Exact Solutions of Einstein and Einstein – Maxwell Equations

Contents

1. Topics

- Mathematical theory of the Inverse Scattering Method
- Mathematical theory of the Integral Equations Method
- Physical applications
- Shells and Membranes in GR

2. Participants

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3. Brief description

3.1. Foreword

The machinery for construction the exact solutions of the Einstein and Einstein-Maxwell equations appeared at the end of 70th with implantation of the Inverse Scattering Method (ISM) into General Relativity and discovery of Gravitational Solitons. Solitons are remarkable solutions of certain nonlinear wave equations which behave in several ways like extended particles. Soliton waves where first found in some two-dimensional nonlinear differential equations in fluid dynamics. In 1960's - 1970's by many authors, following the pioneering discovery of Gardner, Green, Kruskal and Miura, a method for construction the exact solutions of some special classes of non-linear equations, known as the Inverse Scattering Method (ISM) have been developed. In 1978 V.Belinski and V.Zakharov (1; 2) extended the ISM also to General Relativity to solve Einstein equations in vacuum for space-times that admit an orthogonally transitive two-parameter group of isometries. Together with ISM for generation of pure solitonic solutions they also formulated (1) the first version of more general technique: the so-called Integral Equation Method (IEM) for the vacuum gravitational field. In 1980 G.Alekseev (3) generalized ISM to the Einstein-Maxwell equations and in 1985 he developed (4) most general and quite different version of the Integral Equation Method for construction the exact solutions of the coupled Einstein-Maxwell equations (the most detailed and comprehensive account of his approach was given in his later papers (5),(6)). Metrics of this sort depends only on two coordinates (either one time-like and one space-like or both space-like), however such ansatz includes many important physical cases such as number of cosmological models, cylindrically symmetric waves, colliding plane waves and stationary axisymmetric solutions. Some of the solitonic solutions generated by the ISM are most relevant in the gravitational physics, for example, the Kerr, Schwarzschild and Kerr-Newman black holes solutions and their generalizations are solitonic solutions. In a nutshell the ISM procedure involves two main steps. The first step consists in finding for a given nonlinear equation a set of linear "Schrodinger-like" differential system for some "wave function" (so called spectral equations) whose integrability conditions coincide with the original nonlinear equation of interest. The second step consists in finding the special class of solutions of this spectral equations which correspond to the set of poles of the "wave function" in the complex plane of the spectral parameter where from the solutions of the original nonlinear equation, known as solitons, can be extracted. The IEM procedure technically is different and more general, however, in number of cases it permits to find the solitonic solutions by a shorter way with respect to the ISM approach.

3.1.1. Vacuum and electrovacuum fields with two - dimensional Abelian isometry group

For space-times admitting two-dimensional Abelian isometry groups with the orbits which have the signature of internal metric $(+-)$ or $(--)$, the four-dimensional metric can be written in the form

$$ds^2 = f(x^1,x^2)\eta_{\mu\nu}dx^\mu dx^\nu + g_{ab}(x^1,x^2)dx^a dx^b \tag{3.1.1}$$

where $\mu,\nu = 1,2$; $a,b = 3,4$; $f > 0$ and $\eta_{\mu\nu} = \operatorname{diag}\{\epsilon_1,\epsilon_2\}$ with $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$. The Einstein equations for such metrics can be written in the form ($\mathbf{g} \in GL(2,\mathbb{R})$):

$$\left\{\begin{array}{l} \eta^{\mu\nu}\partial_\mu(\alpha\,\partial_\nu\mathbf{g}\cdot\mathbf{g}^{-1}) = 0 \\ \mathbf{g}^T = \mathbf{g}, \quad \det\mathbf{g} \equiv \epsilon\alpha^2 \end{array}\right. \left\| \begin{array}{c} \epsilon \equiv -\epsilon_1\epsilon_2 \\ \partial_\mu\partial_\nu\alpha = 0 \end{array} \right\| \begin{array}{l} \beta: \; \partial_1\beta = \epsilon_1\partial_2\alpha, \\ \qquad \partial_2\beta = -\epsilon_2\partial_1\alpha \end{array} \tag{3.1.2}$$

The real symmetric 2×2-matrix function $\mathbf{g} = \|g_{ab}\|$ depending on some two of the four space-time coordinates (x^1,x^2), should satisfy the nonlinear partial differential equations given above. These equations are equivalent to the "dynamical part" of the Einstein equations. In accordance with these equations, the function $\alpha(x^1,x^2)$, representing the square root of the modulus of determinant of $\mathbf{g}$, should satisfy the linear two-dimensional d'Alembert or Laplace equation for the signatures $(+-)$ or $(--)$ of conformal metric $\eta_{\mu\nu}$ on the orbit space respectively. This "harmonic" property of the function α allows to determine its "harmonically" conjugated function $\beta(x^1,x^2)$. It is convenient to use these functions as new local coordinates on the orbit space of the isometry group. If the metric components which constitute the matrix $\mathbf{g}$ would have been found as the solution of the above equations, the remaining yet unknown function $f(x^1,x^2)$ can be calculated (in principle, at least) from the other ("constraint") part of Einstein equations in quadratures (the corresponding expressions can be found in the papers cited below).

For electrovacuum fields which admit the same types of space-time symmetry, i.e. for the fields with all components and potentials depending on some two of the four space-time coordinates (x^1,x^2) and for both possible signatures $(+-)$ or $(--)$ of the conformal metric $\eta_{\mu\nu}$ on the orbit space of the isometry group, the four-dimensional metric and electromagnrtic potential can be written in the forms

$$ds^2 = f(x^1, x^2)\eta_{\mu\nu}dx^\mu dx^\nu + g_{ab}(x^1, x^2)dx^a dx^b, \qquad A_i = \{A_\mu = 0, A_a\} \tag{3.1.3}$$

The Einstein - Maxwell equations for these electrovacuum fields can be written conveniently in a complex self-dual form of Kinnersley-like equations:

$$\left\{\begin{array}{l} \partial_\mu H_a{}^b = i\alpha^{-1} h_a{}^c \varepsilon_\mu{}^\nu \partial_\nu H_c{}^b \\ \partial_\mu \Phi_a \;= i\alpha^{-1} h_a{}^c \varepsilon_\mu{}^\nu \partial_\nu \Phi_c \\ \partial_\mu H_a{}^b = \partial_\mu h_a{}^b + i\alpha^{-1} h_a{}^c \varepsilon_\mu{}^\nu \partial_\nu h_c{}^b - 2\bar{\Phi}^b \partial_\mu \Phi_a \end{array}\right. \quad \left\| \quad \begin{array}{ll} g_{ab} \equiv h_{ab} & h_a{}^b = \epsilon^{bc} h_{ac} \\ A_a = \mathrm{Re}\, \Phi_a & \varepsilon_\mu{}^\nu = \eta_{\mu\gamma}\varepsilon^{\gamma\nu} \end{array}\right. \tag{3.1.4}$$

where we use the same notations

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}, \quad \det \|h_{ab}\| = \epsilon\alpha^2$$

Both systems of nonlinear equations (3.1.2) and (3.1.4) were found completely integrable. Both of them admit a construction of solitons by appropriate dressing methods and development of some integral equation methods of essentially different structures for construction of solitons as well as of non-soliton types of their solutions.

3.2. Inverse scattering approach to solution of vacuum Einstein equations ("Lambda-solitons" and the Riemann - Hilbert problem)

The integrability of the equations (3.1.2) had been discovered in the papers (1), (2), where the authors developed the methods for construction of solutions using the basic ideas of the Inverse Scattering Transform approach – the dressing method for construction of solitons on arbitrarily chosen vacuum backgrounds and the reduction of the equations (3.1.2) to the classical Riemann - Hilbert problem and corresponding 2×2-matrix linear singular integral equations on the spectral plane for construction of none-soliton types of solutions. Changing a bit the notations of the papers (1), (2), we write here the equations (3.1.2) for $\mathbf{g} \in GL(2, \mathbb{R})$ in terms the coordinates (ξ, η) which are the linear combinations of Weyl-like coordinates (α, β) such that $\xi = \beta + j\alpha$ and $\eta = \beta - j\alpha$, with $j = 1$ for $\epsilon = 1$ and $j = i$ for $\epsilon = -1$. The coordinates (ξ, η), therefore, are real for $\epsilon = 1$ or complex conjugated to each other for

$\epsilon = -1$. In these terms, the equations (3.1.2) take the form

$$\left\{\begin{array}{l} \partial_\xi(\alpha\partial_\eta \mathbf{g}\cdot\mathbf{g}^{-1}) + \partial_\eta(\alpha\partial_\xi \mathbf{g}\cdot\mathbf{g}^{-1}) = 0 \\ \mathbf{g}^T = \mathbf{g}, \quad \det \mathbf{g} = \epsilon\alpha^2, \quad \epsilon = \pm 1. \end{array}\right. \quad \Bigg\| \quad (\xi,\eta) = \left\{\begin{array}{ll} x \pm t, & \epsilon = 1 \\ z \pm i\rho, & \epsilon = -1 \end{array}\right. \tag{3.2.1}$$

In (1), (2), these equations had been presented as the integrability conditions for the overdetermined linear system for a complex 2×2-matrix function $\mathbf{\Psi}(\xi,\eta,\lambda)$ depending (besides the coordinates ξ and η) also on a complex "spectral" parameter λ:

$$\left\{\begin{array}{l} D_\xi \mathbf{\Psi} = \dfrac{\mathbf{V}_\xi}{\lambda - j\alpha}\mathbf{\Psi} \\ D_\eta \mathbf{\Psi} = \dfrac{\mathbf{V}_\eta}{\lambda + j\alpha}\mathbf{\Psi} \end{array}\right. \quad \Bigg| \quad \begin{array}{l} D_\xi = \partial_\xi - \dfrac{\lambda}{\lambda - j\alpha}\dfrac{\partial}{\partial\lambda} \\ D_\eta = \partial_\eta - \dfrac{\lambda}{\lambda + j\alpha}\dfrac{\partial}{\partial\lambda} \end{array} \quad \Bigg| \quad \begin{array}{l} \mathbf{V}_\xi = -j\alpha\,\partial_\xi \mathbf{g}\cdot\mathbf{g}^{-1} \\ \mathbf{V}_\eta = j\alpha\,\partial_\eta \mathbf{g}\cdot\mathbf{g}^{-1} \end{array} \tag{3.2.2}$$

For construction of a dressing method for solution of the above equations a new matrix variable $\chi(\xi,\eta,\lambda)$ was introduced in the above equations instead of $\mathbf{\Psi}(\xi,\eta,\lambda)$. This matrix plays the role of the "scattering" matrix which relates $\mathbf{\Psi}(\xi,\eta,\lambda)$ characterizing the unknown solution $\mathbf{g}$ with $\overset{o}{\mathbf{\Psi}}(\xi,\eta,\lambda)$ characterizing some chosen "background" solution $\overset{\circ}{\mathbf{g}}$ in accordance with the following scheme:

$$\|\overset{\circ}{g}_{ab}\| \quad \to \quad \overset{\circ}{\mathbf{\Psi}} \quad \to \quad \mathbf{\Psi} = \chi\cdot\overset{\circ}{\mathbf{\Psi}} \quad \to \quad \|g_{ab}\|$$

For this new matrix variable the equations (3.2.2) take the form

$$\left\{\begin{array}{l} (\lambda - j\alpha)D_\xi\chi = \mathbf{V}_\xi\cdot\chi - \chi\cdot\overset{\circ}{\mathbf{V}}_\xi \\ (\lambda + j\alpha)D_\eta\chi = \mathbf{V}_\eta\cdot\chi - \chi\cdot\overset{\circ}{\mathbf{V}}_\eta \end{array}\right. \tag{3.2.3}$$

It is important, that these equations for the new unknown matrix variable $\chi(\xi,\eta,\lambda)$ should be supplied by some algebraic constraints providing the solution for the matrix $\mathbf{g}$ to be real and symmetric (see (1), (2) for details):

$$\mathbf{g} = \chi(\lambda)\cdot\overset{\circ}{\mathbf{g}}\cdot\chi^T(\frac{\epsilon\alpha^2}{\lambda}), \qquad \overline{\chi(\overline{\lambda})} = \chi(\lambda) \tag{3.2.4}$$

Another condition imposed on the behaviour of χ at $\lambda = \infty$ allowed to express the solution for metric in a simple form:

$$\chi(\infty) = \mathbf{I} \qquad \Rightarrow \qquad \mathbf{g} = \chi(0)\cdot\overset{\circ}{\mathbf{g}} \tag{3.2.5}$$

3.2.1. Vacuum solitons on arbitrary background

For construction of vacuum N-soliton solutions the following ansatz was used in (1) for the structure of χ on the spectral plane:

$$\chi = \mathbf{I} + \sum_{k=1}^{N} \frac{\mathbf{R}_k(\xi,\eta)}{\lambda - \mu_k(\xi,\eta)}, \qquad \chi^{-1} = \mathbf{I} + \sum_{l=1}^{N} \frac{\mathbf{S}_l(\xi,\eta)}{\lambda - \nu_k(\xi,\eta)} \tag{3.2.6}$$

where the 2×2-matrices $\mathbf{R}_k$ and $\mathbf{S}_l$ and the functions μ_k and ν_l are yet unknown. Substituting of this ansatz into (3.2.3) and conditions (3.2.4) and (3.2.5) and solving of the corresponding (rational with respect to λ) relations, one obtains the following solution. The pole trajectories μ_k and ν_k are determined by simple algebraic equations ($k = 1, 2, \dots, N$):

$$\mu_k^2 + 2(\beta - w_k)\mu_k + \epsilon\alpha^2 = 0, \qquad \nu_k = \frac{\epsilon\alpha^2}{\mu_k}$$

where w_k constitute a set of arbitrarily chosen constants of real ones or/and complex conjugated pairs. The matrices $\mathbf{R}_k$ and $\mathbf{S}_l$ are degenerate and they can be expressed as the products of vector functions which admit an explicit expressions in terms of pole trajectories μ_k, the functions $\mathring{\mathbf{g}}$, $\mathring{\mathbf{\Psi}}$ associated with the background solution and a set of "projective" constant vectors $\mathbf{k}_l$ whose components include arbitrary constants c_l which should satisfy the constraint that they should be real for real w_l and they should be complex and conjugated to each other for complex conjugated w_l:

$$\begin{array}{ll}
\mathbf{R}_k = \mathbf{n_k} \otimes \mathbf{m}_k & \mathbf{m}_k = \mathbf{k}_k \cdot \mathring{\mathbf{\Psi}}^{-1}(\mu_k) \\
\mathbf{S}_l = \mathbf{p_l} \otimes \mathbf{q}_l & \mathbf{n}_k = -\mu_k \sum\limits_{l=1}^{N} \mathring{\mathbf{g}} \cdot \mathbf{m}_l \Gamma_{lk}^{-1} \\
\mathbf{q}_k = \sum\limits_{l=1}^{N} \Gamma_{kl}^{-1} \mu_l \mathbf{m}_l & \mathbf{p}_l = \mathring{\mathbf{g}} \cdot \mathbf{m}_l \\
\Gamma_{kl} = -\mu_k\mu_l \dfrac{(\mathbf{m}_k \cdot \mathring{\mathbf{g}} \cdot \mathbf{m}_l)}{\mu_k\mu_l - \epsilon\alpha^2} & \mathbf{k}_l = \{1,\, c_l\}
\end{array}$$

However, it is necessary to take into account that the corresponding solution of the equations (3.2.3) – (3.2.5) does not satisfy the condition $\det \mathbf{g} = \epsilon\alpha^2$ in (3.2.1). In accordance with (1), this last condition will be satisfied, if we make a special conformal transformation of $\mathbf{g}$ which we give here together with the corresponding transformation for χ of the form (3.2.6) which provides for transformed χ the condition $\det \chi = 1$:

$$\chi \to \prod_{k=1}^{N} \left(\frac{\lambda - \mu_k}{\lambda - \nu_k} \right)^{\frac{1}{2}} \chi, \qquad \mathbf{g} \to \prod_{k=1}^{N} \left(\frac{\mu_k}{\alpha} \right) \mathbf{g}$$

And, the last constraint which should be imposed on this solution tp provide a correct signature of the soliton metrics is that the number of solitons (i.e. the number of poles in (3.2.6)) should be even: $N = 2n$. It can be mentioned here also that in general these soliton solutions can be presented in a compact determinant form, such that for calculation of all metric components, one needs to calculate (instead of all components of $N \times N$- matrix Γ_{kl} and of its inverse) only four determinants of the order $N \times N$.

3.2.2. Formulation of the Riemann-Hilbert problem and its solution

As it was shown in (1), the general solution of the vacuum equations for the matrix $\mathbf{g}$ can be characterized by the matrix $\boldsymbol{\chi}(\xi, \eta, \lambda)$ which is an analytical function on the spectral plane λ outside and inside the contour

$$\Gamma = \{\lambda, \, |\lambda| = \alpha\},$$

where it can be represented as the Cauchy integrals over this contour

$$\boldsymbol{\chi}_{out}^{-1} = \mathbf{I} + \frac{1}{i\pi} \int\limits_{\Gamma} \frac{\boldsymbol{\rho}(\zeta)\, d\zeta}{\lambda - \zeta + i0} + \mathbf{U}(\lambda) \qquad \mathbf{U}(\lambda) = \sum_{k=1}^{N} \frac{S_k}{\lambda - \nu_k}$$
$$\boldsymbol{\chi}_{in}^{-1} = \mathbf{I} + \frac{1}{i\pi} \int\limits_{\Gamma} \frac{\boldsymbol{\rho}(\zeta)\, d\zeta}{\lambda - \zeta - i0} + \mathbf{U}(\lambda)$$

where $+i0$ and $-i0$ mean "outside the contour" and "inside the contour" respectively; the matrix function $\mathbf{U}(\xi, \eta, \lambda)$ characterizes the "soliton part" of the solution and therefore, it should possess the specific structure shown above.

