where, inspired by the features of the mechanism of spontaneous symmetry breaking, the authors sought a solution in which the expectation value of $\phi$ in the fundamental state is given by

$$\langle 0|\phi|0\rangle = \sqrt{\frac{3}{\lambda}}\,\frac{f(\eta)}{a(\eta)}, \tag{E.68}$$

where $\eta$ is the conformal time of an open Friedmann geometry given by

$$ds^2 = a^2(\eta)\left[d\eta^2 - d\chi^2 - \sinh^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{E.69}$$

and $f$ is a function to be determined (see below). For a massless field the equation of motion for the scale factor reduces to

$$\frac{a''}{a} = 1. \tag{E.70}$$

From the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\sigma\phi^4$$

we obtain the equation of the scalar field $\phi$, given by

$$\phi'' + 2\phi'\frac{a'}{a} + 2\sigma a^2\phi^3 = 0. \tag{E.71}$$

[47] This potential interpolates between $p = \rho$ for $\rho << \lambda$, and $p < 0$ for high densities.

Compatibility of these two equations with the assumption in Eqn.(E.68) yields the relation

$$\sigma = \frac{\lambda}{6} \ . \tag{E.72}$$

For the scale factor as function of the Gaussian time $t$ we obtain

$$a(t) = \sqrt{t^2 - L^2}, \tag{E.73}$$

where $L$ is a constant and

$$f'' - f + f^3 = 0. \tag{E.74}$$

By rewriting this equation as a planar autonomous system, it was shown in [286] that the solution $f = 0$ is unstable, while the solutions $f^2 = 1$ are stable under linear homogeneous perturbations. From the equation for $g_{\mu\nu}$ and specializing for $\mu = \nu = 0$ we obtain the value of the constant $L$ in Eqn.(E.73):

$$L^2 = \frac{\kappa}{24\sigma} \tag{E.75}$$

which represents the minimum allowable value of the scale factor. From standard quantum field theory in curved spacetime,

$$G_{\mu\nu} = -\kappa_{(ren)} T_{\mu\nu},$$

it follows that $E_{|0\rangle} = -\frac{3L^2}{a^4} < 0$, which shows explicitly the expected violation of the weak energy condition that causes the absence of a singularity in this model. Note that the gravitational constant in the vacuum state is renormalized:

$$\frac{1}{\kappa_{(ren)}} = \frac{1}{\kappa} - \frac{\phi^2}{6} = \frac{12\sigma t^2 - \kappa/2}{12\sigma\kappa a^2}.$$

It follows that $\kappa_{ren} < 0$ for $t^2 < \frac{\kappa}{24\sigma}$ and $\kappa_{ren} > 0$ for $t^2 > \frac{\kappa}{24\sigma}$, thus showing that a change in the sign of the gravitational constant can be induced by the non-minimal coupling of scalar field with gravity, yielding repulsive gravity.

The phenomenon of repulsive gravity can also be generated at a classical level by means of a non-minimally coupled complex scalar field [368]. The Lagrangian is given by

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - \sigma(\phi^* \phi)^2 - \frac{1}{6} R(\phi^* \phi) + \kappa^{-1} R + \mathcal{L}_m,$$

where $\sigma$ is the constant that measures the auto-interaction of $\phi$, and $\mathcal{L}$ is the matter Lagrangian. The EOM following from this Lagrangian are

$$\Box\phi + 2\sigma\phi^*\phi^2 + \frac{1}{6} R\phi = 0,$$

$$G_{\mu\nu} = -\tilde{\kappa}(\theta_{\mu\nu} + T_{\mu\nu}),$$

where

$$\tilde{\kappa} = \kappa \left(1 - \frac{\kappa}{6}\phi^*\phi\right), \tag{E.76}$$

$$\theta_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi - g_{\mu\nu}(\partial_\rho\phi^*\partial^\rho\phi - \sigma(\phi^*\phi)^2) + \frac{1}{3}g_{\mu\nu}\Box(\phi^*\phi) - \frac{1}{3}(\phi^*\phi)_{;\mu\nu}\right),$$

and $T_{\mu\nu}$ is the energy-momentum tensor associated to matter. From Eqn.(E.76) we see that the gravitational constant is renormalized at the classical level by the scalar field. In fact, as shown in [368], for the open FLRW metric [48] the scalar field has three vacuum solutions: $\phi = 0$, and $\phi = \pm\gamma/a(t)$, where $\gamma$ is a constant. Only the nonzero solutions are stable, and they are also more favorable from the point of view of energy [368]. Since they are inversely proportional to $a$, it may be argued that the scalar field was in a nonzero vacuum in the early universe. Hence,

$$\tilde{\kappa} = \kappa \left[1 - \frac{a_c^2}{a^2}\right]^{-1},$$

where $a_c = (\kappa/12\sigma)^{1/2}$ signals the change of sign of the gravitational interaction. Nonsingular solutions were obtained in [368] for matter given by radiation ($\rho = \epsilon/a^4$):

$$a(t) = \frac{\varpi}{\sqrt{2}}\cosh t,$$

where $\varpi^2 = a_c^2 - \frac{2}{3}\kappa\epsilon$. This case reduces to the case without matter for $\epsilon = 0$.

## E.3.2. Dynamical origin of the geometry

We shall see in this section that a cosmological scenario displaying a bounce arises in an extension of Riemannian geometry called Weyl Integrable Space-Time (WIST) [318].

Let us begin by recalling that one of the central hypothesis of General Relativity is that gravitational processes occur in a Riemannian space-time structure. This means that there exists a metric tensor $g_{\mu\nu}$ and a symmetric connection $\Gamma^\alpha{}_{\mu\nu}$ related by

$$g_{\mu\nu;\alpha} \equiv g_{\mu\nu,\alpha} - \Gamma^\epsilon_{\alpha\mu}\, g_{\epsilon\mu} - \Gamma^\epsilon{}_{\alpha\nu}\, g_{\mu\epsilon} = 0. \tag{E.77}$$

In other words, the connection is metric and can be written in terms of the metric tensor as follows

$$\Gamma^\alpha{}_{\mu\nu} = \left\{{}^{\alpha}_{\mu\nu}\right\} \equiv \frac{1}{2}g^{\alpha\beta}[g_{\beta\mu,\,\nu} + g_{\beta\nu,\,\mu} - g_{\mu\nu,\,\beta}]. \tag{E.78}$$

A direct method to deduce such metricity condition is given by the first order Palatini variation (in which the variation of the metric tensor and of the connection are independent). The starting point is the Hilbert action:

$$\mathcal{S}[g,\Gamma] = \int \sqrt{-g}R[g,\Gamma]d^4x. \tag{E.79}$$

[48] This scenario does not work for the closed case.

In a local Euclidean coordinate system,

$$\delta R_{\mu\nu} = \delta\Gamma^{\alpha}{}_{\mu\alpha;\nu} - \delta\Gamma^{\alpha}{}_{\mu\nu;\alpha}, \tag{E.80}$$

where the covariant derivative represented by a semicolon must be taken in the non-perturbed background geometry. From this equation it follows that

$$\delta\mathcal{L} = (R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\sqrt{-g}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}. \tag{E.81}$$

Correspondingly

$$\begin{aligned} \delta\mathcal{S} &= \int \sqrt{-g}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu} \\ &+ \int \left\{ (\sqrt{-g}g^{\mu\epsilon})_{;\alpha} - \frac{1}{2}(\sqrt{-g}g^{\mu\nu})_{;\nu}\delta^{\epsilon}_{\alpha} - \frac{1}{2}(\sqrt{-g}g^{\nu\epsilon})_{;\nu}\delta^{\mu}_{\alpha} \right\} \delta\Gamma^{\alpha}{}_{\mu\epsilon} \end{aligned} \tag{E.82}$$

Hence,

$$(\sqrt{-g}g^{\mu\epsilon})_{;\alpha} - \frac{1}{2}(\sqrt{-g}g^{\mu\nu})_{;\nu}\delta^{\epsilon}_{\alpha} - \frac{1}{2}(\sqrt{-g}g^{\nu\epsilon})_{;\nu}\delta^{\mu}_{\alpha} = 0, \tag{E.83}$$

and we obtain

$$(\sqrt{-g}g^{\mu\epsilon})_{;\alpha} = 0. \tag{E.84}$$

After some algebra it can be shown that space-time has a Riemannian structure, that is, it obeys the metricity condition,

$$g_{\mu\epsilon;\alpha} = 0. \tag{E.85}$$

The other equation that follows from the variational principle yields Einstein's equations. The lesson we learn from this calculation is that the structure of the manifold associated to space-time is not given *a* priori, but may depend on the dynamics. Surely, we should check whether the addition of matter alters this feature. The answer is not unique: it depends crucially on the way matter couples to gravity. There will be no modification to the precedent structure if we adopt the minimal coupling (that is, if the strong equivalence principle is valid). However, when the interaction is non-minimal, the geometrical structure obtained by the Palatini variation is not Riemannian in general. The simplest way to show this is with an example. Let us take the Lagrangian which describes the non-minimal interaction of a scalar field with gravity in the form:

$$L_{int} = \sqrt{-g}\, R\, f(\varphi). \tag{E.86}$$

Following the procedure sketched above we get:

$$\begin{aligned} \delta S_{int} &= \int \sqrt{-g}\, f\, (R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu} \\ &+ \int \left\{ (\sqrt{-g}\, f\, g^{\mu\epsilon})_{;\alpha} - \frac{1}{2}(\sqrt{-g}\, f\, g^{\mu\nu})_{;\nu}\delta^{\epsilon}_{\alpha} - \frac{1}{2}(\sqrt{-g}\, f\, g^{\nu\epsilon})_{;\nu}\delta^{\mu}_{\alpha} \right\} \delta\Gamma^{\alpha}{}_{\mu\epsilon} \end{aligned} \tag{E.87}$$

and it follows that

$$\left\{\sqrt{-g}f(\varphi)\,g^{\mu\nu}\right\}_{;\epsilon} = 0. \tag{E.88}$$

This equation shows that the covariant derivative of the metric tensor is not zero but

$$g_{\mu\nu;\alpha} = Q_{\mu\nu\alpha}, \tag{E.89}$$

where $Q_{\mu\nu\lambda} = -(\ln f)_{,\lambda}\, g_{\mu\nu}$. Taking the cyclic permutation of Eqn.(E.3.2) yields

$$\Gamma^{\lambda}{}_{\mu\alpha} = \left\{{}^{\lambda}_{\mu\alpha}\right\} - \frac{1}{2}[Q_{\mu}{}^{\lambda}{}_{\alpha} + Q^{\lambda}{}_{\alpha\mu} - Q_{\alpha\mu}{}^{\lambda}]. \tag{E.90}$$

The equation

$$g_{\mu\nu;\alpha} = -(\ln f)_{,\lambda}\, g_{\mu\nu}. \tag{E.91}$$

shows that the structure generated by the Lagrangian (E.86) using the Palatini variation is not Riemannian but, as we shall see in the next section, a special case of Weyl geometry.

### WIST (Weyl Integrable Space Time)

A Weyl geometry is defined by the relation [416]

$$g_{\mu\nu;\alpha} = \varphi_\alpha g_{\mu\nu}. \tag{E.92}$$

This equation implies that there is a variation of the length $\ell_0$ of any vector under parallel transport, given by

$$\Delta\ell = \ell_0 \varphi_\mu \Delta x^\mu. \tag{E.93}$$

This property has the undesirable consequence that the measure of length depends on the previous history of the measurement apparatus, as pointed out by Einstein in the beginning of the past century in a criticism against Weyl's proposal for the geometrization of the electromagnetic field [335]. Einsteins remark led to the abandonment of this type of geometry. However, there is just one particular case in which this problem disappears: the so-called Weyl integrable spacetime (WIST). By definition, a WIST is a particular Weyl spacetime in which the vector $W_\mu$ is irrotational:

$$\varphi_\mu \equiv \partial_\mu \varphi.$$

It follows that in a closed trajectory

$$\oint \Delta\ell = 0, \tag{E.94}$$

which solves the critic raised by Einstein. From the definition given in Eqn.(E.92) it follows that the associated connection is given by

$$C^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - \frac{1}{2}\left(\varphi_\mu \delta^\alpha_\nu + \varphi_\mu \delta^\alpha_\nu - \varphi^\alpha g_{\mu\nu}\right). \tag{E.95}$$

Using this equation we can write the contracted curvature tensor $R^{(W)}_{\mu\nu}$ in terms of the tensor $R_{\mu\nu}$ of the associated Riemann space constructed with the Christoffel symbols $\left\{ {\alpha \atop \mu\nu} \right\}$. We obtain

$$R^{(W)}_{\mu\nu} = R_{\mu\nu} - \varphi_{,\mu;\nu} - \frac{1}{2}\,\varphi_{,\mu}\varphi_{,\nu} + \frac{1}{2}\,\varphi_{,\lambda}\varphi^{,\lambda} g_{\mu\nu} - \frac{1}{2}\,\Box\varphi\, g_{\mu\nu} \tag{E.96}$$

where the covariant derivatives are taken in the associated Riemannian geometry and $\Box$ is the d'Alembertian in the Riemannian geometry. Thus, for the curvature scalar,

$$R^{(W)} = R - 3\Box\,\varphi + \frac{3}{2}\,\varphi_{,\lambda}\varphi^{,\lambda} \tag{E.97}$$

in which $R$ is the curvature scalar of the associated Riemannian spacetime.

The expressions in Eqns.(E.96) and (E.97) are very similar to those obtained by a conformal mapping of a Riemannian geometry as shown in Sec.E.3.4.

### WIST duality: the Weyl map

A Weyl integral spacetime is determined by both a metric tensor and a scalar field. In [416], Weyl introduced a generalization of the conformal mapping, which he called a gauge transformation, given by

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{\chi} g_{\mu\nu}, \qquad \varphi \to \tilde{\varphi} = \varphi + \chi, \tag{E.98}$$

in which $\chi$ is an arbitrary function. Under such transformations the affine connection and the curvature and Ricci tensors are invariant:

$$\begin{aligned} \tilde{C}^{\alpha}{}_{\mu\nu} &= C^{\alpha}{}_{\mu\nu}, \\ \tilde{R}^{(W)\,\alpha}{}_{\beta\mu\nu} &= R^{(W)\,\alpha}{}_{\beta\mu\nu}, \\ \tilde{R}^{(W)}_{\mu\nu} &= R^{(W)}_{\mu\nu}. \end{aligned}$$

Note however that this is not the case for the scalar of curvature, which changes as

$$\tilde{R}^{(W)} = e^{-\chi} R^{(W)}.$$

This property has been used to construct gauge-invariant theories, as we shall see next.

### Invariant Action in WIST

From the behavior of the geometric quantities under a Weyl map, it is not difficult to write an action that is invariant under the transformation given by Eqns.(E.98):

$$S_W = \int \sqrt{-g}\, e^{-\varphi}\, R^{(W)}. \tag{E.99}$$

This Lagrangian can be rewritten in terms of the associated Riemannian quantities as follows:

$$S_W = \int \sqrt{-g} e^{-\varphi} \left( R - 3\Box\,\varphi + \frac{3}{2}\,\varphi_{,\lambda}\varphi^{,\lambda} \right). \tag{E.100}$$

After some algebra, we arrive (up to a total divergence) at the result

$$S_W = \int \sqrt{-g} e^{-\varphi} \left( R - \frac{3}{2}\varphi_{,\mu}\varphi^{,\mu} \right). \tag{E.101}$$

Note that the kinematical term of the scalar field for the scalar field appears with the "wrong" sign. This can be interpreted as a ghost field term hidden in the WIST structure.

### A particular case of WIST Duality

Let us go one step further and add to the above Lagrangian a kinematical term:

$$S_K = \int \sqrt{-g} e^{-\varphi} \varphi_{,\mu}\varphi^{,\mu}. \tag{E.102}$$

If we restrict to the case in which $\chi$ (given in Eqn.(E.98)) is a functional of $\varphi$, it follows that the complete action

$$S = \int \sqrt{-g}\; e^{-\varphi} (R^{(W)} + \beta\,\varphi_{,\mu}\varphi^{,\mu}) \tag{E.103}$$

is invariant under the restricted map

$$\begin{aligned} g_{\mu\nu} &\rightarrow \tilde{g}_{\mu\nu} = e^{-2\varphi} g_{\mu\nu}, \\ \varphi &\rightarrow \tilde{\varphi} = -\,\varphi, \end{aligned} \tag{E.104}$$

which is a special case of the general transformation (E.98). In terms of Riemann variables,

$$S = \int \sqrt{-g}\; e^{-\varphi} \left[ R + \left( \beta - \frac{3}{2} \right) \varphi_{,\mu}\varphi^{,\mu} \right]. \tag{E.105}$$

There are three invariants of dimension (length)$^2$ that can be constructed with the independent quantities of a WIST geometry: $R^{(W)}$, $\varphi^\alpha\,\varphi_\alpha$, and $\varphi^\alpha{}_{;\alpha}$, where $\varphi_\alpha \equiv \varphi_{,\alpha}$. Now, since the covariant derivative ";" in the WIST spacetime can be written in terms of the Riemann covariant derivative (denoted by "||") as

$$\varphi^\alpha_{;\alpha} = \varphi^\alpha_{||\alpha} - 2\varphi^{,\alpha}\varphi_{,\alpha},$$

the three invariants reduce to two. The most general action can then be written as

$$S = \int \sqrt{-g}\,[R^{(W)} + \xi\,\varphi^\alpha_{;\alpha}], \tag{E.106}$$

where $\xi$ is a constant. Independent variation of the metric tensor and the WIST field $\varphi$ yields

$$\Box\,\varphi = 0, \tag{E.107}$$

(the operator $\Box$ is calculated in the Riemannian spacetime) and

$$R^{(W)}{}_{\mu\nu} - \frac{1}{2}\,R^{(W)}\,g_{\mu\nu} + \varphi_{,\mu;\nu} - 2(\xi-1)\varphi_{,\mu}\varphi_{,\nu} + (\xi - \frac{1}{2})g_{\mu\nu}\varphi_{,\alpha}\varphi^{,\alpha} = 0. \tag{E.108}$$

This equation can be rewritten exclusively in terms of the associated Riemannian structure

$$R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} - \lambda^2\,\varphi_{,\mu}\,\varphi_{,\nu} + \frac{\lambda^2}{2}\,\varphi_{,\alpha}\,\varphi^{,\alpha}\,g_{\mu\nu} = 0, \tag{E.109}$$

where

$$\lambda^2 = \frac{1}{2}\,(4\xi - 3). \tag{E.110}$$

### A nonsingular cosmological model in WIST

Let us now show how a nonsingular cosmological scenario in the WIST framework can be constructed, following [318]. We shall work with the standard form of the FLRW metric:

$$ds^2 = dt^2 - a^2(t)\,\left[\frac{dr^2}{1-\epsilon r^2} + r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)\right], \tag{E.111}$$

As in the case of a standard scalar field, the WIST configuration can be represented by a perfect fluid, so that Eq.(E.109) becomes Einsteins equation for a perfect fluid with $v^\mu = \delta^\mu_0$, energy density $\rho_\varphi$ and pressure $p_\varphi$, given by

$$\rho_\varphi = p_\varphi = -\frac{1}{2}\,\lambda^2\,\dot{\varphi}^2 \tag{E.112}$$

In this interpretation, the WIST structure is equivalent to a Riemannian geometry, satisfying the equations of General Relativity with a perfect fluid having negative energy density as a source. The gauge vector $\varphi_\lambda$ for this geometry becomes

$$\varphi_\gamma = \partial_\lambda\varphi(t) = \dot{\varphi}\;\delta^0_\lambda\;, \tag{E.113}$$

where the dot denotes differentiation with respect to the time variable. Use of Eq.(E.107) yields a first integral for the function $\varphi(t)$:

$$\dot{\varphi} = \gamma a^{-3}, \tag{E.114}$$

where $\gamma =$ constant. In turn, EE (E.109) for the Friedman scale factor $a(t)$ are

$$\dot{a}^2 + \epsilon + \frac{\lambda^2}{6}\,(\dot{\varphi}a)^2 = 0, \tag{E.115}$$

$$2a\,\ddot{a} + \dot{a}^2 + \epsilon - \frac{\lambda^2}{2}\,(\dot{\varphi}a)^2 = 0\;, \tag{E.116}$$

where $\epsilon$ is the 3-curvature parameter of the FLRW geometry. From Eqn.(E.115) we see that $\epsilon = -1$. Combining Eqns.(E.114) and (E.115) we get the fundamental dynamical equation

$$\dot{a}^2 = 1 - \left[\frac{a_0}{a}\right]^4 \ , \tag{E.117}$$

with $a_0 = [\gamma^2\lambda^2/6]^{1/4}$. Before entering into the details of the solution of the system of structural and dynamical equations (E.115) and (E.114), let us comment some of the consequences of this cosmological model and list some interesting results.

**Features of the model**

An immediate consequence of Eq.(E.117) is that the scale factor $a(t)$ cannot attain values smaller than $a_0$. Let us consider a time reversal operation and run backwards into the past of the cosmic evolution. As the cosmic radius $a(t)$ decreases, the temperature of the material medium grows. In Hot Big Bang models such increment is unlimited; in the present theory, on the other hand, there is an epoch of greatest condensation in the vicinity of the minimum radius $a_0$. Close to this period, there occurs a continuous "phase transition" in the geometrical background: a Weyl structure is activated, according to Eq.(E.114): the Universe attains the minimum radius $a_0$ at $(t = 0)$, and consequently an unbounded growth of the temperature is inhibited. Notice that since the Universe had this infinite collapsing era to become homogeneous, in the present scenario the horizon problem of standard cosmology does not arise.

For very large times, the scale factor behaves as $a \sim t$. Thus, asymptotically, the geometrical configuration assumes a Riemannian character (since $\dot{\varphi} \to t$) in the form of a flat Minkowski space (in Milne's coordinate system). Consequently, in the present model the evolution of the universe may be started by a primordial instability of Minkowski spacetime at the remote past, due to Weyl perturbations of the Riemann structure through Eq.(E.91). In order to prescribe the behavior of these perturbations, knowledge of the time dependence of the gauge vector $\varphi_\lambda$ is required. Since the WIST function $\dot{\varphi}$ has a maximum at $t = 0$, the largest deviation of the Riemannian configuration corresponds to the epoch of greatest contraction near to the value $a_0$.

**Stability of the solution**

Among the difficult questions concerning bouncing Universes, one may count the problem of their survival with respect to eventual metric perturbations (see Sect.(E.11)). We shall show that during the stage of greatest condensation the WIST model of the Universe is stable. Applying the homogeneous perturbations

$$\varphi \to \varphi + \delta\varphi,$$

$$a \to a + \delta a,$$

to Eqs.(E.114) and (E.115), one obtains

$$\delta\dot{\varphi} = -\frac{3\gamma}{a^4}\,\delta a,$$

$$\delta\dot{a} \sim 2\frac{a_0^4}{a^5\dot{a}}\,\delta a.$$

Hence,

$$\delta\dot{\varphi} = -\frac{9\dot{a}}{\gamma\lambda^2}\,a\delta\dot{a},$$

$$\frac{\delta\dot{a}}{\delta a} \sim a^{-3}[a^4 - a_0^4]^{-\frac{1}{2}}\ .$$

Far from $a_0$ (*i.e.*, for large $t$) we have $a >> a_0$; then,

$$\frac{\delta\dot{a}}{\delta a} \sim a^{-5},$$

$$da \sim dt\ ,$$

so with $(\delta a)_i$ being the initial spectrum of perturbations, one obtains

$$\delta a \sim (\delta a)_i \exp[a^{-4}].$$

The solutions of the system Eqs. (E.114) and (E.115) are therefore stable against metric perturbations in the course of the infinite collapsing phase.

**The exact solution**

No closed solution can be obtained in terms of the cosmological time, so it is convenient to move to conformal time $\eta$, in which case the solution is easily shown to be

$$a(\eta) = a_0\sqrt{\cosh 2(\eta - \eta_0)}, \tag{E.118}$$

where $\eta_0$ is an integration constant. The following qualitative plot shows the difference between this bouncing solution and the radiation-dominated model in standard cosmology. The scale factor has a minimum for $a = a_0$, which corresponds to $\eta = \eta_0$. Thus the Universe had a collapsing era for $\eta < \eta_0$, attained its minimum dimension at $\eta = \eta_0$, and thereafter initiated an expanding era. Both the collapse and the expansion run adiabatically, i.e., at a very slow pace.

The correlate behavior of the Hubble expansion parameter $H \equiv (\dot{a}/a)$ helps to understand the model (Fig. E.2). Indeed, the Hubble parameter $H$ is a smooth function of the conformal time $\eta$ and does not diverge at the origin of the expanding era; quite on the contrary, it vanishes at $\eta = \eta_0$. The corresponding evolution of the Cosmos may be outlined as follows: the Universe stays for a long period in a phase of slow adiabatic contraction, until $H$ attains its minimum value. Then an abrupt transition occurs: a fast compression turns into a fast

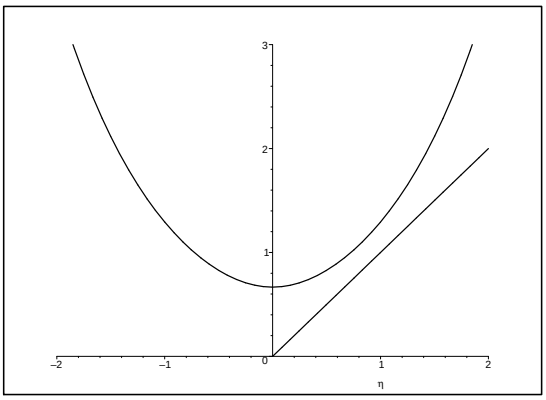


Figure E.1.: The qualitative plot shows (in conformal time) the scale factor for the bouncing model given by Eqn(E.118, and the scale factor for radiation in the SCM, $a \propto \eta$.

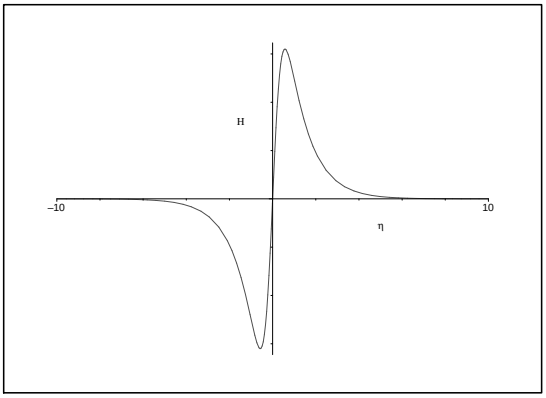


Figure E.2.: Plot of the Hubble parameter in conformal time for $a_0 = 1$.

expansion up to the maximum of $H$, and afterwards the expansion proceeds in an adiabatic slow pace again. While this image supplies a picture of the behavior of an Universe driven by $\varphi(t)$, it is however incomplete, due to the fact that the production of large amounts of matter and entropy has been neglected. This topic will be discussed in Sect.E.3.2).

### The WIST function $\varphi(t)$: structural transitions

According to the basic conception of the scenario presented above, the WIST function $\varphi(t)$ governs the cosmic evolution. Taking into account the solution Eq.(E.118) for the scale factor $a(t)$, the first integral equation (E.114) yields for $\varphi(t)$ the expression

$$\varphi = \frac{\gamma}{2a_0^2} \arccos \left[\frac{a_0}{a}\right]^2 \quad . \tag{E.119}$$

The behavior of $\varphi(t)$ is qualitatively portrayed in Fig. E.3, along with $\dot{\varphi}$. Note that when $a \to \pm\infty$ (i.e., for large times), $\varphi \to \pm\gamma\pi/4a_0^2$ =constant, which is consistent with the assumption that the Universe originated from a Minkowskian "nothing" state. The behavior of the time derivative $\dot{\varphi} = \gamma/a^3$, which appears in Eq.(E.112) of the energy density $\rho_\varphi$ of the "stiff matter" state associated to the WIST field is also shown in Fig.E.3. Since this function has a strong peak

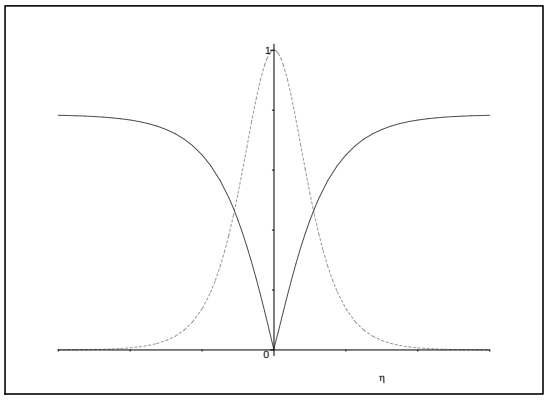


Figure E.3.: Plot of $\varphi$ (full line) and $\dot{\varphi}$ (dotted line) in conformal time for $a_0 = 1$.

in the neighborhood of the minimum radius $a_0$, the greatest deviation from the Riemannian configuration happens at this point. In this sense, a sort of "structural phase transition" takes place when the Universe approaches its maximally condensed state. The increase of the (negative) energy of the WIST "fluid" precludes the collapse to a singularity, reversing the cosmic evolution into an expansion. Note that the "kinky" aspect of the behavior of the WIST function $\varphi(t)$ in Fig.E.3 suggests a similarity between the Weyl structural transition described above and the propagation of instantons in Euclideanized models of quantum creation (see Eqn.(E.117)).

In the derivation of the solution of the WIST structural function $\varphi(t)$ (given by Eq.(E.119)), no attention was paid to the sign of the constant $\gamma$. Since the only information we have about $\gamma$ is that $\gamma^2 = 6a_0^4/\lambda^2$, according to Eqs.(E.114) and (E.117), $\gamma$ can be either positive or negative:

$$\gamma^{(\pm)} = \pm\sqrt{6}\ \frac{a_0^2}{|\lambda|}.$$

Hence, Eqns.(E.114) and (E.119) actually yield two equations, as follows:

$$\varphi^{(\pm)} = \varphi_0^{(\pm)} \arccos\left[\frac{a_0}{a}\right]^2, \tag{E.120}$$

$$\dot{\varphi}^{(\pm)} = \frac{\gamma^{(\pm)}}{a^3}\ , \tag{E.121}$$

in which $\varphi_0^{(\pm)} = \gamma^{(\pm)}/2a_0^2 = \pm\sqrt{3/2}\ |\gamma|^{-1}$. Thus the amplitude of the solutions $\varphi^{(\pm)}$ depends exclusively on the dimensionless parameter $\xi$ (see Eqn.(E.110)). The plot of the WIST functions $\varphi^{(-)}(t)$ and $\dot{\varphi}^{(-)}(t)$ is given by the mirror image of Fig.E.3 with respect to the horizontal axis. Note, however, that the energy density $\rho_\varphi$ of the "stiff matter" state associated with the WIST field $\varphi(t)$ is the same in both cases, since from Eqns. (E.112) and (E.114) we have

$$\rho_\varphi = -\frac{\lambda^2}{2}\ \dot{\varphi}^2 = -3\left[\frac{a_0^4}{a^6}\right]\ . \tag{E.122}$$

Thus, in spite of the fact that the pairs of WIST functions $(\varphi^{(+)}, \dot{\varphi}^{(+)})$ and $(\varphi^{(-)}, \dot{\varphi}^{(-)})$ have different characteristics, they induce the same type of cosmological evolution. Their only distinction, in fact, is connected to length variations, since according to Eq.(E.93) one now has $\Delta L^{(\pm)} = L\dot{\varphi}^{(\pm)}\Delta t$.

It is interesting to observe that the system is invariant with respect to the time reversal operation $t \to (-t)$ if $\varphi^{(+)}$ is concurrently mapped into $\varphi^{(-)}$ and reciprocally. In this sense, the WIST instanton-like functions $\varphi^{(+)}$ and $\varphi^{(-)}$ may be called "WISTon" and "anti-WISTon" solutions, respectively, since an anti-WISTon may be described as a WISTon running backwards in time. According to Eq.E.107, WISTons are defined up to an additive constant.

A closer inspection of the equations governing the behavior of $\varphi(t)$ reveals an instanton-like behavior typical of nonlinear theories of self-interacting scalar fields. Of course, the root of such nonlinearity is the fact that $\varphi(t)$ is taken as the actual source of the curvature of the metric structure, which in turn modifies the D'Alembertian operator $\Box$ due to the introduction of $\varphi$-dependent terms. A direct way to clarify this issue is to make explicit, by means of a change of variables, the hidden nonlinearity of the system of equations of motion involving

the scale factor $a(t)$ and the WIST function $\varphi(t)$. Define the new variable $s(t) \equiv \dot{\varphi}(t)$. Using Eqns. (E.107) and (E.114), we have

$$\begin{cases} \dot{s} + 3\gamma a^{-4}\dot{a} = 0 \ , \\ a^3 - \gamma s^{-1} = 0. \end{cases} \tag{E.123}$$

Taking $s(t)$ to represent a generalized coordinate associated with a one-particle dynamical system yields the conservation equation

$$\frac{1}{2}\ \dot{s}^2 + V(s) = 0 \ , \tag{E.124}$$

in which the associate potential $V(s)$ is given by

$$\begin{aligned} V(s) &= \frac{9}{2\gamma^2}\ \left[a_0^4\ s^4 - \gamma^{\frac{4}{3}} s^{\frac{8}{3}}\right] \\ &= \frac{3\lambda^2}{4}\ \left[s^4 - b^2 s^{\frac{8}{3}}\right] \ , \end{aligned} \tag{E.125}$$

with $b^2 = 6\lambda^{-2}\gamma^{2/3}$. Thus the evolution of field $s$ is equivalent to a unit mass particle moving in a potential with vanishing total energy. Due to the nonlinear character of this potential, the instanton-like aspect of functions $\varphi^{(\pm)}(t)$ is not surprising. Figure E.4 shows the behavior of $V(s)$. The potential vanishes at $s = 0$ and at $s_B^{(\pm)} = \gamma^{(\pm)} a_0^{-3}$ its extrema are at $s = 0$, and at $s_m^{(+)} = (2/3)^{3/4}\gamma^{(\pm)} a_0^{-3}$ (which are minima). However, the system cannot remain at the stable states $V(s_m^{(\pm)}) = \left(-\ \frac{2}{3}\right)\gamma^2 a_0^{-8}$, since in this case $\dot{s} \neq 0$; this in turn implies, of course, a nontrivial, evolving cosmic configuration. This nonlinear scheme provides a

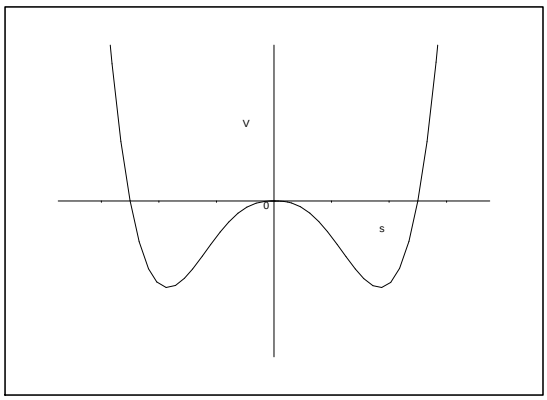


Figure E.4.: Qualitative plot of $V(s)$.

succinct picture of the evolution of the Universe: its development is initiated at $s = 0$ (which corresponds to Minkowski space time at $t \to -\infty$), attains its minimum radius $a(t = 0) = a_0$ at either $s_B^{(+)}$ or $s_B^{(-)}$ and returns back to $s = 0$ (which now corresponds to a Minkowski spacetime at $t \to +\infty$). According

to whether the system proceeds along the right or the left branches (i.e., from $s = 0$ to $s_B^{(+)}$ or $s_B^{(-)}$) of the figure, the cosmic evolution is driven by a WISTon or an anti-WISTon, respectively.

The appearance of instanton-like configurations is a direct consequence of the fundamental dynamical equation (E.117), in combination with the "structural" equation (E.114) which prescribes the degree of "Weylization" of space time.

### Weylization

We shall see next that the "structural transitions" discussed above are equivalent to a quantum tunnelling process in models of quantum creation from "nothing". Consider a generic Einstein equation for a Friedman scale factor,

$$\dot{a}^2 = -\epsilon + \frac{1}{3}\,\rho a^2 \ , \tag{E.126}$$

It was shown in [108] that a semiclassical description of a quantum tunnelling process is given by the bounce solutions of Euclideanized field equations, *i.e.* , of field equations in which the time parameter t is changed into $(-it)$. Applying such an Euclideanization procedure to Eq.(E.126), one obtains

$$\dot{a}^2 = +\epsilon - \frac{1}{3}\,\rho a^2 \ . \tag{E.127}$$

In the case of an $\epsilon = +1$ universe driven by a (positive) cosmological constant $\Lambda = 3\varsigma^2$ this approach was used in [409] to obtain, instead of the classical de Sitter solution, namely

$$a(t) = \frac{1}{\varsigma}\cosh(\varsigma t),$$

the solution

$$a_E(t) = \left(\frac{1}{\varsigma}\right)\cos(\varsigma t) \ , \tag{E.128}$$

corresponding to a de Sitter instanton – a "kink" configuration– propagating with negative classical energy, which bounces at the classical turning point $a = a_0 = (1/\varsigma)$ interpreted as representing the tunnelling to classical de Sitter space from "nothing."

Now consider Eqn.(E.126) in the case of a closed Universe driven by the energy density $\rho = 3[a_0^4/a^6]$. The euclideanized version of Eq.(E.127) gives

$$\dot{a}^2 = 1 - \left[\frac{a_0^4}{a^4}\right] \ .$$

But this is precisely the fundamental dynamical equation (E.117) of the WIST cosmological scenario. In this way, an equivalence is established between the Euclideanization of a closed Universe model driven by a positive energy density and a "structural transition" to a Weyl configuration which results in an *open*

Universe model driven by a "stiff matter" state of negative energy. Just as in models of quantum creation the propagation of an instanton is seen to represent the tunneling of the Universe from a primordial quantum "nothing" state, in the present scenario the propagation of a WISTon (i.e., a deviation of the Riemannian structure) is tantamount to the development of the Universe from a primordial empty Minkowski space.

It has been argued that solutions obtained through Euclideanization are in fact non-realistic, since they are to be interpreted as instantons, field configurations which tunnel across a classically forbidden region. Other authors endorse the view that such solutions correspond to an actual primordial phase of the cosmic evolution in which the basic Lorentzian nature of spacetime is changed into an Euclidean one. According to the present model, a different interpretation may be ascribed to these solutions, since an enlargement of the spacetime structure to a Weyl configuration – in which the geometry is characterized by the pair $(g_{\mu\nu}, \varphi_\lambda)$ of fundamental variables – supplies, at least in a particular case, the same basic behavior. It then becomes possible to reconcile the opposing interpretations of an "abstract soliton configuration" [136] and of a truly observable Euclidean cosmic phase [207]. The WIST solution is observable, whereas its basic nature is always Lorentzian. It is the Riemannian character of spacetime structure that results altered; allegorically, the choice is no longer Euclid or Lorentz, but rather Riemann or Weyl.

### Solution with matter generation

We have mentioned above that the model must be improved by taking into account matter creation. A non-singular solution in WIST that incorporates the effect of the creation of matter on the geometry was studied in [365]. Friedmann equation in conformal time is given by

$$a'^2 - a^2 = -\frac{\lambda^2}{6}(\varphi' a)^2 + \frac{a^4}{3}\rho_m, \tag{E.129}$$

while the second EE is

$$-3\left(2\frac{a''}{a^3} - \frac{a'^2}{a^4} - \frac{1}{a^2}\right) = \rho_m + 3\rho_\varphi. \tag{E.130}$$

The conservation of the stress-energy tensor in the case of ultra-relativistic matter is

$$(a^4\rho_m)' + \frac{1}{a^2}(a^6\rho_\varphi)' = 0. \tag{E.131}$$

A particular solution to these equations that describes creation of relativistic matter only around the bounce, and enters a radiation phase with a constant scalar field in a short time is given by the expression [365]

$$a(\eta) = \beta\sqrt{\cosh(2\eta) + k_0\sinh(2\eta) - 2k_0(\tanh\eta + 1)}, \tag{E.132}$$

with $\beta = a_0/\sqrt{1-k_0}$, and $0 < k_0 < 1/7$. The dependence of $\varphi$ on $\eta$ can be obtained from Eqns.(E.129) and (E.130). An asymmetry is to be expected both in the scale factor and in $\varphi$, since the evolution of this universe starts from the vacuum and enters a radiation dominated epoch. This is pictured in Fig.E.5. Notice that since the scalar field tends rapidly to a constant value, the

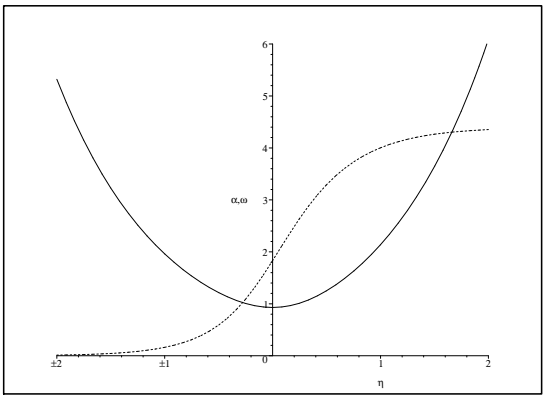


Figure E.5.: Plot of $a$ and $\varphi$ for $k_0 = 1/7$ and $a_0 = 0.93$, values chosen by imposing that the solution in Eqn.(E.132) enters the radiation era for $t \approx 10^{-8}$ sec.

production of matter (controlled by $\varphi'$, see Eqn.(E.131)) stops soon, and the model enters a radiation phase without the need of a potential. In this sense, this solution describes a hot bounce, as opposed to cold bouncing solutions, which do not enter the radiation era unless they are heated up [185]. Another nice feature of this solution is that the scalar field (formally equivalent to the dilaton of string theory) goes automatically to a constant value for $\eta \to \infty$, in such a way that the solution could be taken as the leading order of a perturbative development (as is the case in string theory). Again, no potential was needed in order to display this feature.

## E.3.3. Scalar-tensor theories

Scalar-tensor theories are a generalization of the Brans-Dicke Lagrangian [80], in which the constant appearing in the kinetic term of the scalar field $\phi$ becomes a function of $\phi$. Among the possible Lagrangians to describe these theories, one possibility is [386]

$$\mathcal{L} = -f(\phi)R + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} + 16\pi\mathcal{L}_{\text{matter}}, \tag{E.133}$$

where the scalar field $\phi$ couples non-minimally with the curvature through $f(\phi)$. With the redefinition $\varphi = f(\phi)$, the Lagrangian becomes

$$\mathcal{L} = -\varphi R + \frac{\omega(\varphi)}{\varphi}\varphi_{,\mu}\varphi^{,\mu} + 16\pi\mathcal{L}_{\text{matter}}, \tag{E.134}$$

with $\omega(\varphi) = \frac{1}{2} f/f_\varphi^2$ and $f_\varphi \equiv df/d\varphi$. Brans-Dicke theory is a special case of this Lagrangian, $f(\phi) \propto \phi^2$ which entails $\omega$ =const. This Lagrangian also describes the gravity-dilaton sector of low-energy string theory for $\omega = -1$ [121]. The differences between the two Lagrangians have been analyzed in [256]. Following the results of the discussion presented there, we shall use Eqn.(E.134) as the definition of scalar-tensor theories.

The equations of motion corresponding to Eqn.(E.134) are

$$R_{\mu\nu} = -\frac{1}{\varphi}(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) - \frac{\omega(\varphi)}{\varphi^2}\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{\varphi}\varphi_{,\mu;\nu} - \frac{1}{2\varphi} g_{\mu\nu}\Box\varphi, \tag{E.135}$$

$$[3 + 2\omega(\varphi)]\Box\varphi = T - \omega_\varphi \varphi_{,\mu}\varphi^{,\mu}. \tag{E.136}$$

Eqn.(E.135) suggests that it may be possible to find solutions in which matter satisfies SEC, but the whole r.h.s. is such that $R_{\mu\nu}v^\mu v^\nu \geq 0$ [49]. This implies, via the singularity theorem given in Sect.(E.1.1) that nonsingular solutions may exist in scalar-tensor theories. Using Eqn.(E.135), the inequality $R_{\mu\nu}v^\mu v^\nu \geq 0$ translates for the flat FLRW case and EOS $p = \lambda\rho$ to

$$-\frac{1}{\varphi}(1 + 3\lambda)\rho\,\frac{\omega + 2}{2\omega + 3} - \frac{\dot{\varphi}^2}{\varphi}\left(\frac{\omega}{\varphi} - \frac{\omega'}{2(2 + 3\omega)}\right) - \frac{\ddot{\varphi}}{\varphi} \geq 0. \tag{E.137}$$

Solutions satisfying this constraint, and hence exhibiting a bounce, have been presented in [34], for $\epsilon = 0$ in the cases of vacuum and radiation (for which $T = 0$, see r.h.s. of Eqn.(E.136)) [50]. With these restrictions, Eq.(E.136) written in conformal time takes the form

$$\varphi'' + \frac{2a'}{a}\varphi' = -\frac{\varphi^2\omega_\varphi}{3 + 2\omega}, \tag{E.138}$$

which integrates to

$$\varphi' a^2 = \frac{\sqrt{3}A}{\sqrt{2\omega + 3}}, \tag{E.139}$$

where $A$ is a constant. Introducing the variable $y = \varphi a^2$ and using Eq.(E.139), the Friedmann equation takes the form

$$y'^2 = 4\Gamma y + A^2, \tag{E.140}$$

($\Gamma \geq 0$ is a constant coming from energy conservation) yielding for $y(\eta)$,

$$y(\eta) = A(\eta + \eta_0) \tag{E.141}$$

in the case of vacuum, and

$$y(\eta) = \Gamma(\eta + \eta_0)^2 - \frac{A^2}{4\Gamma} \tag{E.142}$$

[49] The same happens in some wormhole configurations in Brans-Dicke theory. See [13].

[50] A shadow of doubt has been cast on these results in [229], where it was shown that gravitons would still see a singularity, even if the rest of matter does not.

in the case of radiation. Dividing now Eq.(E.139) by $y = \varphi a^2$ we obtain

$$\int \frac{\sqrt{2\omega(\varphi)+3}}{\varphi}\, d\varphi = \sqrt{3}A \int \frac{d\eta}{y(\eta)}. \tag{E.143}$$

If this equation is such that it yields $\varphi = \varphi(\eta)$, we could obtain $a(\eta)$ from $y = \varphi a^2$. To integrate Eq.(E.143), we need to specify the function $\omega(\varphi)$. The choice in [34] was

$$2\omega(\varphi) + 3 = 2\beta \left(1 - \frac{\varphi}{\varphi_c}\right)^{-\alpha}, \tag{E.144}$$

where $\alpha$, $\beta > 0$ and $\varphi_c$ are constants. With this choice of $\omega$, Eq.(E.143) can be solved for $\varphi(\eta)$ in the cases $\alpha = 0$ (which corresponds to Brans-Dicke theory), $\alpha = 1$ and $\beta = -\frac{1}{2}$ (which defines a theory introduced by Barker [30]), and $\alpha = 2$. The latter was studied in [34]. The solutions for the vacuum case are given by

$$a(\eta)^2 = \frac{A(\eta+\eta_0)(1+(\eta+\eta_0)^\lambda)}{\varphi_c(\eta+\eta_0)^\lambda} \tag{E.145}$$

$$\varphi(\eta) = \frac{\varphi_c(\eta+\eta_0)^\lambda}{1+(\eta+\eta_0)^\lambda}, \tag{E.146}$$

with $\lambda = \sqrt{3/2\beta}$. These solutions were shown to be nonsingular for $\beta < 3/2$. Hence the radiation solutions (which approach those for the vacuum for $\eta \to 0$ [34]) are also nonsingular. All the solutions for $\alpha = 2$ approach the FLRW radiation regime at late times because $\varphi$ tends to a constant, and then $\omega(\varphi) \to \infty$, but in order to be in agreement with solar system experiments, $\alpha$ must be greater than 1/2 [34].

The case of stiff matter (defined by $\rho = p$) sourcing the scalar field was studied in [289]. Since the density of a barotropic fluid ($p = (\gamma-1)\rho$) evolves as $\rho \propto a^{-3\gamma}$, this kind of matter is expected to dominate at early times, and the associated solutions give information about the early evolution of the universe. One of the results in [289] is that a necessary condition for $\dot{a} = 0$ when spatial curvature is negligible is $\omega = -6M\varphi/A$, where $A$ and $M$ are positive constants, yielding a negative kinetic term for $\varphi$ (see Eqn.(E.134). A thorough qualitative study of the case in which $\omega(\varphi)$ is a monotonic but otherwise arbitrary function of $\varphi$ was presented in [373], where the existence of nonsingular solutions in theories which agree with GR in the weak field limit was proved.

The first term on the left hand side of Eqn.(E.135) suggests that the gravitational constant is not actually a constant but varies with $\varphi^{-1}$. Based on this idea, a generalization of scalar-tensor theories (the so-called hyper-extended scalar-tensor) was advanced in [398]. The Lagrangian associated to these theories is given by

$$\mathcal{L} = -G(\varphi)^{-1}R + \frac{\omega(\varphi)}{\varphi}\varphi_{,\mu}\varphi^{,\mu} + 16\pi\mathcal{L}_{\text{matter}}, \tag{E.147}$$

which reduces to Eqn.(E.134) when $G(\varphi) = 1/\varphi$. Sufficient conditions on $G(\varphi)$, $\omega(\varphi)$, and their derivatives in order to have bouncing cosmological solutions were given in [159], generalizing the work of [230] for the case of ST theories.

Another descendant of the original ST theory are the multiscalar-tensor (MST) theories [112], which are the generic product of a compactification process of a higher-dimensional theory. The scalar content of a given MST theory depends on the details of the internal manifold that results from the compactification (usually gauge fields are set to zero in cosmological applications). Typically, one or more fields are associated to the size of the extra dimensions. In string theory, the coupling constants depend on the expectation value of massive scalar fields (called moduli fields) also associated with the size and shape of the extra dimensions, the most popular example of them being the dilaton. The moduli are an inescapable ingredient of string theory, hence several problematic issues raised by them must be confronted, such as stabilization, overcritical density, and violations of the Equivalence Principle. Cosmological solutions of low-energy string theories have been extensively studied (see [257] for a review). Needless to say, the results depend on the field content, which in turn depends on the given string theory under scrutiny.

A possible way to parameterize an action of a MST theory is [111]

$$\mathcal{L} = \sqrt{-g}\left[\phi R - \omega\frac{\phi_{,\rho}\phi^{,\rho}}{\phi} - \phi^n\psi_{,\rho}\psi^{,\rho} - \chi_{,\rho}\chi^{,\rho}\right] + \mathcal{L}_{\text{matter}}. \tag{E.148}$$

This Lagrangian represents pure multidimensional theories when $\psi$ =constant, $\chi$ = constant, and $\omega = (1-d)/d$, where $d$ is the number of compactified dimensions (assuming that they have the topology of a torus). The same case but with $\psi \neq$ constant and $n = -2/d + 1$ corresponds to a two-form gauge field in higher dimensions. If this field is conformal, it is associated to a $(d+4)/2$-form, leading to $n = -2/d$. In the case of string theory, $\omega = -1$, and the field $\psi$ is associated to a three-form field $H_{\mu\nu\lambda}$, leading to $n = -1$. The scalar $\chi$ is related to another three-form field coming from the R-R sector of type IIB superstring theory.

The existence of bouncing solutions for this Lagrangian in vacuum and in the presence of radiation for the FLRW geometry for all values of the three-curvature and for arbitrary values of $\omega$ and $n$ has been studied in [111]. The results show that generically there is a bounce for $n < 1$ and $\omega < 0$.

### Corrections coming from String Theory

Superstring theory is a candidate for a unified theory of the fundamental interactions, including gravity [46]. Since the fundamental objects in this theory are at least one-dimensional, geodesics of point particles are replaced by world-volumes. It is a valid question then to ask whether string theory has anything to say about the singularity problem. In this regard, it must be noted that in

string theory, the gravitational excitations are defined on a fixed metric background. Since singularities in general relativity are boundaries of space-time, which are a consequence of the dynamics governing its structure, a fixed manifold is certainly a restriction. Yet another difficulty is the breakdown of string perturbation theory in the regime of interest [66]. However, we have seen in the previous section that the incorporation of the massless degrees of freedom (corresponding to the lowest order EOM), which applies on scales below the string scale and above those where the string symmetries are broken, may smooth out the singularity. One could go further and include higher-order corrections in the action of string theory. There are two types of corrections. First, there are the classical corrections arising from the finite size of the strings, when the fields vary over the string length scale, given by $\lambda_s = \sqrt{\alpha'}$. These terms are important in the regime of large curvature, and lead to a series in $\alpha'$ (the inverse of the tension of the string). Then there are the loop (quantum) corrections. The loop expansion is parameterized by powers of the string coupling parameter $e^{\phi} = g^2_{\text{string}}$, which is a time-dependent quantity in cosmological models. In the so-called strong coupling regime, the dilaton becomes large and quantum corrections are important.

The effective action at the one-loop level is given by (see for instance [18])

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{2} + \frac{1}{4}(\nabla\phi)^2 + \frac{3}{4}(\nabla\sigma)^2 + \frac{1}{16}[\lambda e^{\phi} - \delta\xi(\sigma)]R^2_{GB}\right\}, \quad \text{(E.149)}$$

where $\phi$ is the dilaton, $\sigma$ is a modulus field, and $\lambda = 2/g^2$ ($g$ is the string coupling), $\delta$ is proportional to the 4-d trace anomaly, and $\xi(\sigma) = \ln(2e^{\sigma}\eta^4(ie^{\sigma}))$, where $\eta$ is the Dedekind function. The correction to the gravitational term is given in terms of the Gauss-Bonnet invariant,

$$R^2_{GB} = R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2.$$

The EOM that follow from this action in the case of a FLRW flat spacetime with the metric $g_{\mu\nu} = \text{diag}(1, -e^{2\omega}\delta_{ij})$ are [18] [51]

$$3\dot{\omega}^2 - \frac{3}{4}\dot{\sigma}^2 - \frac{1}{4}\dot{\phi}^2 + 24\dot{f}\dot{\omega}^3 = 0, \quad \text{(E.150)}$$

$$2\ddot{\omega} + 3\dot{\omega}^2 + \frac{3}{4}\dot{\sigma}^2 + \frac{1}{4}\dot{\phi}^2 + 16\dot{f}\dot{\omega}^3 + 8\ddot{f}\dot{\omega}^2 + 16\dot{f}\dot{\omega}\ddot{\omega} = 0, \quad \text{(E.151)}$$

$$\ddot{\sigma} + 3\dot{\omega}\dot{\sigma} + \delta\frac{\partial\xi}{\partial\sigma}\dot{\omega}^2(\dot{\omega}^2 + \ddot{\omega}) = 0, \quad \text{(E.152)}$$

$$\ddot{\phi} + 3\dot{\omega}\dot{\phi} - 3\lambda e^{\phi}\dot{\omega}^2(\dot{\omega}^2 + \ddot{\omega}) = 0, \quad \text{(E.153)}$$

where $f = \frac{1}{16}(\lambda e^{\phi} - \delta\xi(\sigma))$. These equations are not linearly independent due to the conservation of $T_{\mu\nu}$.

[51] See [144] for the case of nonzero spatial curvature.

It was shown in [18] that there are solutions with bounce for $\delta < 0$, which interpolate between an asymptotically flat and a slowly expanding universe with a period of rapid expansion. The bounce is essentially due to the violation of the strong energy condition by the modulus field (the dilaton playing an unimportant role). In a subsequent paper [353] it was shown that non-singular solutions can be obtained under the assumptions that $\xi$ is a smooth function that has a minimum at some point $\sigma_0$, and grows faster than $\sigma^2$ for $\sigma \to \pm\infty$, and $\delta > 0$. However, these solutions were later shown to be generically unstable for tensor perturbations [235]. Less symmetric models (Bianchi I [236] and Bianchi IX [419]) were also studied for this action, confirming the findings of [235].

Another attempt to avoid the singularity is to consider the effect of matter terms to the action of string theory. In [402] an action including dilaton, axion and one modulus field was considered along with matter (radiation or a "stringy" gas) and higher-order dilaton corrections in a flat FLRW background in $d$ dimensions. In this case, the results of [402] show that the energy densities of matter, axion and modulus are strongly suppressed in the inflationary phase driven by the dilaton, and hence the higher-order corrections coming from this field take the system through the graceful exit.

Yet another model inspired in string theory is the so-called ekpyrotic universe and its extension, the cyclic universe which will be discussed in Sect.E.10.2 [52].

### String Pre-Big Bang

A very-well developed example of the string cosmology approach is the so-called "pre-big bang" [179], which we shall call "string pre-big bang" (SPBB), to differentiate it from similar models not coming from string theory (see Sect.E.3.2). There are two properties of string theory that can be expected to play an important role in cosmology [406]. First, in the short-distance regime, a fundamental length $\lambda_s$ is expected to arise, thus introducing an ultraviolet cut-off and bounding physical quantities such as $H^2$ and $a$. Hence a bounce may be expected. Second, as we discussed before, at lower energies, the action of string theory is not Einstein's but a (multi)scalar-tensor theory, where one of the scalar fields is the dilaton, which controls the coupling constants. If these are really constant today (see [404]), the dilaton must be seated at the bottom of its potential, but it may have evolved in cosmological times. The idea of the SPBB is that during the cosmological evolution, the kinetic term of the dilaton drove a period of deflation (or inflation, depending on whether we consider the Einstein frame or the string frame) "before the big bang"(that is, in the contracting phase)[53], which can solve the horizon and flatness problems [172]. In this approach, the universe starts from a perturbative state, passes through a high-curvature and

[52]There are more considerations about singularities and bounces in string theory, to wit AdS/CFT correspondence [117], string gas cosmology [42], and tachyon condensation [377].
[53]This idea was also suggested in [318].

high-coupling stage, and then (hopefully) enters the radiation-dominated FLRW evolution. Duality symmetries present in the low-energy action of string theory are invoked to support this line of reasoning [399]: in the isotropic case, the gravidilaton EOM in the FLRW setting are invariant under a time inversion,

$$t \to -t \Rightarrow H \to -H,$$

$$\dot{\phi} \to -\dot{\phi},$$

and under the duality transformation

$$a \to \tilde{a} = a^{-1},$$

$$\phi \to \tilde{\phi} = \phi - 6 \ln a.$$

(compare with the Weyl transformation, Eqn.(E.98)). These transformations relate four branches of the solution (PBB, and post-big-bang expansion and contraction). In particular, to any expanding solution with decreasing curvature (such as those in the standard cosmological model), duality associates an accelerated contracting solution (see Fig.E.3.3). It is this pairing (which is possible only in the presence of the dilaton) that supports the whole idea of the SPBB. One of the issues of this idea is the joining of the two phases through the

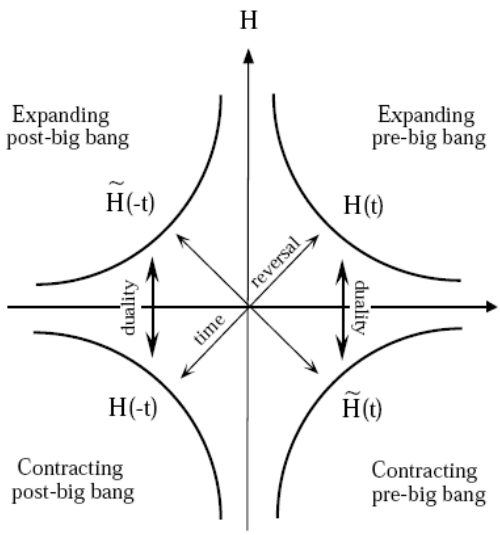


Figure E.6.: The four branches of the low-energy string cosmology backgrounds. Taken from [179].

putative singularity (the graceful exit problem). It has been proved in [228] that the graceful exit transition from the initial phase of inflation to the subsequent standard radiation dominated evolution must take place during a "string phase"

of high curvature or strong coupling is actually required. The corrections to the lowest-order lagrangian can be parameterized as [85]

$$\mathcal{L}_c = \mathcal{L}_{\alpha'} + \mathcal{L}_q,$$

where

$$\frac{1}{2}\mathcal{L}_{\alpha'} = e^{-\phi}\left(\frac{1}{4}R_{GB}^2 - \frac{1}{2}(\nabla\phi)^4\right), \tag{E.154}$$

and $\mathcal{L}_q$ designates the quantum loop corrections. Several forms of $\mathcal{L}_q$ were studied in [85]. The existence of a bounce in the Einstein frame, yielding a solution to the graceful exit problem, was shown by numerical integration of the EOM in [85] for the case $\mathcal{L}_q = -2(\nabla\phi)^4$, $\mathcal{L}_q = -2(\nabla\phi)^4 + R^2/3$, and for the two-loop correction $\mathcal{L}_q = 2e^{\phi}R_{GB}^2$, in all cases by choosing the appropriate sign for the correction.

An even more general form of the corrections was studied in [96], where $\mathcal{L}_c$ was given by

$$\mathcal{L}_c = -\frac{1}{4}e^{-\phi}\left(aR_{GB}^2 + b\phi(\nabla\phi)^2 + cG^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + d(\partial_\mu\phi)^4\right),$$

and $4b + 2c + d = -4a$ ($G^{\mu\nu}$ is the Einstein tensor). The quantum corrections were included by adding a suitable power of the string coupling, so the total effective Lagrangian is given by

$$\mathcal{L} = R + (\partial_\mu\phi)^2 + \mathcal{L}_c + Ae^{\phi}\mathcal{L}_c + Be^{2\phi}\mathcal{L}_c,$$

and the parameters $A$ and $B$ set the scale for the loop corrections. Solutions with graceful exit were found in [96] for a large range of parameters, but it is very hard to obtain the transition in the weak coupling regime, whilst keeping the loop corrections small.

A problem that remains to be solved is the stabilization of the dilaton to a constant value (otherwise there would be violations to the Equivalence Principle and to the observed "constancy of the coupling constants"). This was achieved in the previously mentioned articles in a number of ways: 1) by introducing by hand a friction term in the equation of motion of the dilaton, and then coupling it to radiation in such a way as to preserve overall conservation, 2) by "turning off" by hand the quantum Lagrangian by means of a step function, and 3) by the manipulation of the sign and size of the higher-loop corrections.

## E.3.4. Appendix:Conformal Transformation

Consider the map

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x)\, g_{\mu\nu}(x). \tag{E.155}$$

Then, for the contravariant components:

$$\tilde{g}^{\mu\nu}(x) = \Omega^{-2}(x)\, g^{\mu\nu}(x). \tag{E.156}$$

The conformal transformation of the connection is provided by

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + \frac{1}{\Omega}\left(\Omega_{,\mu}\,\delta^{\alpha}_{\nu} + \Omega_{,\nu}\,\delta^{\alpha}_{\mu} - \Omega_{,\epsilon}\,g^{\epsilon\alpha}\,g_{\mu\nu}\right), \tag{E.157}$$

and for the curvature tensor:

$$\tilde{R}^{\alpha\beta}_{\mu\nu} = \Omega^{-2}\,R^{\alpha\beta}{}_{\mu\nu} - \frac{1}{4}\delta^{[\alpha}{}_{[\mu}\,Q^{\beta]}{}_{\nu]}, \tag{E.158}$$

where

$$Q^{\alpha}{}_{\beta} \equiv 4\,\Omega^{-1}\,(\Omega^{-1})_{,\beta;\lambda}g^{\alpha\lambda} - 2\,(\Omega^{-1})_{,\mu}\,(\Omega^{-1})_{,\nu}\,g^{\mu\nu}\delta^{\alpha}_{\beta}.$$

Contracting Eqn.(E.158) we get

$$\tilde{R}^{\alpha}_{\mu} = \Omega^{-2}\,R^{\alpha}{}_{\mu} - \frac{1}{2}\,Q^{\alpha}{}_{\mu} - \frac{1}{4}Q\delta^{\alpha}_{\mu}, \tag{E.159}$$

and contracting again,

$$\tilde{R} = \Omega^{-2}[\,R + 6\,\Omega^{-1}\,\Box\,\Omega\,]. \tag{E.160}$$

A direct comparison of this conformal scalar of curvature and the Weyl scalar equation (E.97) shows that they coincide (up to a multiplicative factor) if we set

$$\Omega = \exp\left(-\frac{1}{2}\,\varphi\right),$$

and Eqn.(E.160) takes the form

$$\tilde{R} = e^{\varphi}[\,R - 3\Box\varphi + \frac{3}{2}\varphi_{,\mu}\varphi^{,\mu}],$$

which is exactly the transformed of the Ricci scalar for the WIST:

$$\tilde{R} = e^{\varphi}R^{(W)}.$$

# E.4. Maxwellian and Non-Maxwellian Vector Fields

## E.4.1. Introduction

The model described by the FLRW geometry with Maxwell's electrodynamics as its source displays a cosmological singularity at a finite time in the past [247]. However, this is not an intrinsic property of the combined electromagnetic and gravitational fields. Indeed, modifications of Maxwell electrodynamics (or, generically, massless vector field dynamics) can generate non-singular spatially homogeneous and isotropic (SHI) solutions of classical GR. We shall examine here two modifications that are relevant to the singularity problem:

- The non-minimal coupling of the EM field with gravity, and

- the self-interaction of the EM field.

These modifications will be introduced by means of Lagrangians which depend nonlinearly on the field invariants or on the space-time curvature. In both cases, the singularity theorems (see Ch.E.1) are circumvented by the appearance of a large, but nevertheless finite, negative pressure in an early phase of the SHI geometry.

## E.4.2. Einstein-Maxwell Singular Universe

The fact that Maxwell electrodynamics minimally coupled to gravity leads to singular models for the universe in the FLRW framework is a direct consequence of the singularity theorems (see Ch.E.1). Essentially, this can be understood from the examination of the energy conservation law and Raychaudhuri equation, as follows. To be consistent with the symmetries of the SHI metric, an averaging procedure must be performed if electromagnetic fields are to be taken as a source for the EE [396]. As a consequence, the components of the electric $\mathscr{E}_i$ and magnetic $\mathscr{H}_i$ fields must satisfy the following relations:

$$\overline{\mathscr{E}_i} = 0, \qquad \overline{\mathscr{H}_i} = 0, \qquad \overline{\mathscr{E}_i\,\mathscr{H}_j} = 0, \tag{E.161}$$

$$\overline{\mathscr{E}_i\,\mathscr{E}_j} = -\frac{1}{3}\mathscr{E}^2\,g_{ij}, \tag{E.162}$$

$$\overline{\mathscr{H}_i\,\mathscr{H}_j} = -\frac{1}{3}\mathscr{H}^2\,g_{ij}. \tag{E.163}$$

The symmetric energy-momentum tensor associated with Maxwell Lagrangian is given by

$$E_{\mu\nu} = F_{\mu\,\alpha}\,F^{\alpha}{}_{\nu} + \frac{1}{4}\,F\,g_{\mu\nu}, \tag{E.164}$$

in which $F \equiv F_{\mu\nu}\,F^{\mu\nu} = 2(\mathscr{H}^2 - \mathscr{E}^2)$. Using the above average values it follows that the $T_{\mu\nu}$ reduces to a perfect fluid configuration with energy density $\rho_\gamma$ and pressure $p_\gamma$ given by

$$\overline{E_{\mu\nu}} = (\rho_\gamma + p_\gamma)\,v_\mu\,v_\nu - p_\gamma\,g_{\mu\nu}, \tag{E.165}$$

where

$$\rho_\gamma = 3p_\gamma = \frac{1}{2}\,(\mathscr{E}^2 + \mathscr{H}^2). \tag{E.166}$$

The fact that both the energy density and the pressure in this case are positive definite for all values of $t$ implies the singular nature of FLRW universes. In fact, the solution of EE for the above energy-momentum configuration gives for the scale factor the singular form [355]

$$a(t) = \sqrt{a_0^2 t - \epsilon t^2}, \tag{E.167}$$

where $a_0$ is an arbitrary constant. We conclude that the space-time singularity in the Einstein-Maxwell system is unavoidable.

### E.4.3. Non-minimal interaction

Most of the articles concerning the interaction of Electrodynamics with Gravitation assume the principle of minimal coupling, which is a direct application of the strong form of the Equivalence Principle. In the absence of stringent limits from observation, ideally we should keep an open mind and consider other possibilities. Non-minimal coupling of the EM field with gravity has recently been applied in cosmology, following the trend initiated by scalar field theories interacting conformally with gravitation. These opened the way to the examination of more general theories, such as those in which curvature is directly coupled with the fields.

There are seven possible Lagrangians for the interaction of the EM field with Gravity which can be constructed as linear functionals of the curvature tensor. They are divided in two classes. Class I is given by:

$$\begin{aligned}\mathcal{L}_1 &= R\ A_\mu A^\mu,\\ \mathcal{L}_2 &= R_{\mu\nu}A^\mu A^\nu.\end{aligned}$$

These two Lagrangians are gauge dependent but no dimensional constant must be added since they already have the right dimensionality. As shown in [311] the EOM obtained from $\mathcal{L}_2$ in Einstein's gravity with the addition of a kinetic term for $A^\mu$ do not admit a FLRW solution. Thus, in the following we shall limit our analysis to $\mathcal{L}_1$.

In Class II, there are five Lagrangians :

$$\begin{aligned}\mathcal{L}_3 &= R\ F_{\mu\nu}F^{\mu\nu},\\ \mathcal{L}_4 &= R\ F_{\mu\nu}F^{\mu\overset{*}{\nu}},\\ \mathcal{L}_5 &= R_{\mu\nu}\ F^\mu_{\ \alpha}F^{\alpha\nu},\\ \mathcal{L}_6 &= R^{\alpha\beta\mu\nu}\ F_{\alpha\beta}\ F^{\nu\mu},\\ \mathcal{L}_7 &= \overset{*}{W}_{\alpha\beta\mu\nu}\ \ F^{\alpha\beta}\ F^{\mu\nu},\end{aligned} \tag{E.168}$$

where $W^{\alpha\beta\mu\nu}$ is the Weyl tensor and the star in the Weyl tensor means

$$\overset{*}{W}_{\alpha\beta\mu\nu} = W_{\alpha\beta\overset{*}{\mu\nu}} = W_{\overset{*}{\alpha\beta}\mu\nu} = \frac{1}{2}\eta_{\alpha\beta}^{\ \ \rho\sigma}W_{\rho\sigma\mu\nu}.$$

These Lagrangians are gauge independent but they all need the introduction of a length $\ell_0$ in order to have the correct dimensionality.

Another Lagrangian sometimes studied in the literature that is not explicitly contained in this list is

$$\mathcal{L}_8 = R_{\overset{*}{\alpha\beta}\mu\overset{*}{\nu}}\,F^{\alpha\beta}F^{\mu\nu}.$$

However, $\mathcal{L}_8$ is not independent of $(\mathcal{L}_1, ...\mathcal{L}_7)$. Indeed, the double dual $R_{\overset{*}{\alpha\beta}\mu\overset{*}{\nu}}$ satisfies the identity

$$R_{\overset{*}{\alpha\beta}\mu\overset{*}{\nu}} = R_{\alpha\beta\mu\nu} - 2W_{\alpha\beta\mu\nu} - \frac{1}{2}\ R\ g_{\alpha\beta\mu\nu}, \tag{E.169}$$

or, equivalently,

$$\begin{aligned} R_{\overset{*}{\alpha}\beta\mu\overset{*}{\nu}} &= -W_{\alpha\beta\mu\nu} + \frac{1}{2}\left(R_{\alpha\mu}g_{\beta\nu} + R_{\beta\nu}g_{\alpha\mu} - R_{\alpha\nu}g_{\beta\mu} - R_{\beta\mu}g_{\alpha\nu}\right) - \\ &- \frac{1}{3}\, R g_{\alpha\beta\mu\nu}. \end{aligned} \tag{E.170}$$

Thus,

$$\begin{aligned} \mathcal{L}_8 &= -\mathcal{L}_6 - \frac{1}{3}\, R\left(g_{\alpha\mu}g_{\beta\mu} - g_{\alpha\nu}g_{\beta\mu}\right) F^{\alpha\beta}F^{\mu\nu} + \frac{1}{2}\left(R_{\alpha\mu}g_{\beta\nu} + g_{\alpha\mu} - \right. \\ &- \left. R_{\alpha\nu}g_{\beta\mu} - R_{\beta\mu}g_{\alpha\nu}\right) F^{\alpha\beta}F^{\mu\nu}. \end{aligned}$$

Hence, $\mathcal{L}_8 = -\mathcal{L}_6 - \frac{2}{3}\,\mathcal{L}_3 - 2\mathcal{L}_5$.

### E.4.4. An example of a non singular universe

The first example of a nonsingular universe driven by the nonminimal coupling of EM and gravity was presented in [311], using the $\mathcal{L}_1$ of the previous section:

$$\mathcal{L} = R - \frac{1}{4}\, F^{\mu\nu}\, F_{\mu\nu} + \beta\, R\, A_\mu\, A^\mu. \tag{E.171}$$

As mentioned in Sect.E.4.2, in order to obtain a SHI geometry in the realm of General Relativity having a vector field as a source, an average procedure is needed. In the present non-minimal case there is another possibility, which we shall now explore. Since this theory is not gauge-invariant, it is possible to find a non-trivial solution for $A_\mu$ such that $F^{\mu\nu}$ vanishes.

The equations of motion that follow from the Lagrangian (E.171) are:

$$(1 + \beta\, A^2)\left(R_{\mu\nu} - \frac{1}{2}\, R g_{\mu\nu}\right) - \beta\, \Box A^2\, g_{\mu\nu} + \beta\, (A^2)_{;\mu;\nu} + \beta\, R A_\mu A_\nu = -E_{\mu\nu} - T_{\mu\nu}, \tag{E.172}$$

$$F^{\mu\nu}_{;\nu} = -2\beta\, R\, A^\mu. \tag{E.173}$$

From the trace of (E.172) it follows

$$R = -\,3\beta\, \Box A^2,$$

which when inserted in the equation of evolution of the electromagnetic field yields a nonlinear equation:

$$F^{\mu\nu}{}_{;\,\nu} - 6\beta^2\, (\Box\, A^2)\, A^\mu = 0. \tag{E.174}$$

The non-linearity induced by the non-minimal coupling with gravity is a generic feature for any field. To obtain a solution in which the geometry is nonsingular for a SHI geometry without imposing an average on the fields [311] we can consider the case in which $F_{\mu\nu}$ is zero. This is possible due to the explicit

dependence of the dynamical equations on the vector $A_\mu$. We take the vector field $A_\mu$ of the form

$$A_\mu = A(t)\,\delta^0_\mu. \tag{E.175}$$

Defining the quantity $\Omega$ by

$$\Omega(t) \equiv 1 + \beta\, A^2, \tag{E.176}$$

the set of equations (E.172-E.173) in a FLRW geometry reduces to the following:

$$3\,\frac{\ddot{a}}{a} = -\,\frac{\ddot{\Omega}}{\Omega}, \tag{E.177}$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2\,\epsilon}{a^2} = -\frac{\dot{a}}{a}\,\frac{\dot{\Omega}}{\Omega}, \tag{E.178}$$

$$\Box\,\Omega = 0. \tag{E.179}$$

The last equation implies that $a^3\, d\Omega/dt$ is a constant. Thus we set $d\Omega/dt = b\,a^{-3}$. A particular solution of this set of equations for $\epsilon = -1$ is given by [311]

$$A^2(t) \;=\; 1 - \frac{t}{a(t)} \tag{E.180}$$

$$a(t) \;=\; \sqrt{t^2 + \alpha_0^2} \tag{E.181}$$

where $\alpha_0$ is a constant that measures the minimum possible value of the scale factor. When $\alpha_0 = 0$ the system reduces to empty Minkowski space-time in Milne coordinates. For $\alpha_0 \neq 0$ this model represents an eternal universe without singularity and with a bounce [54] Notice that in recent years theories with negative energies have been examined in a cosmological context [326]. One way to achieve this goal is by introducing an *ad-hoc* term in the Lagrangian with the wrong sign. In the case of a scalar field this is given as

$$S = \int \sqrt{-g}\,(R - \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi). \tag{E.182}$$

A fluid with this odd feature can also be obtained by the non-minimal interaction of a vector field with gravity. Indeed, the solution presented in the preceding section can be interpreted as a perfect fluid with negative energy. The equations of motion presented in [311] can be re-written in the form:

$$R_{\mu\nu} = \frac{\Omega_{,\mu;\nu}}{\Omega}, \tag{E.183}$$

were $\Omega$, given by Eqn.(E.176), depends only in time . The structure of the corresponding system of equations is equivalent to the equations of General

[54] This form of the scale factor is similar to Melnikov-Orlov geometry [286], the difference being in the interpretation of the minimum radius $a_0$ and the source of the curvature.

Relativity in the SHI geometry having as its source the energy-momentum tensor of a perfect fluid with negative energy density and pressure given by

$$p = \frac{1}{3}\,\rho = -\,\frac{a_0^2}{a^4} \tag{E.184}$$

In this way, fluids with the "wrong" sign in Einstein's equation can be interpreted as vector fields with non-minimal interaction with gravity.

## E.4.5. Nonlinear electrodynamics

As pointed out in the introduction of this Chapter, linear electromagnetism unavoidably leads to a singularity. This situation changes drastically in the case of non-minimal coupling. In this section, we shall deal with another type of theories, in which it is the nonlinearity of the self-interaction of the EM field that provides the necessary conditions for a cosmological bounce to occur. The theories that will be examined are described by Lagrangians which are arbitrary functions of the invariants $F$ and $G$ that is $\mathcal{L} = \mathcal{L}(F, G)$, where $F = F_{\mu\nu}F^{\mu\nu}$, $G \doteq \frac{1}{2}\eta_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}$. Their corresponding energy momentum tensor, computed from Eqn.(E.5) yields

$$T_{\mu\nu} = -4\,\mathcal{L}_F\,F_\mu{}^\alpha F_{\alpha\nu} + (G\mathcal{L}_G - \mathcal{L})\,g_{\mu\nu}, \tag{E.185}$$

where $\mathcal{L}_A \equiv d\mathcal{L}/dA$, with $A = F, G$. It follows that

$$\rho = -\mathcal{L} + G\mathcal{L}_G - 4\mathcal{L}_F\mathscr{E}^2, \tag{E.186}$$

$$p = \mathcal{L} - G\mathcal{L}_G - \frac{4}{3}(2\mathscr{H}^2 - \mathscr{E}^2)\mathcal{L}_F. \tag{E.187}$$

We shall start our analysis by studying a toy model generalization of Maxwell's electrodynamics generated by a Lagrangian quadratic in the field invariants as in [125], that is:

$$L = -\frac{1}{4}\,F + \alpha\,F^2 + \beta\,G^2, \tag{E.188}$$

where $\alpha$ and $\beta$ are dimensionfull constants [55].

### Magnetic universe

In the early universe, matter behaves to a good approximation as a primordial plasma [391, 94]. Hence, it is natural to limit our considerations to the case in which only the average of the squared magnetic field $\mathscr{H}^2$ survives [137, 391]. This is formally equivalent to put $\mathscr{E}^2 = 0$ in (E.162), and physically means

[55] If we consider that the origin of these corrections come from quantum fluctuations then the value of the constants $\alpha$ and $\beta$ are fixed by the calculations made by Heisenberg and Euler.

to neglect bulk viscosity terms in the electric conductivity of the primordial plasma.

The Lagrangian (E.188) requires some spatial averages over large scales, such as the one given by equations (E.161)–(E.163). If one intends to make similar calculations on smaller scales then either more involved Lagrangians should be used, or some additional magnetohydrodynamical effect [394] should be devised in order to achieve correlation [222] at the desired scale. Since the average procedure is independent of the equations of the electromagnetic field we can use the above formulae (E.161)–(E.163) to arrive at a counterpart of expression (E.165) for the non-Maxwellian case. The average energy-momentum tensor is identical to that of a perfect fluid (E.165) with modified expressions for the energy density $\rho$ and pressure $p$, given by

$$\rho = \frac{1}{2}\,\mathscr{H}^2\,(1 - 8\,\alpha\,\mathscr{H}^2), \tag{E.189}$$

$$p = \frac{1}{6}\,\mathscr{H}^2\,(1 - 40\,\alpha\,\mathscr{H}^2). \tag{E.190}$$

Inserting expressions (E.189)–(E.190) in the conservation equation (E.2) yields

$$\mathscr{H} = \frac{\mathscr{H}_0}{a^2}, \tag{E.191}$$

where $\mathscr{H}_0$ is a constant. With this result, equation (E.4) leads to

$$\dot{a}^2 = \frac{\mathscr{H}_0^2}{6\,a^2}\left(1 - \frac{8\alpha\mathscr{H}_0^2}{a^4}\right) - \epsilon. \tag{E.192}$$

Since the right-hand side of equation (H.32) must not be negative it follows that, for $\alpha > 0$ the scale factor $a(t)$ cannot be arbitrarily small regardless of the value of $\epsilon$. The solution of Eqn.(H.32) is implicitly given as

$$t = \pm\int_{a_0}^{a(t)} \frac{dz}{\sqrt{\frac{\mathscr{H}_0^2}{6z^2} - \frac{8\alpha\mathscr{H}_0^4}{6z^6} - \epsilon}}, \tag{E.193}$$

where $a(0) = a_0$. The linear case described by Eqn.(E.167) can be regained from Eq.(E.193) by setting $\alpha = 0$. For the Euclidean section, expression (E.193) can be solved as [56]

$$a^2 = \mathscr{H}_0\sqrt{\frac{2}{3}\,(t^2 + 12\,\alpha)}. \tag{E.194}$$

From Eqn.(E.191), the average strength of the magnetic field $\mathscr{H}$ evolves in time as

$$\mathscr{H}^2 = \frac{3}{2}\,\frac{1}{t^2 + 12\,\alpha}. \tag{E.195}$$

[56] Nonsingular solutions in Bianchi universes with nonlinear electrodynamics as a source were studied in [170].

Expression (H.33) is singular for $\alpha < 0$, as there exist a time $t = \sqrt{-12\alpha}$ for which $a(t)$ is arbitrarily small. Otherwise, for $\alpha > 0$ at $t = 0$ the radius of the universe attains a minimum value (see Fig.E.7) $a_0$, given by

$$a_0^2 = \mathscr{H}_0 \sqrt{8\,\alpha}, \tag{E.196}$$

which depends on $\mathscr{H}_0$. The energy density $\rho_\gamma$ given by Eqn.(E.189) reaches its maximum value $\rho_{max} = 1/64\alpha$ at the instant $t = t_c$, where

$$t_c = \sqrt{12\,\alpha}. \tag{E.197}$$

For smaller values of $t$ the energy density decreases, vanishing at $t = 0$, while the pressure becomes negative (see Fig.E.8, left panel). Notice that we have a minimum of $a(t)$ along with a minimum of the energy density, entailing a violation of the NEC condition, in accordance with the first row of TableE.1.2.

Only for times $t \lesssim \sqrt{4\alpha}$ the non-linear effects are relevant for the normalized scale-factor, as shown in Figure E.8, left panel. Indeed, the solution (H.33) yields the standard expression (E.167) of the Maxwell case at the limit of large times. Notice that the energy-momentum tensor (E.185) is not trace-free for

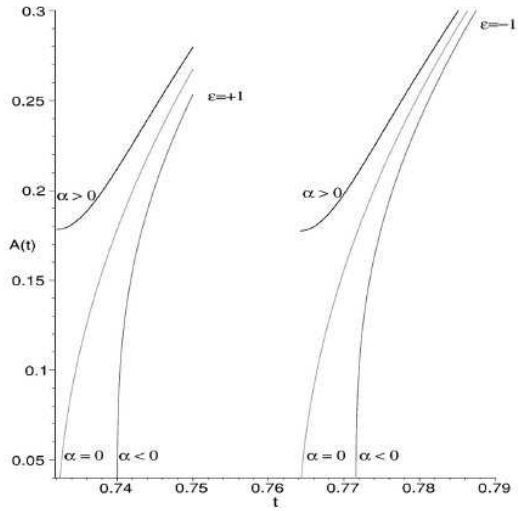


Figure E.7.: Plot of the scale factor as a function of $t$ for different values of $\epsilon$ and $\alpha$. Taken from [125].

$\alpha \neq 0$. Thus, the equation of state $p_\gamma = p_\gamma(\rho_\gamma)$ is no longer that of Maxwell's; it has instead a term proportional to the constant $\alpha$, that is

$$p = \frac{1}{3}\,\rho - \frac{16}{3}\,\alpha\,\mathscr{H}^4. \tag{E.198}$$

This scenario has been generalized in several ways in [93]. First, the general expression for the scale factor was shown to be

$$a(t) = a_0(4\alpha_0^2 t^2 + 4\alpha_0\beta_0 t + 1)^{1/4}, \tag{E.199}$$

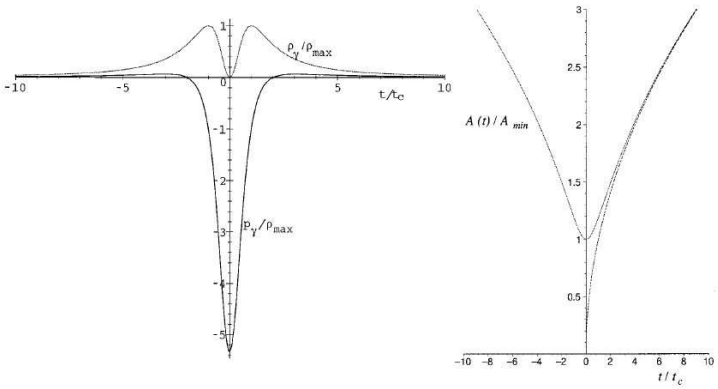


Figure E.8.: Left panel: time dependence of $\rho$ and $p$ for $\epsilon = 0$ and $\alpha > 0$. Right panel: scale factor for $\epsilon = 0$ and $\alpha > 0$ (full line), and for Mawxell's case (dashed line). Taken from [125].

where

$$\alpha_0 = \sqrt{\frac{2}{3}}\mathscr{H}_0, \qquad \beta_0 = \pm\sqrt{1 - 8\alpha\mathscr{H}_0}.$$

Eqn.(H.33) follows as a particular case from Eqn.(E.199), which describes a bounce with

$$a_{\min} = a_0(8\omega\mathscr{H}_0^2)^{1/4}, \qquad t_{\min} = -\beta_0/(2\alpha_0), \qquad \mathscr{H}_{\min} = \frac{1}{2\sqrt{2\alpha}}, \qquad \rho_{\min} = 0.$$

Solutions of this model with the addition of a cosmological constant $\Lambda$ were also discussed in [93]. It was shown that nonsingular solutions are possible both for a constant $\Lambda$, and for certain choices of $\Lambda = \Lambda(t)$.

### Born-Infeld electrodynamics

A widely studied EM theory is that proposed by Born and Infeld, with Lagrangian

$$L_{BI} = \beta^2\left(1 - \sqrt{X}\right) \tag{E.200}$$

where

$$X \equiv 1 + \frac{1}{2\beta^2}F - \frac{1}{16\beta^4}G^2 \tag{E.201}$$

Note that, following Born-Infeld's original work, a constant term has been added in the Lagrangian in order to eliminate a cosmological constant and to set the value of the Coulomb-like field to be zero at the infinity. Using equation (E.186) for the energy density we obtain

$$\rho = \frac{\beta^2}{\sqrt{X}}\left(1 - \sqrt{X} + \frac{\mathscr{H}^2}{\beta^2}\right) \tag{E.202}$$

and for the pressure

$$p = \frac{\beta^2}{\sqrt{X}} \left( \sqrt{X} - \beta^2 + \frac{2}{3} \frac{\mathscr{E}^2}{\beta^2} - \frac{1}{3} \frac{\mathscr{H}^2}{\beta^2} \right) \tag{E.203}$$

A straightforward calculation of $\rho + 3p$ shows that this theory cannot yield a nonsingular universe.

### Bouncing in the Magnetic Universe

The "magnetic universe" displays a very interesting property due to the non-linear dynamics: its energy density can be interpreted as composed of $k$ non-interacting fluids, in the case in which the dynamics is provided by the polynomial

$$\mathcal{L} = \sum_k c_k F^k, \tag{E.204}$$

where $k \in \mathbb{Z}$. The conservation of the energy-momentum tensor projected in the direction of the co-moving velocity $v^\mu = \delta^\mu_0$ yields

$$\dot{\rho} + (\rho + p)\theta = 0. \tag{E.205}$$

From the expression for the energy density and pressure given in Eqns.(E.186) and (E.187) with $\mathscr{E} = 0$ we get that $\rho = \sum_k \rho_k$, and $p = \sum_k p_k$ where

$$\begin{aligned} \rho_k &= -c_k 2^k \mathscr{H}^{2k} \\ p_k &= c_k\, 2^k \mathscr{H}^{2k} \left( 1 - \frac{4k}{3} \right), \end{aligned} \tag{E.206}$$

in such a way that we can associate to each power of $k$ an independent fluid characterized by $\rho_k$ and $p_k$, with an EOS

$$p_k = \left( \frac{4k}{3} - 1 \right) \rho_k.$$

Inserting the total energy density and pressure (from the sum of $\rho_k$ and $p_k$ in Eqns.(E.206) and (E.206)) in the conservation equation (H.12) we obtain

$$\mathcal{L}_F \left[ (\mathscr{H}^2)^{\cdot} + 4\mathscr{H}^2 \frac{\dot{a}}{a} \right] = 0. \tag{E.207}$$

The important result that this equation shows is that each $k$-fluid is separately conserved, since the dependence of the conservation equation on the specific form of the Lagrangian factors out, in such a way that $\mathscr{H}$ evolves with the scale factor as

$$\mathscr{H} = \frac{\mathscr{H}_0}{a^2} \tag{E.208}$$

for any $\mathcal{L}$ of the form given in Eqn.(H.18).

### Two-fluid description

It follows from equations (E.189), (E.190) and (E.191) that in the case of the nonlinear Lagrangian given by Eqn.(E.188) it is not possible to write an equation of state relating the pressure to the energy density. This is a drawback if we want to use a fluid description of the averaged electromagnetic field. In order to circumvent such difficulty a two-fluid description can be adopted, because of the remarkable fact that there exists a separate law of conservation for each component of the fluid, as we saw above. The fact that the dynamical equation for $\mathscr{H}$ factors (see Eqn.(E.207) means that the fluids are conserved independently: the energy-momentum tensor can be separated into two pieces, each representing a perfect fluid which is conserved independently. In other words, there is no interaction between fluids 1 and 2. We shall see in Section E.11.2 that the analysis of the stability of the non-singular universe described in this section is more transparent when using the two-fluid description. This case can be generalized to a multi-component fluid, but we shall restrict here to the 2-fluid application for a pure magnetic field.

In order to get a better understanding of the properties of the cosmic geometry controlled by the magnetic field let us analyze the case in which the spatial section is closed ($\epsilon = 1$). The crucial equations for such analysis are the conservation law, the Raychaudhuri equation for the expansion and the Friedman equation, that is:

$$\dot{\rho} + (\rho + p)\,\theta = 0, \tag{E.209}$$

$$\dot{\theta} + \frac{1}{3}\,\theta^2 = -\frac{1}{2}\,(\rho + 3p). \tag{E.210}$$

$$\rho = \frac{1}{3}\,\theta^2 + \frac{3}{a^2}, \tag{E.211}$$

In the magnetic universe we have

$$\rho = \frac{\mathscr{H}_0^{\,2}}{2a^4}\left(1 - 8\alpha\,\frac{\mathscr{H}_0^{\,2}}{a^4}\right). \tag{E.212}$$

A necessary condition for the existence of a bounce is given by the vanishing of the expansion factor for a given value of $t$. This leads to an algebraic equation of third order in $x \equiv a_b^2$:

$$x^3 - \frac{\mathscr{H}_0^2}{6}\,x^2 + \frac{4}{3}\,\alpha\mathscr{H}_0^{\,4} = 0. \tag{E.213}$$

Using the fact that $\alpha$ is a very small parameter, it can be shown that this equation has three real solutions. Two of them are positive and the third is negative. Thus we retain only the positive solutions which will be called $X_1$ and $X_2$. The important quantity for our analysis is contained in the expression

$$\rho_b + 3p_b = \frac{\mathscr{H}_0^2}{x^4}\,(x^2 - 24\alpha\mathscr{H}_0^{\,2}). \tag{E.214}$$

Thus, at one of the points, say $X_1$ there is a local maximum for the scale factor; and at the other, $X_2$ there is a minimum for $x^2 < 24\alpha\mathscr{H}_0{}^2$. Note that at the bounce (where $\theta = 0$), there is an extremum of the total energy: $\dot{\rho}_b = 0$. The analysis of the second derivative in the bounce depends on the location of $X_2$ through the equations:

$$\ddot{\rho}_b = \frac{1}{3}\frac{\mathscr{H}_0{}^4}{x^8}\left(x^2 - 16\alpha\mathscr{H}_0^2\right)\left(x^2 - 24\alpha\mathscr{H}_0^2\right). \tag{E.215}$$

At $x = X_1$ the density is a minimum. For $x = X_2$ the extremum depends on the location of the bounce with respect to the point in which the quantity $\rho + p$ changes sign. For the case in which $16\alpha\mathscr{H}_0{}^2 < X_2 < 24\alpha\mathscr{H}_0{}^2$, it follows that the density has a maximum at $X_1$. On the other hand if $X_2 < 16\alpha\mathscr{H}_0^2$ it is a minimum. To understand completely the behavior of the energy density the existence of other critical points for $\rho$ must be addressed. This is controlled by equation (E.209). Thus, the extra extremum (which are not bounce or turning points) occur at $x$ such that

$$\rho + p = \frac{2}{3}\frac{\mathscr{H}_0^2}{x^4}\left(x^2 - 16\alpha\mathscr{H}_0^2\right) = 0, \tag{E.216}$$

that is, at points in which the scale factor takes the value $\sqrt{16\alpha\mathscr{H}_0{}^2}$. Direct inspection shows that these are points of maximum density.

Another consequence of nonlinear electromagnetism in cosmology is the occurrence of cyclic universe, as will be discussed in Sect.E.10.2.

## E.4.6. Appendix

### Repulsive gravity

A peculiar result which may provide a framework to generate cosmological scenarios without singularity comes from the nonminimal interaction of EM with gravity, rendering gravity repulsive. The theory is defined by

$$L = \sqrt{-g}\,\left\{R - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \beta R\,A_\mu A^\mu\right\}, \tag{E.217}$$

where $\beta$ is a dimensionless constant. This Lagrangian is not gauge-invariant and can be interpreted in terms of a photon having a mass (and also an additional polarization state) which depends on the curvature of the geometry.