As it was also shown in (1), the limit values of $\boldsymbol{\chi}_{in}$ and $\boldsymbol{\chi}_{out}$ on the contour Γ should satisfy some algebraic relation which includes a 2-matrix $\mathbf{G}$ of a special structure:

$$\Gamma: \quad \boldsymbol{\chi}_{out} = \boldsymbol{\chi}_{in} \cdot \mathbf{G}, \qquad \mathbf{G} = \overset{\circ}{\boldsymbol{\Psi}}\, \mathbf{G}_0(w)\, \overset{\circ}{\boldsymbol{\Psi}}{}^{-1} \tag{3.2.7}$$

where $\mathbf{G}_0(w)$ is an arbitrary 2-matrix function which depends only on the parameter w

$$w = \beta + \frac{1}{2}(\lambda + \frac{\epsilon\alpha^2}{\lambda}) \tag{3.2.8}$$

The problem of construction of the analytical matrix functions $\boldsymbol{\chi}_{in}$ and $\boldsymbol{\chi}_{out}$ which satisfy on the contour Γ the re;ation (3.2.7) is known as the classical matrix Riemann-Hilbert problem. Its solution can be reduced to an equivalent problem of solution of a matrix linear singular integral equation which had been presented in (1) in the following form of the integral equations for the kernel matrix function $\boldsymbol{\rho}(\xi, \eta, \zeta)$

$$\begin{cases} \boldsymbol{\rho}(\zeta) - \mathbf{T}(\zeta)\left\{\mathbf{I} + \frac{1}{i\pi}\fint \frac{\boldsymbol{\rho}(\zeta')}{\zeta - \zeta'}\, d\,\zeta' + \mathbf{U}(\zeta)\right\} = 0 \\ \mathbf{T}(\zeta) = (\mathbf{I} + \mathbf{G})^{-1}(\mathbf{I} - \mathbf{G}) \end{cases}$$

The solution $\boldsymbol{\rho}(\xi, \eta, \zeta)$ of these integral equations should satisfy the supplementary conditions which were imposed in (1) on the corresponding matrices $\boldsymbol{\chi}_{in}$ and $\boldsymbol{\chi}_{out}$ in the form

$$\overline{\boldsymbol{\chi}}_{in,out}(\overline{\lambda}) = \boldsymbol{\chi}_{in,out}(\lambda), \qquad \mathbf{g} = \boldsymbol{\chi}_{out}(\frac{\epsilon\alpha^2}{\lambda})\,\overset{\circ}{\mathbf{g}}\,\boldsymbol{\chi}^{T}_{in}(\lambda)$$

These conditions provide the matrix $\mathbf{g}$ to be real and symmetric, and they allow to express the corresponding solution of vacuum Einstein equations in a simple form

$$\mathbf{g} = \boldsymbol{\chi}_{in}(0)\,\overset{\circ}{\mathbf{g}}$$

For solitons on a given background, one should chose $\mathbf{G} \equiv \mathbf{I}$ and therefore, for this case $\boldsymbol{\rho}(\zeta) \equiv \mathbf{0}$. In this case, $\boldsymbol{\chi} \equiv \boldsymbol{\chi}_{out} = \boldsymbol{\chi}_{in}$ is meromorphic on a Riemann sphere (extended spectral plane) and it has no any jumps on the contour Γ. In this case, a construction of (N-soliton) solutions in terms of $\overset{\circ}{\mathbf{g}}$ and corresponding $\overset{\circ}{\boldsymbol{\Psi}}$ reduces, as it was mentioned above, to pure algebraic steps described in detail in (1), (2) and later in (26).

3.3. Monodromy transform approach to solution of Einstein-Maxwell equations ("Omega-solitons" and the integral equation method)

The analysis of integrability of the symmetry reduced electrovacuum Einstein - Maxwell equations was made in an essentially different way than it was described above for pure vacuum gravitational fields. This analysis was also based on the same idea of consideration of the nonlinear field equations as the integrability conditions of some overdetermined linear system ("spectral" problem). However, a specific structure of the spectral problem for the self-dual equations (3.1.4), constructed in (3) and reformulated in terms of Jordan conditions in (4), suggested a possibility for development of some new approach to solution of these equations called the "monodromy transform" approach (3) – (6). In this approach, any solution can be expressed in quadratures in terms of corresponding solution of a system of linear singular integral equations which arose without a connection with some matrix Riemann - Hilbert problem, but from a simple analysis of the analytical and monodromy properties of the fundamental solution of the associated linear system on the spectral plane. It is worth to note here also, that this approach to solution of Einstein - Maxwell equations, being restricted to pure vacuum case, also can provide us with some new useful tools for solution of vacuum equations in addition to those, which arise from the inverse scattering methods described above.

At first, we recall the structure of the overdetermined linear system and a complete formulation of the corresponding spectral problem whose integrability conditions are equivalent to the complex (Kinnersley-like) self-dual form (3.1.4) of the symmetry reduced Einstein - Maxwell equations. These equations can be represented as the integrability conditions of the linear system for 3×3-matrix function $\mathbf{\Psi}(\xi, \eta, w)$[1] with additional constraints imposed on the ranks and traces of its matrix coefficients which determine uniquely

[1]For simplicity of notations and, on the other hand, trying to keep the notations close enough to those in the original papers, we use everywhere below $\mathbf{\Psi}$ to denote the 3×3-matrix of the fundamental solution of the spectral problem for electrovacuum case, and it should not be confused with the 2×2-matrix function $\mathbf{\Psi}$ of the fundamental solution of an essentially different spectral problem for vacuum case described in the previous section.

the canonical Jordan forms of these complex 3×3-matrices[2]

$$\left\{ \begin{array}{l} 2i(w-\xi)\partial_\xi \mathbf{\Psi} = \mathbf{U}(\xi,\eta)\cdot\mathbf{\Psi} \\ 2i(w-\eta)\partial_\eta \mathbf{\Psi} = \mathbf{V}(\xi,\eta)\cdot\mathbf{\Psi} \end{array} \right. \quad \Bigg\| \quad \begin{array}{ll} \operatorname{rank}\mathbf{U}=1, & \operatorname{tr}\mathbf{U}=i, \\ \operatorname{rank}\mathbf{V}=1, & \operatorname{tr}\mathbf{V}=i, \end{array} \tag{3.3.1}$$

where w is a new spectral parameter whose relation to the spectral parameter λ used in (1) is given by (3.2.8). For an equivalence to the Einstein - Maxwell equations, the integrability conditions of the linear system (3.3.1) together with the Jordan conditions for $\mathbf{U}$ and $\mathbf{V}$ should be supplied with the condition of existence for the above linear system of a Hermitian matrix integral $\mathbf{K}(w)$ of a special structure (see (3)–(6) for details):

$$\left\{ \begin{array}{l} \mathbf{\Psi}^\dagger\cdot\mathbf{W}\cdot\mathbf{\Psi} = \mathbf{K}(w) \\ \mathbf{K}^\dagger(w) = \mathbf{K}(w) \end{array} \right. \quad \Bigg\| \quad \frac{\partial\mathbf{W}}{\partial w} = 4i\mathbf{\Omega}, \qquad \mathbf{\Omega} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.3.2}$$

where $\mathbf{\Psi}^\dagger(\xi,\eta,w) \equiv \overline{\mathbf{\Psi}^T(\xi,\eta,\overline{w})}$ and $\mathbf{W} = \mathbf{W}(\xi,\eta,w)$ is also an unknown function which is linearly dependent on the spectral parameter w with a constant coefficient $4i\mathbf{\Omega}$. This Hermitian 3×3-matrix function plays an important role in the calculation of solutions because its components can be identified directly with the metric and complex vector electromagnetic potential components by the following way:

$$\mathbf{W} = 4i(w-\beta)\mathbf{\Omega} + \mathbf{G}, \qquad \mathbf{G} = \begin{pmatrix} -4h^{ab} + 4\Phi^a\overline{\Phi}^b & -2\Phi^a \\ -2\overline{\Phi}^b & 1 \end{pmatrix} \tag{3.3.3}$$

where the indices a, b are raised similarly to the two-component spinor ones (see (3.1.4)).

3.3.1. Elecrovacuum solitons on arbitrary background

The electrovacuum N-soliton solutions of Einstein - Maxwell equations had been constructed in (3) using the same dressing transformation $\mathbf{\Psi} = \chi\cdot\overset{o}{\mathbf{\Psi}}$, where $\overset{o}{\mathbf{\Psi}}$ denotes the solution of the spectral problem for chosen electrovacuum background. In this case, the equations (3.3.1), (3.3.2) take the form

$$\left\{ \begin{array}{l} 2i(w-\xi)\partial_\xi\chi = \mathbf{U}\cdot\chi - \chi\cdot\overset{\circ}{\mathbf{U}} \\ 2i(w-\eta)\partial_\eta\chi = \mathbf{V}\cdot\chi - \chi\cdot\overset{\circ}{\mathbf{V}} \\ \chi^\dagger\cdot\mathbf{W}\cdot\chi = \overset{o}{\mathbf{W}} \end{array} \right. \quad \Bigg\| \quad \begin{array}{ll} \operatorname{rank}\mathbf{U}=1, & \operatorname{tr}\mathbf{U}=i, \\ \operatorname{rank}\mathbf{V}=1, & \operatorname{tr}\mathbf{V}=i, \end{array} \tag{3.3.4}$$

[2] We note here that some part of necessary supplemetary conditions had been presented in (3) in a not the most convenient form, and they have been formulated in a compact form in terms of given here trace- and rank- conditions a bit later, in (4).

where we have assumed that the gauge transformation $\mathbf{\Psi} \to \mathbf{\Psi} \cdot \mathbf{C}(w)$ is chosen so that the values of the integral (3.3.2) for the solitons and for the background solution coincide:

$$\mathbf{K}(w) = \overset{o}{\mathbf{K}}(w)$$

Following (3), we use now in (3.3.4) the appropriately modified soliton ansatz similar to (1):

$$\mathbf{Ø} = \mathbf{I} + \sum_{k=1}^{N} \frac{\mathbf{R}_k(\xi,\eta)}{w - w_k}, \qquad \mathbf{Ø}^{-1} = \mathbf{I} + \sum_{k=1}^{N} \frac{\mathbf{S}_k(\xi,\eta)}{w - \widetilde{w}_k}. \tag{3.3.5}$$

Here the dressing matrix χ and its inverse χ^{-1} are assumed to have (different) simple poles on the plane of the spectral parameter w. Solving of the corresponding (polynomial with respect to w) relations, one obtains the following solution. The pole locations w_k of χ are arbitrarily chosen complex constants and the poles of χ^{-1} are located at the complex conjugated points:

$$\widetilde{w}_n = \overline{w}_n, \qquad n = 1, 2, \ldots, N$$

The matrices $\mathbf{R}_k$ and $\mathbf{S}_l$ are degenerate and they can be expressed as the products of vector functions which admit an explicit expressions in terms of the functions $\overset{\circ}{\mathbf{\Psi}}$ and $\overset{\circ}{\mathbf{W}}$ of the background solution and two sets of constant vectors $\mathbf{k}_n$ and $\mathbf{l}_n$

$$\begin{array}{ll} \mathbf{R}_k = \mathbf{n_k} \otimes \mathbf{m}_k & \mathbf{m}_k = \mathbf{k}_k \cdot \overset{\circ}{\mathbf{\Psi}}{}^{-1}(w_k) \\ \mathbf{S}_l = \mathbf{p_l} \otimes \mathbf{q}_l & \mathbf{p}_l = \overset{\circ}{\mathbf{\Psi}}(\widetilde{w}_l) \cdot \mathbf{l}_l \end{array}$$

$$\left.\begin{array}{l} \mathbf{n}_k = \sum\limits_{l=1}^{N} (\Gamma^{-1})_{kl} \mathbf{p}_l \\ \mathbf{q}_k = -\sum\limits_{l=1}^{N} \mathbf{m}_l (\Gamma^{-1})_{lk} \end{array} \right\| \quad \Gamma_{kl} = \frac{(\mathbf{m}_l \cdot \mathbf{p}_k)}{w_l - \widetilde{w}_k}$$

where $k, l, \ldots = 1, \ldots, N$. The constant complex 3-dimensional vectors $\mathbf{k}_n$ and $\mathbf{l}_n$ possess a "projective" character in a sense, that the N-soliton solution depends only on the ratios of the components of each of these vectors. The vectors $\mathbf{k}_n$ can be chosen arbitrarily and therefore, their components include $2N$ arbitrary complex constants which we denote as c_n, d_n, while the components of vectors $\mathbf{l}_n$ can be expressed (up to a common constant multiplier) in terms of the the components of $\mathbf{k}_k$ as follows:

$$\mathbf{l}_n = \overset{\circ}{\mathbf{K}}{}^{-1}(\widetilde{w}_n)\mathbf{k}_n^{\dagger}, \qquad \mathbf{k}_n = \{1, c_n, d_n\}$$

All components of metric and complex electromagnetic vector potential of the constructed N-soliton solutions are given by the expressions, where (3.3.3) should be taken into account):

$$\mathbf{G} = \overset{\circ}{\mathbf{G}} - 4i(\mathbf{\Omega} \cdot \mathbf{R} + \mathbf{R}^\dagger \cdot \mathbf{\Omega}) + 4i\beta_* \mathbf{\Omega}, \qquad f = c_0^2 \Gamma \overline{\Gamma} \overset{\circ}{f}$$

$$\mathbf{R} = \sum_{k=1}^{N} \mathbf{n}_k \otimes \mathbf{m}_k = \sum_{k=1}^{N} \Gamma_{kl}^{-1} \mathbf{p}_l \otimes \mathbf{m}_k, \qquad \Gamma = \det \|\Gamma_{kl}\|$$

where β_0 is a real constant which should be chosen to provide the nondiagonal component of metric h_{ab} to be real, and c_0 is an arbitrary real constant. Thus, we obtain the families of N-soliton (for any positive integer N) solutions on arbitrarily chosen electrovacuum background which depend on $3N$ arbitrary complex constant parameters $\{w_n, c_n, d_n\}$.

The electrovacuum solitons described in this section generalize the vacuum solitons constructed in (1) on arbitrary vacuum background which have complex conjugated poles, while the electrovacuum generalization of the vacuum solitons (1) with real poles do not arise in the technique described just above. However, it seems worth to note, that many electrovacuum solutions which generalize vacuum solitons with real poles on some specially chosen backgrounds (e.g., on the Minkowski background) can be constructed as the analytical continuations of soliton solutions with complex poles in the space of their constant parameters. This analytical continuation is quite similar to the known one, which relates the "underextreme "and "overextreme "parts of the Kerr - Newman family of solutions. Such analytical continuations of the soliton solutions in the space of their parameters can be constructed using the monodromy transform approach and the integral equation method described in the next two sections and this represents one of useful applications of this method.

3.3.2. Monodromy data parametrization of the solution space

A remarkable feature of the spectral problem (3.3.1) – (3.3.2) is that its fundamental solution $\mathbf{\Psi}(\xi, \eta, w)$ possess some universal (i.e. solution independent) analytical properties. These properties (found in (4) and described in more details in (5) and (6)) allowed to identify within a general structure of $\mathbf{\Psi}$ a set of four functional parameters $\{\mathbf{u}_\pm(w), \mathbf{v}_\pm(w)\}$ – the coordinate independent functions of the spectral parameter which were called the "monodromy data" because they determine completely the branching (monodromy) properties of $\mathbf{\Psi}$ on the spectral plane. The important property of these functional parameters is that they are defined for any local solution and they take different particular functional values for different solutions. Moreover, as it was proved in the papers cited above, for any particular choice of these functions (holomorphic in some local region of the spectral plane), there exists a uniquely defined local solution of the field equations, such that the corresponding fundamental solution of the spectral problem (3.3.1) – (3.3.2) has such monodromy data. All these mean that these monodromy data functions can play the role of "coordinates" in the infinite-dimensional space of all local solutions of the symmetry reduced Einstein - Maxwell equations. The use of this "coordinate transformation" in the space of solutions was called the "monodromy transform". In what follows, we describe a "direct problem" of this monodromy transform. Namely, we explain, how the monodromy data can be determined (in principle, at leats) for any local solution of the symmetry reduced Einstein - Maxwell equations. In the next section we describe a solution of the "inverse problem" of the monodromy transform, i.e. a construction of solution of the Einstein - Maxwell equations for any particular choice of the monodromy data functions.

To avoid obvious ambiguities in the definitions, we impose some "normalization" conditions on the metric and electromagnetic potential components of any solution as well as on the corresponding fundamental solutions of the spectral problem. For this, we chose some regular space-time point (ξ_0, η_0) for the "initial" point (or the point of normalization) and, without any loss of generality, set there for normalized values (denoted by ˇ): $\check{g}_{ab}(\xi_0, \eta_0) = \epsilon_0 \mathrm{diag}\,\{1, \epsilon\alpha_0^2\}$, $\check{\Phi}_a(\xi_0, \eta_0) = 0$, where $\epsilon_0 = \pm 1$ and $\alpha_0 = (\xi_0 - \eta_0)/2j$. Any fundamental solution $\mathbf{\Psi}$ can be normalized at the initial point by a unit matrix as follows:

$$\check{\mathbf{\Psi}}(\xi, \eta, w) = \mathbf{\Psi}(\xi, \eta, w) \cdot \mathbf{\Psi}^{-1}(\xi_0, \eta_0, w) \quad \Rightarrow \quad \check{\mathbf{\Psi}}(\xi_0, \eta_0, w) = \mathbf{I}$$

As it was shown in (4) –(6), for any solution of the reduced Einstein - Maxwell equations, this normalized 3×3-matrix function $\check{\mathbf{\Psi}}$ possess the four singular (branching) points on the spectral plane: $w = \xi_0$, $w = \xi$, $w = \eta_0$ and $w = \eta$.[3] For given initial values ξ_0 and η_0 and fixed values ξ, η of coordinates, the matrix $\check{\mathbf{\Psi}}$ on the spectral plane is a multi-valued function. For construction of its single-valued branch, it is convenient to chose the cut on the plane w which consists of two opened and non-intersecting contours L_+ and L_-, where L_+ starts at $w = \xi_0$ and ends at $w = \xi$, while L_- starts at $w = \eta_0$ and ends at $w = \eta$. Then this matrix function on the spectral plane outside the cut $L = L_+ + L_-$ will have a holomorphic branch we denote further as $\check{\mathbf{\Psi}}$. At infinity, this holomorphic branch possess the property $\check{\mathbf{\Psi}}(\xi, \eta, w = \infty) = \mathbf{I}$, it has the jumps of certain structures on L_+ and L_- and it branches at their end-points. It is remarkable that the character of this branching on L_+ and L_- can be described by two monodromy matrices[4] $\mathbf{T}_+(w)$ and $\mathbf{T}_-(w)$ respectively,

[3] In this construction, we consider the solutions of reduced Einstein - Maxwell equations in some local region near the initial point, where ξ and η are close enough to ξ_0 and η_0 respectively.

[4] The matrices $\mathbf{T}_+$ and $\mathbf{T}_-$ determine the linear transformations of $\check{\mathbf{\Psi}}$ after its analytical continuation along the paths which join the corresponding points on different edges of

which possess in general the very specific structures:

$$\mathbf{T}_+(w) = \mathbf{I} - 2\frac{\mathbf{l}_+(w)\otimes\mathbf{k}_+(w)}{(\mathbf{l}_+(w)\cdot\mathbf{k}_+(w))}, \quad \mathbf{T}_-(w) = \mathbf{I} - 2\frac{\mathbf{l}_-(w)\otimes\mathbf{k}_-(w)}{(\mathbf{l}_-(w)\cdot\mathbf{k}_-(w))}, \quad \mathbf{T}^2_\pm(w) \equiv \mathbf{I}$$

In accordance with these structures, these monodromy matrices are determined by the independent components of the four "projective" vectors $\mathbf{k}_\pm(w)$ and $\mathbf{l}_\pm(w)$, which can be parametrized, without any loss of generality, as follows:

$$\mathbf{k}_\pm(w) = \{1, \mathbf{u}_\pm(w), \mathbf{v}_\pm(w)\}, \qquad \mathbf{l}_\pm(w) = \{1, \mathbf{p}_\pm(w), \mathbf{q}_\pm(w)\}$$

where the scalar functions $\mathbf{u}_+(w)$, $\mathbf{v}_+(w)$, $\mathbf{p}_+(w)$, $\mathbf{q}_+(w)$ holomorphic on L_+ and $\mathbf{u}_-(w)$, $\mathbf{v}_-(w)$, $\mathbf{p}_-(w)$, $\mathbf{q}_-(w)$ holomorphic on L_- represent a complete set of the monodromy data for the normalized fundamental solution of the linear problem (3.3.1). However, taking into account the additional condition (3.3.2) of our spectral problem, we find that the whole content of this condition can be expressed as a simple relations which determine $\mathbf{l}_+(w)$ in terms of a complex conjugation of $\mathbf{k}_+(w)$ and $\mathbf{l}_-(w)$ in terms of a complex conjugation of $\mathbf{k}_-(w)$ (see (4) for details).. Therefore, a complete set of monodromy data for the fundamental solution of the spectral problem (3.3.1), (3.3.2) consists of the set of four functions $\{\mathbf{u}_\pm(w), \mathbf{v}_\pm(w)\}$, which determine completely the monodromy matrices $\mathbf{T}_\pm(w)$ and vise versa. It is worth to note that the functions $\mathbf{v}_\pm(w)$ are responsible for electromagnetic fields: for pure vacuum case $\mathbf{v}_\pm(w) \equiv 0$ and the set of monodromy data functions in general consists only of two functions $\mathbf{u}_\pm(w)$.

3.3.3. Constructing solutions for arbitrary monodromy data

The structure of the spectral problem (3.3.1), (3.3.2) provides the existence and uniqueness of a local solution of Einstein - Maxwell equations for any choice of the monodromy data functions $\mathbf{u}_\pm(w)$ and $\mathbf{v}_\pm(w)$. In accordance with the analisys given in (4) (more detail explanatins were given in (5), (6)), all components and potentials of a general local solution of electrovacuum Einstein - Maxwell equations can be expressed in quadratures in terms of the monodromy data $\{\mathbf{u}_\pm(w), \mathbf{v}_\pm(w)\}$ and of the corresponding solution of a master system of linear singular integral equations whose kernels and rhs are expressed algebraically in terms of the monodromy data. These expressions can be derived from the following expression, where the structure (3.3.3) should be taken into account:

$$\mathbf{W} = \mathbf{W}_0 - 4i(\mathbf{\Omega}\cdot\mathbf{R} + \mathbf{R}^\dagger\cdot\mathbf{\Omega}), \qquad \mathbf{R} = \frac{1}{i\pi}\int\limits_L [\lambda]_\zeta \mathbf{l}(\zeta)\otimes\boldsymbol{\varphi}(\zeta)d\zeta \qquad (3.3.6)$$

the cut L_+ or L_- respectively.

where $\zeta \in L$ and the contour L on the spectral plane consists of two disconnected opened contours L_+ and L_- with the endpoints (ξ_0, ξ) and (η_0, η) respectively. The notation $[\lambda]_\zeta$ means a jump at the point $\zeta \in L$ of a "standard "branching function $\lambda = \sqrt{(\zeta-\xi)(\zeta-\eta)/(\zeta-\xi_0)(\zeta-\eta_0)}$. $\mathbf{W}_0$ is the initial value of $\mathbf{W}$ and the components of the vector $\mathbf{l}(\zeta)$ are expressed in terms of the monodromy data:

$$\mathbf{W}_0(w) = 4i(w-\beta_0)\mathbf{\Omega} + \text{diag}\,\{-4\epsilon_0\epsilon\alpha_0^2, -4\epsilon_0, 1\},$$

$$\mathbf{l}(\zeta) = \begin{pmatrix} 1 + i\epsilon_0(\zeta-\beta_0)\mathbf{u}^\dagger(\zeta) \\ -i\epsilon_0(\zeta-\beta_0) + \epsilon\alpha_0^2\mathbf{u}^\dagger(\zeta) \\ 4\epsilon_0(\zeta-\xi_0)(\zeta-\eta_0)\mathbf{v}^\dagger(\zeta) \end{pmatrix}$$

where $\epsilon_0 = \pm 1$, $\alpha_0 = (\xi_0 - \eta_0)/2j$, $\beta_0 = (\xi_0 + \eta_0)/2$ and $\mathbf{u}^\dagger(\zeta) = \overline{\mathbf{u}(\overline{\zeta})}$ and $\mathbf{v}^\dagger(\zeta) = \overline{\mathbf{v}(\overline{\zeta})}$. The components of a 3-dimensional vector function $\boldsymbol{\varphi}\,(\xi,\eta,\zeta) = \{\boldsymbol{\varphi}^{[1]}, \boldsymbol{\varphi}^{[\mathbf{u}]}, \boldsymbol{\varphi}^{[\mathbf{v}]}\}$ should satisfy the linear singular integral equations with a scalar kernel $\mathcal{K}$ and the right hand side $\mathbf{k}(\tau) \equiv \{1, \mathbf{u}(\tau), \mathbf{v}(\tau)\}$, both depending on the monodromy data (4):

$$\frac{1}{\pi i}\int\limits_L \frac{\mathcal{K}(\xi,\eta,\tau,\zeta)}{\zeta-\tau} \begin{pmatrix} \boldsymbol{\varphi}^{[\mathbf{1}]}(\xi,\eta,\zeta) \\ \boldsymbol{\varphi}^{[\mathbf{u}]}(\xi,\eta,\zeta) \\ \boldsymbol{\varphi}^{[\mathbf{v}]}(\xi,\eta,\zeta) \end{pmatrix} d\zeta = \begin{pmatrix} 1 \\ \mathbf{u}(\tau) \\ \mathbf{v}(\tau) \end{pmatrix} \tag{3.3.7}$$

where $\tau, \zeta \in L$, and the kernel $\mathcal{K}(\xi,\eta,\tau,\zeta) = -[\lambda]_\zeta(\mathbf{k}(\tau)\cdot\mathbf{l}(\zeta))$. It is easy to see, that these vector equations decouple into three independent equations for each component of the vector function $\boldsymbol{\varphi}\,(\xi,\eta,\zeta)$. It is important to note that each of these three independent equations, in general, is a coupled pair of two integral equations, because each function on the disconnected parts $L_\pm$ of the contour is represented by two indpendent functions, e.g. $\mathbf{u}(\tau)$ should be understood as $\mathbf{u}(\tau) = \mathbf{u}_+(\tau)$ for $\tau \in L_+$ and $\mathbf{u}(\tau) = \mathbf{u}_-(\tau)$ for $\tau \in L_-$ and the same is for $\mathbf{v}(\tau)$, $\boldsymbol{\varphi}^{[1]}(\zeta)$, $\boldsymbol{\varphi}^{[\mathbf{u}]}(\zeta)$, $\boldsymbol{\varphi}^{[\mathbf{v}]}(\zeta)$.

For a conclusion, we note that for different problems the master integral equations (3.3.7) can admit various useful modifications. In particular, for stationary axisymmetric fields, the regularity axis condition implies that the monodromy data are analytically matched: $\mathbf{u}_+(\tau) \equiv \mathbf{u}_-(\tau)$, $\mathbf{v}_+(\tau) \equiv \mathbf{v}_-(\tau)$, and therefore, $\boldsymbol{\varphi}_+^{[1]}(\zeta) = \boldsymbol{\varphi}_-^{[1]}(\zeta)$, $\boldsymbol{\varphi}_+^{[\mathbf{u}]}(\zeta) = \boldsymbol{\varphi}_-^{[\mathbf{u}]}(\zeta)$, $\boldsymbol{\varphi}_+^{[\mathbf{v}]}(\zeta) = \boldsymbol{\varphi}_-^{[\mathbf{v}]}(\zeta)$. This allows to merge L_+ and L_- and reduce (3.3.7) to some more simple scalar forms. In this case, these integral equations can be solved explicitly, if the analytically matched monodromy data are chosen to be arbitrary rational functions of the spectral parameter.

4. Publications

In ICRA and ICRANET group the research on the ISM and IEM methods always have been of a high priority. This field was developed mainly by G.A.Alekseev, V.A.Belinski, A.Paolino, M.Pizzi. The following results were obtained and published:

4.1. Books

1. V. Belinski and E. Verdaguer, "Gravitational Solitons ", Cambridge University Press, Cambridge Monographs on Mathematical Physics, 2001.

   The monograph dedicated to the a self-contained and systematic exposition of the theory of gravitational solitons. The book represents and provides a comprehensive review of exact soliton solutions to the relativistic gravitational equations. The text begins with a detailed discussion of the extension of the Inverse Scattering Method to the theory of gravitation, starting with pure gravity and then extending it to the coupling of gravity with the electromagnetic field. There follows a review of the gravitational soliton solutions based on their symmetries. These solutions include some of the most interesting in gravitational physics such as those describing inhomogeneous cosmological models, gravitational waves and black holes.

4.2. Refereed journals

1. V.Belinski "Gravitational breather and topological properties of gravisolitons ", Phys. Rev., D44, 3109, (1991).

   It was discovered that in general gravitational solitons are topological objects and a topological charge for them can be introduced in rigorous mathematical way. It was shown that for a wide class of solitonic solutions of the vacuum Einstein equations the notions of gravisolitons and antigravisolitons with respect to this charge can be introduced. The presence of attractive forces between two gravitational solitons with the topological charges of opposite signs and repulsive forces between solitons of the same charges was shown. The construction of the exact solution of Einstein equations which have been named "The Gravibreather "as a bound state of the gravisoliton and antigrav-isoliton was described.

2. V.Belinski "On the Equilibrium of Two Charged Masses in General Relativity", Journ. Korean Phys. Soc., 49, 732 (2006).

   It was found the exact solution of Einstein-Maxwell equations representing the equilibrium state of the Reissner-Nordstrom black hole and Kerr-Newman spinning naked singularity object. It was shown that the equilibrium without a strut between such centers is possible only if both of them are critically charged. The conjecture has been made that the last constrain is only due to fact that solutions was obtained under too restrictive constrains and in more general settings the equilibrium state for the two arbitrary charged centers can be constructed.

3. M.Pizzi "Gravitational Field and Electric Force Lines of a new 2-soliton solution", IJMP(D),16, 1087 (2007).

   A new exact solution of the coupled Einstein-Maxwell equations was obtained and studied. It was found using the ISM, adding one soliton to the Schwarzschild background. The solution is stationary and axial-symmetric. The physical interpretation is that it describes a Kerr-Newman naked singularity linked by a "strut "to a charged black hole. On the axis, between the two bodies, it is present an unavoidable anomaly region and a conic singularity. The solution is stationary also in the case with zero total angular momentum. The force lines of the electrical field in a general case, and in the case in which the Kerr-Newman singularity has a much smaller mass than the nearby black hole have been constructed for different distances between the bodies. In spite of the naive interpretation suggested by the mathematical construction of the solution, what one can expect to be a Schwarzschild black hole, appears to be a charged and rotating object. It is possible to interpret this fact as a parameter-mixing phenomenon.

4. G.Alekseev and V.Belinski "Schwarzschild Black Hole Hovering in the Field of a Reissner-Nordstrom Naked Singularity", Nuovo Cimento, 122B, No.2, (2007).

   It was found a three-parametric family of exact static axisymmetric solutions of Einstein - Maxwell equations which describe a Schwarzschild black hole hovering in the field of an over-critically charged Reissner - Nordstrom source (naked singularity). This family of solutions depends on three real (positive) parameters which are the mass of a Schwarzshild black hole, the mass of a naked singularity and the parameter characterizing the distance separating these sources. The charge of the naked singularity providing the equilibrium at this distance and various geometric characteristics of interacting sources in our solutions are functions of these three independent parameters. This configuration is stable with respect to displacements of the sources one towards or apart from another along the axis of symmetry. The existence of such equilibrium state is due to the repulsive forces produced by a naked singularity.

5. G.Alekseev and V.Belinski “Equilibrium configurations of two charged masses in General Relativity”, Phys. Rev. D76, 021501(R), (2007).

   An asymptotically flat static solution of Einstein-Maxwell equations which describes the field of two non-extreme Reissner - Nordström sources in equilibrium is presented. It is expressed in terms of physical parameters of the sources (their masses, charges and separating distance). Very simple analytical forms were found for the solution as well as for the equilibrium condition which guarantees the absence of any struts on the symmetry axis. This condition shows that the equilibrium is not possible for two black holes or for two naked singularities. However, in the case when one of the sources is a black hole and another one is a naked singularity, the equilibrium is possible at some distance separating the sources. It is interesting that for appropriately chosen parameters even a Schwarzschild black hole together with a naked singularity can be ”suspended” freely in the superposition of their fields.

6. A.Paolino and M.Pizzi “Electric Force Lines of the double Reissner-Nordstrom exact solution”, accepted for publication in IJMP(D) (2007).

   The Alekseev-Belinski exact solution of the Einstein-Maxwell equation which describes two Reissner-Nordstrom sources in reciprocal equilibrium (no struts nor strings) have been studied in some detail: examination of coordinate systems used and description of distribution in space of the gravitational and electric fields. In particular, the plots of the electric force lines have been explicitly constructed in three qualitatively different situations: equal-signed charges, opposite-signed charges and the case of a charged naked singularity near a neutral black hole.

7. M.V. Barkov, V.A. Belinski and G.S. Bisnovatyi-Kogan, “An exact General Relativity solution for the Motion and Intersections of Self-Gravitating Shells in the Field of a Massive Black Hole”, JETP 95, 371, (2002); (astro-ph/0210296).

   Finally it should be mention the very interesting and important enough exact solution in the General Relativity for the intersection process of two massive self-gravitating spherically symmetric shells with tangential pressure. This solution was found by the direct integration of the gravitational equations without using any generating solutions technic. It was shown how one can calculate all shell’s parameters after intersection in terms of the parameters before the intersection. The result was quite new, the solution of this kind was known only for the massless shells (Dray and t’Hooft, 1985). The solution have been applied to the analysis of matter ejection effect from relativistic stellar clusters. It was shown that in relativistic case the matter ejection effect is stronger than in Newtonian gravity.

8. V.Belinski, M.Pizzi and A.Paolino "Charged membrane as a source for repulsive gravity", accepted by IJMP(D), in press (2008).

   We demonstrate an alternative (with respect to the ones existing in literature) and more habitual for physicists derivation of exact solution of the Einstein-Maxwell equations for the motion of a charged spherical membrane with tangential tension. We stress that the physically acceptable range of parameters for which the static and stable state of the membrane producing the Reissner-Nordstrom (RN) repulsive gravity effect exists. The concrete realization of such state for the Nambu-Goto membrane is described. The point is that membrane are able to cut out the central naked singularity region and at the same time to join in appropriate way the RN repulsive region. As result we have a model of an everywhere-regular material source exhibiting a repulsive gravitational force in the vicinity of its surface: this construction gives a more sensible physical status to the RN solution in the naked singularity case.

9. M. Pizzi and A. Paolino "Intersection of self-gravitating charged shells in a Reissner-Nordstrom field ",, IJMP(D), submitted, October 2008.