Variation of $g_{\mu\nu}$ and $A_\mu$ yield the equations of motion:

$$\left(\frac{1}{\kappa} + \beta A^2\right)G_{\mu\nu} = \beta g_{\mu\nu}\Box A^2 - \beta A^2{}_{,\mu;\nu} - \beta R A_\mu A_\nu - E_{\mu\nu}, \tag{E.218}$$

$$F^{\mu\nu}_{\;\;;\nu} = -2\beta R A^\mu, \tag{E.219}$$

where $E_{\mu\nu}$ is Maxwell's energy-momentum tensor given by equation (E.164). As will be shown next, this set of equations allows a renormalization of the gravitational constant. Consider for instance the case in which $A_\mu A^\mu = Z =$ constant$\neq 0$. Then

$$\left(\frac{1}{\kappa} + \beta Z\right) G_{\mu\nu} = -\beta R\, A_\mu A_\nu - E_{\mu\nu}. \tag{E.220}$$

Taking the trace of this equation we obtain $R = 0$, and inserting this result back into Eqn.(E.220) we get

$$R_{\mu\nu} = -\tilde{\kappa} E_{\mu\nu},$$

where the renormalized constant $\tilde{\kappa}$ is given by

$$\frac{1}{\tilde{\kappa}} = \frac{1}{\kappa} + \beta Z.$$

Thus, Eqns.(E.218) and (E.219) can be written as

$$R_{\mu\nu} = -\tilde{\kappa} E_{\mu\nu}, \qquad F^{\mu\nu}_{\ \ ;\nu} = 0, \tag{E.221}$$

which are nothing but Maxwell's electrodynamics minimally coupled to gravity with a re-normalized gravitational coupling plus the condition $A_\mu A^\mu = constant = Z$.

The addition of other forms of neutral matter, such that the corresponding energy-momentum tensor is traceless, takes the Lagrangian to

$$L = \sqrt{-g}\ \left\{\frac{1}{\kappa}\ R - \frac{1}{4}\ F_{\mu\nu}F^{\mu\nu} + \beta\, R A_\mu A^\mu + L^{(m)}\right\}, \tag{E.222}$$

where $L^{(m)}$ represents the Lagrangian for all other kinds of matter such that $T^{(m)}_{\mu\nu} g^{\mu\nu} \equiv T^{(m)} = 0$. The equations of motion in this case are given by

$$\left(\frac{1}{\kappa} + \beta\, A^2\right) G_{\mu\nu} = \beta \Box A^2 g_{\mu\nu} - \beta A^2_{,\mu;\nu} - \beta R A_\mu A_\nu - E_{\mu\nu} - T^m_{\mu\nu} \tag{E.223}$$

$$F^{\mu\nu}_{\ \ ;\nu} = -2\beta R A^\mu. \tag{E.224}$$

Taking again the case $A_\mu A^\mu =$ constant, yields $R = 0$. Then Eqns.(E.223-E.224) take the reduced form

$$\begin{aligned} R_{\mu\nu} &= -\tilde{\kappa} E_{\mu\nu} - \tilde{\kappa} T^{(m)}_{\mu\nu}, \\ F^{\mu\nu}_{\ \ ;\nu} &= 0, \end{aligned}$$

where $\tilde{\kappa}$ was given above. Thus, the renormalization of the gravitational constant by the non-minimal coupling represented by the presence of the term $RA_\mu A^\mu$ in the Lagrangian in the state where $A^\mu A_\mu$ is constant is still valid in the presence of matter with null trace [57].

[57] Note that this model provides a mechanism for a bounce, but needs to be modified to account for the correct large-scale behavior.

### Global Dual invariance

While observation must be the ultimate judge of the choice among the possible couplings, if it scarce or not available, we can resort to criteria coming from theoretical considerations. One of them is related to the invariance of the Lagrangian under a given transformation, such as the dual rotation. A dual map is a transformation on the set of the bi-tensors $F_{\mu\nu}$ such that

$$F_{\mu\nu} \to F'_{\mu\nu} = \cos\theta F_{\mu\nu} + \sin\theta F^*_{\mu\nu}. \tag{E.225}$$

Classical Maxwell's electrodynamics is invariant under such transformation only if the angle $\theta$ is constant. In a Minkowskian background it is not possible to implement such invariance for a local map $\theta = \theta(x)$. However, this can be achieved in the case of a non-minimal coupling of the electromagnetic field with the metric of a non-flat geometry. In fact, using the identities

$$\begin{aligned} F_{\mu\alpha}\, F^{\alpha\nu} \quad &- \quad F^*_{\mu\alpha}\, F^{*\,\alpha\nu} = -\,\frac{F}{2}\,\delta^\nu_\mu \\ F_{\mu\alpha}\, F^{*\,\alpha\nu} \quad &= \quad -\,\frac{G}{4}\,\delta^\nu_\mu \end{aligned}$$

it can be shown that the combined Lagrangian:

$$L_{\rm DI} = L_5 - \frac{1}{4}\, L_3 = \left( R_{\mu\nu} - \frac{1}{4}\, R\, g_{\mu\nu} \right) F^\mu_{\ \alpha} F^{\alpha\nu}. \tag{E.226}$$

is invariant under local dual rotations: $\tilde{L}_{\rm DI} = L_{\rm DI}$. This is a remarkable property which has no counterpart in the flat space limit.

## E.5. Viscosity

A full knowledge of the global properties of the universe cannot be achieved without giving a description of the thermodynamics of the cosmic fluid. In the last decades, this task was addressed in three distinct periods. In the first period the universe was treated as a system in equilibrium in which all global processes were described by classical reversible thermodynamics, in such a way that total entropy was conserved. The salient feature of this phase was the development of the standard cosmological model, which comprises the homogeneous and isotropic FLRW geometry, and the characterization of the matter content of the universe as a one-component perfect fluid in equilibrium. In order to solve the EE, the energy density $\rho$ and the pressure $p$ were considered functions of the cosmological time only, and they were related by a linear EOS $p = \lambda\rho$. The FLRW models generated in this way share the common property of having an initial singularity (with $\lambda > -1/3$).

Later, it was realized [135] that the validity of thermal equilibrium near the initial singularity is perhaps too strong an assumption. A second phase then

started, in which the description of the cosmic fluid was improved by allowing viscous processes. Some of the motivations for this alteration are the following:

- The examination of the possible role of viscosity in the dissipation of eventual primordial anisotropies (chaotic cosmologies),
- The effect on the existence and/or the form of the singularity,
- The application in cosmology of results obtained from non-equilibrium thermodynamics.

In 1973 a FLRW cosmological model without singularity was presented [299] (see also [246]), using a viscous fluid as a source. The energy-momentum tensor was given by

$$T_{\mu\nu} = (\rho + p)\, v_\mu\, v_\nu - p\, g_{\mu\nu},$$

in which $p = p_{\rm th} - \zeta\,\theta$; where $p_{\rm th}$ is the thermodynamical pressure, $\zeta$ is a viscous coefficient and $\theta$ is the three times Hubble parameter, which is exactly the case of the energy-momentum tensor representing particle creation [51]. The SEC in this case is given by the inequalities

$$\rho + p_{th} > 0$$

and

$$\rho + 3p_{th} > 0,$$

which are weaker than the correspondent ones in the case of perfect fluid, hence allowing for the absence of singularity. The solution found in [299] is nonsingular, and past-eternal.

More general forms for the dependence of viscous quantities have been investigated for arbitrary Stokesian regimes in which the fluid parameters become more general (for instance nonlinear) functions of the expansion. With these modifications, there are non-singular cosmological solutions, but they may suffer from a possibly worse disease than the initial singularity: they are unstable and display non-causal propagation. In fact, the instability of the model in [299] under homogeneous perturbations was proven by the analysis made in [51]. It was also proved in [51] that the avoidance of the singularity is not generic. In other words, the singularity is not avoided for any type of viscosity (that is, for any dependence of the coefficients of viscosity on the expansion factor).

In this second phase, local equilibrium [346] is still imposed, in such a way that the thermodynamical variables are described as if the dissipative fluxes - e.g. heat flux - do not influence local variables like for instance the entropy, although as a whole the system is not in equilibrium. As another example, a fluid in the regime

$$\tilde{p} = p + \alpha\theta + \beta\theta^2$$

was analyzed in [312], both for $\alpha = \beta =$ constant, and $\alpha = 0$, $\beta = M\rho^m$, with $M$ and $m$ constants. In the second case, nonsingular solutions were found using tools from dynamical systems analysis.

Let us remark that in general, the imposition of local equilibrium leads to causal difficulties, allowing dissipative signals to travel with infinite velocity of propagation. These causal problems were the focus of the third phase, where extended irreversible thermodynamics was used [220]. In this theory, the basic quantities become dependent not only on local variables of classical thermodynamics but also on the dissipative fluxes. This has very important consequences, the most important one being the preservation of causal connections for the whole system. In [127], a FLRW universe was studied in this context, the net consequence of the assumption of extended irreversible thermodynamics being to provide an additional equation of motion for the non-equilibrium pressure $\pi$, with $p = p_{\rm th} + \pi$, given by

$$\tau_0 \dot{\pi} + \pi = -\xi\theta. \tag{E.227}$$

(where $\tau_0$ is the relaxation time) which preserves the causal structure. Thus, contrary to the previous case in which the viscous term is assumed to be a polynomial in $\theta$, here it must obey Eqn.(E.227). The other quantities relevant to thermodynamics (that is, the entropy flux $s^\alpha$ and the particle flux per unit of proper volume $n$) are determined by

$$n\dot{s} = \frac{\pi^2}{\xi T}, \qquad \theta = -\frac{\dot{n}}{n}.$$

Assuming an EOS given by $p_{\rm th} = \lambda\rho$, the cases $\xi =$ constant, and $\xi = \beta\rho$, (with $\beta =$ constant) were analyzed in [127], always with $\tau_0 =$ constant, and nonsingular solutions were discovered in both cases, for $\lambda = 0$ and $\lambda = 1/3$. The relevant equations of this system can be put in the form of an autonomous planar system:

$$\begin{aligned} \frac{d\theta}{dt} &= -3/2(1+\lambda)\theta^2 - \frac{\pi}{2} + \frac{(1+\lambda)}{2}\Lambda, \\ \frac{d\pi}{dt} &= -\frac{1}{\tau_0}(1+3\zeta\,\theta), \end{aligned} \tag{E.228}$$

where $\Lambda$ is the cosmological constant. The set of integral curves of this system was studied in [317], where it was shown that the solution found in [127] is stable.

**Bifurcations in the early cosmos**

Quadratic dissipative processes were analyzed from a new perspective in [314], where it was shown that dissipative processes may lead to the appearance of bifurcations. This is a consequence of the application of a theorem due to Bendixson [17] to the system of EE that describes a universe with curvature

controlled by a dissipative fluid. Indeed, let us consider a planar autonomous system that contains a parameter, say $\sigma$, of the form

$$\begin{aligned} \dot{x} &= F(x, y; \sigma) \\ \dot{y} &= G(x, y; \sigma), \end{aligned} \tag{E.229}$$

where the functions $F$ and $G$ are non-linear and the parameter $\sigma$ has a domain $\mathcal{D}$. Applying methods of qualitative analysis to this system and restricting to the two-dimensional plane $\Gamma$ of all integrals of this system, one arrives to the notion of "elliptical" and "hyperbolic" sectors, that characterize, as the names indicates, the behavior of the integral curves in the neighborhood of a multiple equilibrium point (that is, an isolated points that is a zero of both $F$ and $G$). Let us call $\mathcal{E}$ and $\mathcal{H}$ the number of elliptical and hyperbolic sectors of a given equilibrium point $M \equiv (x_0, y_0)$ of $\Gamma$, respectively. Then the Poincaré index is defined by the formula

$$I_P = \frac{\mathcal{E} - \mathcal{H}}{2} + 1.$$

This is a measure of the topological properties of the integral curves in the phase plane $\Gamma$. If above a certain value $\sigma_c$ of $\mathcal{D}$ the topological properties of the system (E.229) change, then there is an abrupt change of behavior of the physical system in the vicinity of the unstable equilibrium point. The crucial consequence of the above-given theorem is the appearance of indeterministic features. In [314] this theorem was applied to spatially homogeneous and isotropic cosmological models, whose dynamics is described by a planar autonomous system, given by

$$\begin{aligned} \dot{\rho} &= -\gamma\,\rho\,\theta + \alpha\,\theta^2 + \beta\,\theta^3, \\ \dot{\theta} &= -\frac{3\gamma - 2}{2}\,\rho + \frac{3\alpha}{2}\,\theta + \left(\frac{3\beta}{2} - \frac{1}{3}\right)\,\theta^2, \end{aligned} \tag{E.230}$$

where $\sigma$ (referred to in the theorem) can be either $\alpha$, $\beta$ or $\gamma$, and the energy-momentum tensor is

$$T_{\mu\nu} = (\rho + \tilde{p})\; v_\mu\, v_\nu - \tilde{p}\, g_{\mu\nu},$$

where

$$\tilde{p} = p_{th} + \alpha\,\theta + \beta\,\theta^2,$$

with $p_{\rm th} = (\gamma - 1)\,\rho$.

The viscous terms (parameterized by $\alpha$ and $\beta$) can be a phenomenological description of particle creation in a nonstationary gravitational field as proposed in [407] and [426]. Applying the methods of qualitative analysis to the system given in Eqn.(E.230) it was shown in [314] that for $\gamma - 3\beta < 0$, the Poincaré index $I_P(\mathcal{B}) = -1$ (saddle point); for $\gamma - 3\beta \geq 0$, $I_P(\mathcal{B}) = 1$ (two-tangent node). This situation characterizes a bifurcation in the singular point, when $\rho = \theta = \infty$. This bifurcation, caused by dissipative processes involving quadratic viscous terms generates a high degree of indeterminacy in the development of the solution of EE, which enshrouds the past of this model of the universe. In this case, nothing can be stated about the existence of the initial cosmological singularity.

## E.6. Bounces in the braneworld

Theoretical developments coming from string theory have revived the idea that our universe may have more than 4 dimensions (first considerated by Kaluza in the context of unification of gravity and electromagnetism). Among the multidimensional models, those with one or more branes that live in a bulk space have been thoroughly studied recently (see for instance [273]). In these models, the matter fields are typically confined to a 3-brane in $1+3+d$ dimensions, while the gravitational field can propagate also in the $d$ extra dimensions, which need not be small, or even finite, as shown in one of the models introduced by Randall and Sundrum [348], where for $d=1$, gravity can be localized on a single 3-brane even when the fifth dimension is infinite. The Friedmann equation on the brane is modified by high-energy matter terms and also by a term which incorporates the nonlocal effects of the bulk onto the brane [54, 273]:

$$H^2 = \frac{\Lambda}{3} + \frac{\kappa^2}{3}\rho - \frac{\epsilon}{a^2} + \frac{\overline{\kappa}^4}{36}\rho^2 + \frac{1}{3}\left(\frac{\overline{\kappa}}{\kappa}\right)^4 \mathcal{U}_0 \left(\frac{a_0}{a}\right)^4, \tag{E.231}$$

where $\epsilon$ is the 3-curvature, $H = \dot{a}/a$, $\rho$ is the energy density of the matter on the brane, $\overline{\kappa}^2 = 8\pi/\overline{M}^3_{Pl}$, $\overline{M}^3_{Pl}$ is the fundamental 5-dimensional Planck mass, $\kappa^2 = 8\pi/M^2_{Pl}$, and

$$\Lambda = \frac{4\pi}{\overline{M}^3_{Pl}}\left[\overline{\Lambda} + \left(\frac{4\pi}{3\overline{M}^3_{Pl}}\right)\lambda^2\right],$$

where $\lambda$ is the tension of the brane, and $\overline{\Lambda}$ is the 5-dimensional cosmological constant. Finally, $\mathcal{U}_0$ is the constant corresponding to the non-local energy conservation equation. This term comes from the projection of the Weyl tensor of the bulk on the brane [273]. From Eqn.(E.231) we see that a necessary condition to have a bounce with $\rho > 0$ in the $\epsilon = 0, -1$ cases is that either $\Lambda < 0$ or $\mathcal{U} < 0$, or both.

The case that includes matter in the bulk, without cosmological constant for a flat FLRW $d+1$-dimensional was studied in [165]. A neceessary condition in order to have a bounce is that $dH/dt > 0$, with

$$\frac{dH}{dt} = \frac{\kappa^2}{d}(R+P) - \left(\frac{8\pi G_N}{d-1} + \frac{\kappa^4}{4d}\rho\right)(\rho + p) - \frac{d+1}{d(d-1)}E^0_0, \tag{E.232}$$

where (in a notation slightly different from that used in Eqn.(E.231)) $\kappa$ is the bulk gravitational coupling, $G_N$ the effective Newton constant on the $(d+1)$-dimensional brane, $E$ is the projection of the bulk brane Weyl tensor on the brane, and $T^\mu_\nu = (-R, \vec{P})$ is the projection of the bulk energy-momentum tensor on the brane. It follows from this equation that a necessary condition to have a bounce without resorting to exotic forms of matter (that is, matter that violates $\rho > 0$ or $\rho_p > 0$) is a negative $E^0_0$ [165]. This is precisely the approach taken in [233, 295], where a brane evolving in a charged AdS black hole background

was studied. Bouncing solutions were found for both critical ($\Lambda = 0$) and non-critical ($\Lambda \neq 0$) branes, the bounce generically depending on the parameters of the black hole, and on the matter content of the brane [58].

The abovementioned necessary condition was explicitly checked in the case of the dilaton-gravity braneworld [165], and bouncing solutions were obtained for a a flat FLRW brane in a static spherically symmetric bulk [59]. This solution describes (in the string frame) a pre-big bang model where the transition between the branches is realized at low curvature and weak coupling, thus providing an example of succesful graceful exit without resorting to quantum or "stringy" corrections.

Notice that the extra dimension(s) could be spacelike or timelike. The latter case was analyzed in [375]. The usual incantations [273] for the case of an extra timelike dimension and an homogeneous and isotropic brane lead to [375]

$$H^2 + \frac{\epsilon}{a^2} = \frac{\Lambda}{3} + \frac{8\pi G\rho}{3} - \frac{\rho^2}{\bar{M}_{\rm Pl}^6} + \frac{C}{a^4}, \tag{E.233}$$

where $G$ and $\Lambda$ are the effective gravitational and cosmological constant, respectively, and $M$ is the 5-dimensional Planck mass. Notice that the minus sign in front of $\rho^2$ may lead to a bounce instead of a singularity, since this term grows faster than the others, leading to $H = 0$, this feature being independent of the equation of state and also of the spatial curvature of the universe. The simplest of these bouncing universes, described by

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\rho^2}{\bar{M}_{\rm Pl}^6} \tag{E.234}$$

will be discussed in Sec.E.10.2, since it may lead to a cyclic universe.

The case with an extra timelike dimension in this scenario was also extended to Bianchi I universes [375], which exhibit an anisotropic bounce as long as the shear scalar $\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ does not grow faster than $a^{-8}$ as $a$ goes to zero at the end of the contraction phase. All these results were obtained by neglecting the induced curvature on the brane, which can trigger the formation of a singularity at the beginning or at the end of the evolution [375].

Another model along these lines was introduced in [14], where a "test brane" (*i.e.* one that does not modify the ambient geometry) moves in a higher-dimensional gravitational background. Using the thin-shell formalism, in which the field equations are re-written as junction conditions relating the discontinuity in the brane extrinsic curvature to its vacuum energy, the motion of domain walls in de Sitter and anti-de Sitter (AdS) time-dependent bulks was discussed. This motion induces a dynamical law for the brane scale factor, and it was shown

[58] The bounce in the model presented in [295] was analyzed from the point of view of the causal entropy bound in [285], and its stability was put in doubt in [212].

[59] Bouncing solutions for a domain wall in the presence of a Liouville potential were found in [103].

in [14] that in the case of a clean brane the scale factor may describe a non-singular universe. In order to build the class of geometries of interest, two copies of (d+1)-dimensional dS (AdS) spaces $M1$ and $M2$ undergoing expansion were considered. From each of them, one identical $d$-dimensional region $\Omega_i$ $(i = 1, 2)$ was removed. yielding two geodesically incomplete manifolds with boundaries given by the hypersurfaces $\partial\Omega_1$ and $\partial\Omega_2$. Finally, the boundaries were identified up to an homeomorphism $h : \partial\Omega_1 \to \partial\Omega_2$. Hence, the resulting manifold that is defined by the connected sum $M1\#M2$ is geodesically complete. The starting point is the action

$$S = \frac{\ell_{\rm Pl}^{(3-d)}}{16\pi}\int_M d^{d+1}x\,\sqrt{g}\,(R-2\Lambda) + \frac{\ell_{\rm Pl}^{(3-d)}}{8\pi}\int_{\partial\Omega} d^dx\,\sqrt{\gamma}\,\mathcal{K} + \sigma\int_{\partial\Omega} d^dx\,\sqrt{\gamma},$$

where the first term is the usual Einstein-Hilbert action with a cosmological constant $\Lambda$, the second term is the Gibbons-Hawking boundary term, $\mathcal{K}_{MN}$ is the extrinsic curvature, and $\sigma$ is the intrinsic tension of the $d$-dimensional brane. The spatial coordinates on $\partial\Omega$ can be taken to be the angular variables $\phi_i$, which for a spherically symmetric configuration are always well defined up to an overall rotation. Generically, the line element of each patch can be written as

$$ds^2 = -dt^2 + A^2(t)[r^2 d\Omega^2_{(d-1)} + (1-kr^2)^{-1}dr^2],$$

where $\epsilon$ takes the values 1 (-1) for dS (AdS), $\Omega^2_{(d-1)}$ is the corresponding metric on the unit $d-1$-dimensional sphere, and $t$ is the proper time of a clock measured in the higher-dimensional spacetime. In order to analyze the dynamics of the system, the brane is allowed to move radially. Let the position of the brane be described by $x_\mu(\tau, \phi_i) \equiv (t(\tau), a(\tau), \phi_i)$, with $\tau$ the proper time (as measured by co-moving observers on the brane) that parameterizes the motion, and the velocity of a piece of stress-energy at the brane satisfying $u^M u_M = -1$. With these assumptions the brane will have an effective scale factor $\mathcal{A}^2(t) = a^2(t)A^2(t)$. The constraint

$$\frac{d\tau}{dt} = \pm\sqrt{1 - \frac{(A\dot{a})^2}{1-\epsilon a^2}}$$

along with the result of the integration of EE across the boundary (done with the junction conditions) [14] yields two differential equations for $A$ and $a$. For the case of a background composed by two patches of dS undergoing expansion, $A(t) = \ell\cosh(t/\ell)$, and $\epsilon = 1$, where $\ell^2 = d(d-1)/|\Lambda|$ is the dS radius. In this case the EOM for the brane is

$$\frac{4\pi}{L_p^{(3-d)}(d-1)}\,\sigma = \frac{\pm\dot{a}\sinh(t/\ell) + [a\ell\cosh(t/\ell)]^{-1}(1-a^2)}{(1-a^2-[\ell\dot{a}\cosh(t/\ell)]^2)^{1/2}}.$$

Nonsingular analytical solutions of this equation for $\sigma = 0$ can be obtained, while for $\sigma \neq 0$, numerical methods must be used. This latter case also yields bouncing solutions (see Fig.E.6) [60].

[60] Another model along this lines can be found in [374].

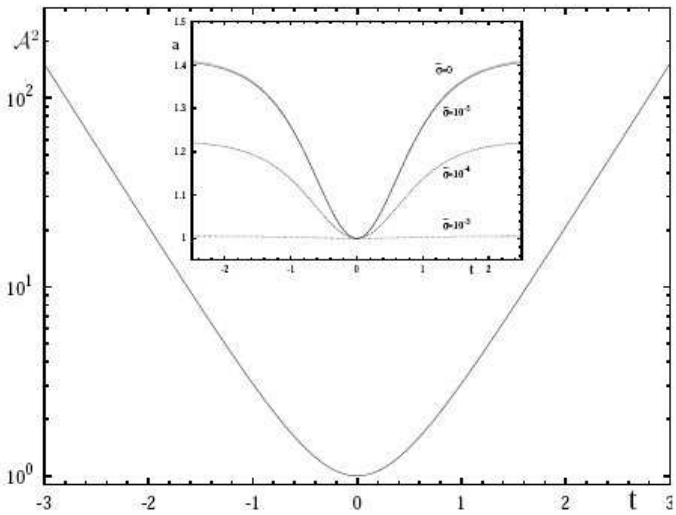


Figure E.9.: Effective scale factor $\mathcal{A}(t)$ plotted for $\bar{\sigma} = 4\pi\sigma/(L_p^{(3-d)}(d-1)) = 10^{-4}$. The insert displays the dependence of $\mathcal{A}$ with $\bar{\sigma}$. Taken from [14].

The motion of a test brane in a background produced by a collection of branes was discussed in [237] (the so-called mirage cosmology). Adopting spherically-symmetric backgrounds, it was shown that although there is a singularity in the evolution of the 4-d brane, the higher-dimensional geometry is regular. The origin of the singularity on the brane is actually the embedding of the brane in the bulk, in such a way that the singularity is smoothed out when the solution is lifted to higher dimensions.

The effect of inflation on a bouncing brane was used in [20] to set limits on the parameters of the braneworld. Specifically, the model consists of a closed FLRW metric embedded in a 5-d conformally flat bulk with one extra timelike dimension, containing a conformally coupled scalar field (the inflaton field) and a radiation fluid, evolving on the brane with corrections due to the bulk. The non-singular bouncing solutions considered were oscillatory and bounded, or initially bounded. They are in principle stable and would never enter an inflationary phase with an exponential growth of the scale factor since they correspond to periodic orbits of the integrable dynamics in the gravitational sector. The introduction of a massive scalar field, even in the form of small fluctuations, turns non-integrable the dynamics of the system [20]. As a consequence, non-linear resonance phenomena are present in the phase space dynamics for certain domains of the parameter space of the models, and the associated dynamical configurations become metastable, allowing the orbits escape to the de Sitter infinity in a finite time. From the conditions for these orbits to happen, limits on the parameters $(\sigma, m, E_0)$ are set, where $\sigma$ is the brane tension, $m$ is the mass

of the scalar field, and $E_0$ is a constant proportional to the total energy of the fluid.

Yet another turn in the mirage model was introduced in [180], where the brane moves in an open orbit around a non-trivial spherically-symmetric background. In this model, the brane is moving on a Calabi-Yau manifold generated by a heap of D3-branes, and the mirage effects dominate the evolution of the Universe only at early time, i.e. when the brane moves in the throat of the background manifold. The new feature is the influence of the angular momentum of the test brane on its motion in the higher-dimensional space. In fact, the effective 4-d metric has two parameters: the energy $U$ and the angular momentum $L$ of the 4-d brane, which determine the form of the orbit. In particular, to have an open orbit in an asymptotically Minkowskian background,

$$L^4 - 4(U+2)U^3 \geq 0.$$

As discussed in [180], the effective metric corresponding to orbits satisfying this constraint display cosmological contraction during the ingoing part of the orbit, expansion during the outgoing part, and a bounce at the turning point [61].

Another model based on the brane scenario is the ekpyrotic universe [241], the cyclic version of which shall be considered in Sect.E.10 [62].

## E.7. Variable cosmological constant

General Relativity allows for the introduction of only one arbitrary constant, the so-called cosmological constant $\Lambda$. At least two attitudes can be taken regarding $\Lambda$ [334]. The first one is to consider it as a derived quantity, that emerges from vacuum fluctuations (see for instance [425]). One way out of the huge disagreement between theory and observation in this case [92] is to assume that $\Lambda$ is actually time-dependent. The second attitude that can be adopted is that $\Lambda$ is, along with $G$, a fundamental parameter of the theory, to be determined by observation [63] [324]. In fact, from a gravitational point of view what matters is the "effective" cosmological constant, since the matter Lagrangian can sometimes contribute with a $\Lambda$-like term, as in the case of the scalar field in the presence of a potential with a minimum:

$$\Lambda_{\text{eff}} = \Lambda + V(\phi_{\text{min}}),$$

where $\Lambda$ is the "bare" cosmological constant. Any change in $\phi_{\text{min}}$ during the evolution leads to changes in the value of $\Lambda_{\text{eff}}$. In fact, the effect of the evolution of the universe on the ground state is to add a temperature dependence,

[61] Further effects of the angular momentum on the motion of the brane, including cyclic universes, were studied in [143].

[62] See [114] for an additional bouncing model using orientifolds.

[63] Notice that this second attitude is somewhat different from Einstein's original ideas leading to GR, since there would be curvature even in the absence of matter, caused by $\Lambda$.

which can be translated into a time dependence [244]. A model along these lines based on a gauge field (instead of a scalar field) was presented in [323] [64]. This is another motivation to consider a variable $\Lambda$, that is not a constant but a function of spacetime coordinates, in such a way that its value is determined by the dynamics of the theory under scrutiny (following the line of reasoning of other "variable constant" theories, see Section E.10.). In fact, a time-dependent cosmological constant has also been called upon to explain the current accelerated expansion and the fact that this phase started in the recent past.

In the case of $\Lambda = \Lambda(t)$, EE for the FLRW metric take the form

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3}\rho + \frac{\Lambda(t)}{3} - \frac{\epsilon}{a^2}, \tag{E.235}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda(t)}{3} - \frac{1}{6}(\rho + 3p), \tag{E.236}$$

and the continuity equation is given by

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = -\dot{\Lambda}. \tag{E.237}$$

As seen from Eqn.(E.237), $\Lambda$ can supply or absorb energy from ordinary matter and radiation. In fact, it follows from this equation that

$$TdS = -Vd\Lambda. \tag{E.238}$$

Hence, $\Lambda$ is a source of entropy. Requiring that $dS/dt > 0$ implies $d\Lambda/da < 0$ through cosmic expansion.

Assuming that only radiation is present, Eqn.(E.237) gives

$$\frac{d\rho}{da} + \frac{d\Lambda}{da} + \frac{4\rho}{a} = 0,$$

which can be integrated to

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^4 - \frac{1}{a^4}\int_{a_0}^{a} A^4 \frac{d\Lambda}{dA} dA, \tag{E.239}$$

where $\rho = \rho_0$ when $a = a_0$, and the subindex 0 denotes quantities evaluated at $t = 0$. Notice that the model is completely determined in this case by providing the function $\Lambda = \Lambda(a)$, since Eqn.(E.239) then yields $\rho = \rho(a)$, and $a = a(t)$ follows from Eqn.(E.235). A cosmological model based on this scenario was discussed in [332], where the dependence of $\Lambda$ on $a$ was fixed by imposing that $\rho = \rho_c$ for all values of $t$, where $\rho_c = 3H^2$ is the critical density. It follows from Eqn.(E.235) that

$$\Lambda = \frac{\alpha\epsilon}{a^2}. \tag{E.240}$$

[64] In fact, any classical nonlinear field theory (such as nonlinear electromagnetism) admits a fundamental state that generates a cosmological constant [325].

The conditions $\dot{\Lambda} \geq 0$ and $\dot{a} \geq 0$ give $\epsilon > 0$, hence $\epsilon = 1$. In the model presented in [332], at $t = 0$ the universe had only a nonzero cosmological constant. With $\rho_0 = 0$, Eqn.(E.239,E.240) give

$$\rho(a) = \frac{\alpha}{a^2}\left(1 - \frac{a_0^2}{a^2}\right). \tag{E.241}$$

Note that $\rho_0 = 0$ implies that $a_0 \neq 0$, in such a way that the singularity at $t = 0$ is absent. An estimation of $a_0$ was made in [332] by assuming that the maximum temperature reached is $T_{max} \sim M_{Pl}$, which gives

$$a_0 \sim \frac{2.5}{\sqrt{N}} \times 10^{-20}(\mathrm{GeV})^{-1},$$

where $N = N(T)$ is the effective number of degrees of freedom at temperature $T$.

The fact that this model does not display a horizon problem was also shown in [332]. In fact, the time $t_c$ at which global causality is established is given by

$$t_c = a_0 \sinh\frac{\pi}{2} \sim 2.3a_0,$$

which indicates that global causal connection was established at a very early time. The model is also free of the monopole problem, but it is worth noting that there is an inflationary period. From Eqn.(E.235) we get

$$a^2 = a_0^2 + t^2. \tag{E.242}$$

A peculiarity of this model is that $a \to \infty$ for $t \to \infty$, even though $\epsilon = 1$. Needless to say, other choices of $\Lambda$ would give a different asymptotic behavior.

The same form of $\Lambda$, namely

$$\Lambda(t) = \frac{\gamma}{a(t)^2}, \tag{E.243}$$

where $\gamma$ is a constant to be determined by observations, was studied in [415], but without the assumption that $\rho = \rho_c$. The conservation equation (E.237) can be solved for dust and radiation. Inserting the solution in Eqns.(E.235) and (E.236) we get

$$\frac{\dot{a}^2}{a^2} + \frac{\Upsilon}{a^2} = \frac{1}{3}\rho^{(i)},$$
$$\frac{\ddot{a}}{a} = -\frac{1}{6}\rho^{(i)},$$

where $\Upsilon = \epsilon - 2\gamma/3$ for radiation, and $\Upsilon = \epsilon - \gamma$ for dust, and $\rho^{(i)}$ is the energy density of dust or radiation for the case $\Lambda = 0$. These equations show that the effect of assuming that $\Lambda \propto a^{-2}$ is to shift the curvature parameter $\epsilon$ by a constant value. A nonsingular cosmological model based on the model presented

in [415] has been analyzed in [2]. Notice that Eqn.(E.243) along with condition $d\Lambda/da < 0$ require that $\gamma$ be positive. A positive $\Lambda$ for all $t$ implies, through Eqn.(E.236) that there may be a zero in $\dot{a}$, and hence the possibility of a bounce. For this to happen we need that $\dot{a}$ be zero at the putative bounce. Supposing there is a bounce, it follows from Eqn.(E.235) evaluated at the bounce that

$$\alpha^{-1}\rho_0 a_0^2 = \epsilon - \gamma.$$

Hence, $\rho_0 > 0$ implies that $\epsilon > \gamma > 0$, and so $\epsilon = 1$. Introducing the *Ansatz* (E.243) in the Friedmann equation, we get

$$a^2\dot{a}^2 = (2\gamma - 1)(a^2 - a_0^2), \tag{E.244}$$

so it follows that $\gamma > 1/2$. Hence, $1/2 < \gamma \leq 1$. This equation can be integrated to get

$$a^2 = (2\gamma - 1)t^2 + a_0^2,$$

which leads to bounded-from-above densities and temperature [65].

Yet another form for the dependence of $\Lambda$, given by

$$\Lambda = \Lambda_1 + \Lambda_2\, a^{-m},$$

where $\Lambda_1$, $\Lambda_2$ and $m$ are constants (with $\Lambda_2 > 0$), was studied in [283]. The analysis of the dynamics was carried out using the analog of the one-dimensional problem of the particle under the influence of the potential $V(a)$ given by

$$V(a) = -\Lambda_1\delta\frac{a^2}{\alpha+2} - \Lambda_2\delta\frac{a^{2-m}}{\alpha-m+2} + ba^{-\alpha},$$

where $\alpha = 1 + 3\lambda$, $\delta = 1 + \lambda$, $b$ is a positive integration constant, and $p = \lambda\rho$. Denoting by $r$ the maximum of the potential, cyclic solutions are obtained for the cases $\epsilon = 1$ with $\Lambda_1$, $\Lambda_2 > 0$, and $r > -1$, and for $\Lambda_1 < 0$, $\Lambda_2 > 0$, and $m \leq 2$, regardless of the sign of $\epsilon$.

The proposal in Eqn.(E.243) was later generalized in [21] to

$$\Lambda = 3\beta H^2 + \frac{3\gamma}{a^2}, \tag{E.245}$$

where $\beta$ and $\gamma$ are dimensionless numbers, and $H = \dot{a}/a$. following [101]. With this *Ansatz*, the Friedmann equation for a radiation-dominated phase can be rewritten as

$$\dot{a}^2 = \frac{2\gamma - \epsilon}{1 - 2\beta} + A_0 a^{-2+4\beta}, \tag{E.246}$$

which allows a bouncing solution at $t = 0$ for $A_0 < 0$, $\beta < 1/2$, $\epsilon = 1$ (with $\rho_0 > 0$). The value $\gamma = 1$ was chosen in [21] so that $dS/da$ is always greater

[65] The evolution of perturbations in this model was studied in [1].

than zero, thus solving the entropy problem. In this case, the model gives $\Omega < 1$ for all $t$.

A thorough review of variable-$\Lambda$ models has been presented in [331]. The models analyzed were power-laws of the different relevant parameters, namely

$$\Lambda_1 = \mathcal{A}t^{-\ell}, \quad \Lambda_2 = \mathcal{B}a^{-m}, \quad \Lambda_3 = \mathcal{C}H^n, \quad \Lambda_4 = \mathcal{D}q^r,$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\ell$, $m$, $n$, and $r$ are constants. Let us state from [331] the relevant results for this review: (1) no bouncing models were found for $\Lambda_1$ with $k = 0$ and $\ell = 1, 2, 3, 4$, irrespectively of the sign of $\mathcal{A}$. (2) For $\Lambda_2$, it was shown (numerically) that there are nonsingular models for dust, $\epsilon = 1$, with $m = 1$, $\Omega_0 = 0.34$, and $0.68 < \Omega_{0\Lambda} < 0.72$, and also with higher values of $m$ and $\Omega_0$. (3) For $\Lambda_3$, the value $n = 2$ admit analytical solution. For this $n$, there are bouncing solutions for $\gamma > 2/3$ and $\epsilon = 1$ with $\mathcal{C} > 3(3\gamma/2 - 1)\Omega_0$, and also for $\gamma > 2/3$ and $\epsilon = -1$, for $\mathcal{C} < 3(3\gamma/2 - 1)\Omega_0$. (4) Only the value $r = 1$ was explored for $\Lambda_4$. Defining $\lambda_0 = -\mathcal{D}q_0/3$, there are closed bouncing solutions for $\lambda_0 > -\Omega_0$, and open bouncing solutions for $\lambda_0 < -\Omega_0$.

The examples given above show that varying-$\Lambda$ scenarios are worth examining because they address a number of pressing problems in cosmology (horizon problem, entropy, initial singularity)[66]. Furthermore, many of them are simple enough to draw definite conclusions about their viability. One of the drawbacks is perhaps the lack of strong motivation for choosing any given form of $\Lambda$. In this regard, let us remember that many of the varying-$\Lambda$ models can be reverse-engineered to scalar-field models with a potential. Unfortunately, in most cases the corresponding models lack predictive power or clear particle physics motivation [334].

## E.8. Past-eternal universes

In this section, we shall examine some models which are nonsingular but do not exhibit a bounce. Historically, perhaps the most important example of these is the Steady-State model [68] [67]. As mentioned in Sect.E.2.1, nonsingular solutions that start from a deSitter state were discussed in [385, 164]. Another example is that discused in [292] in which every contracting and spatially flat, isotropic universe avoids the big crunch by ending up in a deSitter state enforced by the limiting curvature hypothesis.

### E.8.1. Variable cosmological constant

As noted in [260], in all the articles mentioned in Sect.E.7, the dependence of $\Lambda$ on $a$ and $\dot{a}$ was set either from "first principles" (for instance quantum gravity,

[66] Nonsingular cosmological solutions for the case in which the cosmological constant is replaced by a second-rank tensor $\Lambda^{\mu}_{\nu}$ were studied in [81].

[67] For an updated version, see Sect.E.10.2.

as in [415]), or by extrapolating backwards current cosmological data, including the current value of $\Lambda$. However, another view can be taken. Since $\Lambda$ can be considered as a remnant of a period of inflation, a complete model should also describe the era of inflationary expansion. This is precisely the proposal in [260], where $\Lambda$ was taken as

$$\Lambda(H) = 3\beta H^2 + 3(1-\beta)\frac{H^3}{H_\ell}, \tag{E.247}$$

where $H_\ell$ is the timescale of inflation, and $\beta$ is a parameter. Note that when $H = H_\ell$, $\Lambda = 3{H_\ell}^2$, as required by inflation, while $\Lambda \sim 3\beta H^2$ for large cosmological times. In the case of $\epsilon = 0$, and for

$$p = (\gamma - 1)\rho,$$

an equation for the Hubble parameter follows [260]:

$$\dot{H} + \frac{3\gamma(1-\beta)}{2}\, H^2 \left(1 - \frac{H}{H_\ell}\right) = 0,$$

whose solution is

$$H = \frac{H_\ell}{1 + Ca^{3\gamma(1-\beta)/2}},$$

where $C$ is a $\gamma$-dependent integration constant [68]. This equation can be integrated to yield

$$H_\ell t = \ln\left(\frac{a}{a_*}\right) + \frac{2C}{3\gamma(1-\beta)}\, a^{3\gamma(1-\beta)/2},$$

where $a_*$ is an arbitrary value of the scale factor. It follows from this equation that the evolution of the universe starts from a deSitter stage $a \sim e^{H_\ell t}$ for $Ca^{3\gamma(1-\beta)/2} << 1$, and evolves towards a FLRW phase, $a \sim t^{2/3\gamma(1-\beta)}$ for $Ca^{3\gamma(1-\beta)/2} >> 1$.

## E.8.2. Fundamental state for $f(R)$ theories

A novelty in some theories described by Lagrangians that depend only on $R$ is the possibility of the emergence of an intrinsic cosmological constant. This is not the case, however, in theories generated by Lagrangians that are a linear combination of $R^2$ and $R_{\mu\nu}R^{\mu\nu}$ as can be seen by a direct inspection of the EOM (E.35). The proof of this assertion follows from the fact that the tensors $\chi_{\mu\nu}$ and $Z_{\mu\nu}$ appearing in the EOM (E.27) are traceless in the case of a constant curvature scalar ($R_{\mu\nu} = \Lambda\, g_{\mu\nu}$). However, restricting to the $f(R)$ case, Lagrangians that are not linear in $R^2$ can bypass such prohibition. The existence of a deSitter solution in the absence of matter occurs when the function obeys the condition

$$\frac{f'}{f} = \text{constant}. \tag{E.248}$$

[68] Here the value $\epsilon = 0$ was chosen, but this restriction was lifted in [261].

A typical example is provided by the exponential Lagrangian

$$f(R) = \exp\left(\frac{R}{2\Lambda}\right).$$

It follows straightforwardly from Eqn.(E.36) that $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$ is a possible state of the system.

### E.8.3. The emergent universe

Another example of past eternal universe was given in [151]. This model uses general relativity plus a scalar field with a potential, and matter. The relevant equations are

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0,$$

$$\frac{\ddot{a}}{a} = -\left[\frac{1}{2}(1+3\omega)\rho + \dot{\phi}^2 - V(\phi)\right],$$

$$H^2 = \rho + \frac{1}{2}\dot{\phi} + V(\phi) - \frac{\epsilon}{a^2}.$$

From these, it follows that to have a minimum of the scale factor we need to impose the conditions

$$\frac{1}{2}(1+3\omega)\rho + \dot{\phi}^2 < V(\phi),$$

and

$$\frac{1}{2}\dot{\phi}_i^2 + V_i + \rho_i = \frac{\epsilon}{a_i^2},$$

where the subindex $i$ means that the quantities are evaluated at $t_i$, the time at which $a$ is minimum. Assuming positive potentials and energy density, it follows that only $\epsilon = +1$ is allowed. It follows that

$$\frac{1}{2}(1-\omega_i)\rho_i + V_i = \frac{2}{a_i^2},$$

where $V_i = \Lambda_i$, and

$$(1+\omega_i)\rho_i + \dot{\phi}^2 = \frac{2}{a_i^2},$$

so a model can be constructed with $\rho_i = 0$ and constant $\dot{\phi}^2$. This can be achieved in the limit $t \to \infty$ with the potential [151]

$$V(\phi) = V_f + (V_i - V_f)\left[\exp\left(\frac{\phi - \phi_f}{\alpha}\right) - 1\right]^2,$$

where $\phi_f$ is the value of the field for which $V$ is minimum, and $\alpha$ is a constant energy scale. In order to achieve the Einstein universe state in the far past, some

fine-tunning on $a_i$ and $\dot{\phi}_i$ is needed, which is not necessarily a hindrance [151][69]. The choice of such a highly-symmetric state as the initial state is supported by various arguments: it is stable against some types of inhomogeneous linear perturbations [70], it has no horizon problem, it maximizes the entropy within the family of FLRW radiation models, and it is the unique highest symmetry non-empty FLRW model (with a 7 dimensional group of isometries). The model was elaborated further in [153], where it was shown that an explicit form for the potential can be found such that the model leaves the inflationary stage and enters a reheating phase, followed by standard evolution.

## E.9. Quantum Cosmology

As discussed in Sect.E.1, there are reasons to suppose that at very high energies some of the hypotheses of the singularity theorems are rendered invalid: if the universe ever attains this regime, an important role is to be played by quantum gravitational effects, in such a way that a quantum theory of gravitation is needed to have a proper description.

Although there is yet no complete realization of quantum gravity, there are some attempts to tackle the singularity problem in a quantum framework. A standard method of quantizing General Relativity is canonical quantization [199] where the momentum and Hamiltonian constraint equations are interpreted as operators, and it is required that they annihilate the quantum state. The Hamiltonian constraint gives the Wheeler-DeWitt (WdW) equation [417], which depends on the choice of the factor ordering in the products of generalized momenta and "velocities". For some choices of the ordering, the WdW equation turns it into a Klein- Gordon equation on an indefinite DeWitt metric in the infinite-dimensional superspace (space of three-metrics), with a potential term [417]. In addition to the WdW equation, initial conditions must be specified, the two most popular being the "no-boundary" [203], and the "tunnelling" condition [410].

In practice, the infinite degrees of freedom of the superspace are truncated to obtain a minisuperspace model, usually under the assumptions of isotropy and homogeneity. Once a solution to the WdW equation has been found, there is the question of how to interpret it and extract probabilities from it.

Among other issues related to the WdW equation, there is the fact that a suitable initial condition must be chosen to get a solution. It would be desirable that the initial condition be somehow determined by the dynamical law (see for instance [62]). In fact, the most well-accepted proposals mentioned above do not solve the singularity problem [23]. Moreover, in the quantization following the

[69] In particular, the initial scale factor could be chosen in such a way to avoid the quantum gravity regime.

[70] But notice that it is not stable under homogeneous perturbations.

ADM procedure, time is fixed by a gauge choice, and the results are dependent of this choice [340] [71].

As we shall see below, there are other approaches to Quantum Cosmology which may yield a nonsingular universe in the regime where the WdW equation is valid. We shall discuss two possibilities: the Bohm-de Broglie interpretation of QM, and Loop Quantum Cosmology (LQC).

## E.9.1. The ontological (Bohm-de Broglie) interpretation

If the universality of quantum mechanics is assumed, the Universe must be describable by a wave function (furnished by a yet-to-be-discovered quantum theory of gravity and matter fields) in every step of its evolution. Moreover, this description must have a well-defined classical limit. The orthodox interpretation of Quantum Mechanics (the so-called Copenhagen interpretation) [221] is ill-suited for the task of describing the universe, since it assumes the existence of a "classical apparatus" external to the system to solve the measure problem by forcing the collapse of the wave function. Clearly, there is no classical apparatus outside the universe. Therefore, the least we can say is that an alternative to the Copenhagen interpretation is needed. One such alternative that has received some attention recently is that of Bohm and de Broglie (BdB)[59] [72]. In classical physics, the dynamics of a point in configuration space is determined by the principle of extremal action, yielding the classical EOM. According to the BdB interpretation, in quantum physics the evolution of the configuration variables is guided by a quantum wave which obeys Schrödinger's equation. The associated Hamilton-Jacobi equation displays a new term (of quantum origin, see below), that can be interpreted as part of the potential. It should be emphasized that the BdB interpretation furnishes a framework to make predictions based on the wave function of the system, which must be obtained by some means (for instance, through the WdW equation).

Let us briefly review first the quantum mechanics of a single particle in the BdB interpretation, and afterwards the results will be translated, *mutatis mutandis*, to the context of FLRW cosmology. The Schrödinger equation for a non-relativistic particle in a potential $V$ is given by

$$i\hbar\,\frac{d\psi(x,t)}{dt} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)\psi(x,t).$$

With the replacement $\psi = R\exp(iS/\hbar)$, this equation becomes

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0, \tag{E.249}$$

[71] In this regard, it was shown in [126] that a Bianchi I universe, quantized following the ADM recipe with a particular choice of the time coordinate [269] in the presence of dust is nonsingular.