   We describe the equation of motion of two charged spherical shells with tangential pressure in the field of a central Reissner-Nordstrom (RN) source. We solve the problem of determining the motion of the two shells after the intersection by solving the related Einstein-Maxwell equations and by requiring a physical continuity condition on the shells velocities. We consider also four applications: post-Newtonian and ultra-relativistic approximations, a test-shell case, and the ejection mechanism of one shell. This work is a direct generalization of Barkov-Belinski-Bisnovati-Kogan paper.

4.3. Proceedings of international conferences

1. G.Alekseev and V.Belinski "Superposition of fields of two Reissner-Nordstrom sourses", Invited paper for Proceedings of 11 Marcel Grossmann Meeting (Berlin, July 2006); ArXiv:gr-qc/0710.2515 (2007).

   It was described the detailed and systematic derivation (by solving of the linear singular integral equation form of the electro-vacuum Einstein - Maxwell equations) of the 5-parametric family of static asymptotically flat solutions for the superposed gravitational and electromagnetic fields of two Reissner-Nordstrom sources with arbitrary parameters (masses, charges and separating distance). The 4-parametric family of equilibrium configurations of two Reissner-Nordstrom sources (one of which should be a black hole and another one a naked singularity) presented in our previous paper (Phys. Rev. D76, 021501(R), 2007) arises after a restriction of the parameters of the 5-parametric solution by the equilibrium condition which provides the absence in the solution of conical singular points on the symmetry axis between the sources.

2. V.Belinski "Gravitational Topological Charge and Gravibreather", Proceedings of the 10th Italian Conference on General Relativity and Gravitational Physics (Bardonecchia, Italy, September 1-5, 1992), ed. M.Cerdonio et al., page 37, World Scientific, (1994).

3. V.Belinski "Gravitational Topological Charge ", International Conference "Birth of the Universe and Fundamental Physics"(Rome, May 1994), Lectures Notes in Physics, vol. 455, ed. F. Ochionero, Springer, (1995).

4. V.Belinski "Gravitational Topological Charge and the Gravibreather", Proceedings of the Seventh Marcel Grossman Meeting (MG7), Stanford, USA, July 24-30, 1994, ed. R.Jantzen, G. Mac Keiser and R.Ruffini, World Scientific (1996), p. 96.

5. V.Belinski "On the Equilibrium of two Charged Masses in General Relativity", the 9th Italian-Korean Symposium on Relativistic Astrophysics (Seoul, July 19-24, 2005), Journ. Korean Phys. Soc., vol. 49, p.732 (2006)

6. G.Alekseev "Monodromy Transform Approach in the Theory of Integrable Reductions of Einstein's Field Equations and Some Applications", the Eleventh Marcel Grossman Meeting (MG11), Berlin, July 23-29, 2006, Proceedings of MG11, in press.

7. M.Pizzi "Some features of a new 2-soliton solution of the Einstein-Maxwell equations", the Eleventh Marcel Grossman Meeting (MG11), Berlin, July 23-29, 2006, Proceedings of MG11, in press.

8. G.Alekseev and V.Belinski "Superposition of fields of two Reissner-Nordstrom sourses ", Invited paper for Proceedings of 11 Marcel Grossmann Meeting (Berlin, July 23-29, 2006), World Scientific, pub. date September 2008.

9. M.Pizzi and A.Paolino "Electric force lines of the double Reissner-Nordstrom solution ", the 2nd Stueckelberg Workshop on Relativistic Field Theories, (Pescara, September 3-8, 2007), IJMP(A), vol.23(8), March 2008

10. V. Belinski, M.Pizzi, A.Paolino "A Membrane Model of the Reissner-Nordstrom Singularity with Repulsive Gravity ", the 5th Italian-Sino Workshop on Relativistic Astrophysics, (Taiwan, May 28-June 1, 2008), AIP Conference Proceedings, 1059, Relativistic Astrophysics, ed. Da-Shin Lee, Wolung Lee and She-Sheng Xue, page 3 (2008)

11. V. Belinski, M.Pizzi, A.Paolino "Charged masses and repulsive gravity ", The 3rd Stueckelberg Workshop on Relativistic Field Theories, Pescara, 8-18 July, 2008, in press.

12. V.Belinski "Einstein-Maxwell solitons ", Five invited lectures, XIII Brazilian School in Gravitation and Cosmology, Rio de Janeiro, July 20- August 2, 2008, AIP Conference Proceedings, Ed. M.Novello and S. Bergliaffa, in press.

4.4. Talks at international conferences

1. "Gravitational Topological Charge and Gravibreather"

   (V.Belinski)
   Plenary talk at the 10th Italian Conference on General Relativity and Gravitational Physics, Bardonecchia, Italy, September 1-5, 1992.

2. "Gravitational Topological Charge ",

   (V.Belinski)
   Plenary talk at the International Conference "Birth of the Universe and Fundamental Physics", Rome, May 1994.

3. "Gravitational Topological Charge and the Gravibreather"

   (V.Belinski)
   Plenary talk at the Seventh Marcel Grossman Meeting (MG7), Stanford, USA, July 24-30, 1994.

4. "On the equilibrium state for two charged masses in General Relativity"

   (V.Belinski)
   Talk at the 2nd Italian-Sino Workshop on Relativistic Astrophysics, Pescara, June 10-20, 2005.

5. "On the Equilibrium of two Charged Masses in General Relativity"

   (V.Belinski)
   Invited talk at the 9th Italian-Korean Symposium on Relativistic Astrophysics, Seoul, July 19-24, 2005.

6. "Equilibrium configuration of two charged masses in General Relativity"

   (G.Alekseev and V.Belinski)
   The talk at the 3rd Italian-Sino Workshop on Relativistic Astrophysics, Pescara, June 10-20, 2006.

7. "New developments in Einstein-Maxwell Theory: non-perturbative approach "

(G.Alekseev and V.Belinski)
The talk at the 1st Stueckelberg Workshop on Relativistic Field Theories, Pescara, June 25-July 1, 2006.

8. "Monodromy Transform Approach in the Theory of Integrable Reductions of Einstein's Field Equations and Some Applications"

   (G.Alekseev)
   Talk at the Eleventh Marcel Grossman Meeting (MG11), Berlin, July 23-29, 2006.

9. "Some features of a new 2-soliton solution of the Einstein-Maxwell equations"

   (M.Pizzi)
   Talk at the Eleventh Marcel Grossman Meeting (MG11), Berlin, July 23-29, 2006.

10. "The exact solution for the equilibrium configuration of two static Reissner-Nordstrom sources"

    (G.Alekseev and V.Belinski)
    Invited talk at the workshop "Key Problems in Theoretical Cosmology", April 23-28, 2007, Cargese, Institut D'Etudes Scientifiques De Cargese.

11. "The static equilibrium state of two Reissner-Nordstrom sources"

    (G.Alekseev and V.Belinski)
    Talk at the 10th Italian-Korean Symposium on Relativistic Astrophysics, Pescara, June 25-30, 2007.

12. "Interaction of black holes with external gravitational and electromagnetic fields"

    (G.Alekseev)
    Talk at the 10th Italian-Korean Symposium on Relativistic Astrophysics, Pescara, June 25-30, 2007.

13. "The fields of a naked singularity and black hole in mutual equilibrium and the electric force lines in the equilibrium configuration of two Reissner-Nordstrom sources"

    (G.Alekseev, V.Belinski, M.Pizzi (speaker) and A.Paolino)
    Talk at the 4th Italian-Sino Workshop on Relativistic Astrophysics, Pescara, July 20-30, 2007.

14. "Electric force lines of the double Reissner-Nordstrom solution"

    (M.Pizzi)
    Talk at the 2nd Stueckelberg Workshop on Relativistic Field Theories, Pescara, September 3-8, 2007.

15. "A Membrane Model of the Reissner-Nordstrom Singularity with Repulsive Gravity "

    (V. Belinski, M.Pizzi, A.Paolino)
    Talk at the 5th Italian-Sino Workshop on Relativistic Astrophysics, (Taiwan, May 28-June 1, 2008)

16. "Charged masses and repulsive gravity "

    V. Belinski, M.Pizzi, A.Paolino
    Talk at the 3rd Stueckelberg Workshop on Relativistic Field Theories, (Pescara, 8-18 July, 2008).

17. "Einstein-Maxwell solitons "

    V.Belinski
    Five invited lectures at the XIII Brazilian School in Gravitation and Cosmology, Rio de Janeiro, July 20- August 2, 2008

A. Equilibrium configurations of two charged masses in General Relativity

A.1. Introduction

In the Newtonian physics two point-like particles can be in equilibrium if the product of their masses is equal to the product of their charges (we use the units for which $G = c = 1$). In General Relativity, till now the equilibrium condition for two particle-like sources imposed on their physical masses, charges and separating distance was not known in explicit and reasonably simple analytical form which would admit a rigorous analysis without a need of numerical experiments. The only exceptional case was the Majumdar-Papapetrou solution (7; 8), for which the charge of each source is equal to its mass. In this case, the equilibrium is independent of the distance between the sources. For each of the static sources of this sort its outer and inner Reissner-Nordström horizons coincide and such sources are called extreme ones. Accordingly, the sources with two separated horizons are called under-extreme and the sources without horizons – super-extreme.

The problem, which had been under investigation by many researchers and which we solve in the present paper, consists in the search of equilibrium configurations of non-extreme sources. Since the advent of solution generating techniques for stationary axisymmetric Einstein-Maxwell fields, a construction of an exact solution for two charged masses at rest does not represent any principal difficulty. However, in general the asymptotically flat solutions of this kind contain conical singularities on the symmetry axis between the sources which can be interpreted as a presence of some extraneous struts preventing the sources to fall onto or to run away from each other. The equilibrium condition just implies the absence of such struts. Naturally, if the metric is known so is the equilibrium condition. In the static case, the latter means that the product of the metric coefficients g_{tt} and $g_{\rho\rho}$ (in cylindrical Weyl coordinates) should be equal to unity at the axis where $\rho = 0$. However, this equilibrium equation in such general form usually is expressed by a set of formal parameters and it is so complicated that its analytical investigation appears to be very difficult. Therefore, it is desirable to have this equation expressed in terms of physical parameters and in a simple enough form making it accessible for an analytical examination of a possibility of realization of equilibrium. Moreover, this realization should be compatible with a condition of a positive value of the distance between the sources. This task have not been accomplished yet, and up to now there were known only some results achieved by numerical calculations.

The first researches of the equilibrium of non-extreme sources (9) - (14) led to the contradictory conclusions. The authors of the indicated papers used both the exact techniques and pN and ppN approximations. The common opinion expressed in (9; 10) and (12) - (14) was that the equilibrium for non-extreme sources is impossible. Nevertheless, in (13) one can find a remark that the analysis performed was insufficient and the existence of equilibrium configurations for the non-extreme objects can not be excluded. The arguments in favour of such possibility can be found also in (11).

The next step which attracted attention to the problem again have been done by Bonnor in (15), where the equilibrium condition for a charged test particle in the Reissner-Nordström field was analyzed. Examination made there suggested also some plausible assumptions for the exact solutions. As have been indicated in (15) a charged test body can be at rest in the field of the Reissner-Nordström source only if they both are either extreme (for the test particle the degree of its extremality is defined just by the ratio between its charge and mass), balanced irrespective of distance, or one of them is super-extreme and the other is under-extreme, and in this case the equilibrium depends on the distance. There is no way for equilibrium in cases when both sources are either super-extreme or under-extreme. It is worth to mention that in the very recent papers (16) a new perturbative solution describing an equilibrium state of two-body system consisting of a Reissner-Nordström black hole and a super-extreme test particle has been presented. The whole set of combined Einstein-Maxwell equations has been solved there by using the first order perturbation approach developed in (17) and based on the tensor harmonic expansion of both the gravitational and electromagnetic fields adopting the Regge-Wheeler (18) gauge. (The basic equations for combined gravitational and electromagnetic perturbations of the Reissner-Nordström background in the decoupled form were found in another gauges in (19) and in the also decoupled Hamiltonian form in (20)). Both the electromagnetically induced gravitational perturbations and gravitationally induced electromagnetic perturbations (21) due to a mass as well as a charge of the particle have thus taken into account. The expressions in a closed form for both the perturbed metric and electromagnetic field have been explicitly given (16). It is interesting that the equilibrium equation (which arises in this case as a self-consistency condition for the set of differential equations for perturbations) remains the same as of Bonnor (15).

The Bonnor's analysis allows to expect that qualitatively the same can happen also for two Reissner-Nordström sources. For two extreme sources this is indeed the case because it is known that such generalization exists and leads to the Majumdar-Papapetrou solution. Up to 1997 it remained unknown whether the analogous generalization for the non-extreme bodies can be found. The first solid arguments in favour of existence of a static equilibrium configuration for the "black hole - naked singularity" system was presented in (22). These results have been obtained thereby numerical calculations and three examples of numerical solutions of the equilibrium equation have been demonstrated. These solutions can correspond to the equilibrium configurations free of struts. For the complete proof it would be necessary to show that such configurations indeed consist of two sources, separated by physically sensible distance between them. However, in (22) it was pointed out that the distance dependence for the equilibrium state is unknown. The authors of (22) also reported that a number of numerical experiments for two black holes and for two naked singularities showed the negative outcomes, i.e. all tested sets of parameters were not in power to satisfy the equilibrium equation. These findings are in agreement with Bonnor's test particle analysis. One year later the similar numerical analysis was made in (23).

In this paper, we present an exact solution of the Einstein-Maxwell equations which describes the field of two Reissner-Nordström sources in static equilibrium as well as the equilibrium condition itself which turns out to have unexpectedly simple form expressed in terms of physical parameters of the sources. This simplicity permits us to prove a validity of conjectures of the papers (15) and (22) on exact analytical level. It allows also a direct analytical investigation of the physical properties of the equilibrium state of two non-extreme sources.

We precede a description of our results with a few words on the methodology of the derivation of our solution. An application for derivation of this solution of the Inverse Scattering Method for electro-vacuum developed in (24; 25) and described in details in the book (26) leads to not most convenient parametrization of the solution which give rise to some subsequent technical difficulties (although there are no principal obstacles to use this approach). Instead, we used the Integral Equation Method (27; 25) which opens a shorter way to the desirable results. The first step was to construct the solution for the two-pole structure of the monodromy data on the spectral plane with a special choice of parameters providing asymptotical flatness and the static character of the solution. This corresponds also to the two-pole structure of the Ernst potentials (as functions of the Weyl cylindrical coordinate z) on the symmetry axis. Then the expressions for physical masses and physical charges for both sources were found with the help of the Gauss theorem and the notion of distance between these sources was also defined. We stress here that the physical character of masses and charges of the sources follows not only from their definition using the Gauss theorem, but also from the analysis of that limiting case in which one of the sources is a test particle (see the formulae (12), (13) below and the text after them). After that we derived the equilibrium equation in terms of these five physical parameters. The miracle arises if one substitutes this equilibrium equation back into the solution. This results in the impressive simplification of all formulas. Below we expose the final outcome which is ready for using in practical purposes without necessity of knowledge of any details of its derivation.

It is worthwhile to mention that a correctness of our solution has been confirmed also by its direct substitution into the Einstein - Maxwell field equations.

A.2. The solution

For our static solution in cylindrical Weyl coordinates, metric and vector electromagnetic potential take the forms

$$ds^2 = Hdt^2 - f(d\rho^2 + dz^2) - \frac{\rho^2}{H}d\varphi^2, \tag{A.2.1}$$

$$A_t = \Phi, \qquad A_\rho = A_z = A_\varphi = 0, \tag{A.2.2}$$

where H, f and Φ are real functions of the coordinates ρ and z. These functions take the most simple form in bipolar coordinates which consist of two pairs of spheroidal variables (r_1, θ_1), (r_2, θ_2) defined by their relations to the Weyl coordinates:

$$\begin{cases} \rho = \sqrt{(r_1 - m_1)^2 - \sigma_1^2}\sin\theta_1, \\ z = z_1 + (r_1 - m_1)\cos\theta_1, \end{cases}$$
$$\begin{cases} \rho = \sqrt{(r_2 - m_2)^2 - \sigma_2^2}\sin\theta_2, \\ z = z_2 + (r_2 - m_2)\cos\theta_2. \end{cases} \tag{A.2.3}$$

Here and below, the indices $_1$ and $_2$ denote the coordinates and parameters related to the Reissner - Nordström sources located at the symmetry axis at the points $z = z_1$ and $z = z_2$ respectively. A positive constant ℓ defined as

$$\ell = z_2 - z_1 \tag{A.2.4}$$

characterizes the z-distance separating these sources (for definiteness we take $z_2 > z_1$). The constants m_1 and m_2 are physical masses of the sources.

Each of the parameters σ_k ($k = 1, 2$) can be either real or pure imaginary and this property characterizes the corresponding Reissner - Nordström source to be either a black hole or a naked singularity: the real value of σ_k means that this is a black hole whose horizon in Weyl coordinates is $\{\rho = 0,\, z_k - \sigma_k \leq z \leq z_k + \sigma_k\}$ while the imaginary σ_k corresponds to a naked singularity whose critical spheroid $r_k = m_k$ is $\{0 \leq \rho \leq |\sigma_k|,\, z = z_k\}$. So the coordinate distance between two black holes (both σ_1 and σ_2 are real and positive) we define as the distance along z-axis between the nearest points of its intersections with two horizons and this distance is $\ell - \sigma_1 - \sigma_2$. The distance between the black hole located at the point $z = z_2$ and the naked singularity at the point $z = z_1$ (σ_2 is real and positive but σ_1 is pure imaginary) we define as distance between the nearest points of intersections of the symmetry axis with black hole horizon and critical spheroid and this distance is $\ell - \sigma_2$. The distance between two naked singularities (both σ_1 and σ_2 are pure imaginary) is simply ℓ and it is the length of the segment between the nearest points of intersections of two critical spheroids with the z-axis.