[72] Other possibilities (not free of problems, though) are the many-worlds interpretation [156], non-linear quantum mechanics [182], and decoherence [191].

$$\frac{\partial R}{\partial t} + \nabla . \left( R^2 \frac{\nabla S}{m} \right) = 0. \tag{E.250}$$

This last equation suggests that $\nabla S/m$ can be interpreted as a velocity field, leading to the identification $p = \nabla S$, in such a way that Eqn.(E.249) is the Hamilton-Jacobi equation for the particle in the classical potential $V$ plus a "quantum potential" $Q = -\hbar^2 \nabla^2 R/2mR$. The BdB interpretation argues that a quantum system is composed of a particle *and* a field, and that quantum particles follow trajectories $x(t)$, independent on the existence of an ouside observer. These trajectories can be determined from

$$m \frac{d^2 x}{dt^2} = -\nabla V - \nabla Q,$$

or from $p = m\dot{x} = \nabla S$, after $S$ and $R$ are determined using Eqns.(E.249) and (E.250). In practice, since $S$ is the phase of the wave function, it can be read off from the explicit solution of Schrödinger's equation.

Let us analyze an example developed in [110], where the Lagrangian was given by

$$\mathcal{L} = \sqrt{-g} \ \left( R - C_\omega \phi_{,\mu} \phi^{,\mu} \right),$$

where $C_\omega = (\omega + \frac{3}{2})$. From the metric

$$ds^2 = -N^3 dt^2 + \frac{a(t)^2}{1 + (\epsilon/4) r^2} \ \left( dr^2 + r^2 d\Omega^2 \right),$$

and the definitions $\beta^2 = 4\pi \ell_{Pl}^2 / 3V$, $\bar{\phi} = \phi \sqrt{C_\omega / 6}$, we get

$$\mathcal{H} = N \left( -\beta^2 \frac{p_a^2}{2a} + \beta^2 \frac{p_{\bar{\phi}}^2}{2a^3} - \epsilon \frac{a}{2\beta^2} \right),$$

with $p_a = -a\dot{a}/(\beta^2 N)$, $p_{\bar{\phi}} = a^3 \dot{\bar{\phi}}/(\beta^2 N)$. Defining $\tilde{a} = a/\beta$, setting $\beta = 1$ and $\alpha \equiv \ln \tilde{a}$, we get

$$\mathcal{H} = \frac{N}{2 \exp(3\alpha)} \left( -p_\alpha^2 + p_\phi^2 - \epsilon \exp(4\alpha) \right), \tag{E.251}$$

where

$$p_\alpha = -\frac{\dot{\alpha} e^{3\alpha}}{N}, \qquad p_\phi = \frac{\dot{\phi} e^{3\alpha}}{N}.$$

Notice that $p_\phi = \bar{k}$ is a constant of the motion. We shall restrict to the case $\epsilon = 0$ since it is analytically tractable. The classical solutions are given by

$$a = 3\bar{k} t^{1/3}, \qquad \phi = \frac{1}{3} \ln t + c_2,$$

where $c_2$ is an integration constant. Depending on the sign of $\bar{k}$, this solution contracts to or expands from a singularity.

The Wheeler-DeWitt equation corresponding to the Hamiltonian given in Eqn.(E.251) is given by [110]

$$-\frac{\partial^2\Psi}{\partial\alpha^2}+\frac{\partial^2\Psi}{\partial\phi^2}+\epsilon\, e^{4\alpha}\Psi=0.$$

The solution, obtained by separation of variables, reads

$$\Psi(\alpha,\phi)=\int F(\kappa)A_\kappa(\alpha)B_\kappa(\phi)d\kappa,$$

where $\kappa$ is a separation constant, $F(\kappa)$ is an arbitrary function of $\kappa$,

$$A_\kappa(\alpha)=a_1\exp(i\kappa\alpha)+a_2\exp(-i\kappa\alpha),$$

(for $\epsilon=0$), and

$$B_\kappa(\phi)=b_1\exp(i\kappa\phi)+b_2\exp(-i\kappa\phi).$$

A direct application of the formalism sketched for the case of a one-particle system, generalized to several degrees of freedom yields from the Hamiltonian (E.251) [110]

$$Q(\alpha,\phi)=\frac{e^{3\alpha}}{2R}\left(\frac{\partial^2R}{\partial\alpha^2}-\frac{\partial^2R}{\partial\phi^2}\right),$$

with the "guidance relations"

$$\frac{\partial S}{\partial\alpha}=-\frac{e^{3\alpha}\dot\alpha}{N},\qquad \frac{\partial S}{\partial\phi}=\frac{e^{3\alpha}\dot\phi}{N}.$$

A state is now needed to read off from it $S$ and $R$. A Gaussian superposition was chosen in [110], given by

$$\Psi(\alpha,\phi)=\int F_\kappa B_\kappa(\phi)[A_\kappa(\alpha)+A_{-\kappa}(\alpha)]\,d\kappa,$$

with

$$F(\kappa)=\exp\left(-\frac{(\kappa-d)^2}{\sigma^2}\right).$$

and $a_2=b_2=0$. Performing the integration in $\kappa$, we can extract from the result the phase $S$ which, when inserted into the guidance relations (in the $N=1$ gauge) furnishes a planar system:

$$\dot\alpha=\frac{\phi\sigma^2\sin(2d\alpha)+2d\sinh(\sigma^2\alpha\phi)}{\exp 3\alpha\,(2(\cos(2d\alpha)+\cosh(\sigma^2\alpha\phi)))},\tag{E.252}$$

$$\dot\phi=\frac{-\alpha\sigma^2\sin(2d\alpha)+2d\cos(2d\alpha)+2d\cosh(\sigma^2\alpha\phi)}{\exp 3\alpha\,(2(\cos(2d\alpha)+\cosh(\sigma^2\alpha\phi)))}.\tag{E.253}$$

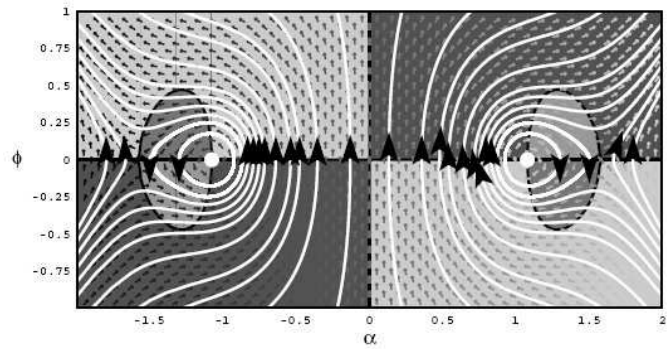


Figure E.10.: Field plot of the planar system (E.252)-(E.253) for $\sigma = d = 1$. Taken from [110].

The plot of this system (see Fig.E.10) shows that there are bouncing trajectories for $\alpha > 0$, and also oscillating universes near the centre points (white points in the plot). The BdB interpretation has been applied to mini-superspace models in Quantum Cosmology (see for instance [6, 340]), and non-singular solutions have been found for models with scalar fields or radiation [157]. The bounce is due to the action of the quantum potential, which generates a repulsive "quantum force", large enough to reverse the collapse.

One of the advantages of this formulation is that, starting from WdW equation, it yields a dynamics that is invariant under time re-parameterizations. Notice however that the results are dependent on the state chosen to represent the system.

## E.9.2. Loop Quantum Gravity

Loop Quantum Gravity is a background-independent, non-perturbative canonical quantization of gravity in which the classical metric and the extrinsic curvature are turned into operators on a Hilbert space [360]. The classical description of space-time is replaced by a quantum counterpart, in such a way that quantum effects are important at very short scales, for instance near putative singularities. In this scenario, the evolution of the universe is divided in three epochs. First there is a quantum epoch with high curvature and energy, described by difference equations for the wave function of the universe. These are a direct consequence of the discreteness of space and time, the step size being dictated by the lowest non-zero eigenvalue of the area operator (see [62]). It is this discreteness that modifies the behavior near the singularity, leading to a theory that is not equivalent to the WdW description (even in the isotropic case), which furnishes a continuous spectrum for the scale factor. A semiclassical epoch follows, with differential equations for matter and geometry modified by non-perturbative quantization effects. Finally, a classical phase is reached,

described by the usual cosmological equations.

Since difference equations are often difficult to analyze or to solve explicitly, and at such a fundamental level, the emergence of space-time in inhomogeneous models with many degrees of freedom from the underlying quantum state is hard to understand, a suitable strategy is to use special models allowing exact solutions. Care must be taken in the extension of results from particular examples to more general cases. In any case, it may be instructive to have a detailed understanding of how the singularity is resolved in some instances.

Yet another convenient simplification is to work in an effective semiclassical theory, which takes into account only some quantum effects. This theory can be understood as governing the motion of a wave packet that solves the difference equation [64], and can be obtained as an asymptotic series of correction terms to the equations of motion in the isotropic case [65]. For instance, in the case of a matter term generated by a scalar field under the influence of a potential, the effective Klein-Gordon equation is [378]

$$\ddot{\phi} = \dot{\phi}\left(-3H + \frac{\dot{D}}{D}\right) - DV'(\phi), \tag{E.254}$$

where

$$D(q) = \left(\frac{8}{77}\right)^6 q^{3/2}\{7[(q+1)^{11/4} - |q-1|^{11/4}] - 11q[(q+1)7/4\} - |q-1|^{7/4}\mathrm{sgn}(q-1)]\}^6,$$

with $q = a^2/a_*^2$ and $a_*^2 = \gamma\ell_{Pl}^2 j/3$, where $\gamma \approx 0.13$, and $j$ is a quantization parameter, which takes half-integer values. This equation represents an approximate expression for the eigenvalues of the inverse volume operator [61]. The function $D$ varies as $a^{15}$ for $a \ll a_*$, has a global maximum at $a \approx a_*$, and falls monotonically to $D = 1$ for $a > a_*$. In turn, the effective Friedmann equation is given by

$$\frac{\dot{a}^2}{a^2} + \frac{\epsilon}{a^2} = \frac{1}{3}\left(\frac{\dot{\phi}^2}{2D} + V(\phi)\right), \tag{E.255}$$

and the effective Raychaudhuri equation is

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\dot{\phi}^2\left(1 - \frac{\dot{D}}{4HD}\right) + \frac{1}{3}V(\phi). \tag{E.256}$$

These approximations are valid for $a_i < a < a_*$, where $a_i = \sqrt{\gamma}\,\ell_{Pl}$. Below $a_i$ the quantum nature of spacetime cannot be replaced by an effective theory, while above $a_*$ we recover classical cosmology. It was shown in [378] that a closed universe with a minimally coupled scalar field will bounce (avoiding the so-called big crunch) as soon as $a \approx a_*$ for any choice of the initial conditions. The bounce in this case is due to the change of sign of the "friction" term in Eqn.(E.254), which becomes frictional for $a << a_*$, freezing the field $\phi$ in some

constant value, and turning the effective EOS into a cosmological constant EOS [378]. Similar results were obtained in the case of anisotropic models [63].

The previous example incorporated quantum gravitational effects on the matter (represented by a scalar field) Hamiltonian, but there may also be modifications of the gravitational Hamiltonian due to quantum geometry. Recently, some calculations illustrating the effects of quantum geometry on both the gravitational and matter Hamiltonians were carried out in the case of a spatially homogeneous, isotropic $\epsilon = 0$ universe with a massless scalar field (a system which is singular both classically and according to the WdW formalism in the Copenhagen interpretation of QM). It was shown in [23] that the singularity is resolved in the sense that a complete set of Dirac observables on the physical Hilbert space remains well-defined throughout the evolution; the big-bang is replaced by a big-bounce in the quantum theory due to the quantum corrections to the geometry; there is a large classical universe on the "other side", and the evolution bridging the two classical branches is deterministic, thanks to the background independence and non-perturbative methods [73]. Notice also that no boundary condition was imposed (it was asked instead that the quantum state be semiclassical at late times) [74].

Surely the major limitation in all the analysis of LQC is that, since a satisfactory quantum gravity theory which can serve as an unambiguous starting point is not available yet, the theory is not developed by a systematic truncation of full quantum gravity. Another limitation is the restriction to isotropy and homogeneity.

## E.9.3. Stochastic approach

A different approach was introduced in [315], which starts from the observation made in [277] that the universe could be enlarged through an "analytic extension". In [277], such an extension is achieved from the geometrical construction of a semiclosed universe, namely a closed Friedmann model extended by gluing a given geometry to the FLRW before the maximum expansion. This gluing can be done in different ways, through the junction conditions. In [277] an asymptotically flat geometry was chosen. A collection of this configuration (called *friedmon* in [315]) was considered in [315], in such a way that each member of the collection perceives the remaining systems as a perturbative effect of random character, as in a stochastic process. Noting that in the case of an open universe, the Friedman equation takes the form of the energy conservation for a harmonic oscillator, namely

$$\dot{a}^2 + \frac{1}{3}\Lambda a^2 = 1,$$

[73] In a subsequent paper the Hamiltonian was modified to forbid the bounce at low densities [24].

[74] An analysis along the same lines was carried out in [405] for the case $\epsilon = -1$, and it was shown that the singularity is avoided too.

a Hamiltonian can be defined by setting $q = a$, $p = \dot{a}$, and the quantum theory of the harmonic oscillator can be developed according to [305]. A straightforward calculation leads to the result

$$E[a^2(t, W)] = a_{Cl}^2 + \frac{1}{2}\sqrt{3}\frac{\hbar}{\Lambda},$$

where $E$ is the expectation value, $a_{Cl}$ is the classical value of $a$, and $W$ is the white noise associated to the stochastic process. One arrives at the result that the net effect of the environment is to preclude the collapse of the model, the minimum of the radius being large if $\Lambda$ is small.

## E.10. Cyclic universes

Oscillating universes have been explored in several contexts in an attempt to solve some problems of the standard cosmological model. The first example of such universes was that presented in the seminal paper by Lemaître [254], who stated that "The solutions where the universe successively expands and contracts, periodically reducing to an atomic system with the dimensions of the solar system, have an incontestable poetic charm, and bring to mind the Phoenix of the legend" [254] [75]. Let us briefly recall some of the issues of the standard model and the solution that oscillating models can provide:

- The flatness problem. The Friedmann equation can be written as

  $$|\Omega_{\rm tot}(t) - 1| = \frac{|\epsilon|}{a^2 H^2},$$

  As already discussed in Section E.1, in a situation in which the universe is dominated by matter or radiation, the difference $|\Omega_{\rm tot}(t) - 1|$ grows as a power of $t$. Since present data indicate that $\Omega_{\rm tot}$ is very close to 1, it must have been incredibly close to one far in the past, if $\Omega_{\rm tot} \neq 1$ initially. This is the so-called flatness problem. As we shall see below, in a cyclic universe $\Omega_{\rm tot}$ starts deviating from 1 only when $a$ approaches its maximum. Since the maximum grows with the number of cycles, in a sufficiently old cyclic universe it may take a long time for $\Omega_{\rm tot}$ to deviate from 1 [138].

- The horizon problem. In the SCM, light signals can propagate only a finite distance between the initial singularity and a given time $t$, provided the energy density changes faster than $a^{-2}$. Hence, microphysics would not have enough time to take the universe to its high degree of homogeneity. In the cyclic model the age of the universe is given by the sum of the duration of all the previous cycles. This would solve the horizon problem, provided correlations safely traverse the bounce.

[75] Note however that Lemaître did not produce an explicit solution for the cyclic universe.

Some implementations of the cyclic model may also solve the so-called "coincidence problem" (why did the universe begin its accelerated expansion only recently?). The model in [420] has its parameters tuned in such a way that the fraction of time that the universe spends in the coincidence state is comparable to the period of the oscillating universe.

Oscillating models have been also used to explain the observed values of the dimensionless constants of nature. In [380], the value of these constants is randomly set after a bounce (see also [290]). In order to see whether cosmological evolution establishes any trend in the behaviour of the "constants", cyclic models were studied in [37] as solutions of varying-constants theories, such as the varying $\alpha$ theory presented in [367], the Brans-Dicke theory, and the variable-speed-of-light theory [275] [76]. The cyclic solutions were studied both for non-interacting and interacting scalar field (which models evolution of the "constant") plus radiation, and the bounce was caused by negative-energy scalar fields. In all three theories, the models showed monotonic changes in the constants from cycle to cycle (the scale factor qualitatively behaving as explained in [35]).

### E.10.1. Thermodynamical arguments

The existence of oscillatory solutions in the FLRW model was shown by Tolman (see [396] and references therein). His argument can be understood from a purely mechanic point of view, by modelling the Friedmann equation as a one-particle system: examination of the effective potential for a closed universe shows that there are oscillatory solutions for some values of the parameters of the model (assuming that there is a mechanism to revert the contraction into expansion before the singularity). These solutions are permitted from a thermodynamical point of view, since the matter term in the FLRW model is a perfect fluid, whose entropy is constant. Hence the expansion is reversible, although at a finite rate. In more realistic models however, entropy generation is inevitable, arising from various sources (such as viscosity effects from particle creation). However notice that, as discussed in [396], the entropy of each element of the fluid need not attain a maximum, as would be the case in an isolated thermodynamical system, because the energy of the fluid element is not constant. In fact, each time a given element of fluid returns to the same volume, its energy density is higher than in the previous passage through the same volume, due to a lag behind equilibrium conditions. The increment in the entropy leads to non-reversibility, which forbids identical oscillations. As a consequence of the raising energy density, the maximum value for $a$ grows in each cycle [77]. This can be easily seen

[76] A word of caution regarding this latter type of theory was issued in [152].

[77] Notice that in these considerations neither the mechanism that allows safe passage through the singularity nor the details of the entropy generation are given.

from Friedmann equation, taking the case $\Lambda = 0$, $\epsilon = 1$ as an example:

$$\dot{a}^2 + 1 = \frac{1}{3}\rho a^2.$$

After one cycle, the 3-volume goes back to a value it had before when $a$ does. Since $\rho$ grows with the number of cycles, this growth can only be attributed to an increment in $\dot{a}$. Hence a sufficiently "old" cycle is strongly peaked, and $\Omega_{tot}$ remains close to 1 until $a$ is very near the maximum, thus yielding a solution to the flatness problem.
Starting from the fact that the entropy of the universe today is finite, and making the reasonable hypothesis that the increment in the entropy through each bounce shares this property, Zeldovich and Novikov [427] among others (see [133]) have estimated the number of cycles back to an initial state (which should not be singular, to keep the idea of a cyclic universe attractive).

To move from qualitative arguments to actual calculations, the key issue is the production of entropy. The irreversible energy transfer from the gravitational field to particle generation was the source of entropy considered in [347], while it was suggested in [193, 124] that black hole evaporation could be responsible for the entropy growth. An analytical study that showed the correctness of Tolman's arguments was presented in [35], where closed Friedmann universes with $\Lambda \neq 0$ were scrutinized, including an *ad-hoc* mechanism of entropy generation, and assuming that there is a bounce, without entering in the details of its realization. The entropy growth was implemented by relating the constant coming from the conservation laws

$$\rho_i a^\alpha = \text{const.} = C_i,$$

where $i$ denotes radiation or dust, and $\alpha = 4$ or 3 respectively, to the expression for the entropy in each case. Let us take the case of radiation, in which

$$S_r = \text{constant} = \frac{8}{3}\pi^2 \beta T^3 a^3,$$

so we can set $T^3 a^3 = \text{const.} = \gamma$. From this equation and the conservation law it follows that

$$C_r = \frac{G\gamma^{1/3}}{\pi c^4}\, S_r,$$

thus linking the increment in entropy to the change in the constant appearing in the solution. In the same way it is shown that $C_m$ is related to $S_m$ through a similar expression. In [35] it was assumed that the entropy is constant within a cycle, but increases at the beginning of each cycle through the increment in the constants $C_r$ and $C_m$. The behaviour of models with different combinations of matter, radiation and cosmological constant were studied for positive and negative $\Lambda$. The results show that for $\Lambda > \Lambda_c$ (where $\Lambda_c = \Lambda_c(C_r, C_m)$) the universe stops its oscillations with increasing maximum and starts an ever-expanding phase (see Fig. E.11). In other words, when the oscillations become

large enough the cosmological constant dominates over the matter and radiation terms, the oscillations cease, and the universe enters a deSitter regime. If $\Lambda < \Lambda_c$, the oscillations are not interrupted. Oscillations in anisotropic models

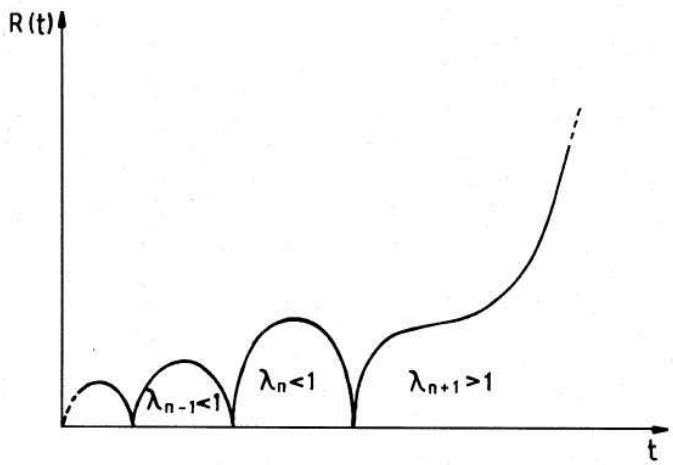


Figure E.11.: The plot shows that transition of a $\Lambda > 0$, radiation-filled universe from the oscillating phase to the ever-expanding phase, due to the growth of entropy (given by the increment in $\lambda_n = \sqrt{4C_{rn}\Lambda/3}$). Taken from [35].

were also studied in [35], paying attention to the question of isotropization after a large number of oscillations. As the entropy increases, the volume of Bianchi I universes with $\Lambda < 0$ oscillates with growing maximum amplitude, while the shear anisotropy vanishes [78].

A more sophisticated model was studied in [106], where FLRW two-fluid out of equilibrium models were considered. Exact solutions were found for a particular cases of the energy exchange, conserving the total energy. In the case of nonzero spatial curvature, cyclic models were shown to exist. The energy exchange between the fluids was modelled by a function $s$ such that

$$\dot{\rho} + 3H\gamma\rho = s, \qquad \dot{\rho}_1 + 3H\Gamma\rho_1 = -s,$$

where $\gamma - 1$ and $\Gamma - 1$ are the EOS parameters of each fluid . Solutions of these equations along with

$$H^2 = \rho + \rho_1 - \frac{\epsilon}{a^2}$$

were found in [106] for different forms of $s$, for the cases radiation and dust, radiation and scalar field, and radiation and negative vacuum energy. In the second case, a new feature appears (as well as the "runaway stage" mentioned

[78] Axisymmetric Bianchi type IX, dust Kantowski-Sachs, Bianchi IX, and some features of inhomogeneous cyclic cosmological models were also studied in [35].

in [35]): the increment in magnitude of the minima in the scale factor as time increases. This was interpreted by the authors as a consequence of the energy exchange: the scalar field reached negative energy values after transferring energy to radiation. Surely this behaviour depends on the specific form of the function $s$. The examples studied in [106] suggest that caution is needed when it is said that cyclic models can solve the flatness problem, since in some of them the cycles cannot become indefinitely large and long-lived, while in others the minimum of the expansion increases.

## E.10.2. Realizations of the cyclic universe

We present in this section some concrete examples of theories that yield cyclic regular solutions (*i.e.* which actually bounce at the minimum of the expansion without presenting singularities), along with some of its successes and conundrums.

### Changes in the matter side of EE

One way to generate a cyclic universe is to add matter that will certainly produce a bounce, and consider next what conditions are to be imposed on it to produce oscillations. A necessary condition that the extrema of the expansion factor must satisfy is given by $H = 0$, with

$$H^2 = \frac{8\pi}{3M_{\rm Pl}^2}(\rho - f(\rho)).$$

This amounts to $\rho - f(\rho) = 0$, where the function $f(\rho)$ is positive. A cyclic universe has been generated along this line in [119], where "wall-like" and "string-like" matter (whose energy scales as $a^{-1}$ and $a^{-2}$ respectively) generate the required $f(\rho)$ [79]. These rather exotic sources can be also thought as originating from scalar fields under the influence of a potential, using the procedure presented in [33]. A modification of the Friedmann equation coming from brane models was used to fix the form of $f(\rho)$ in [82], where

$$H^2 = \frac{8\pi}{3M_p^2}\left(\rho - \frac{\rho^2}{2|\sigma|}\right), \tag{E.257}$$

see Sect.E.6. The dominant component in this model is the so-called "phantom" matter, which has an energy-conditions-violating equation of state characterized by

$$\omega_Q = \frac{p_Q}{\rho_Q} < -1.$$

[79] Earlier attempts along these lines, imposing that $p \propto -a^{-n}$, and $\rho = p \propto -a^{-6}$ are respectively given in [333] and [358]. For a somewhat different approach, see [213, 424].

Since the energy density of matter with state parameter $\omega$ scales with the expansion as

$$\rho = a^{-3(1+\omega)},$$

we see that $\rho$ grows with the expansion. Surely before reaching an infinite energy density, quantum gravity effects will take over the evolution. The somewhat paradoxical situation arises in which very high-density effects must be incorporated in the description of the universe for both very small and very large values of the scale factor. The central idea in [82] is that the same physics causes then the bounce and the turnaround, both governed by Eqn.(E.257). After a bounce, the universe follows the standard evolution until the phantom energy dominates. This energy may erase every trace of structure [92], and dominates the evolution until high-density effects are again important, producing the turnaround. As will be discussed in Sect.E.10.3, one of the problems to be faced in the collapsing phase is the merging of black holes into a "monster black hole". The energy density the universe must reach in order that black holes are torn apart was shown in [82] to be

$$\rho_{\rm br} \propto M_P^4 \left(\frac{M_P}{M}\right)^2 \frac{3}{32\pi} \frac{1}{|1+3\omega_Q|}.$$

This energy density must be reached before the turnaround, characterized by $\rho_{\rm ta} = 2|\sigma|$. The value $\sigma \approx m_{GUT}$ is enough for all but Planck-mass black holes to be torn apart (some of them evaporate before the universe enters the phantom energy stage). These Planck-mass remnants may help in explaining the dark matter puzzle [82] [80]. Some problems still remain in this model. First, the generation of structures in the contracting phase needs to be addressed, to see that the black hole problem does not recur. Second, as stated before, entropy production would lead actually to quasi-cyclic evolution.

A similar model has been studied in [383], given by

$$H^2 = \frac{1}{3}\rho + \nu\rho^2 + \frac{\Lambda}{3}, \tag{E.258}$$

where $\nu$ is a real constant. Analytical solutions of this equation have been found in the case of dust, and their generic feature seems to be the replacement of the initial singularity by a bounce, some solutions displaying also a cyclic behaviour (those for $\Lambda \leq 0$ and $\nu < 0$).

An interesting twist to the entropy problem in cyclic universes was introduced in [43], where a model described by Eqn.(E.258) was studied, with the cosmological constant replaced by a dark energy component with EOS $p = \omega\rho$ and $\omega < -1$, matter and radiation as normal components, and $\nu < 0$. The model takes advantage of the Big Rip phenomenon, where bound systems become unbound and their constituents causally disconnected as a result of the increasing

[80] Details about the evolution of this model and its relation with the so-called coincidence problem can be found in [420].

value of the dark energy density. As a consequence of the Big Rip, the universe would disintegrate in a huge number of disconnected patches. The new ingredient of the model is that the turnaround is placed an instant before the "total Big Rip", when each patch would contain almost no matter at all, and only a small amount of radiation [44] and dark energy. Due to the Big Rip, the huge entropy of the universe is distributed between the enormous number of patches, hence leaving each patch with very low entropy. The subsequent contraction of each patch is free of "formation of structure" problems, and proceeds until a bounce occurs. After the bounce, a normal inflationary phase follows (vastly increasing the entropy), and the cycle starts again.

### Cyclic universes in nonlinear electrodynamics

As discussed in Sect.E.4.5, nonlinear electrodynamics can describe a nonsingular universe. Here it will be shown how a cyclic model arises from the theory given by the Lagrangian [328]

$$\mathcal{L} = -\frac{1}{4}\,F + \alpha\,F^2 - \frac{\gamma^2}{F} \tag{E.259}$$

where $\alpha$ and $\gamma$ are constants, with the dependence of the magnetic field on the scale factor given by $\mathscr{H} = \mathscr{H}_0/a^2$ (see Eqn.(E.208). The time-evolution of the scale factor can be qualitatively described by the effective potential, which arises from Friedmann equation written as a "one-particle" system. For the case at hand, the effective potential is given by

$$V(a) = \frac{A}{a^6} - \frac{B}{a^2} - Ca^6. \tag{E.260}$$

The constants in $V(a)$ are given by

$$A = 4\alpha\mathscr{H}_0^4, \qquad B = \frac{1}{6}\mathscr{H}_0^2, \qquad C = \frac{\gamma^2}{2\mathscr{H}_0^4},$$

and are all positive. The analysis of $V(a)$ and its derivatives implies solving polynomial equations in $a$, which can be reduced to cubic equations through the substitution $z = a^4$. The existence and features of the roots of such equations are discussed in [55]. A key point to the analysis is the sign of $D$, defined as follows. For a general cubic equation

$$x^3 + px = q,$$

the discriminant $D$ is given by

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2.$$

We will denote by $D_V$ the discriminant corresponding to the potential and $D_{V'}$ that of the derivative of V. From the behaviour of the potential and its

derivatives for $a \to 0$ and $a \to \infty$ we see that only one or three zeros of the potential are allowed. The case of interest here (given by $D_V > 0$, $D_{V'} = 0$) is plotted in Fig. E.12, which shows the qualitative behavior of the potential for typical values of the parameters. The model is nonsingular for any value of $\epsilon$,

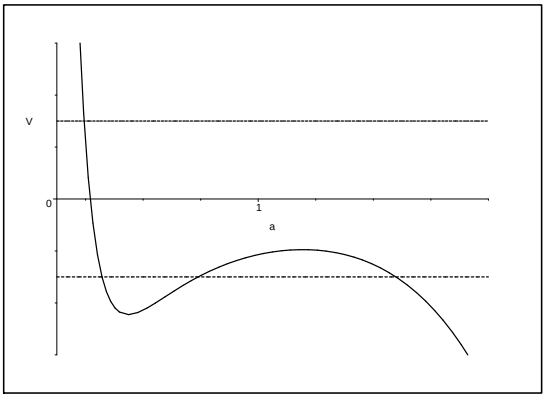


Figure E.12.: Qualitative plot of the effective potential for $D_V > 0$, $D_{V'} = 0$. The lower dotted line corresponds to $\epsilon = 1$.

and a cyclic model is obtained for $\epsilon = 1$.

This setting was generalized in [329], where the Lagrangian

$$\mathcal{L}_T = \alpha^2 F^2 - \frac{1}{4} F - \frac{\mu^2}{F} + \frac{\beta^2}{F^2}. \tag{E.261}$$

was considered, with $\alpha, \beta$ and $\mu$ constants. As shown in [329], four distinct phases can be described with this Lagrangian: a bounce, a radiation era, an acceleration era and a turnaround. This unity of four stages, christened *tetraktys* in [329], constitutes an eternal cyclic configuration. The cyclic behavior is a manifestation of the invariance under the dual map of the scale factor $a(t) \to 1/a(t)$, a consequence of the corresponding inverse symmetry of the Lagrangian (E.261) wrt the electromagnetic field ($F \to 1/F$, where $F \equiv F^{\mu\nu} F_{\mu\nu}$). Restricting to a magnetic universe, as defined in Sect.E.4.5, the Lagrangian $\mathcal{L}_T$ yields for the energy density and pressure given in equations (E.186-E.187):

$$\rho = -\alpha^2 F^2 + \frac{1}{4} F + \frac{\mu^2}{F} - \frac{\beta^2}{F^2}, \tag{E.262}$$

$$p = -\frac{5\alpha^2}{3} F^2 + \frac{1}{12} F - \frac{7\mu^2}{3} \frac{1}{F} + \frac{11\beta^2}{3} \frac{1}{F^2}. \tag{E.263}$$

As we saw in Sect.E.4.5, for any Lagrangian that is a polynomial in $F$,

$$\mathscr{H} = \mathscr{H}_0 \, a^{-2}.$$

As discussed in [329], the combined system of equations of the FLRW metric and the magnetic field described by General Relativity and NLED, are such that the

negative energy density contributions coming from $\mathcal{L}_1$ and $\mathcal{L}_4$ never overcome the positive terms arising from $\mathcal{L}_2$ and $\mathcal{L}_3$. Before reaching undesirable negative energy density values, the universe bounces (for very large values of the field) and bounces back (in the other extreme, that is, for very small values) to precisely avoid this difficulty. These events occur at the values $\rho_B = \rho_{TA} = 0$, which follow from Friedmann's equation in the case $\epsilon = 0$. Notice that this is not an extra condition imposed by hand but a direct consequence of the dynamics described by $\mathcal{L}_T$.

Let us now turn to the generic conditions needed for the universe to have a bounce and a phase of accelerated expansion. From Einstein's equations, the acceleration of the universe is related to its matter content by

$$3\frac{\ddot{a}}{a} = -\frac{1}{2}(\rho + 3p). \tag{E.264}$$

In order to have an accelerated universe, matter must satisfy the constraint $(\rho + 3p) < 0$, which translates into

$$\mathcal{L}_F > \frac{\mathcal{L}}{4\mathscr{H}^2}. \tag{E.265}$$

It follows that any nonlinear electromagnetic theory that satisfies this inequality yields accelerated expansion. In the present model, the terms $\mathcal{L}_2$ and $\mathcal{L}_4$ produce negative acceleration and $\mathcal{L}_1$ and $\mathcal{L}_3$ yield inflationary regimes ($\ddot{a} > 0$). Raychaudhuri's equation imposes further restrictions on $a(t)$ at a bounce. Indeed, the existence of a minimum (or a maximum) for the scale factor implies that at the bounce point $B$ the inequality $(\rho_B + 3p_B) < 0$ (or, respectively, $(\rho_B + 3p_B) > 0$) must be satisfied. Note that, as already mentioned, at any extremum (maximum or minimum) of the scale factor the energy density is zero. Four distinct periods can be identified according to the dominance of each term of the energy density. The early regime (driven by the $F^2$ term); the radiation era (where the equation of state $p = 1/3\rho$ controls the expansion); the third accelerated evolution (where the 1/F term is the most important one) and finally the last era where the $1/F^2$ dominates and in which the expansion stops, the universe bounces back and starts to collapse. The bounce (for an Euclidean section) was discussed in Sect.E.4.5. The standard, Maxwellian term dominates in the intermediate regime. Due to the dependence on $a^{-2}$ of the field, this phase is defined by $\mathscr{H}^2 >> \mathscr{H}^4$ yielding the approximation

$$\begin{aligned} \rho &\approx \frac{\mathscr{H}^2}{2} \\ p &\approx \frac{\mathscr{H}^2}{6} \end{aligned} \tag{E.266}$$

When the universe becomes larger, negative powers of $F$ dominate and the

energy density becomes typical of an accelerated universe, that is:

$$\begin{aligned} \rho &\approx \frac{1}{2}\frac{\mu^8}{\mathscr{H}^2} \\ p &\approx -\frac{7}{6}\frac{\mu^8}{\mathscr{H}^2} \end{aligned} \tag{E.267}$$

In the regime between the radiation and the acceleration eras, the energy content is described by

$$\rho = \frac{\mathscr{H}^2}{2} + \frac{\mu^2}{2}\frac{1}{\mathscr{H}^2},$$

or, in terms of the scale factor,

$$\rho = \frac{\mathscr{H}_0^2}{2}\,\frac{1}{a^4} + \frac{\mu^2}{2\mathscr{H}_0^2}\,a^4. \tag{E.268}$$

For small $a$ it is the ordinary radiation term that dominates. The $1/F$ term takes over only after $a = \sqrt{\mathscr{H}_0}/\mu$, and grows without bound afterwards. Using this matter density in Eqn.(H.22) gives

$$3\frac{\ddot{a}}{a} + \frac{\mathscr{H}_0^2}{2}\,\frac{1}{a^4} - \frac{3}{2}\frac{\mu^8}{\mathscr{H}_0^2}\,a^4 = 0.$$

To get a regime of accelerated expansion, we must have

$$\frac{\mathscr{H}_0^2}{a^4} - 3\frac{\mu^8}{H_0^2}\,a^4 < 0,$$

which implies that the universe will accelerate for $a > a_c$, with

$$a_c = \left(\frac{H_0^4}{3\mu^8}\right)^{1/8}.$$

For very large values of the scale factor, the energy density can be approximated by

$$\rho \approx \frac{\mu^2}{F} - \frac{\beta^2}{F^2} \tag{E.269}$$

and the model goes from an accelerated regime to a phase in which the acceleration is negative. When the field attains the value $F_{TA} = 16\alpha^2\mu^2$ the universe stops expanding and turns to a collapsing phase. The scale factor attains its maximum value

$$a_{max}^4 \approx \frac{\mathscr{H}_0^2}{8\alpha^2\mu^2}.$$

Analytic forms for the scale factor in each regime can be found in [329].

### Cyclic universes in loop quantum gravity

There are realizations of cyclic models in the effective equations for loop quantum gravity (some features of which have been presented in Sect.E.9.2). As discussed in Sect.E.9.2, the Klein-Gordon equation for a scalar field under the influence of a potential, the Friedmann and Raychaudhuri's equations in the semiclassical regime are modified due to quantum gravity effects (see Eqns.(E.254-E.256)). It was shown in [258] that positively curved universes sourced by a massless scalar field can undergo repeated expansions and contractions due to the modifications described above. This was achieved by rewriting Eqns. (E.254-E.256) in the form of the classical FLRW model with the addition of matter described by an effective equation of state, given by

$$\omega \equiv \frac{p_{eff}}{\rho_{eff}} = -1 + \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2DV}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a}\right).$$

A violation of the null energy condition, leading to a bounce, is accomplished when $\omega < -1$, which amounts to $d\ln D/d\ln a > 6$, or $a < 0.914a_*$ [258], with

$$D(q) = \left(\frac{8}{77}\right)^6 q^{3/2}\{7[(q+1)^{11/4} - |q-1|^{11/4}] - 11q[(q+1)7/4\} - |q-1|^{7/4}\mathrm{sgn}(q-1)]\}^6,$$

with $q = a^2/a_*^2$ and $a_*^2 = \gamma\ell_{Pl}^2 j/3$, where $\gamma \approx 0.13$, and $j$ is a quantization parameter, which takes half-integer values. When $V = 0$, $\omega$ is independent of the kinetic energy of the field, and an oscillatory behaviour follows. The addition of a potential leads to the interruption of the cycles as soon as the potential dominates the motion (in analogy to what was discussed in Sect.E.10.1 for the cosmological constant), and a period of inflation may follow [258]. This analysis was later extended to the case of spatially flat universes, with both negative an positive potentials [297, 379].

Yet another realization of a cyclic universe in this scenario is the so-called *emergent universe from a loop* [298]. As mentioned in Sect.E.8.3, the Einstein universe is unstable, so perturbations drive the universe away from this state. This situation partially changes when loop quantum gravity corrections are considered. Using a phase-space analysis, it was shown in [298] that a new static solution appears in the semiclassical regime ($a < a_*$) for positive potentials (for $V < 0$ this is the only solution). This new solution (called loop static, LS) is stable, and the universe oscillates around it, for $V < V_*$, with $V_* = 39/(136\pi l_{Pl}^2 a_*^2)$, while for $V > V_*$, the equilibrium point corresponding to LS merges with that of the Einstein universe. So in the model proposed in [298], the universe is initially at, or in the neighbourhood of the static point LS, with $\phi$ in the plateau region of the potential with $\dot{\phi}_i > 0$. After undergoing a series of non-singular oscillations in a (possibly) past-eternal phase, while the field evolves monotonically along the potential, the cycles are eventually broken as the magnitude of the potential increases, and the universe enters an inflationary epoch. For this

model to work, the potential must be such that $dV/d\phi \to 0$ for $\phi \to -\infty$, and increase monotonically to exit the cycles [81]. An example of a suitable potential is given by

$$V = \alpha \left[\exp(\beta\phi/\sqrt{3}) - 1\right]^2 ,$$

where $\alpha$ and $\beta$ are parameters that may be constrained by the CMB spectrum. As in the case of the classical emergent universe discussed in Sect.E.8.3, there are some fine-tuning issues: the scalar field must start in the asymptotically low-energy region of $V$.

### The cyclic universe based on the ekpyrotic universe

The starting point of the ekpyrotic scenario [241] is five-dimensional heterotic M-theory [211], where the fifth dimension terminates at two boundary $Z_2$ branes, one of which is identified with the visible universe. There are two different versions of the ekpyrotic scenario, the old [240], where there is a bulk brane between the boundary branes and the new [242], where only the boundary branes are present [349]. The initial state in both cases is supposed to be the vacuum state, where the branes are flat, parallel and empty. The branes are drawn together by the action of an attractive potential, and collide inelastically over cosmological times. Part of the kinetic energy is transferred to the branes and used to create matter and radiation. After the collision, the universe enters a "standard" big bang phase, until dark energy domination at the end of the matter era, which causes an accelerated expansion, diluting the content of the universe. The whole process can be described by a 4-d effective theory, with the action (in the Einstein frame) given by

$$S_E = \int d^4x \, \sqrt{-g} \left( \frac{1}{2}R - \frac{1}{2}(\nabla\varphi)^2 - V(\varphi) \right) ,$$

plus higher-order corrections, where the conveniently-tailored potential $V(\varphi)$ is responsible for the main features of the model. The potential is slightly positive for $\varphi > 0$, and goes to zero as $\varphi \to -\infty$. For $\varphi < 0$, the potential has a minimum and is very steep and negative. The minimum corresponds to the close approach of the branes, which happens at such short distances that quantum gravity effects are relevant. The field $\varphi$ moves rapidly through the minimum, and the branes collide as $\varphi \to -\infty$. Both the old and the new model were shown to have problems due to excessive fine-tuning [226], so a cyclic version was introduced [387].

In the cyclic ekpyrotic model, it is assumed that the interbrane potential is the same before and after collision (instead of being zero, as in the non-cyclic model). After the branes bounce and fly apart, the interbrane potential ultimately causes them to draw together and collide again. To ensure cyclic behavior the potential

[81] Other constraints are imposed by succesful reheating.

must vary from negative to positive values [387]. The model may be adjusted in such a way that, at distances corresponding to the present-day separation between the branes, the inter-brane potential energy density is positive and corresponds to the currently observed dark energy, providing roughly 70% of the critical density today. The cosmic acceleration restores the Universe to a nearly vacuous state and as the brane separation decreases, the interbrane potential becomes negative. As the branes approach one another, the scale factor of the Universe, in the conventional Einstein description, changes from expansion to contraction. When the branes collide and bounce, matter and radiation are produced and there is a second reversal transforming contraction to expansion so a new cycle can begin [387]. Figure E.10.2 shows a plot of several forms of the potential that allow for a cyclic universe in this scenario [243]. A quali-

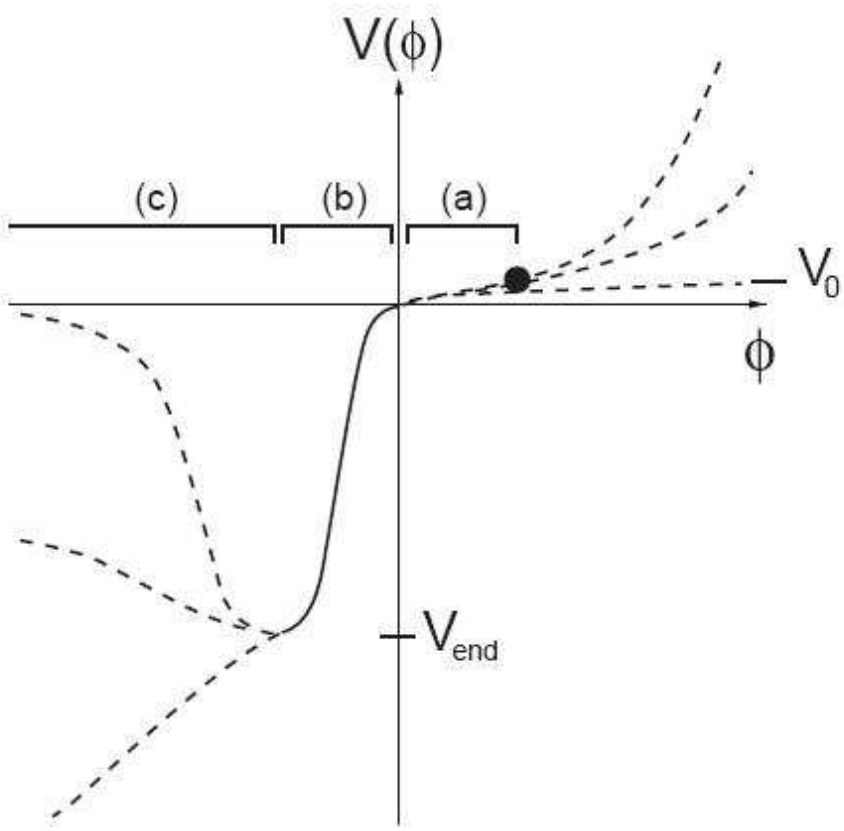


Figure E.13.: Plot of several allowed forms for $V(\varphi)$ .

tative description of the model can be given in terms of this figure as follows. Currently, the field is in region (a), at the point indicated with a dark circle, where the potential is flat and drives cosmic acceleration. Eventually, the field rolls towards negative values of $V$ (region b), where cosmic expansion stops and the universe (being nearly vacuous as a consequence of the acceleration phase) enters a phase of slow contraction, where the spectrum of density perturba-

tions is generated from quantum fluctuations in $\varphi$. In region (c) the kinetic energy of $\varphi$ dominates the energy density. At the bounce, part of this kinetic energy is converted into matter and radiation, while the perturbations in $\varphi$ are imprinted as density fluctuations in the matter/radiation fluid. Meanwhile the field quickly returns back to (a) where it comes to a stop, and the universe enters the radiation-dominated era, so commencing the next cycle. As recognized by its authors, the model presents two weak points (as is the case with many cyclic models): the passage through the would-be singular point, and the propagation of perturbations [82]. It is difficult to achieve the bounce without passing from the semi-classical regime to the high-energy fully quantum regime, where our use of the effective 4-dimensional theory breaks down. The problem is that the kinetic energy and the Hubble rate typically reach Planckian scale as the branes approach. In fact, in the semi-classical regime where loop corrections can be applied, brane collision may be prevented [83].

Recently, a "new ekpyrotic cosmology" was presented in [87], where a NEC-violating ghost condensate was merged with an ekpyrotic phase to generate a non-singular bouncing cosmology. The authors claim to obtain a pre-bounce scale-invariant spectrum using the mechanism of entropy perturbation generation [251]. This is accomplished by having two ekpyrotic scalar fields rolling down their respective negative exponential potentials, and having its own higher-derivative kinetic function. Notice however that the results of this model have been challenged in [227].

### Oscillatory universe from the Steady State model

The Steady State model [68] was proposed as alternative to the Big Bang model, and has fallen into disfavor because the observations of the CMB. However, its authors have advanced a new scenario, called the quasi-steady state model (QSSC, see [214, 363, 303, 304]). In this model, the singularity is avoided by the action of a scalar field $C(x)$, which creates matter in compliance with the Weyl postulate and the cosmological principle, and has negative energy and stresses. The cyclic solutions in the QSSC can be expected from physical grounds as follows [363]. To create a particle, $C(x)$ must have energy-momentum equal or larger than that of the particle. When $C$ is above the threshold, it creates particles and fuels the spacetime expansion (since it has negative stresses). To this overall expansion an oscillation is superimposed. The creation of particles and the expansion set $C$ below the threshold, slowing down the number of created particles, and the expansion. Here, the cosmological constant takes control and causes contraction. The contraction rises the background level of the $C$ field, and the cycle starts again. As shown in [363], a solution to the

[82] This second problem will be discussed in Sec.E.11.
[83] Some other problems of the model were discussed by Linde [262].

EOM of this theory in the FLRW setting that oscillates in this way is given by

$$a(t) = e^{t/P}\left(1 + \eta \cos\theta(t)\right),$$

with $\theta(t) \approx 2\pi t/Q$, where $P$ is the long term "steady state" time scale of expansion, $Q$ is the period of a single oscillation (with $P >> Q$), and $\eta$ is a parameter .

### Other models

Due to the recently discovered dark energy component of the universe, several forms for the dependence of the EOS parameter with the redshift have been analyzed [201]. In fact, some data suggest that $\omega(z)$ evolved from a value larger that $-1$ to a value smaller that $-1$ at some recent redshift. One of the models that describes this crossing is the quintom model [160], where $\omega$ is parameterized as [84]

$$\omega(\ln a) = \omega_0 + \omega_1 \cos[A \ln(a/a_c)], \tag{E.270}$$

with $\omega_0$, $\omega_1$, $A$, and $a_c$ to be fitted by observations [85]. It was shown in [161] that for a certain choice of the parameters, a universe filled with quintom matter (that is, matter with $\omega$ given by Eqn.(E.270) plus radiation and normal matter expands and contracts cyclically, yielding an inflationary period at the beginning of each cycle, and an acceleration period at the end [86].

Perhaps it is convenient at this point to remember that a closed universe has not been discarded by observation yet (and in fact, cannot be discarded with certainty due to the errors inherent to any experiment), though theoretical prejudice and observation tend to favor $\Omega = 1$. As we saw in Chapter E.5, a nonzero bulk viscosity $\zeta$ modifies the fluid pressure according to

$$p = p_0 - 3\zeta H,$$

where $p_0$ is the equilibrium pressure. The asymmetry in the pressure depending on the sign of $H$ causes the increment in energy and entropy, leading to ever-increasing cycles. It was shown in [231] that a similar asymmetry can be caused by scalar fields in a pure non-dissipative setting. Starting from a FLRW setting plus a scalar field under the influence of a potential which displays a minimum, an asymmetry in the pressure, given by $p \approx -\rho$ for $H > 0$, and $p \approx \rho$ for $H < 0$ is generated by the oscillations of the field around the minimum [231]. By imposing appropriate conditions to force a bounce ($a \to a$, $\dot{a} \to -\dot{a}$, $\phi \to \phi$, $\dot{\phi} \to \dot{\phi}$), it was shown that there is an in increment in the maximum radius of expansion of the universe in each cycle, due to conversion of work, done during expansion, into expansion energy. The flatness problem is gradually ameliorated

[84] Constraints on this form of dark energy were studied in [266].
[85] Similar ideas were studied in [29, 309].
[86] Cosmological perturbations of the quintom model were studied in [422].

in this model, since the universe becomes considerably long-lived and more flat after each expansion.

To close this section, other models of a cyclic evolution for the universe are listed next [87]:

- String theory-inspired cyclic universes, starting from the property that there exists a minimal length, $\ell_{Pl}$. See [198].
- A classical spinor field under the influence of a quartic potential in a FLRW background was discussed in [22]. It was shown that $V = \lambda_\psi + m\psi\bar{\psi} - \lambda(\bar{\psi}\psi)^2$ gives rise to oscillations in the scale factor, for certain choices of the parameters [88].
- A cyclic scenario that takes into account matter and radiation evolution if the proton has a finite lifetime was studied in [133].

## E.10.3. Issues of the cyclic models

Cyclic universes are not free of problems. As was put forward in [338], during a matter-dominated cycle, black holes with masses ranging from stellar to galactic will form. During the contracting phase they will coalesce into a "monster black hole" with mass equal to the mass of the universe. Its entropy can be estimated by

$$S = \frac{1}{2}A = 2\pi R^2 = 8\pi M^2 \gtrsim 10^{124},$$

where the mass within one Hubble volume ($\approx 10^{23} M_\odot$) was used. However, the entropy of the radiation in the present Hubble volume is $\approx 10^{87}$, in such a way that black hole formation in a previous cycle would lead to a huge excess of entropy generation. In this scenario, the excess must have somehow been eliminated by the bounce. But there are some ways out of this problem. Sikkema and Israel [376] have suggested that the inner horizon of the monster Kerr black hole absorbs strongly blue-shifted gravitational radiation emitted during the last moments of the collapse. This radiation increases the mass of the core of the black hole by a huge amount, rapidly reaching Planckian values, and correspondingly greatly reduces its specific entropy. If quantum effects produce a bounce, this process would allow the expansion to begin from a state of relatively low disorder Durrer and Laukenmann [138] have noted that the entropy in the radiation we observe today is actually due to the *previous* matter cycle, which may have had shorter duration than the current cycle, leading to less clumping and consequently less entropy production [89].

Another issue of cyclic models was raised in [25], where the evolution of a cosmic string network was considered in a bouncing universe. It was shown that

[87] See also [310].
[88] See also [423].
[89] Gravitational perturbations were also studied in [138].

the string network displays an asymmetric behaviour between the contraction and expansion epochs. In particular, while during expansion a cosmic string network will quickly evolve towards a linear scaling regime, in a phase of collapse it would asymptotically behave like a radiation fluid. A cosmic string network will add a significant contribution, in the form of radiation, to the energy (and hence also entropy) budget of a contracting universe, which will become ever more important as the contraction proceeds. Hence it establishes the need for a suitable entropy dilution mechanism. This process will also operate, *mutatis mutandis*, for other stable topological defects. Conversely, if direct evidence is found for the presence of topological defects (with a given energy scale) in the early universe, their existence alone will impose constraints on the existence and characteristics of any previous phases

## E.11. Perturbations in bouncing universes

As discussed in the Introduction, inflation can solve many of the shortcomings of the SCM, but it also has problems of its own. Bouncing models may provide an alternative (or perhaps a complement) to standard inflation, since in principle the problems of the SCM come from a "shortage of time" for things to happen early after the big bang [179]. The arguments in Sect.E.1 show that an accelerated contraction has the necessary features to solve the problems of the SCM [173]. Let us recall that if in the contracting phase the Hubble radius decreases faster than the physical wavelength corresponding to fixed comoving scales, quantum fluctuations on microscopic scales can be stretched to scales which are cosmological at the present time, exactly as it happens in inflationary models (see for instance [179]). Figure E.14 shows a sketch of the structure of a space-time in which standard inflation starts at $t_i$ and ends at $t_R$. During inflation, the Hubble radius $H^{-1}(t)$ is constant, and it grows linearly afterwards, while the physical length corresponding to a fixed co-moving scale increases exponentially during the period of inflation, and then grows less fast than $H^{-1}(t)$. The figure shows that for a given $k$, the fluctuation can be (causally) produced well inside the Hubble radius, "leave " $H^{-1}(t)$, and "re-enter" in an appropriate way to describe the structures we observe today.

Figure E.15 shows a universe that undergoes a contracting phase, a bounce, and then enters an expanding epoch, assumed to be that of the SCM. In this case, the Hubble radius decreases relative to a fixed comoving scale during the contracting phase, and increases faster in the expanding phase. Fluctuations of cosmological interest today are generated sub- Hubble but propagate outside the Hubble radius for a long time interval. There is however, one main difference with respect to the standard inflationary scenario. In the latter the curvature scale $R \propto H^2$ is (almost) constant, while in the former, it grows until it reaches a maximum and then decreases [90]. This difference may lead to observational con-

[90]This assertion is valid in models in which quantum effects intervene in such a way that

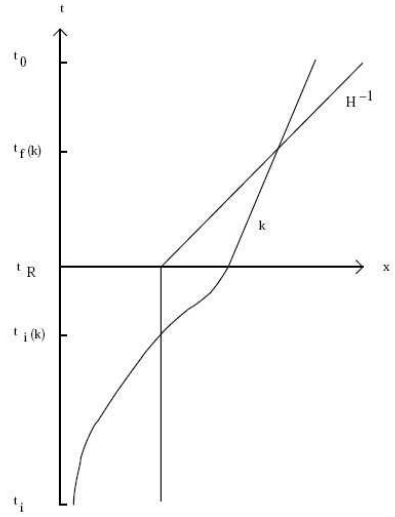


Figure E.14.: Behavior of the comoving scale $k$ and of the Hubble radius $H^{-1}$ as a function of time in inflation. Taken from [79].

sequences [91], particularly regarding the generation of a primordial spectrum of inhomogeneities through parametric amplification of the quantum fluctuations of the background fields in their vacuum state [293]. These, when decomposed in Fourier modes, satisfy a canonical Schrodinger-like equation, whose effective potential is determined by the so-called "pump field", which depends in its turn on the background geometry. There are then two properties of the background in a bouncing universe that can affect the final form of the perturbation spectra [177]: (1) the growth of the curvature scale, and (2) the fields which, together with the gravitational field, determine the background. Property (1) has two important consequences. The first, is that bouncing scenarios may lead to "blue" (*i.e.* growing with frequency) metric perturbation spectra, instead of being flat, or decreasing ("red"), as in standard inflation. A growing spectrum leads to the formation of relic backgrounds whose amplitude is higher at higher frequency, hence more easily detectable. A typical example is that of gravitational waves in SPPB [177] (see Eqn.(E.279)). The second is that the growth of the curvature may also force the comoving amplitude of perturbations to grow (instead of being frozen) outside the horizon (see [84] for this effect in the SPBB) [92].

Regarding Property 2, one of the interesting consequences is the amplifica-

$R_{max} \propto \lambda_{min}^{-2}$, which is the case of loop quantum gravity for instance, where $\lambda_{min} \propto \ell_{Pl}$. For the models in which $H$ reaches a null value, $H^2$ can be replaced by $\ell_c = \sqrt{a^3/a''}$, see Eqn.(E.355).

[91] See [177] for a qualitative discussion of these consequences in the case of string pre-big-bang cosmology.

[92] Consequently, special attention must be taken in the application of linear perturbation theory, see [84].

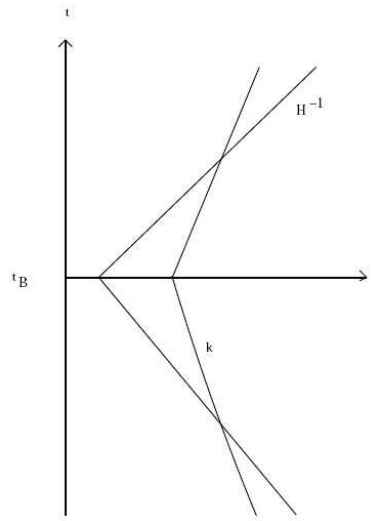


Figure E.15.: Behavior of the comoving scale $k$ and of the Hubble radius $H^{-1}$ as a function of time in a bouncing universe. Taken from [79].

tion of the fluctuations of the EM field, due for instance to the non-minimal coupling with a scalar field (such as the dilaton, or the scalar field in WIST, see Sect.E.11.4). A relic background of scalar particles is also generated, which may be related to dark matter [174].

There is yet another salient feature of the perturbations in a bouncing universe. Since in the far past of this type of models the universe is assumed to be almost flat, one can impose vacuum initial conditions for the perturbations based on simple quantum field theory in flat space [345], instead of having to set initial conditions in a high-curvature regime.

It must be remarked that solving for the perturbations in bouncing models is in principle a nontrivial task, since there are potential ambiguities that may arise at the bounce, not present in standard inflation [93]. Two views can be taken to tackle the study of perturbations in such a scenario. The first one is to devise first a detailed model of the bounce, and then study the properties of the post-bounce perturbations. The problem in this case is that total control of the high-energy physics involved in the bounce is needed, which is not always achieved. It may also happen that the bouncing solution under scrutiny is quite artificial from the physical point of view, as for instance if it is not embedded in any fundamental theory. But in any case some lessons may be extracted from the examples, as we shall see in Sect.E.11.1.

A second attitude is to make some simplifying assumptions and try to work out predictions that are independent of the UV physics that most surely governs

[93] For instance, at the bounce the comoving Hubble scale diverges. Hence all scales are inside the Hubble scale, at least for an instant. However, there are some issues common to both scenarios, such as the transplanckian problem (see for instance [78]).

the bounce. This possibility has led to a great debate [272]. In particular, in order to avoid the specification of the details near the high-curvature regime, matching conditions are used, leading to ambiguities. The dependence of the post-bounce spectrum on the matching conditions has been addressed by many authors, as will be discussed in Sect.E.11.3.

At this point, it is perhaps necessary to say that there are at least two alternative procedures to deal with gravitational perturbations in a relativistic setting. Since Lifshitz's original paper [259], it has been a common practice to start the examination of the theory of perturbations of General Relativity by considering variations of non-observable quantities, such as $\delta g_{\mu\nu}$. The main drawback of this procedure is that it mixes true perturbations and arbitrary (infinitesimal) coordinate transformations, which are unphysical. As shown in [28], [223], [74], [293], this problem can be solved by adopting gauge-independent combinations of the perturbed quantities expressed in terms of the metric tensor and its derivatives. The dynamics of these gauge-independent variables is then provided by the EE.

A second method exists, based on the quasi-Maxwellian (QM) formulation of Einstein's equations. The advantage of this method is that it is gauge-independent from the start, thus dealing only with observable quantities [204], [330], [319, 320, 321] [216]. We shall briefly review both methods in Sect.E.11.5, including a summary of the relation between them.

In the next sections we shall discuss examples of the two approaches. From an observational point of view, the crucial question is whether bouncing models can furnish a nearly-scale invariant spectrum of adiabatic scalar perturbations after the bounce, as demanded by the measurements of the WMAP [381], Sloan survey [392], and 2df [339]. It is also of interest to see if bouncing solutions lead to observable consequences that are markedly different from those of inflation (see Sect. E.12).

## E.11.1. Regular models

In the previous chapters, we have seen that it is possible to generate bouncing models in a wide choice of scenarios, essentially by any of the mechanisms presented in Sect.E.1.1. Obviously, the outcome is very dependent on the choice, but specific models can be sometimes useful in the hope of extracting tendencies of a more general behaviour. In this sense, scalar, vector, and tensor perturbations have been studied in many exact backgrounds displaying a bounce. An incomplete list includes the following:

- General relativity with radiation and a free scalar field having negative energy [341],

- Two scalar fields [200, 9, 134].

- A 5d Randall-Sundrum model with radiation, in which the extra dimension is timelike [40].
- Two perfect fluids [163].
- A nonlinear EM Lagrangian [327].
- A scalar field with higher-order corrections from string theory, with an exponential potential (this case covers the SPBB and the first version of the Ekpyrotic universe) [401].
- A non-canonical scalar field, with Lagrangian $\mathcal{L} = p(X, \phi)$, where $X = 1/2 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ [4, 408].
- Bounce due to quantum cosmological effects using Bohmian solutions of the canonical Wheeler-de Witt equation [344].
- Non-local dilaton potential stemming from string theory [178].

We shall present next a short discussion of scalar, tensor, and vector perturbations in some of these scenarios.

### Scalar perturbations

The evolution of scalar perturbations through a bounce has been a subject of intense debate (see references in [72]). A consensus for the case of a two-component bouncing model in GR seems to have been reached after [72]. This model is described by a flat FLRW metric, and one of the components has negative energy density (to produce the bounce) and is important only near the bounce. The components interact only gravitationally, and the component that dominates away from the bounce has an intrinsic isocurvature mode, in order to describe scalar fields or perfect fluids. The result obtained in [72] is that the spectrum of the growing mode of the Bardeen potential in the pre-bounce is transferred to a decaying mode in the post-bounce [94], [95].

Since the phenomenology associated to the decaying mode in the pre-bounce phase is known to differ from observation [12], we may ask what can be done to lift the negative result of [72]. One possibility is to allow the fluids to interact. Another one is to incorporate in the background solution the decay of the normal component to radiation [96]. Yet another possibility is to consider higher-order corrections. This has been done in several string-inspired models [97], in the gravi-dilaton regime by exploring regular backgrounds (such as those presented

[94] These result is supported by the references cited in [72] and also by the results in [163].
[95] Notice that mode-mixing is possible with $\epsilon = 1$, as for instance in [217].
[96] See Sect.E.3.2 and [365] for an exact solution that has this feature.
[97] The string pre-big-bang model without corrections furnishes a highly blue-tilted spectrum $n_s = 4$ of scalar perturbations [84].

in Sect.E.3.3), as in [98],[400],[401], [100]. The results presented in these articles show that although it may be possible to generate a nearly scale-invariant spectrum in the pre-bounce phase, it corresponds to the decaying mode in the expanding phase [98]. An exception is the model presented in Sect.E.11.1. Another exception may be the ekpyrotic model, where there are results indicating that a scale-invariant spectrum may be obtained in the post-bounce phase [395] [99].

Another set of models comes from the quantum evolution of the universe. As discussed in Sect.E.9.1, bouncing solutions are possible (without the need of a "phantom" field) in the context of the WdW equation, when the Bohm-de Broglie interpretation is used in the mini-superspace approach. A feature of this scenario is that a full quantum treatment of both background and perturbations is possible [342, 343]. The model analyzed in [344] is GR plus a perfect fluid, in which the scalar perturbations can be described in terms of a single degree of freedom, related to the Bardeen potential $\Phi$ (see Appendix). The Bohmian quantum trajectory for the scale factor is given by

$$a(T) = a_0 \left[ a + \left( \frac{T}{T_0} \right)^2 \right]^{\frac{1}{3(1-\omega)}}, \tag{E.271}$$

with $p = \omega\rho$. The normal modes of the scalar perturbation satisfy the equation

$$v_k'' + \left( k^2 - \frac{a''}{a} \right) v_k = 0, \tag{E.272}$$

where a prime means derivative wrt conformal time. Following the usual procedure of expanding the modes for large (negative and positive) values of $T$, matching the expansions, and then transforming to the Bardeen potential, the power spectrum defined by

$$P_\Phi = \frac{2k^3}{\pi^2} |\Phi^2| \propto k^{n_s - 1}, \tag{E.273}$$

yields for the post-bounce phase [344]

$$n_s = 1 + \frac{12\omega}{1 + 3\omega}. \tag{E.274}$$

An analogous calculation for the tensor modes gives

$$n_T = \frac{12\omega}{1 + 3\omega}. \tag{E.275}$$

[98] The SPBB model may yield the right spectrum when axion fluctuations are considered [154].

[99] See ref.[8] for another model in which the growing mode in the contracting phase goes over into the dominant mode in the post-bounce phase.

Notice that a scale-invariant spectrum follows both for the scalar and the tensor perturbations for the case of dust ($\omega = 0$), which is the fluid supposed to dominate the evolution at the time of the matching of the solutions (not necessarily the same governing at the time of the bounce) [344]. An important lesson that follows from this example and the one presented in [163] (see Sect.E.11.1) is that the spectral index is quite insensitive to the details of the bounce, being determined mostly by the dominant component. The example also shows that the bounce is important in the mixing of the modes, which is relevant for the amplitude of the modes in the post-bounce phase.

### Vector perturbations in a contracting background

It is a well-known result of perturbation theory that vector perturbations (VPs) only exhibit decreasing solutions in the context of an expanding Universe (see for instance [293]) [100]. However, as shown in [39], VPs can increase in a contracting flat background, with a perfect fluid as source. Hence, they might provide a signature of a bounce. As shown in the Appendix, the relevant equations are ${S^i}_k = {C^i}_k/a^2$, where $C$ is a constant, and

$$V^i_k \propto \frac{k^2 {C^i}_k}{a^{1-3\omega}}. \tag{E.276}$$

Note that $V^i_k$ increases for $\omega = 0$, and stays constant for radiation, but ${S^i}_k$ always increases for decreasing $a$. As argued in [39], VPs cannot be neglected in the SPBB scenario, in such a way that near a bounce, the metric perturbations may become too large for the use of linear theory (depending on the value of the ${C^i}_k$) [101]. Related results were presented in [186], where it was also shown that the growing vector mode matches with a decaying mode after the curvature bounce, in the context of a low-energy flat gravi-dilaton model [102].

Since many bouncing models are generated by a scalar field, a relevant question is whether VP are important in this type of scenarios. One important point is that VPs are not supported by a scalar field at first order. At second order, the scalar, vector, and tensor modes couple, and VPs can be generated by scalar-scalar mode couplings [287]. Considering exponential potentials and power-law solutions, the ratio of the amplitudes of second order vector perturbations in contracting and expanding phases was studied in [287]. The relative magnitudes of the second order vector perturbations in the two phases depend on the scaling solutions chosen, but at least in one of the examples studied (dust-like collapse, [162]), the observable differences between the collapsing models and the inflationary scenario could be large, assuming that the transition between the two phases does not significantly alter the ratio.

[100] Another interesting result is that the simplest models of inflation do not produce VPs, see for instance [265].

[101] Quantum corrections to the evolution of vector modes were studied in the context of loop quantum gravity in [67]

[102] This is not nececssarily so in multidimensional cosmological models, also analyzed in [186].

### Tensor perturbations

The spectrum of gravitational waves can be a very powerful tool to discriminate between different models of the universe, since gravitational waves decouple very early from matter and travel undisturbed, as opposed to EM waves. In particular, in the context of the SPBB scenario, the amplification of tensor perturbations is greatly enhanced wrt the standard inflationary scenario for large comoving wavenumber $k$ [171]. This result was confirmed in [84], with a gravi-dilaton background solution of the EOM

$$G_\mu^{\ \nu} = \frac{1}{2}\left(\partial_\mu\varphi\partial^\nu\varphi - \frac{1}{2}\delta_\mu^{\ \nu}\partial_\alpha\varphi\right), \tag{E.277}$$

$$\Box\varphi = 0, \tag{E.278}$$

given by

$$a(\eta) = (-\eta)^{1/2}, \qquad \varphi(\eta) = \frac{-3-\sqrt{3}}{1+\sqrt{3}}\ln(-\eta) + \text{const.},$$

the typical amplitude for the normalized vacuum tensor fluctuations outside of the horizon over a scale $k^{-1}$ is given by [84]

$$|\delta_{h_k}(\eta)| \approx \left(\frac{H_1}{M_{Pl}}\right)(k\eta_1)^{3/2}\ln|k\eta|, \tag{E.279}$$

where $H_1 \approx (a_1\eta_1)^{-1}$ is the final contraction scale [103], while the result in the standard inflationary expansion does not have the ln dependence (see for instance [192]). The possible influence of the nonperturbative phase, where the curvature and the dilaton are very large, was studied by imposing a bouncing solution in [169], and by taking into account higher-derivative $\alpha'$ and quantum corrections (see Sect.E.3.3) [176], [97]. The results in these papers show that the low frequency modes, crossing the horizon in the low-curvature regime, are unaffected by higher-order corrections, and also that the shape of the spectrum of the relic graviton background, obtained in the context of the pre-Big Bang scenario, is strongly model-dependent.

This analysis was continued in [428], where cosmological perturbations in the low-energy string effective action with a dilaton coupling $F(\phi)$ were studied, with the addition of a Gauss-Bonnet term, a kinetic term of the type $(\nabla\phi)^4$, and a potential $V(\phi)$. Scale-invariant spectra in the string frame and a suppressed tensor-to-scalar ratio were obtained by imposing slow-roll inflation in the Einstein frame. The results show that it is practically impossible to obtain these conditions without the second-order corrections given by Eq.(E.154), both with and without the Gauss-Bonnet term.

[103] Scalar perturbations of this model were also investigated in [84], and present amplitudes and spectra similar to the tensor perturbations.

Analytic and numerical results for the tensor post-bounce spectrum have been obtained for a two-component model defined by $p_\pm = \omega_\pm \rho_\pm$ [163]. The flat background is given by

$$a(\tau) = a_0 \left(1 + \frac{\tau^2}{\tau_0^2}\right)^\alpha ,$$

with

$$d\tau = \frac{dt}{a^\beta}, \qquad \beta = \frac{3}{2}(2\omega_+ - \omega_- + 1),$$

$$\alpha = \frac{1}{3(\omega_- \omega_+)}, \qquad a_0 = \left(\frac{\gamma_-}{\gamma_+}\right)^\alpha , \qquad \tau_0^2 = \frac{4\alpha^2}{\ell_{\rm Pl}^2}\frac{\gamma_-}{\gamma_+^2},$$

$\gamma_+$ and $\gamma_-$ are constants, with $\gamma_- < 0$, to produce the bounce. The tensor spectrum, assuming that $-1/3 < \omega_+ < 1$, and that the potential that arises from Eqn.(E.355) has only one extremum at $\tau = 0$, is given by [163] $P_h \propto \tilde{k}^{n_T}$, where

$$n_T = \frac{12\omega_+}{1 + 3\omega_+}.$$

Note that the spectral index does not depend on the EOS parameter of the "exotic" fluid (contrary to the case of the spectral index for the scalar perturbations). This was to be expected since large wavelengths are comparable to the curvature scale of the background at a time when the universe is still far from the bounce, so the behaviour obtained in this case can be taken as generic, *i.e.* independent of the details of the bounce.

Yet another example of the calculation of a tensor spectrum in a bouncing model was presented in Sect.E.11.1, based on the quantum evolution (using the Bohmian quantum trajectory) of a universe described by GR plus a perfect fluid. The result is (see the comments after Eqn.(E.275))

$$n_T = \frac{12\omega}{1 + 3\omega}. \tag{E.280}$$

In fact, the tensor-to-scalar ratio in this model was estimated as $T/S \approxeq 5.2 \times 10^{-3}$, and the characteristic bounce length-scale $L_0 \approxeq 1500\ell_{Pl}$, (assuming that $n_s \lesssim 1.01$) which is a value in the range in which quantum effects are expected to be relevant, while at the same time the Wheeler-de Witt equation is valid (without corrections from stringy/loop effects).

### E.11.2. Scalar perturbations in exact models using the quasi-Maxwellian framework

As mentioned in the introduction of this chapter, perturbations can also be studied using the quasi-Maxwellian (quasi-Maxwellian) method. In this section we apply it to two exact bouncing solutions. The first one is generated by the non-minimal coupling of the electromagnetic field with gravity (see Sect.(E.4.4)). As discussed in the Appendix, in the quasi-Maxwellian formalism the scalar

perturbations are completely described by the variables $E$ and $\Sigma$, which obey the equations (E.376)-(E.378):

$$\dot{E} = -\frac{1+\lambda}{2}\,\rho\,\Sigma - \frac{1}{3}\,\theta\,E,$$

$$\dot{\Sigma} = \left[\frac{6\lambda}{1+\lambda}\left(\epsilon + \frac{k^2}{3}\right)\frac{1}{a^2\,\rho} - 1\right]\,E,$$

with $p = \lambda\rho$, and $k$ is the wave number (the subindex $k$ in $E$ and $\Sigma$ has been omited). Combining these, we obtain the equation for the time evolution of the electric part of the perturbed Weyl tensor:

$$\ddot{E} + \dot{E}\left(\frac{4}{3} + \lambda\right)\,\theta + EX = 0, \tag{E.281}$$

where $X$ is a function of the background variables given by

$$X \equiv \lambda\frac{3\epsilon + k^2}{a^2} - \left(\lambda + \frac{2}{3}\right)\,\rho + \frac{2+3\lambda}{9}\,\theta^2.$$

Defining a new function $g(t)$ by $g = E\,a^{-\sigma}$, where $\sigma \equiv -(4+3\lambda)/2$, we obtain from Eqn.(E.281)

$$\ddot{g} + \chi(t)\,g = 0, \tag{E.282}$$

where [104]

$$\chi(t) \equiv \sigma\,\frac{\ddot{a}}{a} - \sigma(\sigma+1)\,\left(\frac{\dot{a}}{a}\right)^2 + X. \tag{E.283}$$

In the case of the bouncing universe given by Eqn.(E.181), we have

$$\left(t^2 + \alpha_0^2\right)^2\,\ddot{g} + \left(\alpha\,t^2 + \beta\,\alpha_0^2\right)\,g = 0, \tag{E.284}$$

where $\alpha \equiv k^2/3 - 7/4$ and $\beta \equiv k^2/3 - 1/2$. With the change of variable $z = 1/2 - it/(2\alpha_0)$, this equation takes the form

$$\frac{d^2g}{dz^2} + I(z)\,g = 0, \tag{E.285}$$

where

$$I(z) = -\frac{\beta}{4z^2\,(z-1)^2} + \frac{\alpha\,(2z-1)^2}{4z^2\,(z-1)^2}. \tag{E.286}$$

After a direct calculation, Eqn.(E.285) can be transformed into a hypergeometric equation

$$z(1-z)\,\frac{d^2\omega}{dz^2} + [c - (a+b+1)z]\frac{d\omega}{dz} - ab\omega = 0, \tag{E.287}$$

[104] Notice that, as shown in the Appendix, this equation is actually a consequence of a transformation that takes the variables $(E, \Sigma)$ (which are not canonically conjugated) into a new pair of variables that are canonically conjugated.

where

$$a = \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}, \tag{E.288}$$

$$b = \frac{1}{2} - \sqrt{\frac{1}{4} - \alpha}, \tag{E.289}$$

$$c = \frac{5}{2}. \tag{E.290}$$

The solution for $g(z)$ is given by

$$g(z) = z^{\frac{c}{2}} (z-1)^{\frac{c-a-b-1}{2}} \omega(z) \tag{E.291}$$

ot, in terms of the hypergeometric function $F(a, b, c; z)$,

$$g(z) = z^{\frac{5}{4}} (z-1)^{-\frac{1}{4}} F\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}, \frac{1}{2} - \sqrt{\frac{1}{4} - \alpha}, \frac{5}{2}; z\right). \tag{E.292}$$

Finally, the solution for the electric part of the Weyl tensor, is given by

$$E_k = s(-4\alpha_0^2)^{-\frac{5}{4}} (z-1)^{-\frac{3}{2}} F\left(\frac{1}{2} + \sqrt{2 - \frac{k^2}{3}}, \frac{1}{2} - \sqrt{2 - \frac{k^2}{3}}, \frac{5}{2}; z\right) \tag{E.293}$$

where $s$ is a constant. Restricting to $z \in \Re$, it follows that this solution is regular for $z < 1$, and can be analytically extended for all values of $z$. Hence, the perturbation is regular.

Notice that the power spectrum of the perturbations can be obtained using (see Appendix)

$$P_k = k^{-1}|E_k|^2. \tag{E.294}$$

The second example we shall study in this section is the model presented in Sect.E.4.5, the perturbation of which was analyzed by the quasi-Maxwellian method in [327]. In this model, the singularity is avoided by the introduction of nonlinear corrections to Maxwell electrodynamics, given by

$$L = -\frac{1}{4} F + \alpha F^2 + \beta G^2, \tag{E.295}$$

where $F = F_{\mu\nu}F^{\mu\nu}, G \doteq \frac{1}{2}\eta_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}$, $\alpha$ and $\beta$ are arbitrary constants. After an average procedure (see Sect.E.4.5), the expression for the scale factor for the "magnetic unverse" with $\epsilon = 0$ is:

$$a(t)^2 = \mathscr{H}_0 \left[\frac{2}{3}(t^2 + 12\alpha)\right]^{1/2}. \tag{E.296}$$

The interpretation of the source as a one-component perfect fluid in an adiabatic regime leads to instabilities [340], which are artificial, as will be seen next. The sound velocity of the fluid in this case is given by [249]

$$\frac{\partial p}{\partial \rho} = \frac{\dot{p}}{\dot{\rho}} = -\frac{\dot{p}}{\theta(\rho + p)}. \tag{E.297}$$

This expression, involving only background quantities, is not defined at the points where the energy density attains an extremum given by $\theta = 0$ and $\rho + p = 0$. In terms of the cosmological time, these points are determined by $t = 0$ and $t = \pm t_c = 12\alpha$. Notice that they are well-behaved regular points of the geometry, indicating that the occurrence of a singularity is in fact caused by an inappropriate description of the source. This difficulty can be circumvented by splitting the part coming from Maxwell's dynamics from the additional non-linear $\alpha-$dependent term in the Lagrangian. As a result, we get two noninteracting perfect fluids:

$$T_{\mu\nu} = T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu}, \tag{E.298}$$

where

$$T^{(1)}_{\mu\nu} = (\rho_1 + p_1)\, v_\mu v_\nu - p_1\, g_{\mu\nu}, \tag{E.299}$$

$$T^{(2)}_{\mu\nu} = (\rho_2 + p_2)\, v_\mu v_\nu - p_2\, g_{\mu\nu}, \tag{E.300}$$

and

$$\rho_1 = \frac{1}{2}\,\mathscr{H}^2, \tag{E.301}$$

$$p_1 = \frac{1}{6}\,\mathscr{H}^2, \tag{E.302}$$

$$\rho_2 = -4\alpha\,\mathscr{H}^4, \tag{E.303}$$

$$p_2 = -\frac{20}{3}\,\alpha\,\mathscr{H}^4. \tag{E.304}$$

From this decomposition it follows that each of the components of the fluid satisfies the conservation equation, thus showing that the source can be described by two non-interacting perfect fluids with equations of state $p_1 = 1/3\,\rho_1$ and $p_2 = 5/3\,\rho_2$. This splitting should be understood only as a mathematical device to allow for a fluid description.

From the considerations presented in Sect.E.11.5 we obtain [327]:

$$\dot{\Sigma}_1 = -\left(\frac{2\lambda_1(3\epsilon + k^2)}{a^2(1+\lambda_1)\rho_1} + 1\right) E_1, \tag{E.305}$$

$$\dot{\Sigma}_2 = -\left(\frac{2\lambda_1(3\epsilon + k^2)}{a^2(1+\lambda_2)\rho_2} + 1\right) E_2, \tag{E.306}$$

$$\dot{E}_1 + \frac{1}{3}\theta E_1 = -\frac{1}{2}\,(1+\lambda_1)\,\rho_1\Sigma_1, \tag{E.307}$$

$$\dot{E}_2 + \frac{1}{3}\theta E_2 = -\frac{1}{2}\,(1+\lambda_2)\,\rho_2\Sigma_2, \tag{E.308}$$

where $k$ is the wave number. As shown in [319], the scalar perturbations can be expressed in terms of the two basic variables $E_i$ and $\Sigma_i$, and the corresponding equations can be decoupled. The result in terms of the $E_i$ is

$$\ddot{E}_i + \frac{4+3\lambda_i}{3}\theta\dot{E}_i + \left\{\frac{2+3\lambda_i}{9}\theta^2\left(\frac{2}{3}+\lambda_i\right)\rho_i\frac{1}{6}(1+3\lambda_j)\rho_j - \frac{(3\epsilon+k^2)\lambda_i}{a^2}\right\}E_i = 0. \tag{E.309}$$

Note that in this expression there is no summation in the indices, and $j \neq i$ , and $\lambda_i = \left(\frac{1}{3}, \frac{5}{3}\right)$. In the first case the equation for the variable $E_1$ becomes

$$\ddot{E}_1 + \frac{5}{3}\theta\dot{E}_1 + \left[\frac{1}{3}\theta^2 - \rho_1 - \rho_2 - \frac{5}{3a^2}\right]E_1 = 0. \tag{E.310}$$

Let us analyze the behavior of the perturbations in the neighborhood of the points where the energy density attains an extremum (*i.e.* the bounce and the point in which $\rho + p$ vanishes). The expansion of the equation of $E_1$ in the neighborhood of the bounce (at $t = 0$) up to second order, is given by:

$$\ddot{E}_1 + at\dot{E}_1 + (b + b_1 t^2)E_1 = 0, \tag{E.311}$$

where the constants $a$ and $b$ are defined as follows

$$a = \frac{5}{2t_c^2}, \tag{E.312}$$

$$b = -\frac{k^2}{\sqrt{6}\mathscr{H}_0 t_c}, \tag{E.313}$$

$$b_1 = -\frac{b}{2t_c^2} - \frac{3}{4t_c^4}. \tag{E.314}$$

Defining a new function $f$ as

$$f(t) = E_1(t)\exp\left\{\left(+\frac{a}{4} - \frac{i}{2}\sqrt{b_1 - \frac{a^2}{4}}\right)t^2\right\}, \tag{E.315}$$

and introducing the coordinate $\xi$ by

$$\xi = -it^2\sqrt{b_1 - \frac{a^2}{4}}, \tag{E.316}$$

we obtain for $f$ the confluent hypergeometric equation [5]

$$\xi\ddot{f} + (1/2 - \xi)\dot{f} + ef = 0, \tag{E.317}$$

where

$$e = \frac{i(b - a/2)}{4(b_1 - a^2/4)^{1/2}} - \frac{1}{2}. \tag{E.318}$$

The solution of this equation is given by

$$f(t) = A\, M\left(d, 1/2, -it^2\sqrt{b_1 - \frac{a^2}{4}}\right), \tag{E.319}$$

where $A$ is an arbitrary constant and $M(d, 1/2, \xi)$ is the confluent hypergeometric function, which is well-behaved in the neighborhood of the bounce. Hence the perturbation $E_1(t)$ is regular and given by

$$\begin{aligned} E_1(t) &= A\, M\left(d, 1/2, -it^2\sqrt{b_1 - \frac{a^2}{4}}\right) \\ &\times \exp\left\{\left(-\frac{a}{4} + \frac{i}{2}\sqrt{b_1 - \frac{a^2}{4}}\right)t^2\right\}. \end{aligned} \tag{E.320}$$

After a similar procedure, the perturbation $E_2$ obeys, in the same neighborhood, the following equation:

$$\ddot{E}_2 + at\dot{E}_2 + (b + b_1 t^2)E_2 = 0. \tag{E.321}$$

This is the same equation we obtained for $E_1$, with different values of $a, b$ and $b_1$ given in this case by

$$a = \frac{9}{2t_c^2}, \tag{E.322}$$

$$b = \frac{3}{2t_c^2} - 5\frac{k^2}{\sqrt{6}\mathscr{H}_0 t_c}, \tag{E.323}$$

$$b_1 = -\frac{5k^2}{t_c^3\mathscr{H}_0\sqrt{6}} - \frac{5}{t_c^4}. \tag{E.324}$$

The solution is given by the real part of

$$\begin{aligned} E_2(t) &= A\, M\left(d, 1/2, -it^2\sqrt{b_1 - \frac{a^2}{4}}\right) \\ &\times \exp\left\{-\left(\frac{a}{4} - \frac{i}{2}\sqrt{b_1 - \frac{a^2}{4}}\right)t^2\right\}, \end{aligned} \tag{E.325}$$

so the perturbation $E_2(t)$ is well-behaved. At the neighborhood of the other critical point, given by $t = t_c$, the equation for the perturbation $E_1$ is given by

$$\ddot{E}_1 + a\dot{E}_1 + (b + b_1 t)\, E_1 = 0, \tag{E.326}$$

with

$$a = \frac{5}{4t_c}, \tag{E.327}$$

$$b = -\frac{3}{4t_c^2} - \frac{\sqrt{3}k^2}{6\mathscr{H}_0 t_c}, \tag{E.328}$$

$$b_1 = \frac{\sqrt{3}}{4t_c^2}\left(\frac{k^2}{3\mathscr{H}_0} - \frac{3}{2t_c}\right), \tag{E.329}$$

By the following variable transformation:

$$E_1(t) = \exp\left(-\frac{at}{2}w(t)\right), \tag{E.330}$$

the differential equation goes to

$$\ddot{w} + \left(b - (a/2)^2 + b_1 t\right) w = 0, \tag{E.331}$$

and the solution is

$$w(t) = w_0 \,\mathrm{Ai}\left(-\frac{b - (a/2)^2 + b_1 t}{b^{2/3}}\right). \tag{E.332}$$

The Airy function Ai is regular near $t = t_c$, and so is $E_1$. Finally we look for the equation of $E_2$ at the neighborhood of $t = t_c$:

$$\ddot{E}_2 + a\dot{E}_2 + (b + b_1 t)\, E_2 = 0, \tag{E.333}$$

where

$$a = \frac{9}{4t_c}, \tag{E.334}$$

$$b = \frac{5}{t_c}\left(\frac{5}{4t_c} - \frac{\sqrt{3}m^2}{6\mathscr{H}_0}\right), \tag{E.335}$$

$$b_1 = \frac{5\sqrt{3}}{2t_c^2}\left(\frac{1}{t_c} - \frac{m^2}{6\mathscr{H}_0}\right). \tag{E.336}$$

This equation differs from Eq.(E.326) only by the numerical values of the parameters $a, b$, and $b_1$ so we obtain the same type of regular solution

$$E_2 = w_0 \,\mathrm{Ai}\left(-\frac{b - (a/2)^2 + b_1 t}{b_1^{2/3}}\right) \exp\left(-\frac{at}{2}\right) \tag{E.337}$$

Hence, it was shown by a direct analysis of a specific nonsingular universe, that in the neighborhood of the special points in which a change of regime occurs, all independent perturbed quantities are well-behaved, and the model is stable with regard to scalar perturbations.

A similar analysis has been carried out for the model described by Eqn.(E.296) in the case of tensor perturbations in [19]. The result shows differences between gravitational waves generated near a singularity and those generated near the bounce. While in the first case the system exhibits a node-focus transition in the $(E, \Sigma)$ plane, independently of the perturbation wavelength $\lambda$, in the bouncing model the trajectories may exhibit a focus-node-focus transition, or no transition at all, depending on the value of $\lambda$.

### E.11.3. Matching

As mentioned in Sect.E.11, another approach to the description of perturbations in a bouncing universe uses the idea of matching a contracting with an expanding phase. The hope here again resides in the fact that some general features can be extracted from given examples, since the matching may be done in such a way as to avoid a very detailed specification of the high curvature phase. Inasmuch as the result depends on the matching conditions, this issue was the subject of a long debate [272]. We shall present next some examples of this technique.

The case of a scalar field with an exponential potential (inspired in the string pre-big bang and the ekpyrotic model) was studied in [139]. A matching between a contracting, scalar field-dominated phase and an expanding, radiation-dominated phase (and also of the corresponding perturbations) was done using the Israel conditions [218]. It was assumed that the slice of spacetime in which high-energy physics takes control is very thin, and can be approximated by a spacelike surface, with a negative surface tension (to be specified by the underlying physics) required by the jump in the extrinsic curvature. Neglecting possible, but subdominant, anisotropic surface stresses [105], and depending on the chosen surface, it was found that a scale-invariant spectrum could be transferred from the contracting to the expanding phase. A similar model has been studied in [162], where it was shown that the value $p = 2/3$ of the power law $a(t) \propto (-t)^p$ was adopted for the scale factor generates a scale-invariant spectrum of adiabatic curvature fluctuations in the collapsing phase. The chosen background corresponds to a contracting Universe dominated by cold matter with null pressure. As a result of the glueing, the spectrum is matched at the bounce to a scale-invariant spectrum during the expanding phase. This model was also shown to generate a scale-invariant spectrum of gravitational waves, as already realized in [271].

It is useful to assume that the physics of the bounce is encoded in the transfer matrix $T$, defined by

$$\begin{pmatrix} D_+ \\ S_+ \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} D_- \\ S_- \end{pmatrix} . \tag{E.338}$$

$T$ gives the degree of mixing between the dominant (D) and sub-dominant (S) modes before and after the bounce for a fixed comoving wave number $k$. Several combinations are possible, such as one for which the spectrum is initially not scale invariant but is turned into it because of a nontrivial $k$ dependence of the transition matrix. Due to the fact that the bounce lasts only a short time, it is conceivable that it does not exert any influence on the large scales that are of astrophysical interest today. This implies that $T$ does not depend on $k$ [139], in such a way that a scale invariant pre-bounce spectrum is transmitted without change to the post-bounce phase. This hypothesis has been tested

[105] This restriction was lifted in [113].

in [280]. It was shown by way of an example (a bouncing solution in general relativity, with positive curvature spatial section, with a scalar field as a source, by using an expansion of the bouncing scale factor around the $\epsilon = 1$ de Sitter-like bouncing solution) that $T$ may depend on $k$, provided that the null energy condition (NEC) is very close to being violated at the bounce, hence affecting the large scale behaviour of the scalar perturbations (see however [129, 282]). Note however that it was shown in [280] that the spectrum of gravitational waves is not affected by the bounce.

The authors of [105] have obtained the most general form of the transfer matrix respecting local causality. In particular, they have shown that no local-causality-respecting matching condition can lead to a scale invariant spectrum for both the pre-big-bang and the ekpyrotic model, in agreement with the result of [118]. They also studied a non-local model based on string theory and showed that under certain conditions a post-bounce SIS is possible.

A different line of attack was pursued in [73] with the central assumption that the bounce in a spatially flat universe is governed by just one physical scale (chosen as $\eta_B$, the cosmological time at which the bounce occurs). Working in GR and incorporating all the eventual new physics in the matter side of EE, the general solution to the problem of the propagation of perturbations through the bounce was presented in [73]. It was shown that the spectrum of the Bardeen potential in the expansion phase depends critically on the relation between the comoving pressure perturbation and the Bardeen potential in the new physics sector of the energy-momentum tensor. Only if the comoving pressure perturbation is directly proportional to the Bardeen potential (rather than its Laplacian, as for any known form of ordinary matter), the pre-bounce growing mode of the Bardeen potential persists in the post-bounce constant mode. This would open the door to models with a scale-invariant spectrum (hence in agreement with observations) for those cases in which there is very slow contraction in the pre-bounce. This result is supported by numerical analysis of a toy model in which $\delta p \propto \Psi$ [73]. Examples of this type of behaviour for the perturbations are given by models with spatial curvature (which cannot be treated however with this approach) and also by models with modifications coming from extra dimensions (such as the one presented in [41]) [73].

## E.11.4. Creation of cosmological magnetic fields

The origin, evolution, and structure of large-scale magnetic fields are amongst the most important issues in astrophysics and cosmology. The standard model for the generation of this fields is the dynamo, which amplifies a small seed field to the current observed values of $1 -$ few $\mu G$. There are several mechanisms to produce these seeds, but the prevalent view is that they have a primordial origin [190]. In particular, the vacuum fluctuations of the EM field may be "stretched" by the evolution of the background geometry to super-horizon scales, and they could appear today as large-scale EM fields. For this to happen, conformal

invariance of the EM equations must be broken. This is the case in models such as dilaton electrodynamics [255] and Weyl integrable spacetime (see Sect.E.3.2, and [366] for a list of references on the subject).

As a previous step in the details of the case of the EM field, let us discuss the creation of massive scalar particles in a bouncing universe with $\epsilon = -1$, following [115]. The expansion factor is given by $a(t) = t^2 + a_0^2$, or $a(\eta) = a_0^2 \cosh \eta$ in conformal time, as in the examples studied in [286, 311]. The EOM for the scalar field is

$$\Box\phi + \left(m^2 + \frac{1}{6}\xi R\right)\phi = 0.$$

With the mode decomposition

$$\phi_k(x) = a(\eta)^{-1/2} Y_{\mathbf{k}}(\vec{x})\chi_k(\eta),$$

where $\mathbf{k} = (k, J, M)$ and the $Y_{\mathbf{k}}(\vec{x})$ are given in terms of the spherical harmonics (see [56]), the function $\chi_k(\eta)$ satisfies the modified Mathieu equation:

$$\frac{d^2\chi_k}{d\eta^2} - (\lambda - 2h^2 \cosh^2 \eta)\chi_k = 0,$$

where $\lambda \equiv -(k^2 + \frac{1}{2}m^2 a_0^2)$, and $h \equiv \frac{1}{2}ma_0$. The number of created quanta in the (asymptotically flat) future can be calculated with the solutions of this equation that have the right asymptotic behaviour, and following standard techniques. In the limit $h << 1$ (*i.e.* when the Compton wavelength of the particle is much greater than $a_0$), the result is [115]

$$|\beta_k|^2 = \frac{1}{2\sinh^2 \pi\tilde{k}}[1 - \cos\left(4\tilde{k}\ln\frac{h}{2}\right) + \varphi],$$

where $\tilde{k}$ is the index in the Mathieu functions $M_{-i\tilde{k}}(\eta, h)$, and is a complicated function of $\lambda$ and $h$, which in the limit for small $h$ reduces to

$$\lambda = -\tilde{k}^2 - \frac{h^4}{2(\tilde{k}+1)} + O(h^8),$$

and $\varphi$ is a phase, independent of $h$. The expression for $|\beta|$ varies from 0 to $4\times \exp(-2\pi\tilde{k})$ for large $k$, and shows that for a given $k$, the particle number depends on the product $ma_0$.