In terms of bipolar coordinates our solution reads:

$$
\begin{aligned}
H &= [(r_1 - m_1)^2 - \sigma_1^2 + \gamma^2 \sin^2\theta_2] \\
&\quad \times [(r_2 - m_2)^2 - \sigma_2^2 + \gamma^2 \sin^2\theta_1]\mathcal{D}^{-2}, && \text{(A.2.5)} \\
\Phi &= [(e_1 - \gamma)(r_2 - m_2) + (e_2 + \gamma)(r_1 - m_1) \\
&\quad + \gamma(m_1 \cos\theta_1 + m_2 \cos\theta_2)]\mathcal{D}^{-1}, && \text{(A.2.6)} \\
f &= [(r_1 - m_1)^2 - \sigma_1^2 \cos^2\theta_1]^{-1} \\
&\quad \times [(r_2 - m_2)^2 - \sigma_2^2 \cos^2\theta_2]^{-1}\mathcal{D}^2, && \text{(A.2.7)}
\end{aligned}
$$

where

$$
\mathcal{D} = r_1 r_2 - (e_1 - \gamma - \gamma\cos\theta_2)(e_2 + \gamma - \gamma\cos\theta_1). \qquad \text{(A.2.8)}
$$

In these expressions the quantities e_1, e_2 represent physical charges of the sources. The parameter γ and the parameters σ_1, σ_2 are determined by the relations:

$$
\begin{aligned}
&\sigma_1^2 = m_1^2 - e_1^2 + 2e_1\gamma, \quad \sigma_2^2 = m_2^2 - e_2^2 - 2e_2\gamma, \\
&\gamma = (m_2 e_1 - m_1 e_2)(\ell + m_1 + m_2)^{-1}.
\end{aligned} \qquad \text{(A.2.9)}
$$

The formulas (A.2.1)-(A.2.9) give the exact solution of the Einstein-Maxwell equations if and only if the five parameters m_1, m_2, e_1, e_2 and ℓ satisfy the following condition

$$
m_1 m_2 = (e_1 - \gamma)(e_2 + \gamma). \qquad \text{(A.2.10)}
$$

The condition (A.2.10) guarantees the equilibrium without any struts on the symmetry axis between the sources.

A.3. Properties of the solution

First of all one can see that the balance equation (A.2.10) do not admit two black holes ($\sigma_1^2 > 0$, $\sigma_2^2 > 0$) to be in equilibrium under the condition that there is some distance between them, that is if $\ell - \sigma_1 - \sigma_2 > 0$. This is in an agreement with a non-existence of static equilibrium configurations of charged black holes proved under rather general assumptions in (28). (To avoid a confusion, we mention here that the results of (28) do not apply in the presence of naked singularities.) The equilibrium is also impossible if one of the sources is extreme and the other is a non-extreme one and a positive distance exists between them, i.e. if $\ell - \sigma_2 > 0$ for the case $\sigma_1 = 0$ and $\sigma_2^2 > 0$ (a negative value for σ_2^2 is forbidden at all if $\sigma_1 = 0$) [1]. The condition (A.2.10) implies also that σ_1^2 and σ_2^2 never can be both negative, that is the equilibrium of two naked singularities is impossible. So, for separated sources an equilibrium may exist either between a black hole and a naked singularity or between two extreme sources. The latter case can be realized only if $\sigma_1 = \sigma_2 = 0$, $\gamma = 0$ and it is easy to see that the formulas (A.2.1)-(A.2.9) reduce for this case to the Majumdar-Papapetrou solution.

At spatial infinity the variables r_1, r_2 coincide and one can choose any of them as the radial coordinate. In this region the fields, as can be seen from (A.2.5) and (A.2.6), acquire the standard Reissner-Nordström asymptotical form with the total mass $m_1 + m_2$ and the total charge $e_1 + e_2$.

At the symmetry axis $\cos^2\theta_1 = \cos^2\theta_2 = 1$ and the formulas (A.2.5), (A.2.7) show that the condition $fH = 1$ is satisfied there automatically, i.e. there are no conical singularities. Besides the singularities inherent to the sources themselves, any other kind of singularities (such as, for example, the off-axis singularities found in the double-Kerr solution in (29)) are also absent in our solution.

The constant γ vanishes in the limit $\ell \to \infty$ whence it follows from (A.2.10) that the equilibrium condition asymptotically reduces to the Newtonian form $m_1 m_2 = e_1 e_2$ for a large distance between the sources.

If one of the sources disappears, e.g. $m_1 = e_1 = 0$, our solution reduces to the exact Reissner-Nordström solution with the mass m_2 and the charge e_2 in the standard spherical coordinates r_2, θ_2.

Let us turn now to that limiting case in which one of the sources can be considered as a test particle. For this we assume that m_1 and e_1 are infinitesimally small but the ratio e_1/m_1 is finite. In this case, in the first non-vanishing order with respect to the constants m_1 and e_1 the equilibrium condition (A.2.10) gives:

$$(\ell + m_2)(m_1 m_2 - e_1 e_2) = (m_1 e_2 - m_2 e_1) e_2. \qquad \text{(A.3.1)}$$

We introduce instead of m_1 a new parameter μ_1 defined by the relation:

$$\begin{aligned} m_1 = {} & \mu_1 [1 - 2m_2(\ell + m_2)^{-1} + e_2^2(\ell + m_2)^{-2}]^{1/2} \\ & + e_1 e_2 (\ell + m_2)^{-1}. \end{aligned} \qquad \text{(A.3.2)}$$

[1]Non-separated objects for which the horizons overlap each other or the horizon intersects with the critical spheroid also may be possible but such cases are not in the scope of this communication.

Now the relation (A.3.1) takes the form:

$$\begin{aligned} & m_2 - e_2^2(\ell + m_2)^{-1} \\ & = e_1 e_2 \mu_1^{-1}[1 - 2m_2(\ell + m_2)^{-1} + e_2^2(\ell + m_2)^{-2}]^{1/2}. \end{aligned} \tag{A.3.3}$$

This last equation is nothing else but the Bonnor's balance condition (15) for the test particle of the rest mass μ_1 and the charge e_1 in the Reissner-Nordström field of the mass m_2 and the charge e_2. The particle is at rest on the symmetry axis at the point $R = \ell + m_2$ where R is the radius of the standard spherical coordinates of the Reissner-Nordström solution. If we calculate from (A.2.6) the potential Φ in the linear approximation with respect to the small parameters m_1 and e_1 for the particular case $e_2 = 0$ (i.e. for the Schwarzschild background) the result will coincide exactly with the potential which have been found first in (30) - (32) in the form of multipole expansion and then in (33) in closed analytical form.

The relation (A.3.2) is important since it exhibits clearly the physical nature of the mass m_1 and gives its correct interpretation. This relation shows that the parameters m_1, m_2 are not the rest masses but they represent the total relativistic energy of each source in the external field produced by its partner.

Finally it is worth to mention that our exact solution remains physically sensible also in the case $e_2 = 0$. This corresponds to a Schwarzschild black hole of the mass m_2 hovering freely in the field of a naked singularity of the mass m_1 and the charge e_1. Such configuration exists due to the repulsive nature of gravity in the vicinity of the naked Reissner-Nordström singularity.

B. Superposition of fields of two Reissner-Nordstrom sources

B.1. Introduction

In our recent short paper (34) we presented (in a surprisingly simple form) an exact 4-parametric family of static asymptotically flat solutions of electrovacuum Einstein-Maxwell equations which describes the equilibrium configurations of two (nonrotating) charged massess in General Relativity. In this paper we present a more general, 5-parametric solution of these equations which represent a nonlinear superposition of fields of two Reissner-Nordström sources with arbitrary mass and charge parameters and arbitrarily chosen separating distance and describe the procedure for a construction of a superposition of these fields. In the subsequent sections, we use a special divergent form of the reduced Einstein-Maxwell equations to derive the Komar-like integrals for the total gravitational mass and charge of this field and calculate the physical masses and charges of the sources defined as the additive inputs of each source in the total gravitational mass and total charge of the system. The expression of our solution in terms of these physical parameters simplifies it considerably. Then we determine the constraint which should be imposed on the parameters of this superposition of fields to provide the absence in the remaining 4-parametric solution of any non-physical singularities such as the conical points on the axis outside the sources. *This constraint plays the role of the condition for equilibrium of these sources in their common gravitational and electromagnetic fields.*

The 5-parametric solution, presented in this paper, was constructed using the monodromy transform approach (35; 36) which give rise to a reformulation of the Einstein-Maxwell equations for stationary axisymmetric fields in terms of equivalent system of linear singular integral equations. As it is explained below, the functional parameters (monodromy data) in the kernels of these integral equations for our solution are chosen as analytically matched, rational functions of the spectral parameter which have two simple poles and vanish at infinity. A solution of the integral equations (35) for this kind of the monodromy data was described in (36). Moreover, our solution can be identified as the particular one within an infinite hierarchy of families of electrovacuum solutions corresponding to arbitrary rational (analytically matched) monodromy data, whose general explicit (determinant) form was first presented in (37) and later described in (38). Besides that, we note that the static solution for superposition of two Reissner-Nordström sources with arbitrary charges, masses and separating distance (and therefore, including the struts) which we present in this paper, is closely related with a general two-soliton electrovacuum solution which was constructed using the soliton generating technique (39) and first presented in an explicit form in (40) and later in (36). Namely, this two-soliton twelve-parametric solution describes a superposition of fields of two Kerr-Newman sources with arbitrary masses, angular momenta, NUT parameters, electric and magnetic charges and the separating distance, coincides in its static limit with that part of our 5-parametric static solution which corresponds to a superposition of fields of two Reissner-Nordström naked singularities, while the black hole - black hole part and mixed part of our 5-parametric family of solutions arise as the analytical continuations of the part with two naked singularities in the space of parameters. However, the interplay of the twelve parameters in general two-soliton solution seemed us from the beginning so complicate that we preferred to construct our 5-parametric solution using the alternative way based on the integral equations solving the inverse problem of the monodromy transform which allow from the beginning to make such choice of the monodromy data which would guarantee that the solution will be static and include all parts related by the analytical continuations in the spaces of their parameters.

Thus, in this paper, after some historical remarks concerning various methods for solution of integrable reductions of Einstein - Maxwell equations, we start from the results of the papers (35; 36) and (37) and describe step by step a construction of our solution using appropriate choice of the parameters which leads to so compact form of our solution and makes it well adapted for a detail analysis. [1]

[1]Our choice of the monodromy data means that the solution possess *rational axis data*, i.e. that the values of all components of metric and electromagnetic potential, expressed in terms of cylindrical Weyl coordinates (ρ, z), on the axis of simmetry $\rho = 0$ are rational functions of z. This means also that our solution virtually is the soliton solution on the Minkowski background in the sense, that it consists of a pure soliton part and its analytical continuation in the space of parameters. (This analytical continuation is similar to that which connects different parts of the Kerr-Newman family of solutions corresponding to the field of a naked singularity and the black hole solution.) These three classes of vacuum and electrovacuum stationary axisymmetric solutions — the solitons on the Minkowski background, solutions with rational axis data and the solutions with rational (analytically matched) monodromy data were constructed by different authors in the framework of very different approaches. The corresponding hierarchies of solutions coincide in the main, but very different forms in which they have been derived and presented in the literature, and an absence of a correct comparative analysis in later publications and discussions, gave rise to some misunderstandings, confusions and dupplications of the results in later publications. Having no the purpose to give in this paper a corresponding detailed survey, we confine ourselves here by some historical remarks concerning various methods and corresponding citations of the original results. In particular, we would like

On the integral equation methods

Different approaches to solution of integrable space-time symmetry reductions of the Einstein's field equations, which began to develop about thirty years ago, gave rise to different formulations (generally non-equivalent to each other) of the integral equation methods, which allow to calculate the solutions of these field equations solving some systems of linear singular integral equations.

The first reformulation of symmetry reduced Einstein equations in terms of a system of linear singular integral equations for vacuum gravitational fields (together with a construction of vacuum solitons) was proposed in the framework of the *inverse scattering approach* (41). This construction was based on a formulation of equivalent Riemann-Hilbert problem for the 2×2-matrix functions on the spectral plane which gave rise to some system of linear singular integral equations. It is important, that the solutions of this equations possess the character of *solution generating transformations*, because the matrix kernel of these integral equations includes, as functional parameters, the components of an arbitrarily chosen vacuum metric which serves as the background for solitons, while the generating solution itself can be considered as describing some nonlinear perturbation of this background.

In the framework of another, *group-theoretic approach*, for construction of the solution generating transformations corresponding to the elements of the infinite dimensional algebra of internal symmetries of electrovacuum Einstein-Maxwell equations for stationary axisymmetric fields (found by Kinnersley and Chitre (44)), Hauser and Ernst (42) reduced these equations to a homogeneous Hilbert problem for 3×3-matrix functions of an auxiliary complex parameter and then, to the corresponding 3×3-matrix linear singular inte-

to mention that the solitons on the Minkowski background are a small part of the largest known class of explicitly calculating solutions - the solitons generated on arbitrary chosen vacuum (41) or electrovacuum (39) backgrounds. The main steps of an algorithm for a direct construction of solutions with rational axis data for the Ernst potentials were described and illustrated for the already known solutions in the papers of Hauser and Ernst (42) and Sibgatullin (43). In the last one and a half decades this algorithm was actively used by Sibgatullin and others for a formal calculation of numerous examples of asymptotically flat solutions. Calculation of solutions for arbitrary rational (analytically matched) monodromy data have been described in (36), while a complete class of these solutions in a general explicit form was presented in (37), where any asymptotically flat as well as asymptotically non-flat solutions with rational axis data were included in a unified manner and the result was presented in a determinant form. A bit later the equivalent forms of this class of solutions were published by Sibgatullin et al and by Hauser and Ernst. However, it is necessary to note, that all these formal calculations of solutions and using of such (formally) explicit general forms do not lead immediately to the solutions in a compact form which would be best adapted for detail analysis of the corresponding physical and geometrical properties of the solutions, while in each particular case a special choice of parameters most adequate to a particular physical situation under consideration can simplify significantly the intermediate steps of calculations and, was is more important, the final form of the corresponding solution.

gral equations. This construction assumed an additional constraint imposed on the class of solutions, which means that only those solutions are considered which possess (locally) a regular behaviour of fields near the axis of symmetry. The integral equations (42) have rather complicate matrix kernel which construction includes a calculation of matrix exponents of the elements of Kinnersley and Chitre algebra, represented by some algebraically defined holomorphic 3×3-matrix functions of the mentioned above auxiliary complex parameter. Later, fixing the choice of the seed solution by the simplest one, Sibgatullin (43) reduced the matrix integral equations (42) of Hauser and Ernst to a simpler scalar linear singular integral equation with some "normalization" condition imposed additionally on its solutions. The kernel of this scalar integral equation was expressed explicitly in terms of the values of the Ernst potentials on the axis of symmetry. This integral equation was actively used during the last one and a half decades by Sibgatullin and others for mostly formal calculation of asymptotically flat solutions for various particular choices of (asymptotically flat) rational axis data for the Ernst potentials.

The *monodromy transform approach*(35; 36) does not follow the ideology of the matrix Riemann-Hilbert problems, however, it is also based on some ideas of the modern theory of integrable systems, analytical theory of differential equations and the theory of linear singular integral equations. For physically different classes of vacuum and electrovacuum fields with two commuting isometries (stationary axisymmetric fields, plane and cylindrical waves, inhomogeneous cosmological solutions and some others) this approach suggests rather simple general construction of the coordinate-independent functional parameters (called as monodromy data) which characterize uniquely every local solution. The problem of constructing solutions for given monodromy data (the inverse problem of the monodromy transform) gave rise to a system of linear singular integral equations which differs essentially from the integral equations mentioned above. A specific structures of the (scalar) kernels of these integral equations and of the integration path on the spectral plane allow to describe all degrees of freedom of the gravitational and electromagnetic fields and make these integral equations *equivalent* to the symmetry reduced Einstein-Maxwell equations. For stationary axisymmetric fields satisfying the regularity axis condition, the integral equations (35; 36) simplify considerably. A general scheme for constructing of solutions with any *rational analytically matched monodromy data* (or, equivalently, of solutions for any rational, not only asymptotically flat, axis data) was described earlier in detail (see Refs. (36; 38) and the references therein).

Solitons on the Minkowski background and rational axis data.

A discovery of existence of pure gravitational solitons and of the ways for their generating on arbitrarily chosen vacuum backgrounds—the dressing methods(41)(see also Ref. (45)), together with initiated by these results developments of the similar methods for Einstein-Maxwell fields(39), gave us a powerful tool for construction of a large variety of solutions of Einstein's field equations and for nonlinear superposition of fields of certain kinds of sources with various external fields.

On the other hand, later developments of the integral equation methods briefly described above suggested a simple idea to calculate the solutions of Einstein and Einstein-Maxwell equations using these integral equations with the simplest choice of functional parameters (the contour data, or the axis data for the Ernst potentials, or analytically matched monodromy data) in their kernels as rational functions of their arguments. This also leads to a construction of large families of exact solutions with an arbitrary (finitely large) number of free parameters, but without any freedom in the choice of the background solution. This gives rise to obvious questions concerning a comparison of these two constructions.