The creation of magnetic fields in a bouncing universe in models that break the conformal invariance with a coupling to a scalar field was studied in [365, 185, 366]. In the latter, canonical quantization was applied to the model given by

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\; f(\omega)F_{\alpha\beta}\; F^{\alpha\beta},$$

where $\omega$ is the scalar field, and $F_{\alpha\beta}$ an abelian field, with $f(\omega) = \exp(-2\omega)$. The modes of the potential $\mathcal{A}_\mu = e^{-\omega} A_\mu$ satisfy the equation

$$\mathcal{A}''^{(\sigma)}_{k\alpha}(\eta) + (k^2 - V(\eta))\mathcal{A}^{(\sigma)}_{k\alpha}, \tag{E.339}$$

where $\sigma = +, -$ designates the base of travelling waves, $\alpha = 1, 2$ describes the two transverse degrees of freedom, and $V(\eta) = -\omega'' + \omega'^2$. For the background described in Eqn.(E.118),

$$V(\eta) = \frac{2\sigma \sinh(2\eta) + \sigma^2}{\cosh^2(2\eta)}, \tag{E.340}$$

where $\sigma \equiv \sqrt{6}/\lambda$, where $\lambda^2$ is the coupling constant of the scalar field to gravity. The mode equation (E.339) admits analytical solutions in terms of hypergeomet-

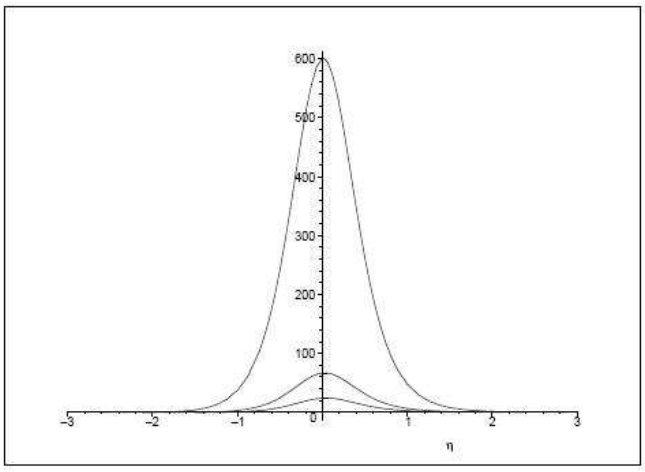


Figure E.16.: Plot of the potential (see Eq.(E.340)), for $\lambda = 0.1, 0.3, 0.5$ (solid, dotted, dashed line respectively).

ric functions, in terms of which the Bogolubov coefficients, and the expression for the energy density of the magnetic field $\rho_m$ can be calculated [366]. The amplification factor with respect to the conformal vacuum peaks for the modes with momenta such that $k \approx 1.31$, and is given by

$$\frac{\rho_m}{(\rho_m)_{cf}} \propto \exp\left(\frac{\pi\sqrt{6}}{\lambda}\right), \tag{E.341}$$

for $\eta >> 1$. The conditions for the spectrum to be greatly amplified today are [366]

$$a_0 << ct_r, \qquad \lambda << 1,$$

where $t_r$ is the time at which the scalar field is negligible, in such a way that the EM field is free again.

At a comoving scale of about 10 kpc, the strength of conformal vacuum fluctuations is of the order of $10^{-55}$ G. To reach the strength required to feed the galactic dynamo, $B_{seed} \propto 10^{-20}$ G, , which is a conservative estimate, we get from Eqn.(E.341) that $\lambda \approx 0.1$. Taking for the comoving scale the size of the universe ($\approx 4 \times 10^6$ kpc), the amplification factor becames $10^{46}$, and we need $\lambda \approx 0.07$. So the strength needed in both cases can be achieved by a modest value of $\lambda$, the coupling constant of $\omega$ to gravity.

These results were obtained in a model that did not take into account the effect of the creation of matter by the decay of the scalar field. The solution presented in Sect.(E.3.2), namely

$$a(\eta) = \beta\sqrt{\cosh(2\eta) + k_0 \sinh(2\eta) - 2k_0(\tanh\eta + 1)}, \tag{E.342}$$

with $\beta = a_0/\sqrt{1-k_0}$, and $0 < k_0 < 1/7$. incorporates this feature, and its influence on the creation of photons was discussed in [365]. The result, displayed in Fig.(E.11.4), shows that there is a substantial increment in the number of photons if we take into account the effect of matter creation.

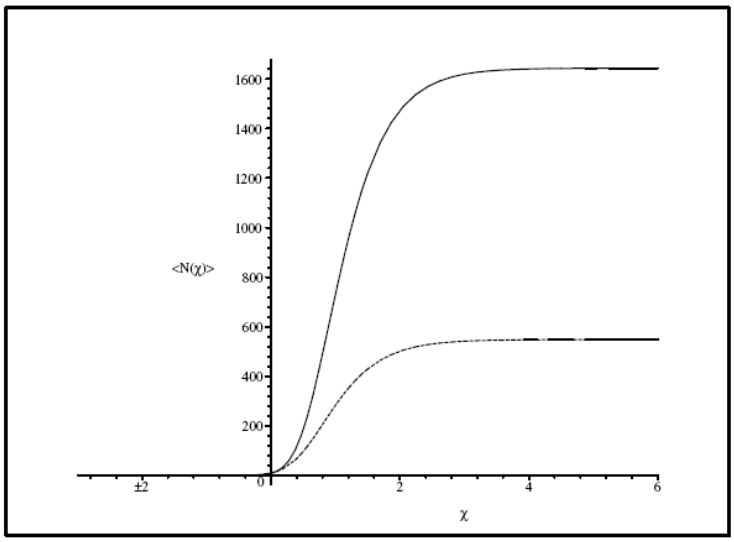


Figure E.17.: Plot of the mean number of photons as a function of the conformal time for $m = 20$, in the case without matter (dashed line) and for the case with matter creation (full line), for $\lambda = 1$.

## E.11.5. Appendix

In this appendix, we give a short summary of two gauge-invariant methods that can be applied to study the perturbations in cosmological scenarios.

### Perturbations using Bardeen variables

The fluctuations of the metric tensor can be classified by their properties under spatial rotations into scalar, vector and tensor perturbations. In the linear theory, their evolution is decoupled. In the case of scalar perturbations, the perturbed metric of a homogeneous and isotropic spacetime can be written as

$$ds^2 = a^2(\eta)\left\{(1+2\phi)d\eta^2 - 2B_{;i}d\eta\, dx^i - [(1-2\psi)\gamma_{ij} + 2E_{,i;j}dx^i dx^j]\right\}, \tag{E.343}$$

where $\gamma_{ij}$ is the metric of the 3-space. We shall sketch the case of hydrodynamical perturbations of a perfect fluid [106] with energy-momentum tensor

$$T^\alpha_\beta = (\rho + p)u^\alpha u_\beta - p\delta^\alpha_\beta. \tag{E.344}$$

Following [28], it is convenient to build, from the four variables appearing in (E.343), two gauge-invariant quantities, given by

$$\Phi = \phi + \frac{[(B-E')a']'}{a}, \qquad \Psi = \psi - \frac{a'(B-E')}{a}.$$

In terms of these, the gauge-invariant perturbed EE are

$$-3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2\Psi + 3k\Psi = \frac{1}{2}a^2\delta T_0^{(gi)0}, \tag{E.345}$$

$$(\mathcal{H}\Phi + \Psi')_{,i} = \frac{1}{2}a^2\delta T_i^{(gi)0}, \tag{E.346}$$

$$[(2\mathcal{H}' + \mathcal{H}^2\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' - k\Psi + \frac{1}{2}\nabla^2(\Phi - \Psi)]\delta^i_j - \frac{1}{2}\gamma^{ij}(\Phi-\Psi)_{|kj} = -\frac{1}{2}a^2\delta T_j^{(gi)i}, \tag{E.347}$$

where the $\delta T^{(gi)\alpha}_\beta$ are gauge invariant combinations of the $\delta T^\mu_\nu$, $B$, and $E$ (see [293] for details).

In the case of hydrodynamical matter, the most general form of the perturbation can be written in terms of the perturbed energy $\delta\rho$, the perturbed pressure $\delta p$, the potential $\mathcal{V}$ of the 3-velocity $v^i(t, \vec{x})$, and the anisotropic stress $\sigma$ as follows [28]:

$$(\delta T^\mu_\nu) = \begin{pmatrix} \delta\rho & -(\rho_0 + p_0)a^{-1}\mathcal{V}_{,i} \\ (\rho_0 + p_0)a\mathcal{V}_{,i} & -\delta p\delta_{ij} + \sigma_{ij} \end{pmatrix}.$$

For the case of a perfect fluid, with energy-momentum tensor given by Eqn.(E.344), $\sigma_{ij} = 0$.

The pressure perturbation can be split into its adiabatic and entropy components as

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_S \delta\rho + \left(\frac{\partial p}{\partial S}\right)_\rho \delta S \equiv c_s^2\delta\rho + \tau\delta S. \tag{E.348}$$

[106] For other cases, such as s scalar field, see [293].

Entropy perturbations may be important in the case of two-component systems, such as plasma and radiation.

The gauge-invariant perturbations of the energy-momentum tensor can be expressed in terms of the gauge-invariant energy density, pressure, and velocity perturbation:

$$\delta T_0^{(gi)0} = \delta\rho^{(gi)}, \qquad \delta T_i^{(gi)0} = (\rho_0 + p_0)a^{-1}\delta u_i^{(gi)}, \qquad \delta T_j^{(gi)i} = -\delta p^{(gi)}\delta^i_{\ j},$$

with

$$\delta\rho^{(gi)} = \delta\rho + \rho_0'(B - E'), \qquad \delta p^{(gi)} = \delta p + p_0'(B - E'), \qquad \delta u_i^{(gi)} = \delta u_i + a(B - E')_{|i}.$$

From Eqns.(E.345)-(E.347) applied to this case, it follows that $\Phi = \Psi$. Using Eqn.(E.348), the system can be written as

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2\mathcal{H}' + (1 + 3c_s^2)(\mathcal{H}^2 - k)]\Phi = \frac{1}{2}a^2\tau\delta S. \quad \text{(E.349)}$$

$$(a\Phi)'_{,i} = \frac{1}{2}a^2(\rho_0 + p_0)\delta u_i^{(gi)}. \quad \text{(E.350)}$$

For adiabatic perturbations, Eqn.(E.349) yields $\Phi$, which determines $\delta\rho^{(gi)}$ through Eqn.(E.345), and $\delta u_i^{(gi)}$ through Eqn.(E.350).

Eqn.(E.349) can be simplified with the change of variables

$$\Phi = \sqrt{\frac{1}{2}}\sqrt{\frac{\mathcal{H}^2 - \mathcal{H}' + k}{a^2}}\, u,$$

yielding

$$u'' - c_s^2\nabla^2 u - \frac{\theta''}{\theta}\, u = \mathcal{N},$$

with

$$\theta = \frac{1}{a}\left(\frac{\rho_0}{\rho_0 + p_0}\right)^{1/2}\left(1 - \frac{3\epsilon}{a^2\rho_0}\right)^{1/2},$$

$$\mathcal{N} = a^2(\rho_0 + p_0)^{-1/2}\tau\delta S.$$

**Vector perturbations**

The most general perturbed metric including only vector perturbations is given by [107]

$$(\delta g_{\mu\nu}) = \begin{pmatrix} 0 & -S^i \\ -S^i & F^i_{\ ,j} + F^j_{\ ,i} \end{pmatrix},$$

where the vectors $S$ and $F$ are divergenceless. From their transformation properties, it can be shown that

$$\sigma^i = S^i + \dot{F}^i$$

[107] The results quoted in this section are taken from [293].

(where the dot means derivative w.r.t. conformal time) is a gauge-invariant quantity. For the perturbations of the stress-energy tensor, we have

$$(\delta T^{\alpha}_{\beta}) = \begin{pmatrix} 0 & -(\rho_0 + p_0)V^i \\ (\rho_0 + p_0)(V^i + S^i) & p_0(\pi^i{}_{,j} + \pi^j{}_{,i}) \end{pmatrix},$$

where $V^i$ and $\pi^i$ are divergenceless. $V^i$ is related to the perturbation of the 4-velocity by

$$(\delta u^{\mu}) = \begin{pmatrix} 0 \\ V^i/a \end{pmatrix}.$$

The gauge-invariant quantities are given in this case by $\theta^i = V^i - \dot{F}^i$ and $\pi^i$. Adopting the Newtonian gauge (in which $F = 0$), from the perturbed EE we get

$$-\frac{1}{2a^2}\nabla^2 S^i = (\rho + p)V^i, \tag{E.351}$$

$$-\frac{1}{2a^4}\nabla_t(a^2(S^j{}_{,i} + S^i{}_{,j})) = p(\pi^i{}_{,j} + \pi^j{}_{,i}), \tag{E.352}$$

where $\nabla^2$ is the spatial Laplacian. From Eqn.(E.351) we get

$$V^i_k = \frac{1}{2a^2(\rho + p)}k^2 S^i_k, \tag{E.353}$$

for the Fourier modes of $V$ and $S$. Assuming that there are no anisotropic stresses, as in the case of pressureless dust, we get from Eqn.(E.352),

$$\nabla_t(a^2 S^i{}_k) = 0.$$

Hence $S^i{}_k = C^i{}_k/a^2$, where $C$ is a constant. From this and Eqn.(E.353), we get

$$V^i_k \propto \frac{k^2 C^i{}_k}{a^{1-3\omega}}. \tag{E.354}$$

Note that $V^i_k$ increases for $\omega = 0$, and stays constant for radiation, but $S^i{}_k$ always increases for decreasing $a$.

**Tensor perturbations**

These perturbations are built using a symmetric 3-tensor $h_{ij}$ which satisfies the constraints

$$h^i{}_i = 0 \qquad h^{|j}_{ij} = 0,$$

in such way that the metric for tensor perturbations is

$$(\delta g^{(t)}_{\mu\nu}) = -a^2(\eta)\begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}.$$

From the perturbed EE we find (see for instance [294])

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \triangle h_{ij} = 2a^2\delta T^{(gi)T}_{ij},$$

where $\delta T_{ij}^{(gi)T}$ is the gauge-invariant "pure tensor" part of $\delta T_{\mu\nu}$. In Fourier space, and introducing the rescaled variable $h_{ij} = e_{ij}v/a$, we have

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0. \tag{E.355}$$

### The quasi-Maxwellian method

The QM method has its roots in the formulation of Jordan and his collaborators [225] and uses the Bianchi identities to propagate initial conditions. The basic idea is to identify gauge invariant quantities from the beginning, using Stewart's lemma [389], which guarantees that the perturbation of an object $Q$ is gauge-invariant if $Q$ is either constant or a linear combination of $\delta^\mu_\nu$ with constant coefficients. In conformally flat models, the Weyl tensor (defined below) is identically zero, so its perturbation is a true perturbation, and not a gauge artifact. We shall see below how to obtain a minimum set of variables to completely characterize a perturbation, along with their evolution equations.

**Definitions and notation**

The Weyl conformal tensor is defined by means of the expression

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - M_{\alpha\beta\mu\nu} + \frac{1}{6} R g_{\alpha\beta\mu\nu},$$

where

$$g_{\alpha\beta\mu\nu} \equiv g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}, \tag{E.356}$$

and

$$2M_{\alpha\beta\mu\nu} = R_{\alpha\mu} g_{\beta\nu} + R_{\beta\nu} g_{\alpha\mu} - R_{\alpha\nu} g_{\beta\mu} - R_{\beta\mu} g_{\alpha\nu}. \tag{E.357}$$

The 10 independent components of the Weyl tensor can be separated in the electric and magnetic parts, defined (in analogy with the electromagnetic field) for an observer with 4-velocity $v^\mu$ as:

$$E_{\alpha\beta} = -W_{\alpha\mu\beta\nu} v^\mu v^\nu, \tag{E.358}$$

$$H_{\alpha\beta} = -W^*_{\alpha\mu\beta\nu} v^\mu v^\nu. \tag{E.359}$$

The dual operation was performed with the completely skew-symmetric Levi-Civita tensor $\eta_{\alpha\beta\mu\nu}$. From the symmetry properties of the Weyl tensor it follows that the operation of taking the dual is independent on the pair in which it is applied.

It follows from these definitions that the tensors $E_{\mu\nu}$ and $H_{\mu\nu}$ are symmetric, traceless and belong to the three-dimensional space orthogonal to the observer with 4-velocity $v^\mu$, that is:

$$E_{\mu\nu} = E_{\nu\mu}, \qquad E_{\mu\nu} v^\mu = 0, \qquad E_{\mu\nu} g^{\mu\nu} = 0, \tag{E.360}$$

and similar relations for $H_{\mu\nu}$. The metric $g_{\mu\nu}$ and the vector $v^\mu$ (tangent to a timelike congruence of curves $\Gamma$) induce a projector tensor $h_{\mu\nu}$ which separates any tensor in terms of quantities defined along $\Gamma$ plus quantities defined on the 3-dimensional space orthogonal to $v^\mu$. The tensor $h_{\mu\nu}$, defined on this 3-dimensional space is symmetric and a true projector, that is

$$h_{\mu\nu}h^{\nu\lambda} = \delta_\mu{}^\lambda - v_\mu\, v^\lambda = h_\mu{}^\lambda. \tag{E.361}$$

We shall work with the FLRW geometry written in the standard Gaussian coordinate system:

$$ds^2 = dt^2 + g_{ij}dx^i dx^j \tag{E.362}$$

where $g_{ij} = -a^2(t)\gamma_{ij}(x^k)$. The 3-dimensional geometry has constant curvature and thus the corresponding Riemannian tensor ${}^{(3)}R_{ijkl}$ can be written as

$${}^{(3)}R_{ijkl} = \epsilon\gamma_{ijkl}.$$

The covariant derivative in the 4-dimensional space-time will be denoted by the symbol ";" and the 3-dimensional derivative will be denoted by "$\|$".

The irreducible components of the covariant derivative of $v^\mu$ are given in terms of the expansion scalar ($\theta$), shear ($\sigma_{\alpha\beta}$), vorticity ($\omega_{\mu\nu}$) and acceleration ($A_\alpha$) by the standard definition:

$$v_{\alpha;\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta} + \omega_{\alpha\beta} + A_\alpha v_\beta, \tag{E.363}$$

where

$$\begin{aligned}
\sigma_{\alpha\beta} &= \tfrac{1}{2}h^\mu_{(\alpha}h^\nu_{\beta)}v_{\mu;\nu} - \tfrac{1}{3}\theta h_{\alpha\beta},\\
\theta &= v^\alpha{}_{;\alpha},\\
\omega_{\alpha\beta} &= \tfrac{1}{2}h^\mu_{[\alpha}h^\nu_{\beta]}v_{\mu;\nu},\\
A_\alpha &= v_{\alpha;\beta}v^\beta.
\end{aligned} \tag{E.364}$$

We also define

$$\theta_{\alpha\beta} \equiv \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}. \tag{E.365}$$

**Quasi-Maxwellian equations of gravity and their perturbation**

We shall present in this subsection a sketch of the deduction of the equations that govern the perturbations in the quasi-Maxwellian formalism. The details of the calculations in this section can be found in [319]. Using Einstein's equations and the definition of Weyl tensor, Bianchi identities can be written in an equivalent form as

$$\begin{aligned}
W^{\alpha\beta\mu\nu}{}_{;\nu} &= \frac{1}{2}R^{\mu[\alpha;\beta]} - \frac{1}{12}g^{\mu[\alpha}R^{,\beta]}\\
&= -\frac{1}{2}T^{\mu[\alpha;\beta]} + \frac{1}{6}g^{\mu[\alpha}T^{,\beta]}.
\end{aligned}$$

The quasi-Maxwellian equations of gravity are obtained by projecting these equations (*i.e.* the Bianchi identities are taken as true dynamical equations which describe the propagation of gravitational disturbances). The evolution equation for the perturbations for $\delta\theta$, $\delta\sigma_{\mu\nu}$, and $\delta\omega^{\mu}$, as well as 3 constraint equations, are obtained projecting and perturbing the equation

$$v_{\mu;\alpha;\beta} - v_{\mu;\beta;\alpha} = R_{\mu\omega\alpha\beta} v^{\omega}$$

which follows from the definition of the curvature tensor. Finally we get two more equations by projecting the conservation law $T^{\mu\nu}{}_{;\nu} = 0$. Adding up, we have a set of twelve equations which when perturbed yield (after straighforward manipulations) the coupled differential equations needed to give a complete description of the perturbation. In a general case, the variables are

$$\mathcal{V} = \{\delta E_{ij}, \delta H_{ij}, \delta\omega_{ij}, \delta\sigma_{ij}, \delta\pi_{ij}, \delta A_i, \delta q_i, \delta\rho, \delta\theta, \delta V_0, \delta V_k\},$$

where $\delta q_i$ is the perturbation of the heat flux. From now on we will concentrate on the case of scalar irrotational perturbations. As shown in [259], it is useful to develop the perturbed quantities in the spherical harmonics basis. It is enough for our purposes to work only with scalar quantities, denoted by $Q^{(k)}(x^i)$ (with $\partial Q^{(k)}/\partial t = 0$) and the vector and tensor quantities that follow from it, defined by $Q_i^{(k)} \equiv Q_{,i}^{(k)}$, $Q_{ij}^{(k)} \equiv Q_{,i;j}^{(k)}$. The scalar $Q^{(k)}$ obeys the eigenvalue equation defined in the 3-dimensional background space by:

$$\bigtriangledown^2 Q^{(k)} = k Q^{(k)}, \tag{E.366}$$

where $k$ is the wave number, and the symbol $\bigtriangledown^2$ denotes the 3-dimensional Laplacian:

$$\bigtriangledown^2 Q \equiv \gamma^{ij} Q_{,i\|j} = \gamma^{ij} Q_{,i;j}. \tag{E.367}$$

Since the modes do not mix at the linear order, we will drop the superindex $(k)$ from $Q$. The traceless operator $\hat{Q}_{ij}$ is defined as

$$\hat{Q}_{ij} = Q_{ij} + \frac{k^2}{3} Q \gamma_{ij}, \tag{E.368}$$

and the divergence of $\hat{Q}_{ij}$ is given by

$$\hat{Q}^{ij}{}_{;j} = -2\left(\epsilon + \frac{k^2}{3}\right) Q^i. \tag{E.369}$$

Due to Stewart's lemma, the good (since they are gauge-invariant and null in the background) objects in the list $\mathcal{V}$ are $\delta E_{ij}$, $\delta\Sigma_{ij}$, $\delta\pi_{ij}$, $\delta a_i$, and $\delta q_i$. According to causal thermodynamics the evolution equation of the anisotropic pressure is related to the shear through [219]

$$\tau\dot{\Pi}_{ij} + \Pi_{ij} = \xi\sigma_{ij} \tag{E.370}$$

in which $\tau$ is the relaxation parameter and $\xi$ is the viscosity parameter. For simplicity we will take the case in which $\tau$ can be neglected and $\xi$ is a constant [108]. Eq.(E.370) then gives

$$\Pi_{ij} = \xi \sigma_{ij}, \tag{E.371}$$

and the associated perturbed equation is

$$\delta\Pi_{ij} = \xi\, \delta\sigma_{ij}. \tag{E.372}$$

We shall decompose the four independent and gauge-invariant perturbations as [109]

$$\begin{aligned}
\delta E_{ij} &= \sum_k E^{(k)}(t)\hat{Q}^{(k)}_{ij}, \\
\delta\Sigma_{ij} &= \sum_k \Sigma^{(k)}(t)\hat{Q}^{(k)}_{ij}, \\
\delta A_i &= \sum_m \psi^{(m)}(t)Q^{(m)}_i, \\
\delta q_i &= \sum_m q^{(m)}(t)Q^{(m)}_i.
\end{aligned}$$

It can be shown that $\psi$ is a function of $\Sigma$ and $E$ [319]. It follows that, restricting to the case $q = 0$ (no energy flux) [110], $E(t)$ and $\Sigma(t)$ constitute the fundamental pair of variables in terms of which the dynamics for the perturbed FLRW geometry is completely characterized. Indeed, the evolution equations for these two quantities (which follow from Einstein's equations) generate a dynamical system involving only $E$ and $\Sigma$ (and background quantities) which, when solved, contains all the necessary information for a complete description of all remaining perturbed quantities of the FLRW geometry.

The evolution equations are given by [319]

$$\dot{\Sigma} = -E - \frac{1}{2}\,\xi\,\Sigma - k^2\,\psi, \tag{E.373}$$

$$\begin{aligned}
\dot{E} &= -\frac{(1+\lambda)}{2}\rho\,\Sigma - \left(\frac{\theta}{3} + \frac{\xi}{2}\right) E \\
&- \frac{\xi}{2}\left(\frac{\xi}{2} + \frac{\theta}{3}\right)\Sigma - \frac{k^2}{2}\,\xi\,\psi.
\end{aligned} \tag{E.374}$$

[108] In the general case $\xi$ and $\tau$ are functions of the equilibrium variables, for instance $\rho$ and the temperature $T$ and, since both variations $\delta\Pi_{ij}$ and $\delta\sigma_{ij}$ are expanded in terms of the traceless tensor $\hat{Q}_{ij}$, it follows that the above relation does not restrain the kind of fluid we are examining. However, if we consider $\xi$ as time-dependent, the quantity $\delta\Pi_{ij}$ must be included in the fundamental set $\mathcal{M}_{[A]}$.

[109] In fact, $\sqrt{\delta E_{ij}\delta E^{ij}}$ is the only quantity that characterizes without ambiguity a true perturbation of the Debever invariants [319].

[110] We further assume an equation of state relating the pressure and the energy density, i.e. $p = \lambda\rho$, which is preserved under arbitrary perturbations.

As mentioned before, $\psi$ can be expressed in terms of $E$ and $\Sigma$ [111]:

$$(1+\lambda)\,\rho\,\psi = 2\left(1+\frac{3\epsilon}{k^2}\right)\,a^{-2}\,[-\lambda E + \frac{1}{2}\,\lambda\,\xi\,\Sigma + \frac{1}{3}\,\xi\,\Sigma]. \tag{E.375}$$

Thus the set of perturbed equations reduces to a time-dependent dynamical system in the variables $E$ and $\Sigma$:

$$\dot{\Sigma} = F_1(\Sigma, E), \qquad \dot{E} = F_2(\Sigma, E), \tag{E.376}$$

with

$$F_1 \equiv -E - \frac{1}{2}\xi\,\Sigma - k^2\,\psi, \tag{E.377}$$

and

$$\begin{aligned} F_2 \;\equiv\; & -\left(\frac{1}{3}\theta + \frac{1}{2}\xi\right)\,E - \frac{k^2}{2}\,\xi\,\psi \\ & -\left(\frac{1}{4}\xi^2 + \frac{(1+\lambda)}{2}\rho + \frac{1}{6}\xi\theta\right)\,\Sigma \end{aligned} \tag{E.378}$$

where $\psi$ is given in terms of $E$ and $\Sigma$ by Eqn.(E.375), so the system (E.376) can be written as

$$\begin{pmatrix} \dot{E} \\ \dot{\Sigma} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E \\ \Sigma \end{pmatrix}, \tag{E.379}$$

where

$$\alpha \equiv -\frac{\theta}{3}, \quad \beta \equiv -\frac{1+\lambda}{2}\rho, \quad \delta = 0, \quad \gamma = \frac{6\lambda}{1+\lambda}\left(\epsilon + \frac{k^2}{3}\right)\frac{1}{a^2\rho} - 1.$$

Since

$$\frac{\partial \dot{E}}{\partial E} + \frac{\partial \dot{\Sigma}}{\partial \Sigma} = -\frac{\theta}{3},$$

the system (E.379) is not Hamiltonian due to the expansion of the universe. Nonetheless, new variables $(Q, P)$ can be introduced in such a way that the system (E.379) is Hamiltonian. Defining

$$Q \equiv a^m \sigma, \qquad P = a^n E,$$

it is easily shown from the Poisson brackets that the otherwise arbitrary powers $m$ and $n$ must satisfy the relation $m + n = 1$ for the variables $Q$ and $P$ to be canonically conjugated. It follows that

$$\ddot{P} = \mathcal{M}_1 P + \mathcal{M}_2 Q.$$

The choice $n = 3\lambda/2 + 2$ yields $\mathcal{M}_2 = 0$, and $P$ satisfies the equation

$$\ddot{P} + \mu(t)P = 0,$$

[111] Except when $(1+\lambda) = 0$, see [319] for this case.

with

$$\mu(t) = \left(\frac{5}{4}\lambda + \frac{2}{3}\right)\rho + \frac{1}{a^2}\left[\frac{3\lambda}{2}\left(\frac{3\lambda}{2}\right)\epsilon - \lambda k^2\right],$$

which is equivalent to Eqn. (E.282).

This method can be extended to vector and tensor perturbations in the FLRW model [320]. In the first case, the observable quantities are described in terms of the vorticity and the shear, while the electric and magnetic parts of the Weyl tensor suffice for the gravitational waves [112]. The three types of perturbation are describable in Hamiltonian form, thus paving the way to canonical quantization [321], which was performed for scalar, vectorial, and tensor perturbations using the squeezed states formalism in [321]. In fact, in the case of scalar perturbations, the Hamiltonian in terms of the $(Q, P)$ variables (with the choice $m = 0$) is given by

$$H = \frac{h_1}{2}Q^2 + \frac{h_2}{2}P^2 + \frac{h_3}{Q}P,$$

with

$$h_1 = \frac{1+\lambda}{2}\frac{\rho}{a}, \quad h_2 = \frac{6\lambda}{1+\lambda}\left(\epsilon + \frac{k^2}{3}\right)\frac{1}{a\rho} - a, \quad h_3 = 0.$$

### E.11.6. Relation between the two methods

The Bardeen variables $(\Phi, \Psi)$ are related to the quasi-Maxwellian variables $(E, \Sigma)$. For instance, in the case of scalar perturbations the relation between $E$ and $\Phi$ (for a perfect fluid) is given by [189, 224]

$$E = -k^2\Phi,$$

from which the relation for the spectrum given in Eqn.(E.294) follows.

## E.12. Conclusion

The idea of a bouncing universe has been considered since the early days of relativistic cosmology, as shown in this review. However, only a few analytical solutions describing a nonsingular universe served as a starting point to build a complete cosmological scenario. The main reason for this neglect by the majority of the physics community in the last 30 years of the 20th century was the strong influence of the singularity theorems, which led to the belief that some sort of singularity was inevitable in gravitational processes. The situation should have changed with the recently discovered positive acceleration of the universe since, in the realm of GR, the accelerated expansion means that the matter content must satisfy the condition $\rho + 3p < 0$, which is precisely one of the conditions needed to have a bounce in Einstein's gravity. This violation

[112] Perturbations in the Kasner solution were studied in [322].

of the SEC was already accepted in the early 80's in order to have a phase of inflationary expansion, and nowadays several systems are known which do not satisfy the inequality $\rho + 3p > 0$ (see for instance [27]). Hence, there is mounting evidence against one of the main theoretical prejudices forbidding bouncing universes in GR. Surprisingly, nonsingular models have not attracted the interest that should be expected based on the preceding considerations [113].

Almost contemporaneous to the discovery of the accelerated expansion was the gradual advent of a handful of cosmological models based on nonsingular solutions. These models aimed at solving the most stringent problems of the (pre-inflationary) cosmological standard model: the initial singularity, the isotropy and homogeneity of the currently-observed universe, the horizon problem, the flatness problem[114], and the formation of structure [115]. Bouncing universes have partially met these challenges. The singularity is obviously absent, and its avoidance requires any of the assumptions listed in Sect.E.1.1, which range from the violation of SEC (in GR) to quantum gravitational effects.

As explained in Sect.E.1, a phase of accelerated contraction may solve the flatness problem in GR, and may also get rid of particle horizons (see for instance [179]) [116].

Finally, the amplification of primordial seeds (a problem prior to the formation of structure) in bouncing universes has been intensely debated recently (see Sect.E.11). The asymptotic behavior of these universes is markedly different from that of the SCM or inflation. The universe at past infinity starts to collapse from a flat empty structure-less state that at past infinity can be approximated by Minkowski geometry written in terms of Milne coordinates [117]. The transmission of the quantum fluctuations from this initial state to the post-bounce phase is strongly model-dependent, but there are some models which yield a scale-invariant spectrum for the scalar perturbations in the post-bounce phase (see Sect.E.11.1).

An offspring of the bouncing models are the cyclic universes (see Sect.E.10). The cyclic models also attempt to solve the above-mentioned problems, and also may offer a new view on the initial conditions: since by definition, there

[113] It may be argued that this lack of interest is due to the fact that the bounce is expected to involve scales where quantum effects render GR inapplicable. But this is true also of the singularity theorems, as was known already in the early 70's. Moreover, there is no evidence against the possibility of a bounce in the classical regime [316], as follows from some of the models presented in Sect.E.11.1, see also [158].

[114] Note that the flatness problem may in principle not be a problem in gravitational theories other than GR (see Sect.E.2.2).

[115] In spite of its historical importance, the so-called monopole problem is not included in this list, since there is still room for it to be be considered as a problem of field theory first, and then (perhaps) of the standard cosmological model, see for instance [264, 122].

[116] See however the concerns in [89] about the efficiency of some bouncing models in erasing possible initial inhomogeneities.

[117] We have also seen that there are eternal (non-bouncing) universes, that start in a de Sitter regime.

is neither a beginning nor an end of time in these models, there is no need to specify initial conditions. Generically, cyclic universes share the problems of the universes that bounce only once. In addition, they must assure that the large scale structure present in one cycle (generated by the quantum fluctuations in the preceding cycle) is not endangered by perturbations or structure generated in earlier cycles, and will not interfere with structure generated in later cycles. One of the latest cyclic models, presented in [155], claims to have successfully faced these issues (however see [265]).

As compelling a scenario may (or may not) seem, the ultimate judge is observation, so we can ask if there are any that may point to the occurrence of a bounce. As far as we know, there are two possibilities [118]:

- As discussed in Sect.E.11.1, the tensor spectrum of a nonsingular universe has a unique feature. As an example, the SPBB models predict a stochastic spectrum of gravitational waves whose amplitude increases as a function of frequency in some frequency ranges (see Sect.E.11.1), hence avoiding the bounds due to the CMB, pulsar timing, and Doppler tracking [276]. The parameter space of the "minimal" SPBB model [175] was limited using LIGO results in [276]. Notice also that nonsingular universes may produce vector perturbations (see Sect.E.11.1).

- The bounce may cause oscillations, that will be superimposed on the power spectrum of scalar perturbations. These oscillations would also appear in the WMAP data, linked to the spectrum through the multipole moments which are in turn defined through the two-point correlation function of the temperature fluctuations. [281]. Let us note however, that such oscillations may be due not only to a bounce, but also to transplanckian effects [281] or to non-standard initial conditions in the framework of hybrid inflation [90].

We would like to close by pointing out that although they do not yet give a complete description of the universe, a better understanding of bouncing models in classical GR should be attempted since they are inevitably imposed upon us by the apparently observed violation of the strong energy condition. It must also be noted that there are at least two more reasons to attempt this task. First, the current solution to the problems of the standard cosmological models (namely inflation) is successful, but has several problems (see Sect.E.1). Second, even if bouncing models do not succeed in yielding a complete description of the universe (thus offering an alternative to inflation [119]), they may throw light upon the singularity problem (an issue in which inflation has nothing to say).

[118] Some bouncing models in GR were severely restricted in [390], using SNIa data, CMB analysis, nucleosynthesis, and the age of the oldest high-redshift objects.

[119] The comparison of bouncing models with the inflationary scenario has been undertaken in several articles (see for instance Refs.[179] and [263]).

Summing up, we have seen in this review that bouncing universes have some attractive features, but they are not complete yet: much work is needed to achieve a stage in which their predictions can match those of the cosmological standard model. Therefore, we hope this review encourages further developments in nonsingular cosmologies.

## E.13. Acknowledgements

The authors would like to thank all the participants of the *Pequeno Seminário* at CBPF for interesting discussions on some parts of this review, and J. Salim and specially N. Pinto-Neto for the reading of some chapters and discussions. MN would like to acknowledge support from CNPq and FAPERJ. The authors would like to acknowledge support from ICRANet-Pescara for hospitality during some stages of this work.

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## E.14. BSCG (Brazilian School of Cosmology and Gravitation): 30 years

**MARIO NOVELLO**

### A Revolution in Science: The Expansion of Cosmology

In the second half of the 1970's, the attention of physicists was drawn to processes of a global nature, namely cosmic processes. This was ensued by intense activity throughout the community of physicists in various areas, many of whom were led to migrate to Cosmology. Such a broad and intense displacement, involving so many scientists, requires proper sociological analysis of the scientific practice in order to provide insights into the transformation Cosmology was going through and to the changes in the traditional mode cosmological studies had been conducted until then.

This activity produced numerous proposals of solutions to some cosmological problems and prompted a reformulation of traditional questions of Physics, thanks to the reliability that could be attributed to the cosmic way of investigating nature, a fact acknowledged by the international scientific community

Up until the late 1960s, Cosmology attracted very little interest, apart from a small group of scientists working in the area. There are several reasons one could attribute to the causes of this lack of interest. Though dissemination of activities in Cosmology had started in that decade, the 1970s could be considered the split between one attitude and the other, and the popularization of Cosmology in the overall community of physicists was achieved in the 1980s. In fact, it was in this decade that major conferences brought cosmologists, astronomers, relativist astrophysicists (traditionally, those who dealt with the Universe in its totality), and theoretical high energy physicists (who examined the microcosmos of elementary particles) together in a single event. One remarkable example was the US Fermilab 1983 Conference, which was given the suggestive title of Inner Space / Outer Space.

There have been concerted reasons contributing for this growth in Cosmology, some of which are intrinsic to this science while others are totally independent of it. This is not the place for such an inventory, but, just for clarifying purposes, one could give two examples. One, internal to Cosmology, is related to the success of the new telescopes and space probes, which yielded a huge amount of highly-reliable new data. Another, of an extrinsic nature, was the crisis of elementary particles physics in the 1970s, which, for the purposes of its own development, required the construction of huge and extraordinarily expensive high-energy accelerators, which faced political hindrances in Europe and in the United States.

The evolutionary character associated to the geometry discovered by Russian mathematician A. Friedmann, who described a dynamic expanding Universe, was thus the territory of choice to substitute in the minds of high-energy physicists, for the lack of particle accelerators, machines that could not be accomplished due to financial reasons. Such displacement was associated with the successes of Cosmology. Indeed, the standard model of the Universe was based on the existence of a configuration that described its material content as a perfect fluid in thermodynamic balance, whose temperature scaled as the inverse of the expansion; that is, the smaller the Universes total spatial volume, the greater the temperature. So, in the early times of the current expansion phase,

the Universe would have experienced fantastically high temperatures, thereby exciting particle states and requiring the knowledge of the behavior of matter in situations of very high energies for its description. And, most conveniently, for free, without costs: all it took was to look at the skies.

It was within this context that the Brazilian School of Cosmology and Gravitation, BSCG, became a national and international endeavor , promoting the interaction between different physicists communities, involving astronomers, relativists, cosmologists, and theoretical high-energy physicists. It may not be an overstatement to say that the history of Cosmology in our country may be revealed through the analysis of the history of the BSCG.

## Moving Toward a Second Copernican Revolution?

The booming interest for Cosmology, as recorded in the past few decades, has yielded several consequences, but perhaps the most remarkablethough not yet recognized as such shall be that it is inducing an effort to re-found Physics. To mention but one example that can help us understand the meaning of this re-founding, we could refer to Electrodynamics.

The success of Maxwells linear theory in describing electromagnetic processes was remarkable along the 20th Century. The application of this theory to the Universe, within the standard scenario of spatial homogeneity and isotropy, produced a number of particular features , including some unexpected ones. Among the latter, the one with most formidable consequences was the demonstration that the linear theory of Electromagnetism inevitably leads to the existence of a singularity in our past. That is, the Universe would have had a finite time to evolve and reach its current state.

This was the single most important characteristic of the linear theory since it led to the acceptance, in the scientists imagination, that the so-called theorems of singularity discovered in the late 1960s would, in effect, be applicable to our Universe.

However, in the following decade, a slightly more profound criticism changed this interpretation, thus rendering the consequences of theorems less imposing. This involved a lengthier r analysis of the mode through which the electromagnetic field is affected by the gravitational interaction. That it is affected, there had been no doubt, because this property was at the basis of the very theory of General Relativity, given that the field carries energy. What was yet to be learned, in detail, , was how to describe this action and which qualitative differences t the participation of the gravitational field could provoke. It soon came out that there was no single mode to describe this interaction. This is due to the vectorial and tensorial nature of electromagnetic and gravitational fields, respectively. Several proposals for this interaction were then examined.

One of these changes to Electromagnetism, motivated by the gravitational field, seemed to be somehow unrealistic because it could be naively interpreted as if the field transporter, the photon, acquired a mass in this process of interaction

with the geometry of space-time, through its curvature. Moreover: this mass would depend on the intensity of this curvature. In fact, to adhere to the technical terminology, it was a non-minimal coupling between both fields: a mode of interaction that does not allow the behavior of the electromagnetic field to be reducedby using the Principle of Equivalenceto the structure that this field possesses in the idealized absence of a gravitational field. , This coupling radically changes the properties of the geometry of the Universe in the spatially homogeneous and isotropic scenario. Just to mention a new and remarkable characteristic, the electromagnetic field, under this mode of interaction with the gravitational field, produces an Eternal Universe, without singularity, without beginning, extending indefinitely to the past. It is not difficult to show that this interaction also generates a non-linearity of the electromagnetic field.

This property led the way to think about other non-linear feature of the electromagnetic field where this form of interaction with gravitation was not dominant. These features did not correspond to non-linear corrections to Maxwells Electromagnetism such as those obtained by Euler and Heisenberg, of quantum origin, though they could contain them. Regardless of these possibilities allowed by the quantum world, physicists started to think about other origins for the non-linearity: they should be thought as if Maxwells equations with which Electromagnetism had been treated this far would be nothing more than approximations of a more complex form associated to a non-linear description. This non-linearity should appear as a cosmic mode of the field, where linearity is locally an approximation, thereby inverting the traditional way of thinking non-linearity as corrections to the basic linear theory.

This simple example allows for the introduction of a fantastic situation that Cosmology would be producing and that we can synthesize in a small sentence of great formal consequences: the extrapolation of terrestrial Physics to the entire Universe should be reviewed.

The old generalization mode is a rather natural and common procedure among scientists. Thus, by extrapolation, even in conditions that have never been tested before, we go on legislating until new physics can stop, block, limit this extension of the local scientific knowledge .

In other words, the considerations above seem to point to the need of a new Copernican criticism. Not quite the one that removed us from the center of the Universe, but another, arguing against the extrapolation scientists have been resorting to. That is, to think that a global characteristic should not be attributed to the Laws of Physics and that, from this perspective, the action of discarding global cosmological processes in building a complete theory of natural phenomena would be legitimate.

That is, these Laws may take forms and modes that are different from those with which, in similar but not the same situations, "terrestrial Physics was successfully developed. This analysis, that may lead to a description different from that physicists are used to, which becomes more and more necessary, even indispensable, is what we refer to as re-founding Physics through Cosmology.

We may quote English physicist P.A.M. Dirac and Brazilian physicist C. Lattes as recent precursors of this way of thinking. Unfortunately, the practical mode they proposed for a particular re-foundation was too simple, thus allowing for a powerful reaction that shunned these ideas to the bordering and swampy terrain of speculation. Recent and formidable advances in Observational Cosmology allow us to accept that the time is coming when an analysis of this re-foundation, slightly more sophisticated than that simple modification of the fundamental constants as Dirac and others intended, may be seriously undertaken.

## Antecedents of the Brazilian School of Cosmology and Gravitation

At the end of January, 1971, my post-doctorate supervisor in Oxford, the renowned scientist Denis Sciama, invited me for a meeting at the All Souls College to which some scientists who worked in his research group were also invited (R. Penrose, S. Hawking, G. Ellis, W. Rindler, among others–Apart from myself, of all these, only Penrose and Rindler showed up). The goal was to informally discuss some major issues of Physics, particularly those related to a science that was experiencing intense activity back then: Cosmology. In a given moment of that meeting, Sciama said he considered it important that we participated in the first major School of Cosmology that the French were organizing for the coming summer, possibly July, in a beautiful place in the Mediterranean, in the small island of Cargse, Corsica.

It was a very special situation and it came at a crucial moment of my decision to dedicate myself to Cosmology. One week before, when I had participated in a conference at the International Center for Theoretical Physics (ICTP) in Trieste, I had talked to a CBPF physicist who had just arrived from Brazil and made some comments on my decision to dedicate to Cosmology that caused me to become apprehensive. His comments were that a decision had been made that it would be very important for Brazil and the CBPF that I shifted my interests and started a program to guide my research efforts to a more useful area for the country, such as some sector of solid state physics. And, he added, renewal of my doctorates scholarship could depend on my decision. This type of action was not uncommon in those days. I dont know whether such interference would happen today. At least, not with that lack of subtlety! My decision had already been made and my scholarship was renewed, particularly thanks to a Brazilian physicist who worked in Geneva like myself, though he was not at the Geneva Universitys Institut de Physique but rather at CERN: Roberto Salmeron. After learning of the evolution of my dissertation work, he told me he would be supporting my decision to choose a path that looked totally estranged from the major motivation of most scientists, that is, Cosmology. If I allow myself to wander a bit into this incident, it is just to show the general state of affairs a scientist had to overcome back then in order to address Cosmology. Curiously enough, less than ten years later, Cosmology started a formidable phase of expansion, and has attracted an ever bigger number of scientists since

then. Having said that, let us go back to Corsica.

The Cargèse School was an enormous success. In attendance were great names of Cosmology coming not only from England and the United States, such as Schucking, Silk, Steigman, Harrison, Rees, Ellis, and others, but also some European ones, particularly professor Hagedorn, who was very successful at the time with his theory that postulated the existence of a maximum temperature inducing a new perspective on the singularity of the standard model.

For various reasons, the big names missing in that meeting were the representatives of the Soviet School of Cosmology who, nevertheless, attracted my attention because their approach seemed to be more imaginative than the conventional proposals by European and American physicists. However, they were the ones who eventually commanded the thoughts of the western community of scientists for the coming decades, with some beautiful exceptions.

A simple and superficial exam of the list of lecturers that participated in the BSCG shows that this Russian School has been really active, from the very first meeting to date. Thus, the BSCG have managed to popularize, especially among Brazilian scientists, many ideas from those Soviet physicists and, later, from the Russian community. The peculiarity and originality of this Russian School have marked this unique participation and often allowed it to become the main outlet for ideas that are alternative to the ones dominating the panorama of Cosmology. To share a particular and extremely relevant example, suffice it to mention the course program offered in 1979 at the II BSCG, in Joo Pessoa, by Professor Evgeni Lifshitz who, based on his previous efforts with V. Belinski and I. Khalatnikov, addressed the way in which the Universe behaved in the vicinities of a singularity, raising a daring hypothesis of the existence of a primordial anisotropic phase. Nearly thirty years later, in the most recent Marcel Grossmann Congress held in 2006 in Berlin, one of the plenary sessions conducted by the French physicist Thibault Damour attempted to revive the original ideas by Belinski-Lifshitz-Khalatnikov , adapting them to modern proposals of cosmological investigation.

The Cargèse School lasted two wonderful weeks, under the happy and casual coordination of Professor E. Schatzman. Himself an enthusiast of scientific communication, he brought together the young participants , sometimes at the beach and others at tiny Cargèse's downtown area, for some beautiful starry evenings of explanations to awed locals about recent discoveries in Astrophysics and Cosmology . After a brief introduction to the behavior and structure of stars and galaxies, our coordinator urged listeners to ask questions of all sorts to the scientists. Those questions were never limited to Astrophysics, Cosmology, and Physics in general; they rather and inevitably overflowed into a scientists social role, a theme Schatzman was passionate about.

In one such evening, feeling the cold breeze from the sea, concentrated around a small bonfire, I told him that the meeting had been so exciting to me, so informative, and such a unique experience, that I would try to organize similar meetings as soon as I got back to my country. Being so kind and heedful of

others, as usual, he committed himself by saying that I could certainly count on his support, adding one question about the number of scientists working in that area in Brazil. I answered that though there were d a few physicists working in isolation who could follow up on the development of modern properties of the gravitation theory, there was nothing systematic going on in my country. He then added that if the idea was to be successful, I should try to create first a small core composed by young scientists who were to receive solid training in the theory of gravitation and r one or two years of Cosmology studies. When I returned to CBPF, in the second semester of 1972, that was exactly what I did, creating the Gravitation and Cosmology Group of CBPF, which turned up to be the seed of todays Institute of Cosmology Relativity and Astrophysics (ICRA).

**First School: Success of the Teacher-Student Interaction**

During the year of 1976, the Brazilian Center for Research in Physics went through a radical change. Aware of the constant difficulties posed to a special institution such as the CBPF, focused on fundamental research, the federal government finally accepted to integrate this center to a federal agency. The CBPF thus became the first physics research institute to be directly incorporated to the National Research Council (CNPq), currently the National Council for Scientific and Technological Development.

The CBPF started its new phase with the arrival of Antonio Csar Olinto, designated as head of the new CBPF/CNPq. It was within this framework of renewal that Cosmology conquered its space and came forth as a new area of the endeavors of CBPF. The history of this period is rich in debates between personalities who built the history of Physics in Brazil, but I will talk about it in another occasion. Of our interest here is only the outcome, as the head of the CBPF agreed to grant financial and institutional support to the First Brazilian School of Cosmology and Gravitation, which would later be known as Brazilian School of Cosmology and Gravitation when it went international, therefore acquiring the acronym BSCG.

This School was divided into two parts, involving basic programs that lasted a full week, and advanced seminars whose classes could be limited to one up to three sessions at most. Interestingly enough, the BSCG is structured as such, to date.

The budget of the School was very small, as it was basically funded by the CBPF. However, the enthusiasm of the students was such that turned it into a major success, contrary to the pessimistic forecast of various colleagues. To mention but one example of this important student co-participation, I recall their performance in organizing the School texts. Though the faculty had carefully prepared their class notes, we had no possibility to print them. The solution was then offered by the students themselves: they mimeographed the notes, created a strongly-yellow-colored cover and manually bound all of the texts!

This willpower on the part of the students greatly encouraged the staff, who

then spent the entire School in permanent activity, thus producing a student-teacher interaction that lingered on as a hallmark and operated as a trigger for CBPFs director to convince the relevant authorities (CNPq, Capes) to provide the funds in the subsequent year for the 2nd School, much more complete and administratively more organized than the 1st .

Both the 1st and the 2nd School (held respectively in 1978 and 1979) were a means to consolidate the basic structure of Gravitational Theory for our young physicists, as well as the crucial mathematical tools and techniques for a better understanding of the General Theory of Relativity. Besides this basic endeavor, some crucial concepts of theories that are correlated with Gravitation and the General Theory of Relativity involving rudiments of the Unified Theories and some basic aspects of Relativistic Astrophysics were discussed. This may be confirmed with an overview of the course programs offered for the 2nd School.

In the 3rd and 4th Schools (held in 1982 and 1984, respectively), notions of Astrophysics presented in the previous Schools were elaborated. Furthermore, there was a focus on the study of the Theory of Elementary Particles and its last association with the so-called Standard Model of Cosmology, identified with the notion of an explosive and hot start for the Universe (known in the literature as the Hot Big Bang Hypothesis).

**The Internationalization**

In 1987, the 5th School of Cosmology and Gravitation could increase the knowledge base and the analysis presented in the previous Schools, thus evolving to a broader and deeper debate of the feasible potential alternatives to explain the large scale behavior of the Universe. Back then, courses based on the Standard Model were presented, as well as several talks dealing with the idea of an Eternal Universe, without beginning or end. Besides these specific approaches, the relation between Quantum Physics and Gravitation was examined in detail. Though this union is still far from being complete, the basic ideas involving quantum principles of gravitation were presented in the 5th School that were later developed in the 6th School.

The 5th School was also the first one opened to the international scientific community: researchers and students from twenty-four (24) countries were enrolled and, from that 5th edition onward, the Schools name became international and it was then renamed as the Brazilian School of Cosmology and Gravitation. The lectures presented there also reflected this internationalization.

The ideas preliminarily presented in the previous School were developed during the two weeks of the 6th School of Cosmology and Gravitation, in 1989. The courses underlined the emphasis given to quantum processes in Cosmology. That fact is a natural evolution of the previous events, reflecting the important role played, even then, by the examination of quantum processes in Cosmology. Besides these course programslasting a week eachsmall working meetings were held as parallel courses. Amongst these additional events, two were par-

ticularly important: the opening of a session of student-participant seminars, thus promoting greater interaction between them and exhibitors; the start of an extraordinary debate session where the ten presenting professors individually exposed their ideas on the main current issues of Cosmology and related areas. This experience was so satisfactory that it was integrated in the organization of subsequent Schools.

In 1991, due to financial difficulties of the countrys Science/Technology system, the periodicity of the Brazilian Schools of Cosmology and Gravitation could not be maintained. Nevertheless, in order not to hinder an entire generation of young scientists, a small meeting was held at CBPF: A Crash-Course on July 15-26, 1991, whose program was as follows: Cosmology: M. Novello Gravitation: I.D.Soares Relativist Thermodynamics: J. M. Salim Hamiltonian Formulation of Gravitation: N. Pinto Neto Quantum Theory of Fields with Curved Spaces: N. F. Svaiter This crash-course was attended by 79 student/grantees of different Brazilian universities and was an important basis for later studies and projects.

**New models on the creation of the Universe**

In 1993, the 7th Brazilian School of Cosmology and Gravitation was again held in two weeks. Besides presenting an overall panorama of the main conquests and unresolved issues of Cosmology today, this School enabled the continued discussion on a most formidable issue : the creation of the Universe. The main novelty was due to a general change in the scientists behavior concerning the remote past of our Universe: whereas up until recently the role of an explanation generator for all of natures ulterior processes was attributed to an inaccessible initial explosion, back then several competing proposals started to appear in search for access to the issue of creation, both the classical and the quantum ones. So, models of the Eternal Universe without singularity were discussed in this School, at various moments. There was, however, general consensus that the Universe would have been through an extremely hot period. It means that either a process of quantum tunneling or a previous classical collapsing phase should provide the conditions for a likely moment of tremendous concentration of matter/energy. Different proposals of that sort were examined in the courses and seminars of this School.

The 8th Brazilian School of Cosmology and Gravitation, held in 1995, consolidated the international nature of the School, not only for the fact that it involved professors who enjoyed high prestige in the international scientific community but also, and mostly, because of the large number of student-participants coming from other countries. In this School, special emphasis was given to quantum processes and their consequences in an expanding Universe. Not only quantum processes of matter in classical background (semi-classical approach) were examined but also different proposals for quantum treatment of the very gravitational field were proposed. The recent attempts to explain the existence and formation of major structures (galaxies, clusters etc.) were also examined and

discussed either from a more observational and classical perspective or through elementary quantum processes.

Two round-tables were also organized: Loss of Information from Black Holes (coordinated by Prof. W.Israel) and Time Machines (coordinated by Prof. A.Starobinsky). Furthermore, a number of seminars on other topics of interest to Cosmology and related areas were included.

### A Speaker is given the Nobel Prize

The 9th BSCG happened in 1998. Its international nature appears when we list of the countries where participating scientists came from: Brazil, Argentina, Canada, Denmark, France, Israel, Italy, Mexico, Portugal, Russia, Spain, United States, and Venezuela. In this School, we commemorated twenty years of its existence. On the occasion, Professor Yvonne Choquet-Bruhat was honored with a tribute pronounced by Prof. Werner Israel. Special emphasis was given to localized astrophysical processes, particularly e to properties of black holes. A series of lectures on CMBR was delivered by Professor G. Smoot, who was subsequently awarded the Nobel Prize, precisely for his endeavors in that area. The theory of the gravitational field and the analysis of field theories on the light cone and on geometries representing expanding universes were also presented.

The 10th School was held in July, 2002, and involved scientists from 16 countries: Brazil, Germany, Bolivia, Canada, Chile, Denmark, France, England, Ireland, Italy, Mexico, Poland, Russia, United States, and Turkey. At this moment, the BSCG consolidated its tendency to open the exam of non-conventional issues not only in Cosmology but also in related areas. A brief examination of the topics therein is enough to underline this fact. This tendency continued on in the other Meetings.

### Some scientist's comments on the BSCG

In 1988, CBPFs Group of Cosmology and Gravitation intended to give a permanent role to the Schools by creating a Cosmology Center, under the Ministry of Science and Technology. At that time, several physicists (at the request of the minister) were asked for their opinions on the group, as transcribed below. Particular attention should be paid to the support I received from great Brazilian scientist Csar Lattes. Whenever Lattes came to Rio, we often talked about this possibility. On these occasions, Lattes would air his ideas, similar to Paul Diracs, on local effects of the properties of the evolution of the Universe, saying he had solid arguments to show how Physics very interactions would depend on the Universes state of evolution. Years earlier, Vitrio Canuto had presented an extensive review of Diracs ideas in the School and, in the early 1970s, my CERN collaborator P. Rotelli and I had produced an alternative to Diracs proposal on the cosmic dependence of weak interactions. Lattess ideas did not possess similar development to Diracs, and were very close to mine, that being the reason

why we started to write the draft (for a yet unpublished paper) together.

Lattes used to think it was totally unnecessary to write about his support to my idea of transforming the Schools of Cosmology and Gravitation into a permanent and continuous forum entirely focused on cosmological issues. I eventually convinced him that this letter of his could be important to openly communicate his opinion.

We have reproduced the content of letters by some professors where their opinions on the School are recorded.

- YVONNE CHOQUET-BRUHAT (19/9/1988) (Professor at the University of Paris VI; Director of the Relativist Mechanics Laboratory and Fellow of the French Academy of Sciences):

  () The Brazilian Schools of Cosmology and Gravitation that you have organized since 1978 have proved extremely successful both for the advancement of science at an international level, and for the development of a remarkably good Brazilian group in these fields. Having myself attended two of these Schools, I have been able to appreciate their excellent organization, the high level course programs on the most up-to-date topics by the best specialists in the field, a fruitful experience to all by the active participation of many in the audience, from the Director of the School to the youngest colleagues. These meetings have certainly contributed to obtaining many results in the fields of Cosmology and Gravitation, which have given your group the high reputation that it enjoys internationally.

- RUBEN ALDROVANDI (29/09/1988) (So Paulo Institute of Theoretical Physics):

  Although I think you know my opinion on the CBPF Group of Cosmology and Gravitation and on the Brazilian School it has been organizing for so many years, this seems to be a good opportunity to put it down in written words. The Group is the only one worthy of this name in Brazil, as other people working on those subjects never really seem to get their act together. I have very high regards for the quality, coherence andin Brazil this is essentialendurance shown during all the difficult times the Group has been in existence. As to the School: I have been in many Schools, and most are fairly good, but have never met one that is better organized than this. (...) Such an institution would give stability to the School and, I am convinced, greatly contribute to the development of activities in the sister sciences of Cosmology and Gravitation. For the reasons given above, it is a matter of course that the CBPF Group and its School are the ideal nucleation centre for the Institute.

- EDWARD W. KOLB (23/09/1988) (Professor of Astronomy and Astrophysics at the University of Chicago and at the FERMILAB):

(...) As you know, I had the opportunity of attending the 4th and 5th Schools as a lecturer. I cannot express the student's view, but from my perspective they were both great successes. I benefited a great deal from the lectures by the many distinguished scientists and from questions and discussions with students. CBPFs Gravitation and Cosmology Group is large and active. The people already present at CBPF could easily serve as a nucleus for a more ambitious program. An Institute with a larger scope would be beneficial to Brazilian science in two ways: It would attract to Rio the best people in the international scientific community to share recent developments in general relativity and cosmology; and it would afford the opportunity for the rest of the world to learn about the great work done in Rio by Brazilian scientists. I can think of no better use of resources available to help the development of science in Brazil. I would be happy to do anything I can to help your initiative. Good luck with your efforts.

- VITORIO CANUTO (31/10/1988) (Member of NASA, Goddard Institute for Space Studies):

  (...) In all of Latin America, Brazil is the country that, thanks to your efforts, has taken the leadership in the field of General Relativity and Cosmology, as witnessed by the success of the several Schools that you have convened in the last ten years. From both the scientific and the organizational points of views, I believe they were a remarkable success. Cosmology is about to be reborn thanks to launching the Space Telescope next year. The wealth of new data available in the near future will dramatically change the field, and the fact that your Schools have already prepared young researchers in this field represents an investment on which this Institute can confidently be built. For these reasons, I firmly believe that an Institute of Research in Cosmology and Gravitation will be an outstanding Brazilian contribution not only to the development of science in Latin America but to future generations of young scientists. As can be seen from the excerpts above, even back then the Brazilian Schools of Cosmology and Gravitation already had aninternationally recognizedtradition of providing young researchers and students with easy access, and as thorough as possible, to the current state of research in some sectors of Cosmology, Gravitation, Astrophysics, and related areas. The following passages have been taken from scientists who participated in the Schools of Cosmology and Gravitation at different times.

- BAHRAN MASSHOOM (Missouri, EUA), 1993:

  The organization of the School was excellent: a rigorous schedule of lectures combined with evening seminars. There was ample time, however, to get to know the participants and to have lengthy discussions of scientific issues of mutual interest that arose in the course of lectures and seminars.

(...) On the administrative side, I can only have high praise for the professionalism and dedication of the staff combined with a pleasant human touch that added warmth to the atmosphere of the School. The quality of the School was outstanding. I was also impressed with the excellent quality of graduate students at the School.

- BERNARD JONES (Copenhagen, Denmark), 1993:

  The organization of the School was in fact one of the best I have ever encountered. In fact, it was so good I never noticed it, since everything seemed to work like clockwork and, most important, the organizing team exhibited a remarkable degree of flexibility. You, evidently, have the organization of this kind of meeting down to an art-form. I made many contacts among the young people at the School and I am currently looking into the question of partially financing a bi-lateral cooperation on the subjects of mutual interest. I have contacted our Ministry of Education and will see other relevant groups over the next couple of months. I am hopeful we will be able to invite people to spend some time here.

- VITALY MELNIKOV (Head of CSVRs Department of Fundamental Interaction and Metrology; President of the Russian Gravitational Association, Moscow, Russia), 1993:

  The scientific level of the VII Brazilian School of Cosmology and Gravitation was on a good international level. Practically all modern problems on cosmology and gravitation were discussed at the School. Lecturers were renowned scientists from Europe, USA, and Brazil. It is very nice that among lecturers were some scientists representing Russian schools in basic sciences: Prof. A.Dolgov, I.Tyutin (seminar), Gitman (seminar), and myself. It may contribute to further cooperation and interaction between Brazilian and Russian basic sciences in the field of cosmology and gravitation. There were interesting discussions on the cosmological constant problem and inflationary models, as well as discussions concerning each lecture. The fact that nearly all the Brazilian groups were represented at the School and also many scientists from Argentine, Mexico, other Latin American countries, and even some people from Europe makes this School in essence an international one. The scientific organization of the School was excellent: strict time-table, full attendance, copying of the lectures, work of secretaries, conditions to work, discussions, etc. The fact that all participants lived in one compact and nearly isolated place is very good for productive interaction between all the participants and lecturers. I should like to note that it is a very good practice that all participants had their accommodations paid for by the Organizing Committee, where the scientific merit was the only reason for choosing the attendants. It is the same practice that is used in many other renowned schools like Les Houches, in France, Erice, in Italy, etc. Especially I should like to stress

the great role of Prof. Mrio Novello in the preparation and organization of the work of the School. Due to his attitude, the atmosphere was very friendly and creative. Conditions of living and meals were also good. As to suggestions for future schools I should like to point out that some topics may be represented more widely like quantum cosmology and quantum gravity and also experimental problems of gravitation. In general, I think the traditional interaction of Brazilian and Russian scientists in cosmology and gravitation should be kept and enhanced. And, of course, the best traditions of the Brazilian School of Cosmology and Gravitation, which already were present at the VII School, must be kept.

- A.DOLGOV (Theoretical Astrophyiscs Center - TAC, Copenhagen, Denmark), 1998:

  The Brazilian Schools of Cosmology and Gravitation already have a long and glorious history. They started 20 years ago and, ever since, remain as one of the leading schools on the subject, not only in Brazil but in the world. It is difficult to overstate their educational and scientific value. The level of lecturers is always first rate. The scientific programs each year contain most interesting, important, and up-to-date subjects. In parallel to the main courses of lectures, more brief scientific seminars are organized, where original works by the local and visiting physicists are presented. This makes the Schools not only educationally important but also plays an essential role in the recognition of Brazilian scientific achievements. I would also like to stress the great, excellent, and difficult work done by Professor M.Novello in organizing these Schools.

- IGOR NOVIKOV (Director, Theoretical Astrophysics Center, Copenhagen, Denmark), 1998: I am writing in connection with the great tradition of Brazilian physicists: a series of scientific meetings called the Brazilian Schools of Cosmology and Gravitation (BSCG). (...) The BSCG have taken place approximately every two years starting from 1978. In this year of 1998, the IX BSCG was held in which I had the privilege to participate as an invited lecturer. The main goals of the Schools are to provide the possibility to present and discuss the new achievements in cosmology, general theory of relativity, astrophysics, quantum field theory and in related areas. I have learned these Schools from my colleagues and from Proceedings of the Schools for many years. This year as a participant of the IX BSCG I personally observed the highest scientific and organizational level of the School. The unique format of the BSCG and very friendly working atmosphere provided many fruitful discussions both in pure science and in scientific education. It leads to a real progress in physics and is especially important and competitive at a world class level, and the list of lecturers at BSCG is a who's who of the leaders of cosmology and physics of the international level. I believe that the outstanding BSCG is the result of

enormous work of the talented organizers of the School under the leadership of the Head of BSCG, Prof. M.Novello. It would be very important both for Brazilian physics and for the world physics community to continue the Brazilian Schools of Cosmology and Gravitation in the future.

- EDWARD W. KOLB (Theoretical Astrophysics, FERMILAB; The University of Chicago, EUA), 1998:

  I have had the pleasure of attending two of the Brazilian Schools of Cosmology and Gravitation. In addition to an enthusiastic audience for my lectures, I learned a great deal from the other fine lectures at the Schools. The Schools were exceptionally well run and well balanced. I believe that the Schools have had many benefits for Brazilian science. Not only are the students exposed to ideas and research of leading scientists from the entire world, but scientific leaders from throughout the world are exposed to the very fine young Brazilian researchers. There are many talented young scientists who would otherwise not be easily noticed outside Brazil. Because of the contacts made during my visits to Brazil to attend the Schools, several young scientists have been invited to spend long periods visiting our group at Fermi National Accelerator Laboratory. I am sure that we benefitted from their visits, and I believe that they benefitted from visiting us as well. Nowadays it is difficult to provide continuity even to successful projects. In spite of difficulties you may face, I would like to encourage you to do whatever it takes to continue with the Brazilian Schools of Cosmology and Gravitation. The benefits of the School are quite considerable.

- J. NARLIKAR (Inter-University Centre for Astronomy and Astrophysics - IUCAA, India), 1998:

  I am writing this letter to give my impressions on the Schools of Cosmology and Gravitation conducted by your group in Brazil over the last 20 years. I recall participating in one of the schools in 1987 as a resource person. It was indeed an exhilarating experience to meet the students who were attracted not only from Brazil but also from other countries. The resource persons were also from many different countries and enjoyed international reputation. The School which I attended and lectured in certainly ????? went a long way in bringing to the student community the latest ideas in cosmology and astrophysics. Knowing that many of the students would normally miss the lectures that are routinely delivered in schools held in Europe or the United States, I think the BSCG is playing a very vital role in this field. I do hope that you will continue this activity and possibly expand upon it if your funding agency so permits. You have established a tradition which has to be continued, and I hope that it will.

- FANG LI-ZHI (University of Arizona, Tucson, USA), 1998:

(...) Gravitational theory and cosmology are two of the most fundamental fields of physics. It could not exist without strong public support. However, given the small number of researchers in gravitation and cosmology, these fields make unexpectedly large contributions to formal and informal science education. In the current world, more and more countries recognize that the synergistic, educational, and cultural contributions of the study of cosmology and gravitation are worthy. Therefore, not only big and rich countries attach importance to these fields, but also many others. For instance, even under the current Asian financial crisis, the programs of cosmological and gravitational research in Korea, Vietnam, and Taiwan have firmly been funded by their own authorities. I had the honor to be invited as a lecturer at the BSCG in 1984. Since then I have kept in touch with colleagues of the BSCG. I would like to evaluate the BSCG to be the first rank of schools in the field. All lecturers are influential, and all lectures delivered at the BSCG are on the frontier of gravitation and cosmology research. In addition, the BSCG provides unusual opportunities for international exchange and cooperation of colleagues from Brazil and Latin America with the rest of the world. Therefore, I strongly recommend support to the BSCG School, and its activity should be regular and permanent.

- G.F.R.ELLIS (University of Cape Town, Department of Mathematics and Applied Mathematics, South Africa ), 1998:

  This letter is to state that the series of scientific meetings called the Brazilian Schools of Cosmology and Gravitation (BSCG) have been a significant series of meetings, pulling together high quality lecturers from around the world, and resulting from time to time in good quality publications of significant merit. I therefore believe that continuation of these schools on a regular basis will be a very worthwhile project, and will make a significant contribution to the development of relativity and cosmology not merely to Brazil, but in the whole of Latin America. I am therefore pleased to support your request that funding for these schools should be continued.

- VLADIMIR MOSTEPANENKO (A.Friedmann Laboratory for Theoretical Physics, Moscow, Russia; Visiting Professor, UFPb, Joo Pessoa), 1998:

  Let me express my gratitude for your kind invitation to take part in the IX Brazilian School of Cosmology and Gravitation and to give the lectures there. The School of Cosmology and Gravitation has become a traditional event in Brazil. During twenty years it has gathered the most qualified lecturers on the subject from all over the world and the most promising young Brazilian researchers working in the field of cosmology and gravitation. It is a great honor to Brazil that this country considers it possible to support this field of fundamental physics research. Giving seemingly small contribution to technologies, Cosmology and Gravitation investigate

and solve the most profound problems of the structure and evolution of our Universe. These problems have attracted the most prominent scientists from different countries during all the history of mankind. Now both Gravitation and Cosmology are the experimentally based exact sciences with great perspectives. I hope that the tradition of the Brazilian Schools of Cosmology and Gravitation will be prolonged giving significant contribution to education and science in Brazil.

- YVONNE CHOQUET-BRUHAT (Universit Pierre et Marie Curie, Gravitation et Cosmologie Relativistes, Paris, France), 1998:

  The Brazilian School of Cosmology and Gravitation has held regular meetings - or rather summer schools - since 1978. The list of speakers at these schools is an impressive assembly of internationally renowned names of specialists covering the broad area of General Relativity and Cosmology. I myself have been fortunate enough to participate in two of these schools. I have learned greatly from the lectures of colleagues working in fields distinct but related to mine (which is mainly mathematical problems in General Relativity). The school was also attended by a member of graduate students. The solid background as well as the advanced view points that they received there was certainly a great asset for their future. The Brazilian School of Cosmology and Gravitation has an international reputation, enhanced and perpetuated by the volumes of its proceedings. This school totally deserves to be supported.

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# G. Non linear Electrodynamics

M. NOVELLO, J M SALIM AND S E P BERGLIAFFA

## G.1. Introduction

In recent years, there has been a growing interest in models that mimic in the laboratory some features of gravitation. The actual realization of these models relies on systems that are very different in nature: ordinary non-viscous fluids, super-fluids, flowing and non-flowing dielectrics, non-linear electromagnetism in vacuum, and Bose-Einstein condensates. The basic feature shared by these systems is that the behavior of the fluctuations around a background solution is governed by an "effective metric". More precisely, the particles associated to the perturbations do not follow geodesics of the background space-time but of a Lorentzian geometry described by the effective metric, which depends on the background solution. It is important to notice that only some kinematical aspects of general relativity can be imitated by this method, but not its dynamical features.

By use of this analogy, the geometrical tools of General Relativity can be used to study some condensed matter systems. More importantly perhaps is the fact that the analogy has permitted the simulation of several configurations of the gravitational field, such as wormholes and closed space-like curves for photons, and warped spacetimes for phonons. Particular attention has been paid to analog black holes, because these would emit Hawking radiation exactly as the gravitational black holes do, and are obviously much easier to generate in the laboratory. The fact that analog black holes emit thermal radiation was shown first by Unruh in the case of dumb black holes, and it is the prospect of observing this radiation (thus testing the hypothesis that the thermal emission is independent of the physics at arbitrarily short wavelengths) that motivates the quest for a realization of analog black holes in the laboratory. Let us emphasize that the actual observation of the radiation is a difficult task from the point of view of the experiment, if only because of the extremely low temperatures involved. In the case of a quasi one-dimensional flow of a Bose-Einstein condensate for instance, the temperature of the radiation would be around 70

nK, which is comparable but lower than the temperature needed form the condensate.

We shall begin by presenting the basics of the idea of the effective geometry by studying a simple case: nonlinear electromagnetism. Later on we shall analyze another example: photons in a flowing dielectric medium. We shall see that, in analogy to the most general nonlinear electromagnetic case, the photons experience bi-refringence and bi-metricity. Then we show that is possible to build a static and spherically symmetric analog black hole, generated by a *flowing* isotropic dielectric that depends on an applied electric field. We give a specific example, in which the radius of the horizon and the temperature depend on three parameters (the zeroth order permittivity, the charge that generates the external field, and the linear susceptibility) instead of depending only on the zeroth order permittivity. As we shall show another feature of this black hole is that there is a new term in the surface gravity (and hence in the temperature of Hawking radiation), in addition to the usual term proportional to the acceleration of the fluid. This new term depends exclusively on the dielectric properties of the fluid, and it might give an opportunity to get Hawking radiation with temperature higher than that reported up to date.

## G.2. The effective metric

Historically, the first example of the idea of effective metric was presented by W. Gordon in 1923. In modern language, the wave equation for the propagation of light in a moving nondispersive medium, with slowly varying refractive index $n$ and 4-velocity $u^\mu$:

$$\left[\partial_\alpha \partial^\alpha + (n^2 - 1)(u^\alpha \partial_\alpha)^2\right] F_{\mu\nu} = 0.$$

Taking the geometrical optics limit, the Hamilton-Jacobi equation for light rays can be written as $g^{\mu\nu} k_\mu k_\nu = 0$ where

$$g^{\mu\nu} = \eta^{\mu\nu} + (n^2 - 1) u^\mu u^\nu \tag{G.1}$$

is the effective metric for this problem. It must be noted that only photons in the geometric optics approximation move on geodesics of $g^{\mu\nu}$: the particles that compose the fluid couple instead to the background Minkowskian metric.

Let us study now in detail the example of nonlinear electromagnetism. We start with the action

$$S = \int \sqrt{-\gamma}\, L(F)\, d^4x, \tag{G.2}$$

where $F \equiv F^{\mu\nu} F_{\mu\nu}$ and $L$ is an arbitrary function of $F$. Notice that $\gamma$ is the determinant of the background metric, which we take in the following to be that of flat spacetime, but the same techniques can be applied when the background

is curved. Varying this action w.r.t. the potential $A_\mu$, related to the field by the expression

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu},$$

we obtain the Euler-Lagrange equations of motion (EOM)

$$(\sqrt{-\gamma}\, L_F F^{\mu\nu})_{;\nu} = 0, \tag{G.3}$$

where $L_F$ is the functional derivative $L_F \equiv \frac{\delta L}{\delta F}$. In the particular case of a linear dependence of the Lagrangian with the invariant $F$ we recover Maxwell's equations of motion.

As mentioned in the Introduction, we want to study the behavior of perturbations of these EOM around a fixed background solution. Instead of using the traditional perturbation method, we shall use a more elegant method set out by Hadamard. In this method, the propagation of low-energy photons are studied by following the evolution of the wave front, through which the field is continuous but its first derivative is not. To be specific, let $\Sigma$ be the surface of discontinuity defined by the equation

$$\Sigma(x^\mu) = constant.$$

The discontinuity of a function $J$ through the surface $\Sigma$ will be represented by $[J]_\Sigma$, and its definition is

$$[J]_\Sigma \equiv \lim_{\delta\to 0^+} \left( J|_{\Sigma+\delta} - J|_{\Sigma-\delta} \right).$$

The discontinuities of the field and its first derivative are given by

$$[F_{\mu\nu}]_\Sigma = 0, \qquad\qquad [F_{\mu\nu,\lambda}]_\Sigma = f_{\mu\nu} k_\lambda, \tag{G.4}$$

where the vector $k_\lambda$ is nothing but the normal to the surface $\Sigma$, that is, $k_\lambda = \Sigma_{,\lambda}$.

To set the stage for the nonlinear case, let us first discuss the propagation in Maxwell's electrodynamics, for which $L_{FF} = 0$. The EOM then reduces to $F^{\mu\nu}_{;\nu} = 0$, and taking the discontinuity we get

$$f^{\mu\nu} k_\nu = 0. \tag{G.5}$$

The other Maxwell equation is given by $F^{*}_{\mu\nu}{}^{;\nu} = 0$ or equivalently,

$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0. \tag{G.6}$$

The discontinuity of this equation yields

$$f_{\mu\nu} k_\lambda + f_{\nu\lambda} k_\mu + f_{\lambda\mu} k_\nu = 0. \tag{G.7}$$

Multiplying this equation by $k^\lambda$ gives

$$f_{\mu\nu} k^2 + f_{\nu\lambda} k^\lambda k_\mu + f_{\lambda\mu} k^\lambda k_\nu = 0, \tag{G.8}$$

where $k^2 \equiv k_\mu k_\nu \gamma^{\mu\nu}$. Using the orthogonality condition from previous equation it follows that

$$f^{\mu\nu} k^2 = 0 \tag{G.9}$$

Since the tensor associated to the discontinuity cannot vanish (we are assuming that there is a true discontinuity!) we conclude that the surface of discontinuity is null w.r.t. the metric $\gamma^{\mu\nu}$. That is,

$$k_\mu k_\nu \gamma^{\mu\nu} = 0. \tag{G.10}$$

It follows that $k_{\lambda;\mu} k^\lambda = 0$, and since the vector of discontinuity is a gradient,

$$k_{\mu;\lambda} k^\lambda = 0. \tag{G.11}$$

This shows that the propagation of discontinuities of the electromagnetic field, in the case of Maxwell's equations (which are linear), is along the null geodesics of the Minkowski background metric.

Let us apply the same technique to the case of a nonlinear Lagrangian for the electromagnetic field, given by $L(F)$. Taking the discontinuity of the EOM, we get

$$L_F f^{\mu\nu} k_\nu + 2\eta \, L_{FF} \, F^{\mu\nu} k_\nu = 0, \tag{G.12}$$

where we defined the quantity $\eta$ by $F^{\alpha\beta} f_{\alpha\beta} \equiv \eta$. Note that contrary to the linear case in which the discontinuity tensor $f_{\mu\nu}$ is orthogonal to the propagation vector $k^\mu$, here there is a complicated relation between the vector $f^{\mu\nu} k_\nu$ and quantities dependent on the background field. This is the origin of a more involved expression for the evolution of the discontinuity vector, as we shall see next. Multiplying equation (G.8) by $F^{\mu\nu}$ we obtain

$$\eta \, k^2 + F^{\mu\nu} f_{\nu\lambda} k^\lambda k_\mu + F^{\mu\nu} f_{\lambda\mu} k^\lambda k_\nu = 0. \tag{G.13}$$

Now we substitute in this equation the term $f^{\mu\nu} k_\nu$ from Eqn.(G.12), and we arrive at the expression

$$\eta k^2 - 2 \frac{L_{FF}}{L_F} \eta (F^{\mu\lambda} k_\mu k_\lambda - F^{\lambda\mu} k_\mu k_\lambda), \tag{G.14}$$

which can be written as $g^{\mu\nu} k_\mu k_\nu = 0$, where

$$g^{\mu\nu} = L_F \gamma^{\mu\nu} - 4 L_{FF} \, F^{\mu\alpha} F_\alpha{}^\nu. \tag{G.15}$$

We then conclude that

> The low-energy photons of a *nonlinear* theory of electrodynamics with $L = L(F)$ do not propagate on the null cones of the background metric but on the null cones of an *effective* metric, generated by the self-interaction of the electromagnetic field.

This statement is always true in case of Lagrangians depending only of the invariant $F$. For Lagrangians that depend also of $F^*$, there may be some special cases in which the propagation coincides with that in Minkowski. Another feature of the more general case $L = L(F, F^*)$ is that bi-refringence is present. That is, the two polarization states of the photon propagate in a different way. In some special cases, there is also bi-metricity (one effective metric for each state). Even more special cases (such as Born-Infeld electrodynamics) exhibit only a single metric. Some of these features are present in our next example.

## G.3. Effective metric in flowing fluids with zero vorticity

Another example in which an effective metric arises naturally is that of fluid dynamics for inviscid fluids. The equations decribing this system are the continuity equation,

$$\partial_t \rho + \vec{\nabla}.(\rho \vec{v}) = 0$$

and Euler's equation,

$$\rho(\partial_t \vec{v} + (\vec{v}.\vec{\nabla})\vec{v}) = -\vec{\nabla} p - -\rho \vec{\nabla} \Phi.$$

If we assume that assuming that there is no vorticty, the velocity of the fluid can be expressed in terms of a potential:

$$\vec{v} = -\vec{\nabla}\psi.$$

If we also assume that the fluid is barotropic, that is

$$\vec{\nabla} h = \frac{1}{\rho}\vec{\nabla} p,$$

Euler eqn. reduces to

$$-\partial_t \psi + h + \frac{1}{2}(\vec{\nabla}\psi)^2 + \psi + \Phi = 0 \tag{G.16}$$

Linearize the EOM around some assumed background using

$$\rho = \rho_0 + \epsilon \rho_1 + O(\epsilon^2)$$

and similar developments for $p$ and $\psi_1$ (the background quantities have a 0 subindex).

Keeping up to first order in $\epsilon$, we get from the linearized EOM:

$$-\partial_t \left( \frac{\partial \rho}{\partial p} \rho_0 (\partial_t \psi_1 + \vec{v}_0.\vec{\nabla}\psi_1) \right) + \vec{\nabla}. \Big( \rho_0 \vec{\nabla}\psi_1 - \frac{\partial \rho}{\partial p} \rho_0 \; \vec{v}_0 (\partial_t \psi_1 + \vec{v}_0.\vec{\nabla}\psi_1) \Big) = 0$$

Introducing the velocity of sound $c_{\rm s}^{-2} = \frac{\partial\rho}{\partial p}$, and the metric

$$g_{\mu\nu} = \frac{\rho_0}{c_{\rm s}} \begin{pmatrix} -(c_s^2 - v_0^2) & \vdots & -v_0^j \\ \cdots & . & \cdots \\ -v_0^j & \vdots & \delta_{ij} \end{pmatrix}$$

We can write the wave equation

$$\triangle\psi_1 = 0, \tag{G.17}$$

where $\triangle$ is the d'Alembertian in the geometry $g_{\mu\nu}$:

$$\triangle\psi_1 = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\; g^{\mu\nu}\partial_\nu\psi_1), \tag{G.18}$$

*The scalar field $\psi_1$ moves in an effective curved spacetime, in which the geometry depends on the background fluid.*

Many of the notions of GR (like horizon and ergosphere) can be applied in this context. In particular, it is rather easy to generate an analog black hole in this model, and it can be shown that this analog black hole emits Hawking radiation.

### G.3.1. Effective metric(s) in the presence of a dielectric

We now move to another interesting case where the effective geometry is useful to study the motion of low-energy photons. We shall analyze the propagation of such photons in a nonlinear medium. Let us define first the antisymmetric tensors $F_{\mu\nu}$ and $P_{\mu\nu}$, which are convenient to represent the electromagnetic field when material media are present. These tensors can be expressed in terms of the strengths ($E$, $H$) and the excitations ($D$, $B$) of the electric and magnetic fields as

$$\begin{aligned} F_{\mu\nu} &= v_\mu E_\nu - v_\nu E_\mu - \eta_{\mu\nu}{}^{\alpha\beta} v_\alpha B_\beta, \\ P_{\mu\nu} &= v_\mu D_\nu - v_\nu D_\mu - \eta_{\mu\nu}{}^{\alpha\beta} v_\alpha H_\beta. \end{aligned}$$

where $v_\mu$ represents the 4-velocity of an arbitrary observer (which we will take later as co-moving with the fluid). The Levi-Civita tensor introduced above is defined in such way that $\eta^{0123} = +1$ in Cartesian coordinates. Since the electric and magnetic fields are space-like vectors, the notation $E^\alpha E_\alpha \equiv -E^2, H^\alpha H_\alpha \equiv -H^2$ will be used. We will consider here media with properties determined only by the tensors $\epsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$ (*i.e.* media with null magneto-electric tensor), which relate the electromagnetic excitations to the field strengths by the constitutive laws,

$$D_\alpha = \epsilon_\alpha{}^\beta(E,H)E_\beta, \qquad B_\alpha = \mu_\alpha{}^\beta(E,H)H_\beta. \tag{G.19}$$

In order to get the effective metric, we shall use Hadamard's method as in the previous section. By taking the discontinuity of the field equations ${}^*F^{\mu\nu}{}_{;\nu} = 0$ and $P^{\mu\nu}{}_{;\nu} = 0$, and assuming that

$$\epsilon^{\mu\beta} = \epsilon(E)(\gamma^{\mu\beta} - v^\mu v^\beta), \tag{G.20}$$

and

$$\mu^{\mu\beta} = \mu_0(\gamma^{\mu\beta} - v^\mu v^\beta), \tag{G.21}$$

with $\mu_0 = \text{const.}$, we get the following equations:

$$\epsilon(k.e) - \frac{\epsilon'}{E}(E.e)(k.E) = 0, \tag{G.22}$$

$$\mu_0(k.h) = 0, \tag{G.23}$$

$$\epsilon(k.v)e^\mu - \frac{\epsilon'}{E}E^\alpha e_\alpha(k.v)E^\mu + \eta^{\mu\nu\alpha\beta}k_\nu v_\alpha h_\beta = 0, \tag{G.24}$$

$$\mu_0(k.v)h^\mu - \eta^{\mu\nu\alpha\beta}k_\nu v_\alpha e_\beta = 0, \tag{G.25}$$

where $k^\mu$ is the wave propagation vector, $\epsilon'$ is the derivative of $\epsilon$ w.r.t. $E$, and

$$[E_{\mu,\lambda}]_\Sigma = e_\mu\, k_\lambda, \qquad [H_{\mu,\lambda}]_\Sigma = h_\mu\, k_\lambda.$$

Note in particular that previous equation shows that the vectors $k^\mu$ and $e^\mu$ are not always orthogonal, as would be the case if $\epsilon'$ was zero. Substituting in the previous equation, we get

$$Z^{\mu\beta}e_\beta = 0, \tag{G.26}$$

where the matrix $Z$ is given by

$$Z^{\mu\beta} = \left[k^2 + (k.v)^2(\mu_0\epsilon - 1)\right]\gamma^{\mu\beta} - \mu_0\frac{\epsilon'}{E}(k.v)^2E^\mu E^\beta + (v.k)(v^\mu k^\beta + k^\mu v^\beta) - \left[\epsilon\mu_0(k.v) + k^2\right]\ v^\mu v^\beta - k^\mu \tag{G.27}$$

Non-trivial solutions can be found only for cases in which $\det\left|Z^{\mu\beta}\right| = 0$ ( this condition is a generalization of the well-known Fresnel equation).

This equation can be solved by expanding $e_\nu$ as a linear combination of the four linearly independent vectors $v_\nu$, $E_\nu$, $k_\nu$ and $\eta_{\alpha\beta\mu\nu}v^\alpha E^\beta k^\mu$ (the particular case in which the vectors $v_\nu$, $E_\nu$ and $k_\nu$ are coplanar will be examined below). That is,

$$e_\nu = \alpha E_\nu + \beta\eta_{\alpha\lambda\mu\nu}v^\alpha E^\lambda k^\mu + \gamma k_\nu + \delta v_\nu. \tag{G.28}$$

Notice that taking the discontinuity of $E^\mu{}_{,\lambda}$ we can show that $(e.v) = 0$. This restriction imposes a relation between the coefficients of Eqn.(G.28):

$$\delta = -\gamma(k.v)$$

With the expression given in Eqn.(G.28), Eqn. (G.26) reads

$$\begin{aligned}
\alpha\left[k^2 - (1 - \mu_0\,(\epsilon\,E)')\,(k.v)^2\right] - \gamma\left[\mu_0(k.v)^2\frac{1}{E}\,\epsilon'^\alpha k_\alpha\right] &= 0,\\
\alpha E^\mu k_\mu + \gamma(1 - \mu_0\epsilon)(k.v)^2 + \delta(k.v) &= 0,\\
\alpha(k.v)E^\mu k_\mu + \gamma(k.v)k^2 + \delta\left[k^2 + \mu_0\epsilon\,(k.v)^2\right] &= 0,\\
\beta\left[k^2 - (1 - \mu_0\epsilon)(k.v)^2\right] &= 0.
\end{aligned}$$

The solution of this system results in the following dispersion relations:

$$k_-^2 = (k.v)^2 \left[1 - \mu_0(\epsilon\ E)'\right] + \frac{1}{\epsilon E}\ \epsilon'^{\alpha} E^{\beta} k_\alpha k_\beta, \tag{G.29}$$

$$k_+^2 = [1 - \mu_0 \epsilon(E)](k.v)^2. \tag{G.30}$$

They correspond to the propagation modes

$$e_\nu^- = \rho^- \left\{\mu_0\ \epsilon(k.v)^2 E_\nu + E^\alpha k_\alpha [k_\nu - (k.v) v_\nu]\right\}, \tag{G.31}$$

$$e_\nu^+ = \rho^+\ \eta_{\alpha\lambda\mu\nu} v^\alpha E^\lambda k^\mu, \tag{G.32}$$

where $\rho^-$ and $\rho^+$ are arbitrary constants. The labels "+" and "−" refer to the ordinary and extraordinary rays, respectively. Eqns. that govern the propagation of photons in the medium characterized by $\mu = \mu_0$ =const., and $\epsilon = \epsilon(E)$. They can be rewritten as $g_\pm^{\mu\nu} k_\mu k_\nu = 0$, where we have defined the effective geometries

$$g_{(-)}^{\mu\nu} = \gamma^{\mu\nu} - \left[1 - \mu_0\ (\epsilon\ E)'\right] v^\mu v^\nu - \frac{1}{\epsilon E}\ \epsilon'^{\mu} E^\nu, \tag{G.33}$$

$$g_{(+)}^{\mu\nu} = \gamma^{\mu\nu} - [1 - \mu_0\ \epsilon] v^\mu v^\nu. \tag{G.34}$$

The metric given above was derived previously, while the second metric very much resembles the metric derived by Gordon. The difference is that in the case under consideration, $\epsilon$ is a function of the modulus of the external electric field, while Gordon worked with a constant permeability.

We see then that in this example each polarization state has its own dispersion relation, so there is bi-refringence. There is also bi-metricity, because each type of photon moves according a different metric.

Let us discuss now a particular instance in which the vectors used as a basis in previous Eqn. are not linearly independent. If we assume that

$$E^\mu = a k^\mu + b v^\mu, \tag{G.35}$$

then vectors $e^\mu$, $k^\mu$, and $v^\mu$ are coplanar. In this case, the basis chosen is not appropriate. Notice however that if we assume that $e^\mu$ is a combination of vectors that are perpendicular to $k^\mu$, so that $(e.k) = 0$, then $(E.e) = 0$. The converse is also true: if $(E.e) = 0$, then $(k.e) = 0$. For this particular case, in which $e^\mu$ is perpendicular to $v^\mu$, $k^\mu$ (and consequently to $E^\mu$), imply that

$$\left[k^2 + (k.v)^2(\mu_0\epsilon - 1)\right]\ e^\mu = 0$$

We see then that in the case in which $E_\mu = a k_\mu + b v_\mu$, Fresnel's equation determines that the polarization of the photons is perpendicular to the direction of propagation and to the velocity of the fluid. Moreover, the motion of these photons is governed by the metric $g_+^{\mu\nu}$. For instance, if the electric field, the velocity of the fluid, and the direction of propagation are all radial, then the polarization is in the plane perpendicular to the propagation, and the two polarization modes feel the same geometry.

## G.4. The Analog Black Hole

We shall show in this section that the system described by the effective metrics given above can be used to produce an analog black hole. It will be convenient to rewrite at this point the inverse of the effective metric using a different notation:

$$g_{\mu\nu}^{(-)} = \gamma_{\mu\nu} - \frac{v_\mu v_\nu}{c^2}(1-f) + \frac{\xi}{1+\xi}\, l_\mu l_\nu, \tag{G.36}$$

where we have defined the quantities

$$f \equiv \frac{1}{c^2\mu_0\epsilon(1+\xi)}, \qquad \xi \equiv \frac{\epsilon' E}{\epsilon}, \qquad l_\mu \equiv \frac{E_\mu}{E}.$$

Note that $\epsilon = \epsilon(E)$. We have introduced here the velocity of light $c$, which was set to 1 before. Taking a Minkowskian background in spherical coordinates, and

$$v_\mu = (v_0, v_1, 0, 0), \qquad E_\mu = (E_0, E_1, 0, 0), \tag{G.37}$$

we get for the effective metric,

$$g_{00}^{(-)} = 1 - \frac{v_0^2}{c^2}\,(1-f) + \frac{\xi}{1+\xi}\, l_0^2, \tag{G.38}$$

$$g_{11}^{(-)} = -1 - \frac{v_1^2}{c^2}\,(1-f) + \frac{\xi}{1+\xi}\, l_1^2, \tag{G.39}$$

$$g_{01}^{(-)} = -\frac{v_0 v_1}{c^2}\,(1-f) + \frac{\xi}{1+\xi}\, l_0\, l_1, \tag{G.40}$$

and $g_{22}^{(-)}$ and $g_{33}^{(-)}$ as in Minkowski spacetime. The vectors $v_\mu$ and $l_\mu$ satisfy the constraints

$$v_0^2 - v_1^2 = c^2, \tag{G.41}$$

$$l_0^2 - l_1^2 = -1, \tag{G.42}$$

$$v_0 l_0 - v_1 l_1 = 0. \tag{G.43}$$

This system of equations can be solved in terms of $v_1$, and the result is

$$v_0^2 = c^2 + v_1^2, \tag{G.44}$$

$$l_0^2 = \frac{v_1^2}{c^2}, \qquad l_1^2 = \frac{c^2 + v_1^2}{c^2}. \tag{G.45}$$

Now we can rewrite the metric in terms of $\beta \equiv v_1/c$, a definition which coincides with the usual one for small values of $v_1$. The explicit expression of the metric coefficients is:

$$g_{00}^{(-)} = \frac{1 - \beta^2(c^2\mu_0\epsilon - 1)}{c^2\mu_0(\epsilon + \epsilon' E)}, \tag{G.46}$$

$$g_{01}^{(-)} = \beta\sqrt{1+\beta^2}\ \frac{1-c^2\mu_0\epsilon}{c^2\mu_0(\epsilon+\epsilon' E)}, \tag{G.47}$$

$$g_{11}^{(-)} = \frac{\beta^2 - c^2\mu_0\epsilon(1+\beta^2)}{c^2\mu_0(\epsilon+\epsilon' E)}. \tag{G.48}$$

¿From Eqn.(G.46) it is easily seen that, depending on the function $\epsilon(E)$, this metric has a horizon at $r = r_h$, given by the condition $g_{00}(r_h) = 0$ or, equivalently,

$$\left(c^2\mu_0\epsilon - \frac{1}{\beta^2}\right)\Bigg|_{r_h} = 1. \tag{G.49}$$

The metric given above resembles the form of Schwarzschild's solution in Painlevé-Gullstrand coordinates:

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 \pm 2\sqrt{\frac{2GM}{r}}\ dr\ dt - dr^2 - r^2 d\Omega^2. \tag{G.50}$$

With the coordinate transformation

$$dt_P = dt_S \mp \frac{\sqrt{2GM/r}}{1-\frac{2GM}{r}}\ dr, \tag{G.51}$$

the line element given in above equation can be written in Schwarzschild's coordinates. The "+" sign covers the future horizon and the black hole singularity.