First of all, it is clear that the asymptotically flat rational-data solutions can be only a very special case of soliton solutions corresponding to a particular choice of the background for solitons. This can be the Minkowski space-time or any other soliton solution on this background. Indeed, a direct calculation shows that vacuum solitons(41) and electrovacuum solitons (39), both generated on the Minkowski background, represent the asymptotically flat solutions with rational axis data. These solitons have as many of free parameters per one simple pole of the dressing matrix, as it is necessary to describe an arbitrary chosen rational asymptotically flat axis data with a twice lower number (for the vacuum case) or with the same number of poles in this data.[2] Of course, the coincidence of the number of parameters itself does not mean that these classes of solutions coincide. Moreover, some doubts in this coincidence can arise from that fact that the construction(39) of electrovacuum solitons (in contrast to the vacuum soliton generating technique(41)) allows one to generate the solitons with arbitrarily located complex poles of the dressing matrix and with the complex conjugated poles of the inverse matrix, while the solitons with real poles do not arise in this technique. Of course, one can try to obtain the electrovacuum solitons with real poles as limiting cases of solitons with complex poles, however, this leads to solutions with coinciding real poles of the dressing matrix and its inverse, which have a fewer number of free parameters per pole than the solitons with complex poles.

To clarify this situation, it is worth considering a one-soliton solution on the Minkowski background, which coincides with the over-extreme part of the well known Kerr-Newman solution corresponding to a naked singularity. The under-extreme part of this solution, which has a horizon and cor-

[2] It is necessary to note, however, that the poles of the dressing matrices for solitons coincide with the poles of the solutions of the integral equations, but they do not in general coincide with the poles of the axis data for the Ernst potentials.

responds to a black hole, can be described in terms of the soliton generating technique by a dressing matrix with one real pole. However, as is well known, the under-extreme part of the Kerr-Newman family can be obtained by a simple analytical continuation from its over-extreme part in the space of its parameters. It is very likely that a similar analytical continuation is possible not only for a one-soliton solution on the Minkowski background, but for any number of solitons as well. Indeed, solving the integral equations for the (static) two-pole monodromy data, we have found that the part of our static 5-parametric solution which corresponds to a pair of naked Reissner-Nordström singularities coincides with the static subfamily of two-soliton solutions on the Minkowski background, while the remaining part of this family is connected with this two-soliton solution by analytical continuations in the space of parameters similar to the ones used in the one-soliton case to connect different parts of the Kerr-Newman family of solutions. The motivation given above shows that it is most reasonable to expect that the class of solutions of the integral equations for asymptotically flat rational axis data consists of the already known solitons on the Minkowski background together with their analytical continuations in the space of parameters (which should be supplied also, for completeness, by certain limiting solutions corresponding to multiple poles in the dressing matrices). On the one hand, in view of the above considerations, this means that there is not any reason to consider every formally calculated solution of the integral equations for rational asymptotically flat axis data as some new one describing the "extended" or "generalized" solitons, because it would be curiously enough to call similarly the under-extreme part of the Kerr-Newman family of solutions which is a simple analytically continuation in the parameter space of its over-extreme part.

On the other hand, the above motivation does at all not mean that it has no sense to use the integral equation methods for calculation of these soliton solutions starting directly from some asymptotically flat rational axis data. It is worth to mention here, that the application of the soliton generating technique leads to some specific parameterization of the constructing solutions which differs significantly from that which arise for the solutions constructed from the corresponding solutions of the integral equations. Each of these parameterizations can occur to be more or less useful in different considerations of this class of soliton (or rational-data) solutions. For example, it seems, that just the existence of the analytical continuation in the parameter space discussed above can be proved for N-soliton solution more easily if we use its expression which arises from the solution of the integral equations, but considering the general twelve-parametric two-soliton solution, we find that it has the most compact form just in the soliton parameterization(36).

This concludes our very brief recollection of some fragments of the history of the methods and of some interrelations between the solitons and solutions with rational axis data derived from the integral equation methods. At the end of this introduction, we sketch out the monodromy transform approach and the integral equation method which we use for construction of our 5-parametric solution for a superposition of fields of two Reissner-Nordström sources. In all necessary details, a theory of this method and a general scheme for construction of solutions with rational, analytically matched monodromy data, were developed long ago and were described in the papers cited above with some additional useful references therein.

Monodromy transform approach

This approach is based a) on the parametrization of the space of local solutions of the symmetry reduced Einstein-Maxwell equations by the monodromy data—a set of coordinate independent functions of a spectral parameter, which determine the branching properties on the spectral plane of the fundamental solution of associated linear system, and b) on the reformulation of these equations in terms of a system of linear singular integral equations. For stationary axisymmetric fields, this approach allows to construct nonlinear superpositions of electrovacuum fields of different sources characterized by analytically matched, rational monodromy data (36) (see also the Appendix in Ref. (38)). In the following sections, we demonstrate at first that the external field of a single Reissner-Nordström source is characterized just by this kind of the monodromy data functions which have on the spectral plane one simple pole and vanish at infinity. For superposition of fields of two such sources we choose the rational monodromy data functions as the sums of two poles with such coefficients which guarantee that the solution is static. Given this monodromy data, we describe step by step the construction of the corresponding solution.

The space of local solutions and its parameterization by monodromy data

For electrovacuum Einstein-Maxwell fields depending only on two space-time coordinates, in the entire space of local solutions, which are analytical near some initial point and take at this point (the point of "normalization") some "standard" values, every local solution with the metric g_{ik} and electromagnetic potential A_i can be characterized uniquely by the monodromy data which consist of four coordinate-independent holomorphic functions of the spectral parameter w:

$$\{g_{ik}(x^1,x^2), A_i(x^1,x^2)\} \quad \longleftrightarrow \quad \{\mathbf{u}_\pm(w), \mathbf{v}_\pm(w)\} \qquad \text{(B.1.1)}$$

This monodromy data are defined as a complete set of independent functions which characterize the branching properties of the corresponding fundamental solution of the associated linear system on the spectral plane at its four singular points. This data are defined uniquely for any local solution and for arbitrary choice of these functions there always exists a unique local solution of electrovacuum Einstein-Maxwell equations with given monodromy data.

This construction is useful as far as some effective methods can be developed for explicit constructing of local solutions for given monodromy data. Then, given an explicit local solution, the corresponding global solution can be determined using its analytical continuation to other space-time regions where this solution may reveal also some regular behaviour or approaches various types of singularities.

The master system of linear singular integral equations

The key point of the use of the monodromy data is the existence of some system of linear singular integral equations whose kernels and right hand sides are expressed algebraically in terms of the monodromy data and whose solution determines (by means of some quadratures) all components of metric and electromagnetic potential.

In general, the structure of this system of linear singular integral equations is rather complicate. The singular integrals are defined on the contour L on the spectral plane which consists of two disconnected parts $L = L_+ + L_-$. The locations of their endpoints depend on the space-time coordinates and coordinates of the initial point which enter also the integrands as parameters. In particular, L_+ goes from $w = \xi_o$ to $w = \xi$ and L_- goes from $w = \eta_o$ to $w = \eta$, where, for example, for stationary axisymmetric fields in terms of Weyl coordinates $\xi = z + i\rho$, $\eta = z - i\rho$, but for plane waves $\xi = x + t$, $\eta = x - t$) and (ξ_o, η_o) correspond to the initial point.

The monodromy data as well as the unknown functions in these integral equations are defined in two disconnected regions of the spectral plane—the neighbourhoods of L_+ and L_-, where they are represented by pairs of functions $(\mathbf{u}_+, \mathbf{u}_-)$, $(\mathbf{v}_+, \mathbf{v}_-)$ and $(\boldsymbol{\varphi}_+, \boldsymbol{\varphi}_-)$ respectively. It is clear that these equations cannot be solved for arbitrarily chosen monodromy data. However, for some classes of fields, such as, for example, stationary axisymmetric fields with a regular axis of symmetry considered in this paper, these integral equations can be simplified considerably and admit infinite hierarchies of multiparametric families of explicit solutions.

Monodromy data for stationary fields with a regular symmetry axis

For stationary axisymmetric fields, it is typical that physical and geometrical formulations of various problems (as, for example, in the case of asymptotically flat fields) imply that at least some part of the axis of symmetry is free of the field sources and therefore, a behaviour of metric and matter fields in the neighbourhood of this part of the axis should be regular. For such fields, if we choose the initial point (the point of normalization) of a solution on such regular part of the axis, the initial points of the contours L_+ and L_- coincide and instead of two disconnected contours we obtain one simple curve $L = L_+ + L_-$. Changing preliminary the direction of integration on L_-, we obtain one contour where the system of singular integral equations is defined. It starts at the point $w = \eta$, goes through the initial point $w = z_0$ and ends at $w = \xi$. On this contour the monodromy data and the unknown variable in the integral equations are represented by twice lower number of holomorphic functions because for these fields we have

$$\begin{array}{ll} \mathbf{u}_+(w) = \mathbf{u}_-(w) \equiv \mathbf{u}(w), & \boldsymbol{\varphi}_+(w) = \boldsymbol{\varphi}_-(w) \equiv \boldsymbol{\varphi}(w) \\ \mathbf{v}_+(w) = \mathbf{v}_-(w) \equiv \mathbf{v}(w), & \end{array} \tag{B.1.2}$$

Usually, we call these conditions as the regularity axis condition, however we note that these conditions guarantee only a regular local behaviour of fields near the axis, but they do not exclude the presence of some non-curvature singularities, such as conical points on the axis or closed time-like curves near it. Thus, for stationary axisymmetric electrovacuum fields with a regular axis of symmetry the monodromy data are represented only by two arbitrary holomorphic functions $\mathbf{u}(w)$ and $\mathbf{v}(w)$. We recall also that $\mathbf{v}(w)$ is "responsible" for a presence of electromagnetic field, so that for vacuum $\mathbf{v}(w) \equiv 0$ and the space of solutions of vacuum stationary axisymmetric fields near the regular part of the axis of symmetry is parameterized by the monodromy data consisting of one holomorphic function $\mathbf{u}(w)$.

Another important property of the stationary axisymmetric electrovacuum fields with the regular axis of symmetry is that any solution can be characterized by the finite values of its metric components and potentials (e.g., of the complex Ernst potentials $\mathcal{E}(\rho, z)$, $\Phi(\rho, z)$) on the regular part of the axis of symmetry. In this case, the monodromy data (B.1.2) can be related to the values $\mathcal{E}(z)$, $\Phi(z)$ on the axis:

$$\mathcal{E}(\rho = 0, z) = \mathcal{E}_0 - 2i(z - z_0)\mathbf{u}(w = z), \quad \Phi(\rho = 0, z) = \Phi_0 + 2i(z - z_0)\mathbf{v}(w = z), \tag{B.1.3}$$

where z is the Weyl coordinate along the axis, z_0 is a coordinate of the initial point on this axis, $\mathcal{E}_0$ and Φ_0 are the "normalized" values of the Ernst potentials at the initial point, for which we usually put $\mathcal{E}_0 = 1$ and $\Phi_0 = 0$.

The conditions for asymptotically flat and static fields

The expressions (B.1.3) allow us to relate the structure of the monodromy data (B.1.2) with some physical and geometrical properties of fields. In particular, for any asymptotically flat field $\mathbf{u}(w)$ and $\mathbf{v}(w)$ should be holomorphic at

$w = \infty$ and

$$\mathbf{u}(w) \to 0 \qquad \text{and} \qquad \mathbf{v}(w) \to 0 \qquad \text{for} \qquad w \to \infty.$$

Moreover, in this case, the coefficients of expansions of these functions in the inverse powers of w for $w \to \infty$ can be simply related to the multipole moments of this asymptotically flat field. Therefore, the multipole structure of the field can be determined in advance by the appropriate choice of the monodromy data.

For static fields, $\mathcal{E}$ should be real, while Φ should be real for pure electric fields and imaginary for pure magnetic fields and therefore, $\mathbf{u}(w)$ and $\mathbf{v}(w)$ should satisfy

$$\mathbf{u}^\dagger(w) = -\mathbf{u}(w) \qquad \text{and} \qquad \mathbf{v}^\dagger(w) = \mp\mathbf{v}(w)$$

where $\mathbf{u}^\dagger(w) \equiv \overline{\mathbf{u}(\overline{w})}$ and $\mathbf{v}^\dagger(w) \equiv \overline{\mathbf{v}(\overline{w})}$ and a bar means a complex conjugation.

Exact solutions with rational monodromy data

An infinite hierarchies of solutions of the Einstein-Maxwell equations can be calculated explicitly if we choose the analytically matched monodromy data (B.1.2) to be rational functions of the spectral parameter:

$$\mathbf{u}(w) = \frac{U(w)}{Q(w)}, \qquad \mathbf{v}(w) = \frac{V(w)}{Q(w)}$$

where the functions $U(w)$, $V(w)$ and $Q(w)$ are some polynomials. A general algorithm for solution of the integral equations for these polynomials of arbitrary orders was described in (36; 38). This algorithm leads to explicit form of solutions in a unified, but rather complicate form, and, as we shall see below, a large careful work is necessary for finding of appropriate choice of physical parameters which can simplify significantly the constructed solutions.

The linear singular integral equations

For stationary axisymmetric fields outside their sources the metric and electromagnetic vector potential can be considered in cylindrical coordinates in the form

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 - f(d\rho^2 + dz^2) \\ A_i &= \{A_t, 0, 0, A_\varphi\}, \end{aligned} \tag{B.1.4}$$

where $x^i = \{t, \rho, z, \varphi\}$ and the metric components g_{tt}, $g_{t\varphi}$, $g_{\varphi\varphi}$ and f as well as the components A_t and A_φ of the electromagnetic potential are functions of the coordinates ρ and z only. It is well known, that for the metric and electromagnetic fields (B.1.4) the symmetry reduced electrovacuum Einstein-Maxwell field equations decouple into two parts. One of these parts is a closed system of "dynamical" equations-the nonlinear partial differential equations for the functions g_{tt}, $g_{t\varphi}$, $g_{\varphi\varphi}$, A_t and A_φ. (We recall here that the Weyl cylindrical coordinates are defined so that $g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 = -\rho^2$ and therefore, only two of these three metric components are unknown functions). Another part is a pair of constraint equations which allow to determine in quadratures the conformal factor f provided the solution for g_{tt}, $g_{t\varphi}$, $g_{\varphi\varphi}$, A_t and A_φ is already known. Though an explicit calculation of these quadratures for the conformal factor f represent usually a large technical difficulty, the principal problem is a construction of the solution of the "dynamical" equations with wanted physical and geometrical properties. That is why we concentrate below mainly on the construction of solution of the "dynamical" equations.

In accordance with a general scheme (35; 36), the components of metric and electromagnetic potential (B.1.4), which satisfy the electrovacuum Einstein-Maxwell equations, can be expressed in the form

$$\begin{aligned} g_{tt} &= 1 - i(R_t{}^\varphi - \overline{R}_t{}^\varphi) + \Phi_t\overline{\Phi}_t & \mathcal{E} &= 1 - 2iR_t{}^\varphi \\ g_{t\varphi} &= -i(z - z_0) + i(R_t{}^t + \overline{R}_\varphi{}^\varphi) + \Phi_t\overline{\Phi}_\varphi & \begin{pmatrix}\Phi_t \\ \Phi_\varphi\end{pmatrix} &= 2i\begin{pmatrix}R_t{}^* \\ R_\varphi{}^*\end{pmatrix} \\ g_{\varphi\varphi} &= i(R_\varphi{}^t - \overline{R}_\varphi{}^t) + \Phi_\varphi\overline{\Phi}_\varphi \end{aligned} \tag{B.1.5}$$

where $\{\Phi_t, \Phi_\varphi\}$ are the components of a complex electromagnetic potential and $\mathrm{Re}\Phi_t = A_t$, $\mathrm{Re}\Phi_\varphi = A_\varphi$; the functions R_t^t, R_t^φ, R_t^*, R_φ^t, R_φ^φ, R_φ^* constitute a matrix which is determined by the integral over the contour L on the spectral plane

$$\begin{aligned} \mathbf{R} &\equiv \begin{pmatrix} R_t{}^t & R_t^\varphi & R_t^* \\ R_\varphi{}^t & R_\varphi^\varphi & R_\varphi^* \end{pmatrix} \\ &= \frac{1}{i\pi}\int\limits_L [\lambda]_\zeta \begin{pmatrix} 1 + i(\zeta - z_0)\mathbf{u}^\dagger(\zeta) \\ -i(\zeta - z_0) \end{pmatrix} \otimes \{\boldsymbol{\varphi}^{[\mathbf{1}]}(\zeta), \boldsymbol{\varphi}^{[\mathbf{u}]}(\zeta), \boldsymbol{\varphi}^{[\mathbf{v}]}(\zeta)\} d\zeta \end{aligned} \tag{B.1.6}$$

Here $\zeta \in L$ and the contour L (unlike the general case) is a simple curve which starts from $w = \eta \equiv z - i\rho$, goes through the initial point $w = z_0$ and ends at the point $w = \xi \equiv z + i\rho$, where ρ and z are the well known cylindrical Weyl coordinates in which $g_{tt}g_{\varphi\varphi} - g_{t\varphi} = -\rho^2$, and z_0 determines the location of the point of normalization on the axis of symmetry $\rho = 0$. In the above integral, $[\lambda]_\zeta$ denotes a jump (i.e. a half of the difference between left and right limits) at the point $\zeta \in L$ of a "standard" branching function

$$\lambda = \sqrt{(\zeta - z - i\rho)(\zeta - z + i\rho)/(\zeta - z_0)^2}, \qquad \lambda(\rho, z, \zeta = \infty) = 1. \tag{B.1.7}$$

The functions $\boldsymbol{\varphi}^{[\mathbf{1}]}(\zeta)$, $\boldsymbol{\varphi}^{[\mathbf{u}]}(\zeta)$, $\boldsymbol{\varphi}^{[\mathbf{v}]}(\zeta)$ should satisfy to the decoupled linear singular integral equations with the same scalar kernels and different right hand sides (35; 36):

$$-\frac{1}{\pi i} \fint_L \frac{[\lambda]_\zeta \,\mathcal{H}(\tau,\zeta)}{\zeta - \tau} \begin{pmatrix} \boldsymbol{\varphi}^{[\mathbf{1}]}(\zeta) \\ \boldsymbol{\varphi}^{[\mathbf{u}]}(\zeta) \\ \boldsymbol{\varphi}^{[\mathbf{v}]}(\zeta) \end{pmatrix} d\zeta = \begin{pmatrix} 1 \\ \mathbf{u}(\tau) \\ \mathbf{v}(\tau) \end{pmatrix} \tag{B.1.8}$$

where $\zeta, \tau \in L$; there is a Cauchy principal value integral in the left hand side, and the kernel function $\mathcal{H}(\tau,\zeta)$ in its integrand is

$$\mathcal{H}(\tau,\zeta) = 1 + i(\zeta - z_0)\left[\mathbf{u}^\dagger(\zeta) - \mathbf{u}(\tau)\right] + 4(\zeta - z_0)^2 \mathbf{v}(\tau)\mathbf{v}^\dagger(\zeta)$$

and everywhere below, we can put $z_0 = 0$ without any loss of generality. The coordinates ρ and z enter the equations (B.1.8) as the parameters which determine the location of the endpoints of the contour L and as the arguments of the function λ, however, for simplicity we have not shown explicitly in (B.1.8) the dependence of the unknown functions $\boldsymbol{\varphi}^{[\mathbf{1}]}$, $\boldsymbol{\varphi}^{[\mathbf{u}]}$ and $\boldsymbol{\varphi}^{[\mathbf{v}]}$ on these coordinates. We note also, that in general, $\boldsymbol{\varphi}^{[\mathbf{1}]}$, $\boldsymbol{\varphi}^{[\mathbf{u}]}$ and $\boldsymbol{\varphi}^{[\mathbf{v}]}$ as well as the right hand sides of (B.1.8) constitute the row-vectors, however, for a convenience we write (3.3.7) in a transposed form.