The effective metric looks like the metric in Eqn.(G.50). In fact, it can be written in Schwarzschild's coordinates, with the coordinate change

$$dt_{PG} = dt_S - \frac{g_{01}(r)}{g_{00}(r)} dr. \tag{G.52}$$

Using this transformation with the metric coefficients given in Eqns.(G.46) and (G.47), we get the expression of $g_{11}^{(-)}$ in Schwarzschild coordinates:

$$g_{11}^{(-)} = -\frac{\epsilon(E)}{(1-\beta^2[c^2\mu_0\epsilon(E)-1])(\epsilon(E)+\epsilon(E)' E)}. \tag{G.53}$$

Note that $g_{01}^{(-)}$ is zero in the new coordinate system, while $g_{00}^{(-)}$ is still given by Eqn.(G.46). Consequently, the position of the horizon does not change, and is still given by Eqn.(G.49).

Working in Painlevé-Gullstrand coordinates, we have shown that the metric for the "−" polarization describes a Schwarzschild black hole if Eqn.(G.49) has a solution. Afterwards we have rewritten the "−" metric in the more familiar Schwarzschild coordinates. Let us consider now photons with the other polarization. They "see" the metric given by Eqn.(G.34), whose inverse is given by:

$$g_{\mu\nu}^{(+)} = \gamma_{\mu\nu} - \frac{v_\mu v_\nu}{c^2}\left(1 - \frac{1}{c^2\mu_0\epsilon(E)}\right). \tag{G.54}$$

Using this equation and Eqns.(G.44) and (G.45) it is straightforward to show that

$$g_{00}^{(+)} = 1 - \left(1+\beta^2\right)\left(1 - \frac{1}{c^2\mu_0\epsilon(E)}\right), \tag{G.55}$$

$$g_{01}^{(+)} = -\beta\sqrt{1+\beta^2}\left(1 - \frac{1}{c^2\mu_0\epsilon(E)}\right), \tag{G.56}$$

$$g_{11}^{(+)} = -1 - \beta^2\left(1 - \frac{1}{c^2\mu_0\epsilon(E)}\right). \tag{G.57}$$

This metric also corresponds to a Schwarzschild black hole, for some $\epsilon(E)$ and $\beta$. Comparing Eqns.(G.46) and (G.55) we see that the horizon of both analog black holes is located at $r_h$, given by Eqn.(G.49).

By means of the coordinate change defined by Eqn.(G.52), we can write this metric in Schwarzschild's coordinates. The relevant coefficients are given by

$$g_{00}^{(+)} = \frac{1+\beta^2(1-c^2\mu_0\epsilon(E))}{c^2\mu_0\epsilon(E)}, \tag{G.58}$$

$$g_{11}^{(+)} = -\frac{1}{1+\beta^2(1-c^2\mu_0\epsilon(E))}. \tag{G.59}$$

It is important to stress then that *the horizon is located at $r_h$ given by Eqn.(G.49) for photons with any polarization.* Moreover, the motion of the photons in both geometries will be qualitatively the same, as we shall show below.

## G.5. An example

We have not specified up to now the functions $\epsilon(E)$ and $E(r)$ that determine the dependence of the coefficients of the effective metrics with the coordinate $r$. From now on we assume a linear $\epsilon(E)$, a type of behaviour which is exhibited for instance by electrorheological fluids. Specifically, we take

$$\epsilon(E) = \epsilon_0(\overline{\chi} + \chi^{(2)}E(r)), \tag{G.60}$$

with $\overline{\chi} = 1 + \chi^{(1)}$. The nontrivial Maxwell's equation then reads

$$\left(\sqrt{-\gamma}\,\epsilon(r)F^{01}\right)_{,1} = 0. \tag{G.61}$$

Taking into account that $(F^{01})^2 = \frac{E^2}{c^2}$, we get as a solution of Eqn.(G.61) for a point source in a flat background in spherical coordinates

$$F^{01} = \frac{-\overline{\chi} \pm \sqrt{\overline{\chi}^2 + 4\chi^{(2)}Q/\epsilon_0 r^2}}{2c\chi^{(2)}}. \tag{G.62}$$

Let us consider a particular combination of parameters: $\chi^{(2)} > 0$, $Q > 0$ and the "+" sign in front of the square root in $F^{01}$, in such a way that $E > 0$ for all $r$. To get more manageable expressions for the metric, it is convenient to define the function $\sigma(r)$:

$$E(r) \equiv \frac{\overline{\chi}}{2\chi^{(2)}}\, \sigma(r) \tag{G.63}$$

where

$$\sigma(r) = -1 + \frac{1}{r}\sqrt{r^2 + q} \tag{G.64}$$

and

$$q = \frac{4\chi^{(2)} Q}{\epsilon_0 \overline{\chi}^2}. \tag{G.65}$$

In terms of $\sigma$, the metrics take the form

$$\begin{aligned} ds^2_{(-)} = & \frac{2 - \beta^2 \left[\, \overline{\chi}\, (\sigma(r) + 2) - 2\right]}{2\, \overline{\chi}\, (1 + \sigma(r))}\, d\tau^2 - \\ & \frac{2 + \sigma(r)}{\left[2 - \beta^2 \left(\overline{\chi}\, (\sigma(r) + 2) - 2\right)\right] (1 + \sigma(r))}\, dr^2 - r^2 d\Omega^2, \end{aligned} \tag{G.66}$$

$$\begin{aligned} ds^2_{(+)} = & \frac{2 - \beta^2 \left[\, \overline{\chi}\, (\sigma(r) + 2) - 2\right]}{\overline{\chi}\, (2 + \sigma(r))}\, d\tau^2 - \\ & \frac{2}{2 + \beta^2 \left[2 - \overline{\chi}\, (\sigma(r) + 2)\right]}\, dr^2 - r^2 d\Omega^2. \end{aligned} \tag{G.67}$$

Notice that the $(t, r)$ sectors of these metrics are related by the following expression:

$$ds^2_{(+)} = \Phi(r)\, ds^2_{(-)} \tag{G.68}$$

where the conformal factor $\Phi$ is given by:

$$\Phi = 2\, \frac{1 + \sigma(r)}{2 + \sigma(r)}$$

We shall study next some features of the effective black hole metrics. It is important to remark that up to this point, the velocity of the fluid $v_1$ is completely arbitrary; it can even be a function of the coordinate $r$. We shall assume in the following that $v_1$ is a constant. This assumption, which will be lifted later on, may seem rather restrictive but it helps to display the main features of the effective metrics in an easy way.

To study the motion of the photons in these geometries, we can use the technique of the effective potential. Standard manipulations show that in the case of a static and spherically symmetric metric, the effective potential is given by

$$V(r) = \varepsilon^2 \left(1 + \frac{1}{g_{00}(r)\, g_{11}(r)}\right) - \frac{L^2}{r^2 g_{11}(r)} \tag{G.69}$$

where $\varepsilon$ is the energy and $L$ the angular momentum of the photon.

In terms of $\sigma(r)$, and of the impact parameter $b^2 = L^2/\varepsilon^2$, the "small" effective potential $v(r) \equiv V(r)/\varepsilon^2$ for the metric Eqn.(G.66) in Schwarzschild coordinates can be written as follows:

$$v^{(-)}(r) = 1 - \frac{2(1+\sigma(r))^2}{2+\sigma(r)} - \frac{b^2}{r^2}\frac{(2-\beta^2\sigma(r))(1+\sigma(r))}{2+\sigma(r)} \tag{G.70}$$

A short calculation shows that $v^{(-)}$ is a monotonically decreasing function of $\beta$. $b = 1, 3, 5$ (starting from the lowest curve), and $\beta = 0.5$.

The effective potential for the Gordon-like metric can be obtained in the same way. From Eqns.(G.69) and (G.67) we get

$$v^{(+)}(r) = 1 - \frac{2+\sigma(r)}{2} + \frac{b^2}{2r^2}\,[2-\beta^2\sigma(r)]. \tag{G.71}$$

We see that, in the case of a constant flux velocity, the shape of the effective potential for both metrics qualitatively agrees with that for photons moving on the geometry of a Schwarzschild black hole.

## G.6. Surface gravity and temperature

Let us now go back to the more general case of $\beta = \beta(r)$, and calculate the "surface gravity" of our analog black hole. We present first the results for the constant permittivity case. By setting $\epsilon'(E) \equiv 0$ in the metrics Eqns.(G.33) and (G.34), we regain the example of constant index of refraction It is easy to show that the horizon of the black hole in this case is given by

$$\beta^2(r_h) = \frac{1}{\bar{\chi}-1}. \tag{G.72}$$

The "surface gravity" of a spherically symmetric analog black hole in Schwarszchild coordinates is given by

$$\kappa = \frac{c^2}{2}\lim_{r\to r_h}\frac{g_{00,r}}{\sqrt{|g_{11}|\,g_{00}}}. \tag{G.73}$$

For the metrics Eqns.(G.33) and (G.34) with $\epsilon = \epsilon_0\bar{\chi}$ and $r_h$ given by Eqn.(G.72), the analog surface gravity is

$$\kappa = -\frac{c^2}{2}\frac{1-\bar{\chi}}{\sqrt{\bar{\chi}}}\,(\beta^2)'\big|_{r_h}. \tag{G.74}$$

This equation can be rewritten in terms of the velocity of light in the medium and the refraction index, respectively given by

$$c_m^2 = \frac{1}{\mu_0\epsilon}, \qquad n = \frac{c}{c_m}. \tag{G.75}$$

The result is

$$\kappa = \frac{c^2}{2}\,\frac{1-n^2}{n}\,(\beta^2)_{,r} \tag{G.76}$$

In this expression we can see the influence of the dielectric properties of the fluid (through the index of refraction of the medium) and also of its dynamics through the physical acceleration in the radial direction, given by

$$a_r\big|_{r_h} = \frac{c^2}{2}\,(\beta^2)'\big|_{r_h}\,,$$

for $\beta^2(r_h) \ll 1$. This acceleration is a quantity that must be determined solving the equations of motion of the fluid [1].

Going back the the more general case of a linear permittivity, described by the metrics given above and considering that $\beta(r_h) \ll 1$, the radius of the horizon is [2]:

$$r_h^2 = \frac{q\bar{\chi}^2}{4}\beta^4(r_h). \tag{G.77}$$

Using the expressions given above, the result for the surface gravity of the "$-$" black hole for $\beta(r_h) \ll 1$ is

$$\kappa^{(-)} = \frac{c^2}{\beta}\left(\frac{1}{\bar{\chi}\sqrt{q}} - \frac{1}{2}(\beta^2)'\right)\Bigg|_{r_h}. \tag{G.78}$$

This equation differs from the surface gravity of the case of constant permittivity (Eqn.(G.74)) by the presence of a new term that does not depend on the acceleration of the fluid. To see where this new term comes from, we can go back to the definition of the surface gravity given in Eqn.(G.73), and use the fact that in the high frequency limit the velocity of light and the index of refraction in a medium of variable $\epsilon$ are still given by Eqn.(G.75), replacing the constant permittivity by $\epsilon = \epsilon(E)$. The result is

$$\kappa = \left(\frac{c^2}{2}\,\frac{1-n^2(E)}{n(E)}\,(\beta^2)_{,r} + \frac{n(E)\epsilon(E)}{\epsilon(E)+\epsilon(E)'E}\,(c_m^2)_{,r}\right)\Bigg|_{r_h} \tag{G.79}$$

In this expression, the first term is the generalization of the case $\epsilon = \text{const.}$ (compare with Eqn.(G.76)), which mixes the acceleration of the fluid with its dielectric properties. On the other hand, the second term, which is the new term displayed in Eqn.(G.78), is related to the radial variation of the velocity of light in the medium. It is important to point out that the result exhibited in Eqn.(G.79) is parallel to that of dumb holes: Unruh found in that case that the surface gravity for constant speed of sound is proportional to the acceleration of the fluid (as in the first term of Eqn.(G.79)). This was generalized by Visser, who

[1] If we set $\beta \equiv 0$ in Eqns.(G.46)-(G.48), we cease to have a black hole.

[2] Notice that we cannot take the limit $q \to 0$ in this expression or in any expression in which this one has been used.

showed that for a position-dependent velocity of sound a second term appears, coming from the gradients of the speed of sound, in analogy with the second term of Eqn.(G.79).

It is easy to show that the these results also apply to the black hole described by the Gordon-like metric. This is not surprising though, because of the conformal relation between the two metrics, given by Eqn.(G.68) .

Let us remark once more that the concept of temperature, and indeed that of effective geometry is valid in this context only for low-energy photons, *i.e.* photons with wavelengths long compared to the intermolecular spacing in the fluid. For shorter wavelengths, there would be corrections to the propagation dictated by the effective metric. However, results for other systems (such as dumb black holes and Bose-Einstein condensates) suggest that the phenomenon of Hawking radiation is robust (*i.e.* independent of this "high-energy" physics). Consequently, it makes sense to talk about the temperature of the radiation in these systems.

At first sight it may seem that by choosing an appropriate material and a convenient value of the charge we could obtain a high value of the temperature of the radiation, given by

$$T \equiv \frac{\hbar}{2\pi k_B c}\,\kappa \approx 4\times 10^{-21}\,\kappa\ \mathrm{Ks}^2/\mathrm{m}. \tag{G.80}$$

However, the equation for the surface gravity can be rewritten as [3]

$$\kappa = c^2\left(\frac{\beta}{2r} - \beta_{,r}\right)\Bigg|_{r_h}.$$

We see then that, because $\beta(r_h) \ll 1$, the new term appearing in $\kappa$ is bound to be very small. In spite of this result, the emergence in the surface gravity of the term due to the variable velocity of light suggests that it may be worth to study if some media with nonlinear dependence on an external electromagnetic field can be used to generate analog black holes whose Hawking radiation could be measured in laboratory.

[3]Note that this equation depends on $\chi^{(2)}$ through the expression for $r_h$, Eqn.(G.77).

# H. Cosmological effects of non linear Electrodynamics

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Recent works have shown the important role that Nonlinear Electrodynamics (NLED) can have in two crucial questions of Cosmology, concerning particular moments of its evolution for very large and for low-curvature regimes, that is for very condensed phase and at the period of acceleration. We present here a a toy model of a complete cosmological scenario in which the main factor responsible for the geometry is a nonlinear magnetic field which produces a FRW homogeneous and isotropic geometry. In this scenario we distinguish four distinct phases: a bouncing period, a radiation era, an acceleration era and a re-bouncing. It has already been shown that in NLED a strong magnetic field can overcome the inevitability of a singular region typical of linear Maxwell theory; on the other extreme situation, that is for very weak magnetic field it can accelerate the expansion. The present model goes one step further: after the acceleration phase the universe re-bounces and enter in a collapse era. This behavior is a manifestation of the invariance under the dual map of the scale factor $a(t) \to 1/a(t)$, a consequence of the corresponding inverse symmetry of the electromagnetic field ($F \to 1/F$, where $F \equiv F^{\mu\nu}F_{\mu\nu}$) of the NLED theory presented here. Such sequence collapse-bouncing-expansion-acceleration-re-bouncing-collapse constitutes a basic unitary element for the structure of the universe that can be repeated indefinitely yielding what we call a Cyclic Magnetic Universe.

## H.1. Introduction

In the last years there has been increasing of interest on the cosmological effects induced by Nonlinear Electrodynamics (NLED). The main reason for this is related to the drastic modification NLED provokes in the behavior of the cosmological geometry in respect to two of the most important questions of standard cosmology, that is, the initial singularity and the acceleration of the scale factor. Indeed, NLED provides worthwhile alternatives to solve these two problems in a unified way, that is without invoking different mechanisms for each one of them separately. Such economy of hypotheses is certainly welcome.

The partial analysis of each one of these problems was initiated in previous paper. Here we will present a description of a new cosmological model.

The most general form for the dynamics of the electromagnetic field, compatible with covariance and gauge conservation principles reduces to $L = L(F)$, where $F \equiv F^{\mu\nu}F_{\mu\nu}$. We do not consider here the other invariant $G \equiv F^{\mu\nu}F^{*}_{\mu\nu}$ constructed with the dual since in our scenario the average of the electric field vanishes in a magnetic universe as we shall see in the next sections. Thus, the Lagrangian appears as a regular function that can be developed as positive or negative powers of the invariant $F$. Positive powers dominate the dynamics of the gravitational field in the neighborhood of its moment of extremely high curvatures. Negative powers control the other extreme, that is, in the case of very weak electromagnetic fields. In this case as it was pointed out previously it modifies the evolution of the cosmic geometry for large values of the scale factor, inducing the phenomenon of acceleration of the universe. The arguments presented by Lemoine make it worth considering that only the averaged magnetic field survives in a FRW spatially homogeneous and isotropic geometry. Such configuration of pure averaged magnetic field combined with the dynamic equations of General Relativity received the generic name of Magnetic Universe.

The most remarkable property of a Magnetic Universe configuration is the fact that from the energy conservation law it follows that the dependence on time of the magnetic field $H(t)$ is the same irrespective of the specific form of the Lagrangian. This property allows us to obtain the dependence of the magnetic field on the scale factor, without knowing the particular form of the Lagrangian $L(F)$. Indeed, as we will show later on, from the energy-momentum conservation law it follows that $H = H_0\, a^{-2}$. This dependence is responsible for the property which states that strong magnetic fields dominates the geometry for small values of the scale factor; on the other hand, weak fields determines the evolution of the geometry for latter eras when the radius is big enough to excite these terms.

In order to combine both effects, here we will analyze a toy model. The symmetric behavior of the magnetic field in both extremes – that is for very strong and very weak regimes – allows the appearance of a repetitive configuration of the kind exhibited by an eternal cyclic universe.

Negative power of the field in the Lagrangian of the gravitational field was used by Carroll and others in an attempt to explain the acceleration of the scale factor of the universe by modification of the dynamics of the gravitational field by adding to the Einstein-Hilbert action a term that depends on negative power of the curvature, that is

$$S = \frac{M_{\rm Pl}^2}{2} \int \sqrt{-g} \left( R - \frac{\alpha^4}{R} \right) d^4x,$$

This modification showed not to be a good candidate to describe a local gravitational field. However, as a by-product of such proposal, one could envisage the possibility to deal with a new symmetry between strong and weak fields. In a

paper by Novello et al, a model assuming this idea was presented and its cosmological consequences analyzed. In this model, the action for the electromagnetic field was modified by the addition of a new term, namely

$$S = \int \sqrt{-g} \left( -\frac{F}{4} + \frac{\gamma}{F} \right) d^4x. \tag{H.1}$$

This action yields an accelerated expansion phase for the evolution of the universe, and correctly describes the electric field of an isolated charge for a sufficiently small value of parameter $\gamma$. The acceleration becomes a consequence of the properties of this dynamics for the situation in which the field is weak.

In another cosmological context, in the strong regime, it has been pointed out in the literature that NLED can produces a bouncing, altering another important issue in Cosmology: the singularity problem. In this article we would like to combine both effects improving the action given in Eqn.(H.1) to discuss the consequences of NLED for both, weak and strong fields.

It is a well-known fact that under certain assumptions, the standard cosmological model unavoidably leads to a singular behavior of the curvature invariants in what has been termed the Big Bang. This is a highly distressing state of affairs, because in the presence of a singularity we are obliged to abandon the rational description of Nature. It may happen that a complete quantum cosmology could describe the state of affairs in a very different and more complete way. For the time being, while such complete quantum theory is not yet known, one should attempt to explore alternatives that are allowed and that provide some sort of phenomenological consequences of a more profound theory.

It is tempting then to investigate how NLED can give origin to an unified scenario that not only accelerates the universe for weak fields (latter cosmological era, for latter times) but that is also capable of avoiding an initial singularity as a consequence of its properties in the strong regime.

Scenarios that avoid an initial singularity have been intensely studied over the years. As an example of some latest realizations we can mention the pre-big-bang universe and the ekpyrotic universe. While these models are based on deep modifications on conventional physics (assuming the important role of new entities as scalar fields, string theory or branes) the model we present here relies instead on the electromagnetic field. The new ingredient that we introduce concerns the dynamics that is rather different from that of Maxwell in distinct regimes. Specifically, the Lagrangian we will work with is given by

$$L_T = \alpha^2 F^2 - \frac{1}{4} F - \frac{\mu^2}{F} + \frac{\beta^2}{F^2}. \tag{H.2}$$

The dimensional constants $\alpha, \beta$ and $\mu$ are to be determined by observation. Thus the complete dynamics of electromagnetic and gravitational fields are governed by Einstein equations plus $L_T$.

We shall see that in Friedmann-Robertson-Walker (FRW) geometry we can distinguish four typical eras which generate a basic unity – which we will call

***tetraktys*** – that repeat indefinitely[1]. The whole cosmological scenario is controlled by the energy density $\rho$ and the pressure $p$ of the magnetic field. Each era of the tetraktys is associated with a specific term of the Lagrangian. As we shall see the conservation of the energy-momentum tensor implies that the field dependence on the scale factor yields that the invariant $F$ is proportional to $a^{-4}$. This dependence is responsible by the different dominance of each term of the Lagrangian in different phases. The first term $\alpha^2 F^2$ dominates in very early epochs allowing a bouncing to avoid the presence of a singularity. Let us call this the ***bouncing era***. The second term is the Maxwell linear action which dominates in the ***radiation era***. The inverse term $\mu^2/F$ dominates in the ***acceleration era***. Finally the last term $\beta^2/F^2$ is responsible for a ***re-bouncing***. Thus the tetraktys universe can be described in the following way:

- The bouncing era: There exists a collapsing phase that attains a minimum value for the scale factor $a_B(t)$;
- The radiation era: after the bouncing, $\rho+3p$ changes the sign; the universe stops its acceleration and start expanding with $\ddot{a} < 0$;
- The acceleration era: when the $1/F$ factor dominates the universe enters an accelerated regime;
- The re-bouncing era: when the term $1/F^2$ dominates the acceleration changes the sign and starts a phase in which $\ddot{a} < 0$ once more; the scale factor attains a maximum and re-bounces starting a new collapsing phase and entering a bouncing era once more.

This unity of four stages, the tetraktys, constitutes an eternal cyclic configuration that repeats itself indefinitely.

The plan of the article is as follows. In section II we review the Tolman process of average in order to conciliate the energy distribution of the electromagnetic field with a spatially isotropic geometry. Section III presents the notion of the Magnetic Universe and its generic features concerning the dynamics of electromagnetic field generated by a Lagrangian $L = L(F)$. Section IV presents the conditions of bouncing and acceleration of a FRW universe in terms of properties to be satisfied by $L$. In section V we introduce the notion of inverse symmetry of the combined electromagnetic and gravitational fields in a cosmological context. This principle is used to complete the form of the Lagrangian that guides the combined dynamics of the unique long-range fields yielding a spatially homogeneous and isotropic nonsingular universe. In sections VI and VII we present a complete scenario consisting of the four eras: a bouncing, an expansion with negative acceleration, an accelerated phase and a re-bouncing.

[1]This term was taken from Pithagoras who represented the unity of the world constituted by four basic elements by a geometrical figure called tetratkys.

We end with some comments on the form of the scale factor and future developments. In appendix we present the compatibility of our Lagrangian with standard Coulomb law and the modifications induced on causal properties of nonlinear electrodynamics.

## H.2. The average procedure and the fluid representation

The effects of a nonlinear electromagnetic theory in a cosmological setting have been studied in several articles.

Given a generic gauge-independent Lagrangian $L = L(F)$, written in terms of the invariant $F \equiv F_{\mu\nu}F^{\mu\nu}$ it follows that the associated energy-momentum tensor, defined by

$$T_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta L \sqrt{-\gamma}}{\delta \gamma^{\mu\nu}}, \tag{H.3}$$

reduces to

$$T_{\mu\nu} = -4\, L_F\, F_\mu{}^\alpha\, F_{\alpha\nu} - L\, g_{\mu\nu}. \tag{H.4}$$

In the standard cosmological scenario the metric structure of space-time is provided by the FLRW geometry. For compatibility with the cosmological framework, that is, in order that an electromagnetic field can generates a homogeneous and isotropic geometry an average procedure must be used. We define the volumetric spatial average of a quantity $X$ at the time $t$ by

$$\overline{X} \equiv \lim_{V \to V_0} \frac{1}{V} \int X \sqrt{-g}\, d^3x, \tag{H.5}$$

where $V = \int \sqrt{-g}\ d^3x$ and $V_0$ is a sufficiently large time-dependent three-volume. In this notation, for the electromagnetic field to act as a source for the FLRW model we need to impose that

$$\overline{E_i} = 0, \quad \overline{H_i} = 0, \quad \overline{E_i H_j} = 0, \tag{H.6}$$

$$\overline{E_i E_j} = -\frac{1}{3} E^2 g_{ij}, \quad \overline{H_i H_j} = -\frac{1}{3} H^2 g_{ij}. \tag{H.7}$$

With these conditions, the energy-momentum tensor of the EM field associated to $L = L(F)$ can be written as that of a perfect fluid,

$$T_{\mu\nu} = (\rho + p) v_\mu v_\nu - p\, g_{\mu\nu}, \tag{H.8}$$

where

$$\begin{aligned} \rho &= -L - 4 L_F E^2, \\ p &= L - \frac{4}{3} \left(2H^2 - E^2\right) L_F, \end{aligned} \tag{H.9}$$

where $L_F \equiv dL/dF$.

## H.3. Magnetic universe

A particularly interesting case occurs when only the average of the magnetic part does not vanishes and $E^2 = 0$. Such situation has been investigated in the cosmological framework yielding what has been called *magnetic universe.* This should be a real possibility in the case of cosmology, since in the early universe the electric field is screened by the charged primordial plasma, while the magnetic field lines are frozen. In spite of this fact, some attention was devoted to the mathematically interesting case in which $E^2 = \sigma^2 H^2 \neq 0$.

An interesting feature of such magnetic universe comes from the fact that it can be associated with a four-component non-interacting perfect fluid. Let us give a brief proof of the statement that in the cosmological context the energy-content that follows from this theory can be described in terms of a perfect fluid. We work with the standard form of the FLRW geometry in Gaussian coordinates provided by (we limit the present analysis to the Euclidean section)

$$ds^2 = dt^2 - a(t)^2 \left(dr^2 + r^2 d\Omega^2\right). \tag{H.10}$$

The expansion factor, $\theta$ defined as the divergence of the fluid velocity reduces, in the present case, to the derivative of logarithm of the scale factor

$$\theta \equiv v^{\mu}_{;\mu} = 3\,\frac{\dot{a}}{a} \tag{H.11}$$

The conservation of the energy-momentum tensor projected in the direction of the co-moving velocity $v^{\mu} = \delta^{\mu}_0$ yields

$$\dot{\rho} + (\rho + p)\theta = 0 \tag{H.12}$$

Using Lagrangian $L_T$ in the case of the magnetic universe yields for the density of energy and pressure given in equations (H.9):

$$\rho = -\,\alpha^2\,F^2 + \frac{1}{4}\,F + \frac{\mu^2}{F} - \frac{\beta^2}{F^2} \tag{H.13}$$

$$p = -\,\frac{5\alpha^2}{3}\,F^2 + \frac{1}{12}\,F - \frac{7\mu^2}{3}\,\frac{1}{F} + \frac{11\beta^2}{3}\,\frac{1}{F^2} \tag{H.14}$$

Substituting these values in the conservation law, it follows

$$L_F\left[(H^2)^{\cdot} + 4\,H^2\,\frac{\dot{a}}{a}\right] = 0. \tag{H.15}$$

where $L_F \equiv \partial L/\partial F$.

The important result that follows from this equation is that the dependence on the specific form of the Lagrangian appears as a multiplicative factor. This property shows that any Lagrangian $L(F)$ yields the same dependence of the

field on the scale factor irrespective of the particular form of the Lagrangian. Indeed, equation (H.15) yields

$$H = H_0\, a^{-2}. \tag{H.16}$$

This property implies that for each power $F^k$ it is possible to associate a specific fluid configuration with density of energy $\rho_k$ and pressure $p_k$ in such a way that the corresponding equation of state is given by

$$p_k = \left(\frac{4k}{3} - 1\right) \rho_k. \tag{H.17}$$

We restrict our analysis in the present paper to the theory provided by a toy-model described by the Lagrangian

$$\begin{aligned} L_T &= L_1 + L_2 + L_3 + L_4 \\ &= \alpha^2\, F^2 - \frac{1}{4}\, F - \frac{\mu^2}{F} + \frac{\beta^2}{F^2} \end{aligned} \tag{H.18}$$

where $\alpha, \beta, \mu$ are parameters characterizing a concrete specific model. For latter use we present the corresponding many-fluid component associated to Lagrangian $L_T$. We set for the total density and pressure $\rho_T = \sum \rho_i$ and $p_T = \sum p_i$ where

$$\begin{aligned} \rho_1 &= -\alpha^2\, F^2 \ , \ p_1 = \frac{5}{3}\rho_1 \\ \rho_2 &= \frac{1}{4}\, F \ , \ p_2 = \frac{1}{3}\rho_2 \\ \rho_3 &= \frac{\mu^2}{F} \ , \ p_3 = -\frac{7}{3}\, \rho_3 \\ \rho_4 &= -\frac{\beta^2}{F^2} \ , \ p_4 = -\frac{11}{3}\, \rho_4. \end{aligned} \tag{H.19}$$

Or, using the dependence of the field on the scale factor equation (H.16),

$$\begin{aligned} \rho_1 &= -\,4\alpha^2\, H_0^4\, \frac{1}{a^8} \\ \rho_2 &= \frac{H_0}{2}\, \frac{1}{a^4} \\ \rho_3 &= \frac{\mu^2}{2H_0^2}\, a^4 \\ \rho_4 &= -\frac{\beta^2}{4H_0^4}\, a^8. \end{aligned} \tag{H.20}$$

Let us point out a remarkable property of the combined system of this NLED generated by $L_T$ and Friedman equations of cosmological evolution. A simple look into the above expressions for the values of the density of energy exhibits

what could be a possible difficulty of this system in two extreme situations, that is, when $F^2$ and $1/F^2$ terms dominate, since if the radius of the universe can attain arbitrary small and/or arbitrary big values, then one should face the question regarding the positivity of its energy content. However, as we shall show in the next sections, the combined system of equations of the cosmic metric and the magnetic field described by General Relativity and NLED, are such that a beautiful conspiracy occurs in such a way that the negative contributions for the energy density that came from terms $L_1$ and $L_4$ never overcomes the positive terms that come from $L_2$ and $L_3$. Before arriving at the undesirable values where the density of energy could attain negative values, the universe bounces ( for very large values of the field) and re-bounces (in the other extreme, that is, for very small values) to precisely avoid this difficulty. This occurs at the limit value $\rho_B = \rho_{RB} = 0$, as follows from equation

$$\rho = \frac{\theta^2}{3}. \tag{H.21}$$

We emphasize that this is not an extra condition imposed by hand but a direct consequence of the dynamics described by $L_T$. Indeed, at early stages of the expansion phase the dynamics is controlled by the approximation Lagrangian $L_T \approx L_{1,2} = L_1 + L_2$. Then

$$\rho = \frac{F}{4}\,(1 - 4\alpha^2\,F).$$

Using the conservation law (H.12) we conclude that the density of energy will be always positive since there exists a minimum value of the scale factor given by $a^4_{mim} = 8\alpha^2 H_o^2$. A similar conspiracy occurs in the other extreme where we approximate $L_T \approx L_{2,3} = L_2 + L_3$, which shows that the density remains positive definite, since $a(t)$ remains bound, attaining a maximum in the moment the universe makes a re-bounce. These extrema occurs precisely at the points where the total density vanishes. Let us now turn to the generic conditions needed for the universe to have a bounce and a phase of accelerated expansion.

# H.4. Conditions for bouncing and acceleration

## H.4.1. Acceleration

¿From Einstein's equations, the acceleration of the universe is related to its matter content by

$$3\frac{\ddot{a}}{a} = -\frac{1}{2}(\rho + 3p). \tag{H.22}$$

In order to have an accelerated universe, matter must satisfy the constraint $(\rho + 3p) < 0$. In terms of the quantities defined in Eqn. (H.9),

$$\rho + 3p = 2(L - 4H^2 L_F). \tag{H.23}$$

Hence the constraint $(\rho + 3p) < 0$ translates into

$$L_F > \frac{L}{4H^2}. \tag{H.24}$$

It follows that any nonlinear electromagnetic theory that satisfies this inequality yields accelerated expansion. In our present model it follows that terms $L_2$ and $L_4$ produce negative acceleration and $L_1$ and $L_3$ yield inflationary regimes ($\ddot{a} > 0$).

For latter uses we write the value of $\rho + 3p$ for the case of Lagrangian $L_T$ :

$$\rho + 3p = -6\alpha^2 F^2 + \frac{F}{2} - \frac{6\mu^2}{F} + \frac{10\beta^2}{F^2}.$$

### H.4.2. Bouncing

In order to analyze the conditions for a bouncing it is convenient to re-write the equation of acceleration using explicitly the expansion factor $\Theta$, which is called the Raychaudhuri equation:

$$\dot{\theta} + \frac{1}{3}\theta^2 = \frac{1}{2}(\rho + 3p) \tag{H.25}$$

Thus besides condition (H.24) for the existence of an acceleration a bounce needs further restrictions on $a(t)$. Indeed, the existence of a minimum (or a maximum) for the scale factor implies that at the bouncing point $B$ the inequality $(\rho_B + 3p_B) < 0$ (or, respectively, $(\rho_B + 3p_B) > 0$) must be satisfied. Note that at any extremum (maximum or minimum) of the scale factor the density of energy vanishes. This is a direct consequence of the first integral of Friedman equation which, in the Euclidean case, reduces to equation (H.21).

## H.5. Duality on the Magnetic Universe as a consequence of the inverse symmetry

The cosmological scenario that is presented here deals with a cyclic FRW geometry which has a symmetric behavior for small and big values of the scale factor. This scenario is possible because the behavior of its energy content at high energy is the same as it has in its weak regime. This is precisely the case of the magnetic universe that we are dealing with here. To obtain a perfect symmetric configuration for our model we will impose a new dynamical principle:

- **The inverse symmetry principle:**

  *The NLED theory should be invariant under the inverse map*

$$F \to \tilde{F} = \frac{4\mu^2}{F}.$$

This restricts the number of free parameters from three to two, once a direct application of this principle implies that $\beta^2 = 16\alpha^2\mu^4$. This symmetry induces a corresponding one for the geometry. Indeed, the cosmological dynamics is invariant under the associated dual map

$$a(t) \to \tilde{a}(t) = \frac{H_0}{\sqrt{\mu}} \frac{1}{a} \tag{H.26}$$

It is precisely this invariance that is at the origin of the cyclic property of this cosmological scenario.

Let us point out that the above map is a conformal transformation. Indeed, in conformal time, the geometry takes the form

$$ds^2 = a(\eta)^2 \left(d\eta^2 - dr^2 - r^2 d\Omega^2\right). \tag{H.27}$$

Thus making the conformal map

$$\tilde{g}_{\mu\nu} = \Omega^2 \, g_{\mu\nu}$$

where $\Omega = \lambda/a^2$, and $\lambda \equiv H_0/\sqrt{\mu}$. Note that although the Lagrangian $L_T$ is not invariant under a conformal transformation, the average procedure used to make compatible the dynamical system of the electomagnetic field and the Friedman equation is invariant. Indeed, we have

$$\tilde{F} = \tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} \, F_{\alpha\mu}F_{\beta\nu} = \frac{4\mu^2}{F}.$$

## H.6. A complete scenario

There is no doubt that electromagnetic radiation described by a maxwellian distribution has driven the cosmic geometry for a period. Now we would like to analyze the modifications introduced by the non linear terms in the cosmic scenario. The simplest way to do this is to combine the previous lagrangian with the dependence of the magnetic field on the scale factor. We set

$$L_T = \alpha^2 \, F^2 - \frac{1}{4} \, F - \frac{\mu^2}{F} + \frac{\beta^2}{F^2} \tag{H.28}$$

where $\beta$ is related to the other parameters $\alpha$ and $\mu$ by the inverse symmetry principle, as displayed above.

## H.7. Potential

It will be more direct to examine the effects of the magnetic universe controlled by the above lagrangian if we undertake a qualitative analysis using an analogy with classical mechanics. Friedman's equation reduces to the set

$$\dot{a}^2 + V(a) = 0 \tag{H.29}$$

where

$$V(a) = \frac{A}{a^6} - \frac{B}{a^2} - Ca^6 + Da^{10} \tag{H.30}$$

is a potential that restricts the motion of the localization $a(t)$ of the "particle". The constants in $V$ are given by

$$A = \frac{4\alpha^2 H_0^4}{3}, \quad B = \frac{H_0^2}{6}, \quad C = \frac{\mu^2}{6H_0^2}, \quad D = \frac{4\alpha^2\mu^4}{3H_0^4},$$

and are positive.

We can then synthesize the properties of the magnetic universe the dynamics of which is given by $L_T$. We recognize that the dependence of the field as $H = H_0/a^2$ implies the existence of four distinct epochs, which we will analyze now.

The derivative $dL/dF$ has three zeros in points $a, b, c$. In these points $\rho + p$ vanishes. In the case of pure magnetic universe the value of $F$ is always positive. We distinguish the following eras:

## H.8. The four eras of the Magnetic Universe

The dynamics of the universe with matter density given by Eqn.(H.39) can be obtained qualitatively from the analysis of Einstein's equations We distinguish four distinct periods according to the dominance of each term of the energy density. The early regime (driven by the $F^2$ term); the radiation era (where the equation of state $p = 1/3\rho$ controls the expansion); the third accelerated evolution (where the 1/F term is the most important one) and finally the last era where the $1/F^2$ dominates and in which the expansion stops, the universe re-bounces and enters in a collapse era.

## H.9. Bouncing era

In the strong field limit the value of the scalar of curvature is small and the volume of the universe attains its minimum, the density of energy and the pressure are dominated by the terms coming from the quadratic lagrangian $F^2$ and is approximated by the forms

$$\begin{aligned} \rho &\approx \frac{H^2}{2}\,(1 - 8\alpha^2\,H^2) \\ p &\approx \frac{H^2}{6}\,(1 - 40\alpha^2\,H^2) \end{aligned} \tag{H.31}$$

Using the dependence $H = H_o/a^2$, leads to

$$\dot{a}^2 = \frac{kH_o^2}{6\,a^2}\left(1 - \frac{8\alpha^2 H_o^2}{a^4}\right) - \epsilon. \tag{H.32}$$

We remind the reader that we limit our analysis here to the Euclidean section ($\epsilon = 0$). As long as the right-hand side of equation (H.32) must not be negative it follows that, the scale-factor $a(t)$ cannot be arbitrarily small. Indeed, a solution of (H.32) is given as

$$a^2 = H_o \sqrt{\frac{2}{3}\,(t^2 + 12\,\alpha^2)}. \tag{H.33}$$

The linear case can be achieved by setting $\alpha = 0$. The average strength of the magnetic field $H$ evolves in time as

$$H^2 = \frac{3}{2}\,\frac{1}{t^2 + 12\,\alpha^2}. \tag{H.34}$$

Note that at $t = 0$ the radius of the universe attains a minimum value at the bounce:

$$a_B^2 = H_o\,\sqrt{8\,\alpha^2}. \tag{H.35}$$

Therefore, the actual value of $a_B$ depends on $H_o$, which - for given $\alpha$, $\mu$ turns out to be the sole free parameter of the model. The energy density $\rho$ reaches its maximum for the value $\rho_B = 1/64\alpha^2$ at the instant $t = t_B$, where

$$t_B = \sqrt{12\,\alpha^2}. \tag{H.36}$$

For smaller values of $t$ the energy density decreases, vanishing at $t = 0$, while the pressure becomes negative. Only for very small times $t < \sqrt{4\alpha^2/k}$ the non-linear effects are relevant for cosmological solution of the normalized scale-factor. Indeed, solution (H.33) fits the standard expression of the Maxwell case at the limit of large times.

## H.10. Radiation era

The standard, Maxwellian term dominates in the intermediary regime. Due to the dependence on $a^{-2}$ of the field, this phase is defined by $H^2 >> H^4$ yielding the approximation

$$\begin{aligned} \rho &\approx \frac{H^2}{2} \\ p &\approx \frac{H^2}{6} \end{aligned} \tag{H.37}$$

This is the phase dominated by the linear regime of the electromagnetic field. Its properties are the same as described in the standard cosmological model.

## H.11. The accelerated era: weak field drives the cosmological geometry

When the universe becomes larger, negative powers of $F$ dominates and the distribution of energy becomes typical of an accelerated universe, that is:

$$\begin{aligned} \rho &\approx \frac{1}{2}\frac{\mu^8}{H^2} \\ p &\approx \frac{-7}{6}\frac{\mu^8}{H^2} \end{aligned} \tag{H.38}$$

In the intermediate regime between the radiation and the acceleration regime the energy content is described by the combined form

$$\rho = \frac{H^2}{2} + \frac{\mu^2}{2}\frac{1}{H^2},$$

or, in terms of the scale factor,

$$\rho = \frac{H_0^2}{2}\,\frac{1}{a^4} + \frac{\mu^2}{2H_0^2}\,a^4. \tag{H.39}$$

For small $a$ it is the ordinary radiation term that dominates. The $1/F$ term takes over only after $a = \sqrt{H_0}/\mu$, and would grows without bound afterwards. In fact, the curvature scalar is

$$R = T^{\mu}_{\ \mu} = \rho - 3p = \frac{4\mu^2}{H_0^2}\,a^4,$$

showing that one could expect a curvature singularity in the future of the universe for which $a \to \infty$. We shall see, however that the presence of the term $1/F^2$ changes this behavior.

Using this matter density in Eqn.(H.22) gives

$$3\frac{\ddot{a}}{a} + \frac{H_0^2}{2}\,\frac{1}{a^4} - \frac{3}{2}\frac{\mu^8}{H_0^2}\,a^4 = 0.$$

To get a regime of accelerated expansion, we must have

$$\frac{H_0^2}{a^4} - 3\frac{\mu^8}{H_0^2}\,a^4 < 0,$$

which implies that the universe will accelerate for $a > a_c$, with

$$a_c = \left(\frac{H_0^4}{3\mu^8}\right)^{1/8}.$$

## H.12. Re-Bouncing

For very big values of the scale factor the density of energy can be approximated by

$$\rho \approx \frac{\mu^2}{F} - \frac{\beta^2}{F^2} \tag{H.40}$$

and we pass from an accelerated regime to a phase in which the acceleration is negative. When the field attains the value $F_{RB} = 16\alpha^2\mu^2$ the universe changes its expansion to a collapse. The scale factor attains its maximum value

$$a_{max}^4 \approx \frac{H_0^2}{8\alpha^2\mu^2}.$$

## H.13. Positivity of the density of energy

The total density of energy of the tetraktys universe is always positive definite (see equation H.21). In the bouncing and in the re-bouncing eras it takes the value $\rho_B = \rho_{RB} = 0$. At these points the density is an extremum. Actually, both points are minimum of the density. This is a direct consequence of equations (H.12) and (H.21). Indeed, derivative of (H.12) at the bouncing and at the re-bouncing yields

$$\ddot{\rho}_B = \frac{3}{2}\, p_B^2 > 0.$$

Thus there must exists another extremum of $\rho$ which should be a maximum. This is indeed the case since there exists a value on the domain of the evolution of the universe between the two minima such that

$$\rho_c + p_c = 0.$$

At this point we have

$$\ddot{\rho} + \dot{p}_c\, \theta_c = 0$$

showing that at this point $c$ the density takes its maximum value.

## H.14. The behavior of the scale factor

Let us pause for a while and describe the form of the scale factor as function of time in the four regimes. To simplify such description let us separate in three parts:

Phase A: Bouncing-Radiation

Phase B: Radiation-Acceleration

characterized respectivelly by the dynamics controlled by: $L_A = L_1 + L_2; L_B = L_2 + L_3; L_C = L_3 + L_4$. It is straightforward to obtain an analytical expression for each one of these periods. We obtain for phase $A$ :

$$a(t)_{BR} = \sqrt{H_0}\left(\frac{2}{3}t^2 + 12\,\alpha^2\right)^{\frac{1}{4}} \tag{H.41}$$

The inverse symmetric phase $C$ is given by

$$a(t)_{AR} = Constant\left((t-t_c)^2 + \frac{8\alpha^2\mu^4 H_0^2}{\mu^2}\right)^{-\frac{1}{4}} \tag{H.42}$$

For the case of phase $B$ it is convenient to use an auxiliary coordinate $\Psi$ and write A specific form is provided by

$$\begin{aligned} t &= \frac{\sqrt{3}}{2\sqrt{\mu}}\,F(arcos\Psi, \frac{\sqrt{2}}{2}) \\ \Psi &= \frac{1-na^4}{1+na^4} \end{aligned} \tag{H.43}$$

where $n \equiv \mu/H_0^2$, and $F$ is a first kind elliptic function.

## H.15. Some general comments

Although we have analyzed a simplified toy model it displays many regular properties that should be worth of further investigation. In particular, it provides a spatially homogeneous and isotropic FRW geometry which has no Big Bang and no Big Rip. It describes correctly the radiation era and allows for an accelerated phase without introducing any extra source.

The particular form of NLED described here is based on a new principle that states an intimate relation between strong and weak field configurations. This inverse-symmetry principle reduces the number of arbitrary parameters of the theory and allows for the regular properties of the cosmical model. The universe is a cyclic one, having its main characteristics synthesized in the following steps:

- Step 1: The universe contains a collapsing phase in which the scalar factor attains a minimum value $a_B(t)$;
- Step 2: after the bouncing the universe expands with $\ddot{a} < 0$;
- Step 3: when the $1/F$ factor dominates the universe enters an accelerated regime;

- Step 4: when $1/F^2$ dominates the acceleration changes the sign and starts a phase in which $\ddot{a} < 0$ once more, the scale factor attains a maximum and re-bounces starting a new collapsing phase;

- Step 5: the universe repeats the same behavior passing steps 1, 2, 3 and 4 again and again, indefinitely.

The particular form of the dynamics of the magnetic field is dictated by the inverse principle, which states that the behavior of the field is invariant under the mapping $F \to \tilde{F} = \frac{4\mu^2}{F}$. This reflects on the symmetric behavior of the geometry by the dual map $a \to \tilde{a} = \frac{constant}{a}$.

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