B.2. The Kerr-Newman field as a one-pole solution

We begin our description of solution of the integral equation (B.1.8) with a simple case of a one-pole structure of the monodromy data functions:

$$\mathbf{u}(w) = \frac{u_0}{w - h}, \qquad \mathbf{v}(w) = \frac{v_0}{w - h}. \tag{B.2.1}$$

where u_0, v_0 and h are arbitrary complex constants. For these data the kernel function $\mathcal{H}(\tau,\zeta)/(\zeta - \tau)$ can be split into the singular and regular parts:

$$\frac{\mathcal{H}(\tau,\zeta)}{\zeta - \tau} = \frac{1}{(\tau - h)(\zeta - \overline{h})} \left[\frac{P(\zeta)}{\zeta - \tau} + R(\zeta)\right] \tag{B.2.2}$$

where the polynomials $P(\zeta)$ and $R(\zeta)$ possess the expressions ($z_0 = 0$):

$$\begin{aligned} P(\zeta) &= \zeta^2[1 - i(u_0 - \overline{u_0}) + 4v_0\overline{v_0}] - \zeta[(1 + i\overline{u_0})h + (1 - iu_0)\overline{h}] + h\overline{h} \\ R(\zeta) &= -(1 + i\overline{u_0})\zeta + \overline{h} \end{aligned} \tag{B.2.3}$$

Assuming that $h \neq 0$, what means that the pole is not located on the integration path L, we present the polynomial $P(w)$ in a factorized form

$$P(w) = P_0(w - w_1)(w - \widetilde{w}_1), \qquad P_0 = 1 - i(u_0 - \overline{u_0}) + 4v_0\overline{v_0} \tag{B.2.4}$$

The coefficients of $P(w)$ always are real and therefore, its roots w_1 and $\widetilde{w}_1$ are real or complex conjugated to each other. We parameterize these roots as

$$w_1 = z_1 + \sigma_1, \qquad \widetilde{w}_1 = z_1 - \sigma_1 \tag{B.2.5}$$

where z_1 is a real parameter while σ_1 can be real or pure imaginary. A comparison of (B.2.3) and (B.2.4) allows to express z_1 and σ_1 in terms of u_0, v_0 and h, however, it is more convenient to use z_1 and σ_1 as new parameters and express some of the parameters u_0, v_0 and h as functions of z_1, σ_1 and others. We give the explicit expressions later, but now we concentrate on solving of the integral equations (B.1.8).

As it can be concluded from the structure of the singular integral equations, their solutions for the monodromy data (B.2.1) should have the form

$$\left\{\boldsymbol{\varphi}^{[\mathbf{1}]}(w),\ \boldsymbol{\varphi}^{[\mathbf{u}]}(w),\ \boldsymbol{\varphi}^{[\mathbf{v}]}(w)\right\} = \frac{(w - \overline{h})}{P(w)}\left\{w + X_0,\ Y_0,\ Z_0\right\} \tag{B.2.6}$$

where X_0, Y_0 and Z_0 are independent of the spectral parameter w, but they can depend on the coordinates ρ and z. As we see from these expressions, the solutions of the integral equations (B.1.8) are rational functions of the spectral parameter and they have the poles coinciding with the roots of the polynomial $P(w)$.

To calculate the values of X_0, Y_0 and Z_0 explicitly, we substitute (B.2.1), (B.2.2) and (B.2.6) into the integral equations (B.1.8) and obtain the linear algebraic equations for X_0, Y_0 and Z_0 with rather complicate coefficients. These coefficients are linear combinations of the usual or singular Cauchy principal value integrals of the form

$$\frac{1}{\pi i}\fint_L \frac{[\lambda]_\zeta\, \zeta^k}{\zeta - \tau}\, d\zeta, \qquad \frac{1}{\pi i}\int_L \frac{[\lambda]_\zeta\, \zeta^k}{P(\zeta)}\, d\zeta$$

where $\tau, \zeta \in L$ and $k \geq 0$ is some integer. Any integral of these types can be calculated explicitly using the elementary theory of residues. Indeed, the integrands in these integrals can be expressed as the jumps of analytical functions $\lambda(w)\,w^k/(w-\tau)$ and $\lambda(w)\,w^k/P(w)$ respectively. Therefore, these integrals can be expressed as the integrals over the closed curves $\mathfrak{L}$ surrounding the contour L:

$$\frac{1}{\pi i}\int\limits_L \frac{[\lambda]_\zeta\,\zeta^k}{\zeta-\tau}\,d\zeta=\frac{1}{2\pi i}\int\limits_{\mathfrak{L}}\frac{\lambda(\chi)\,\chi^k}{\chi-\tau}\,d\chi,\quad \frac{1}{\pi i}\int\limits_L \frac{[\lambda]_\zeta\,\zeta^k}{P(\zeta)}\,d\zeta=\frac{1}{2\pi i}\int\limits_{\mathfrak{L}}\frac{\lambda(\chi)\,\chi^k}{P(\chi)}\,d\chi \tag{B.2.7}$$

where $\chi\in\mathfrak{L}$ and $\mathfrak{L}$ surrounds L in negative direction (i.e. so that the interior is to the right) and close enough to L so that no poles of $P(w)$ are inside $\mathfrak{L}$. The function $\lambda(w)\,w^k/(w-\tau)$ is analytical outside the contour L and it can have only the pole at $w=\infty$, while the function $\lambda(w)\,w^k/P(w)$ is analytical outside L, besides the poles which arise from zeros of $P(w)$ and possible poles at $w=\infty$. This allows us to transform the closed integration path $\mathfrak{L}$ into a one approaching $w=\infty$ taking into account the inputs from the finite poles and therefore, in (B.2.7), each of the integrals over $\mathfrak{L}$ is equal to the sum of residues of its integrand at finite poles plus the residue at $w=\infty$. Thus, for the integrals (B.2.7) we obtain

$$\begin{aligned}
&\frac{1}{\pi i}\int\limits_L \frac{[\lambda]_\zeta\,\zeta^k}{\zeta-\tau}\,d\zeta=-\sum_{m=0}^{k}(\lambda)_{k-m}\,\tau^m,\\
&\frac{1}{\pi i}\int\limits_L \frac{[\lambda]_\zeta\,\zeta^k}{P(\zeta)}\,d\zeta=\frac{\lambda(w_1)\,w_1^k}{P'(w_1)}+\frac{\lambda(\widetilde{w}_1)\,\widetilde{w}_1^k}{P'(\widetilde{w}_1)}-\left(\frac{\lambda(w)\,w^k}{P(w)}\right)_{-1}
\end{aligned} \tag{B.2.8}$$

where $(\lambda)_{m-k}$ means the coefficient in front of $1/w^{k-m}$ in the inverse powers expansion of the function $\lambda(w)$ at $w=\infty$ and similarly, $(\ldots)_{-1}$ means the coefficient in front of $1/w$ in the inverse powers expansion of a function at $w=\infty$. The parameters w_1 and $\widetilde{w}_1$ are the roots of the polynomial $P(w)$, while $P'(w_1)$ and $P'(\widetilde{w}_1)$ mean a derivative of $P(w)$ with respect to w at $w=w_1$ or $w=\widetilde{w}_1$ respectively.

The substitution of (B.2.6) and (B.2.2) into (B.1.8) and the subsequent calculation of the contour integrals leads to the equations whose left and right hand sides, after multiplication by $(\tau-h)$, become polynomial functions of τ. Equating the coefficients of these polynomials, we obtain the linear algebraic equations for X_0, Y_0 and Z_0 whose solution allows to obtain the explicit expressions for $\boldsymbol{\varphi}$:

$$\left\{\boldsymbol{\varphi}^{\,[\mathbf{1}]}(w),\ \boldsymbol{\varphi}^{\,[\mathbf{u}]}(w),\ \boldsymbol{\varphi}^{\,[\mathbf{v}]}(w)\right\}=\frac{(w-\overline{h})}{P(w)}\left\{w+z-\frac{h+\Delta_1}{\Delta_0},\ \frac{u_0}{\Delta_0},\ \frac{v_0}{\Delta_0}\right\} \tag{B.2.9}$$

where Δ_0 and Δ_1 possess the expressions

$$\begin{aligned}
\Delta_0 &= -\frac{\lambda(w_1)}{P'(w_1)}R(w_1) - \frac{\lambda(\widetilde{w}_1)}{P'(\widetilde{w}_1)}R(\widetilde{w}_1) + 1 - \frac{1+i\overline{u_0}}{P_0} \\
\Delta_1 &= -\frac{\lambda(w_1)}{P'(w_1)}R(w_1)(w_1+z) - \frac{\lambda(\widetilde{w}_1)}{P'(\widetilde{w}_1)}R(\widetilde{w}_1)(\widetilde{w}_1+z) \\
&\quad - \frac{1+i\overline{u_0}}{P_0^2}\left[\overline{h}(1-iu_0) + h(1+i\overline{u_0})\right] + \frac{\overline{h}}{P_0}.
\end{aligned} \tag{B.2.10}$$

and the polynomials $P(\zeta)$ and $R(\zeta)$ were defined in (B.2.3). Now we have to substitute these solutions of the integral equations into the integrands of quadratures (B.1.6) and calculate these quadratures using the same methods for calculation of these type of integrals as it was described above. The components of metric and of electromagnetic potential, as well as the Ernst potentials for the constructing solution can be calculated then algebraically from the expressions (B.1.5).

The formal calculations for the monodromy data (B.2.1) described above lead us to the explicit but very complicate form of the solution. This solution can be simplified considerably using (i) a new and most appropriate set of parameters, (ii) some global gauge transformations—$SL(2,R)$-rotations and rescalings of the Killing vectors ξ_t and ξ_φ with the corresponding linear transformations of t and φ coordinates and (iii) a set of more convenient coordinates (such as the prolate or oblate ellipsoidal coordinates) instead of the Weyl coordinates ρ and z. In the remaining part of this section we describe these simplifications and show that this solution represents nothing more but the well known Kerr-Newman family.

At first, comparing the explicit expression (B.2.3) for $P(w)$ with its factorized form (B.2.4) and using (B.2.5), we obtain a set of relations between z_1, σ_1 and the original set of independent parameters u_0, v_0 and h. Solving these relations we express u_0, v_0 and h in terms of a new set of parameters which consists of the real parameters m, a, b, z_1 and a complex parameter e:

$$\begin{aligned}
u_0 &= -i + \frac{ih}{w_1\widetilde{w}_1}(z_1+ia), & h &= z_1 + m - i(a+b) \\
v_0 &= \frac{ie}{2h}\sqrt{P_0}, & P_0 &= \frac{h\overline{h}}{w_1\widetilde{w}_1},
\end{aligned} \tag{B.2.11}$$

and the parameter σ_1 is is related to the parameters m, a, b, z_1 and e by the equation

$$\sigma_1^2 = m^2 + b^2 - a^2 - e\overline{e}$$

Now, for the choice of new coordinates we note, that besides a presence of ρ and z in the solution explicitly, its dependence of these coordinates also comes from the functions $\lambda(\rho,z,w)$ which should be calculated at the points

$w = w_1$ and $w = \widetilde{w}_1$, i.e. at the roots of $P(w)$. Let us express these functions in the form

$$\lambda(\rho, z, w_1) = \frac{x_1 - \sigma_1 y_1}{z_1 + \sigma_1}, \qquad \lambda(\rho, z, \widetilde{w}_1) = \frac{x_1 + \sigma_1 y_1}{z_1 - \sigma_1}$$

where x_1 and y_1 will be considered as new coordinates. If we compare these expressions with the definition (B.1.7), we obtain the relations between ρ, z and x_1, y_1:

$$\rho = \sqrt{x_1^2 - \sigma_1^2}\sqrt{1 - y_1^2}, \qquad z = z_1 + x_1 y_1$$

Finally, we have to make some gauge transformation of our solution:

$$\begin{array}{lll} h_{tt} \to \dfrac{h_{tt}}{P_0} & \mathcal{E} \to 1 + \dfrac{\mathcal{E} - \mathcal{E}_0 + 2\overline{\Phi}_0(\Phi - \Phi_0)}{P_0} & \mathcal{E}_0 = 1 - 2iu_0 \\ h_{t\varphi} \to h_{t\varphi} & \Phi \to \dfrac{\Phi - \Phi_0}{\sqrt{P_0}} & \Phi_0 = 2iv_0 \\ h_{\varphi\varphi} \to P_0 h_{\varphi\varphi} & \widetilde{\Phi} \to \sqrt{P_0}\widetilde{\Phi} & \end{array}$$

After this reparametrization, change of coordinates and gauge transformations we obtain the solution in a simple form. In particular, we obtain the Ernst potentials

$$\begin{aligned} \mathcal{E} &= 1 - 2\frac{(-i\epsilon_0 u_0 + 4\epsilon_0 v_0 v_0^\dagger)}{P_0 \Delta_0} = 1 - \frac{2(m - ib)}{x_1 + iay_1 + m - ib}, \\ \Phi &= -\frac{2iv_0}{\sqrt{P_0}\Delta_0} = \frac{e}{x_1 + iay_1 + m - ib} \end{aligned}$$

which can be identified with those for the well known Kerr-Newman solution. The coordinates x_1 and y_1 are connected directly with the polar spheroidal coordinates:

$$x_1 = r_1 - m, \qquad y_1 = \cos\theta_1$$

and the real parameters m, a, b, z_1, real and imaginary part of e can be identified respectively with the mass, angular momentum, NUT-parameter, electric and magnetic charges of the Kerr-Newman source (a black hole or a naked singularity).

The Reissner-Nordström field as a static one-pole solution

As it was already mentioned above, to obtain a static solution with the monodromy functions having only one pole, it is not necessary to make all of the calculations described above, but it would be enough to restrict from the beginning the monodromy data functions by the case, when these functions would take pure imaginary values on the real axis on the spectral plane. For this, it is necessary to choose the parameter h to be real and the parameters u_0 and v_0 pure imaginary. In this case, as one can see from (B.2.11), we would have

$$h = \overline{h}, \quad u_0 = -\overline{u_0}, \quad v_0 = -\overline{v_0} \qquad \Longleftrightarrow \qquad a = 0, \quad b = 0, \quad e = \overline{e}$$

i.e. we obtain the Reissner-Nordström solution.

B.3. Superposing the fields of two Reissner - Nordström sources

In this section we describe the key points of the calculation of solutions for monodromy data having two simple poles on the spectral plane. This monodromy data can be represented as a superposition of two one-pole terms and therefore, in accordance with the previous section, it is naturally to expect that the corresponding solution will describe the superposition of fields of two Reissner-Nordström sources.

Structure of the monodromy data

Thus, we begin with the choice of the monodromy data functions in the form

$$\mathbf{u}(w) = \frac{u_1}{w - h_1} + \frac{u_2}{w - h_2}, \qquad \mathbf{v}(w) = \frac{v_1}{w - h_1} + \frac{v_2}{w - h_2}. \tag{B.3.1}$$

where u_1, u_2, v_1, v_2, h_1 and h_2 are arbitrary complex constants. However, to obtain a static solution, we specify this data imposing the following constraints:

$$\begin{array}{lll} u_1 = -\overline{u_1} & v_1 = -\overline{v_1} & h_1 = \overline{h_1} \\ u_2 = -\overline{u_2} & v_2 = -\overline{v_2} & h_2 = \overline{h_2} \end{array} \tag{B.3.2}$$

Structure of the kernel and a new set of parameters

In this case, the kernel $\mathcal{H}(\tau,\zeta)/(\zeta - \tau)$ splits into the singular and regular parts:

$$\frac{\mathcal{H}(\tau,\zeta)}{\zeta - \tau} = \frac{1}{(\tau - h)(\zeta - \overline{h})} \left[\frac{P(\zeta)}{\zeta - \tau} + R(\tau,\zeta) \right]$$

where the polynomials $P(\zeta)$, $R(\tau,\zeta)$ are defined as (everywhere below $z_0 = 0$):

$$
\begin{aligned}
P(\zeta) = {} & h_1^2 h_2^2 - 2h_1h_2[(1-iu_1)h_2 + (1-iu_2)h_1]\zeta + [(h_1+h_2)^2 + 2h_1h_2 \\
& - 4(h_1v_2+h_2v_1)^2 - 2i(h_1^2u_2 + h_2^2u_1) - 4ih_1h_2(u_1+u_2)]\zeta^2 \\
& + [2h_1(-1+iu_1+2iu_2+4v_1v_2+4v_2^2) + 2h_2(-1+2iu_1+iu_2 \\
& + 4v_1v_2+4v_1^2)]\zeta^3 + \left[1-2i(u_1+u_2)-4(v_1+v_2)^2\right]\zeta^4 \\
R(\tau,\zeta) = {} & R_0(\zeta) + R_1(\zeta)\tau \\
R_0(\zeta) = {} & h_1h_2(h_1+h_2) \\
& - \left[h_1^2 + 3h_1h_2 + h_2^2 - ih_1(h_1+2h_2)u_2 - ih_2(2h_1+h_2)u_1\right]\zeta \\
& + \left[2(h_1+h_2)(1-iu_1-iu_2-2v_1v_2) - ih_1u_2 - ih_2u_1 - 4h_1v_2^2 - 4h_2v_1^2\right]\zeta^2 \\
& + \left[-1+2i(u_1+u_2)+4(v_1+v_2)^2\right]\zeta^3 \\
R_1(\zeta) = {} & -h_1h_2 + (h_1+h_2-ih_1u_2-ih_2u_1)\zeta + (-1+iu_1+iu_2)\zeta^2
\end{aligned}
\tag{B.3.3}
$$

Assuming that the pole of the monodromy data is not located on the integration path L and therefore, $h_1h_2 \neq 0$, we present $P(w)$ in a factorized form

$$P(w) = P_0(w-w_1)(w-\widetilde{w}_1)(w-w_2)(w-\widetilde{w}_2) \tag{B.3.4}$$

The coefficients of the polynomial $P(w)$ always are real and therefore, its roots w_1, $\widetilde{w}_1$ and w_2, $\widetilde{w}_2$ are real or complex conjugated in pairs. We parameterize them as

$$
\begin{array}{llll}
w_1 = z_1 + \sigma_1 & w_2 = z_2 + \sigma_2 & h_1 = z_1 + \widetilde{m}_1 & \\
\widetilde{w}_1 = z_1 - \sigma_1 & \widetilde{w}_2 = z_2 - \sigma_2 & h_2 = z_2 + \widetilde{m}_2 & z_2 = z_1 + \ell
\end{array}
\tag{B.3.5}
$$

where z_1 and z_2 (and therefore, ℓ) are real parameters while σ_1 as well as σ_2 can be real or pure imaginary. A comparison of coefficients of different powers of the spectral parameter in (B.3.3) and in (B.3.4) leads to a number of relations of the form

$$
\begin{aligned}
h_1^2h_2^2 &= P_0\mathfrak{I}_4, \\
2h_1h_2(h_1+h_2-ih_1u_2-ih_2u_1) &= P_0\mathfrak{I}_3, \\
h_1^2+h_2^2+4h_1h_2-2i(h_1+h_2)(h_1u_2+h_2u_1)-2ih_1h_2(u_1+u_2) & \\
-4(h_1v_2+h_2v_1)^2 &= P_0\mathfrak{I}_2, \\
2(h_1+h_2)-2i(h_1u_2+h_2u_1)-2i(h_1+h_2)(u_1+u_2) & \\
-8(h_1v_2+h_2v_1)(v_1+v_2) &= P_0\mathfrak{I}_1, \\
1-2i(u_1+u_2)-4(v_1+v_2)^2 &= P_0.
\end{aligned}
\tag{B.3.6}
$$

where we have introduced the notations

$$\begin{aligned}\mathcal{I}_1 &= w_1 + \widetilde{w}_1 + w_2 + \widetilde{w}_2\\ \mathcal{I}_2 &= w_1\widetilde{w}_1 + w_1 w_2 + w_1\widetilde{w}_2 + \widetilde{w}_1 w_2 + \widetilde{w}_1\widetilde{w}_2 + w_2\widetilde{w}_2\\ \mathcal{I}_3 &= \widetilde{w}_1 w_2\widetilde{w}_2 + w_1 w_2\widetilde{w}_2 + w_1\widetilde{w}_1\widetilde{w}_2 + w_1\widetilde{w}_1 w_2\\ \mathcal{I}_4 &= w_1\widetilde{w}_1 w_2\widetilde{w}_2\end{aligned}$$

In terms of the parameters (B.3.5) these functions possess the expressions:

$$\begin{aligned}\mathcal{I}_1 &= 2(z_1 + z_2) & \mathcal{I}_3 &= 2z_1 z_2(z_1 + z_2) - 2z_2\sigma_1^2 - 2z_1\sigma_2^2\\ \mathcal{I}_2 &= z_1^2 + 4z_1 z_2 + z_2^2 - \sigma_1^2 - \sigma_2^2 & \mathcal{I}_4 &= (z_1^2 - \sigma_1^2)(z_2^2 - \sigma_2^2)\end{aligned}$$

The relations (B.3.6) allow to express the roots of $P(w)$ (or the new parameters z_1, z_2 and σ_1, σ_2 in terms of u_1, u_2, v_1, v_2 and h_1, h_2, however, it is more simple and convenient to use z_1, z_2 and σ_1, σ_2 as new parameters and express P_0 and some of the parameters u_1, u_2, v_1, v_2 and h_1, h_2 as functions of z_1, z_2, σ_1, σ_2 and others. We present these expressions in the form

$$\begin{aligned}u_1 &= \frac{2ih_1h_2^2\widetilde{e}_1\widetilde{e}_2P_1P_2 + 2ih_1^3\widetilde{e}_2^2P_2^2 + ih_1^2(h_1-h_2)(h_2^3\mathcal{I}_1 - 2h_2^2\mathcal{I}_2 + 3h_2\mathcal{I}_3 - 4\mathcal{I}_4)}{2(h_1-h_2)^3\mathcal{I}_4},\\ u_2 &= \frac{-2ih_1^2h_2\widetilde{e}_1\widetilde{e}_2P_1P_2 - 2ih_2^3\widetilde{e}_1^2P_1^2 + ih_2^2(h_1-h_2)(h_1^3\mathcal{I}_1 - 2h_1^2\mathcal{I}_2 + 3h_1\mathcal{I}_3 - 4\mathcal{I}_4)}{2(h_1-h_2)^3\mathcal{I}_4},\\ v_1 &= -\frac{ih_2\widetilde{e}_1P_1}{2(h_1-h_2)\sqrt{\mathcal{I}_4}}, \qquad v_2 = -\frac{ih_1\widetilde{e}_2P_2}{2(h_1-h_2)\sqrt{\mathcal{I}_4}}.\end{aligned} \tag{B.3.7}$$

where the parameters $\widetilde{e}_1$, $\widetilde{e}_2$ and P_1, P_2 are defined as follows. Solving (B.3.6) leads to the expressions $P(h_1)$ and $P(h_2)$ which must be non-negative:

$$\begin{aligned}P(h_1) &\equiv (h_1 - w_1)(h_1 - \widetilde{w}_1)(h_1 - w_2)(h_1 - \widetilde{w}_2) \geq 0\\ P(h_2) &\equiv (h_2 - w_1)(h_2 - \widetilde{w}_1)(h_2 - w_2)(h_2 - \widetilde{w}_2) \geq 0\end{aligned} \tag{B.3.8}$$

Assuming for the case of real roots of $P(w)$ that they are numbered so that $\widetilde{w}_1 < w_1 < \widetilde{w}_2 < w_2$, and that in any case, $z_1 < z_2$, we conclude from (B.3.8) that h_1 and h_2 should be located outside the interval $(\widetilde{w}_1, w_1)$ if w_1 (and therefore, $\widetilde{w}_1$) is real and outside the interval $(\widetilde{w}_2, w_2)$ if w_2 (and therefore, $\widetilde{w}_2$) is real. This means that

$$(h_1 - w_1)(h_1 - \widetilde{w}_1) \equiv \widetilde{m}_1^2 - \sigma_1^2 \geq 0, \qquad (h_2 - w_2)(h_2 - \widetilde{w}_2) \equiv \widetilde{m}_2^2 - \sigma_2^2 \geq 0$$

and therefore, there exist the real parameters $\widetilde{e}_1$ and $\widetilde{e}_2$ such that

$$\widetilde{e}_1^2 = \widetilde{m}_1^2 - \sigma_1^2, \qquad \widetilde{e}_2^2 = \widetilde{m}_2^2 - \sigma_2^2 \tag{B.3.9}$$

and we allow to each of the parameters $\widetilde{e}_1$ and $\widetilde{e}_2$ to be positive or negative in order to cover all opportunities to have in the expressions different signs in front of the square roots: $\pm\sqrt{P(h_1)}$ and $\pm\sqrt{P(h_2)}$. The parameters P_1 and P_2 in (B.3.7) are defined as positive parameters which have the following explicit expressions:

$$\begin{aligned} P_1 &= \sqrt{(h_1 - w_2)(h_1 - \widetilde{w}_2)} = \sqrt{(\ell - \widetilde{m}_1)^2 - \sigma_2^2} \\ P_2 &= \sqrt{(h_2 - w_1)(h_2 - \widetilde{w}_1)} = \sqrt{(\ell + \widetilde{m}_2)^2 - \sigma_1^2} \end{aligned} \qquad \text{(B.3.10)}$$

The expressions (B.3.7) allow us to use instead u_0, u_1, v_0, v_1 and h_1, h_2 the parameters $z_1, z_2, \widetilde{m}_1, \widetilde{m}_2, \widetilde{e}_1, \widetilde{e}_2$ (with $\ell = z_2 - z_1 > 0$) which can take arbitrary real values, provided $P_1^2 > 0$ and $P_2^2 > 0$, in view of (B.3.10) and (B.3.9). Without loss of generality, we assume that if all roots are real, then $\sigma_1 > 0$, $\sigma_2 > 0$, and for the sources to be separated, we assume also that $\ell > \sigma_1 + \sigma_2$.

Formal construction of the solution

For solving of the integral equations (B.1.8) for the monodromy data (B.3.1)-(B.3.2), we substitute this data into the integral equation (B.1.8) and obtain the equations whose solution has the structure which is very similar to that for the one-pole case (B.2.6):

$$\begin{pmatrix} \boldsymbol{\varphi}^{[\mathbf{1}]}(w) \\ \boldsymbol{\varphi}^{[\mathbf{u}]}(w) \\ \boldsymbol{\varphi}^{[\mathbf{v}]}(w) \end{pmatrix} = \frac{(w-h_1)(w-h_2)}{P(w)} \begin{pmatrix} X_0 + X_1 w + X_2 w^2 \\ Y_0 + Y_1 w + Y_2 w^2 \\ Z_0 + Z_1 w + Z_2 w^2 \end{pmatrix} \qquad \text{(B.3.11)}$$

where the coefficients X_k, Y_k and Z_k ($k = 0, 1, 2$) are independent of the spectral parameter w and they are functions of coordinates and constant parameters. To find these coefficients explicitly, we have to substitute (B.3.11) back into the integral equations, to calculate the corresponding integrals using the rules (B.2.8) and solve the linear algebraic equations for X_k, Y_k and Z_k. Using this solution, we find explicitly (using again the rules (B.2.8)) all components (B.1.6) of the matrix $\mathbf{R}$ and calculate then pure algebraically the Ernst potentials and all metric and electromagnetic potential components in accordance with their general expressions (B.1.5).

Weyl cylindrical and bipolar coordinates

During the calculation of the contour integrals (B.2.7), the coordinate dependence of the solution arises from calculation of residues of the integrands. It is easy to see, that the residue at infinity gives rise to the terms which are polynomial functions of Weyl coordinates ρ and z, while the residues at finite poles — the roots of $P(w)$ give rise to the terms which are proportional to the values of the function $\lambda(\xi,\eta,w)$ at these roots: $\lambda_1 = \lambda(\xi,\eta,w_1)$, $\widetilde{\lambda}_1 = \lambda(\xi,\eta,\widetilde{w}_1)$, $\lambda_2 = \lambda(\xi,\eta,w_2)$ and $\widetilde{\lambda}_2 = \lambda(\xi,\eta,\widetilde{w}_2)$. To have deal with these functions, we introduce bipolar coordinates (x_1,y_1) and (x_2,y_2) related to the corresponding pairs of spherical-like coordinates (r_1,θ_1) and (r_2,θ_2) centered on the symmetry axis $\rho = 0$ at $z = z_1$ and $z = z_2$, so that $x_1 = r_1 - \widetilde{m}_1$, $y_1 = \cos\theta_1$, $x_2 = r_2 - \widetilde{m}_2$, $y_2 = \cos\theta_2$. The coordinates (x_1,y_1) and (x_2,y_2) are defined by the relations to λ_1, $\widetilde{\lambda}_1$, λ_2 and $\widetilde{\lambda}_2$

$$\lambda_1 = \frac{x_1 - \sigma_1 y_1}{z_1 + \sigma_1} \qquad \widetilde{\lambda}_1 = \frac{x_1 + \sigma_1 y_1}{z_1 - \sigma_1} \qquad \begin{cases} \rho = \sqrt{x_1^2 - \sigma_1^2}\sqrt{1 - y_1^2} \\ z = z_1 + x_1 y_1 \end{cases}$$

$$\lambda_2 = \frac{x_2 - \sigma_2 y_2}{z_2 + \sigma_2} \qquad \widetilde{\lambda}_2 = \frac{x_2 + \sigma_2 y_2}{z_2 - \sigma_2} \qquad \begin{cases} \rho = \sqrt{x_2^2 - \sigma_2^2}\sqrt{1 - y_2^2} \\ z = z_2 + x_2 y_2 \end{cases}$$

while their relations to the Weyl coordinates given just above follow from the definition of the function $\lambda(\xi,\eta,w)$.

Physical parameters of the solution

The calculations described above lead to rather complicate form of the solution expressed in terms of functions of bipolar coordinates x_1, x_2, y_1, y_2, the formal mass and charge parameters $\widetilde{m}_1$, $\widetilde{m}_2$, $\widetilde{e}_1$, $\widetilde{e}_2$ and the parameters z_1 and z_2 which characterize the location of the sources on the symmetry axis. However, it is more convenient to describe the structure of this solution using certain combinations of these parameters, which, as it will be shown later, represent physical parameters—the masses of the sources m_1, m_2, their charges e_1, e_2 calculated using the Gauss theorem and therefore, expressed in terms of the Komar-like integrals. These physical parameters are determined by the expressions

$$m_1 = \widetilde{m}_1 + \frac{\widetilde{e}_1\widetilde{e}_2(\ell - \widetilde{m}_1 + \widetilde{m}_2)}{P_1P_2 + \widetilde{e}_1\widetilde{e}_2} \qquad e_1 = \frac{\widetilde{e}_1 P_2(\ell - \widetilde{m}_1 + \widetilde{m}_2)}{P_1P_2 + \widetilde{e}_1\widetilde{e}_2}$$

$$m_2 = \widetilde{m}_2 - \frac{\widetilde{e}_1\widetilde{e}_2(\ell - \widetilde{m}_1 + \widetilde{m}_2)}{P_1P_2 + \widetilde{e}_1\widetilde{e}_2} \qquad e_2 = \frac{\widetilde{e}_1 P_1 + \widetilde{e}_2 P_2}{\ell - \widetilde{m}_1 + \widetilde{m}_2} - e_1$$

where P_1, P_2 were introduced in (B.3.10). The inverse relations are (γ is defined below):

$$\widetilde{m}_1 = \frac{(\ell + m_1 + m_2)^2 - K_0}{2(\ell + m_1 + m_2)} \qquad \widetilde{m}_2 = \frac{(m_1 + m_2)^2 - \ell^2 + K_0}{2(\ell + m_1 + m_2)}$$
$$\widetilde{e}_1^2 = \frac{1}{2}\left[(\ell + m_2)^2 - m_1^2 - K_0\right] + e_1(e_1 + e_2)$$
$$\widetilde{e}_2^2 = \frac{1}{2}\left[(\ell - m_1)^2 - m_2^2 - K_0\right] + (e_2 + 2\gamma)(e_1 + e_2) + \frac{(m_1 + m_2)K_0}{\ell + m_1 + m_2}$$
$$P_1^2 = \frac{1}{2}\left[(\ell - m_1)^2 - m_2^2 + K_0\right] + (e_2 + 2\gamma)(e_1 + e_2) - \frac{(m_1 + m_2)K_0}{\ell + m_1 + m_2}$$
$$P_2^2 = \frac{1}{2}\left[(\ell + m_2)^2 - m_1^2 + K_0\right] + e_1(e_1 + e_2)$$
$$\widetilde{e}_1\widetilde{e}_2 = -P_1P_2\left(\frac{(\ell + m_2)^2 - m_1^2 - K_0}{(\ell + m_2)^2 - m_1^2 + K_0}\right)$$
$$K_0^2 \equiv [(\ell + m_2)^2 - m_1^2]^2 + 4e_1(\ell + m_1 + m_2)[e_2(\ell - m_1 + m_2) + 2e_1m_2]$$

The parameters σ_1 and σ_2 also can be expressed in terms of m_1, m_2, e_1, e_2 and ℓ:

$$\begin{aligned}\sigma_1^2 &= \widetilde{m}_1^2 - \widetilde{e}_1^2 = m_1^2 - e_1^2 + 2e_1\gamma,\\ \sigma_2^2 &= \widetilde{m}_2^2 - \widetilde{e}_2^2 = m_2^2 - e_2^2 - 2e_2\gamma,\end{aligned} \qquad \text{where} \qquad \gamma = \frac{m_2e_1 - m_1e_2}{\ell + m_1 + m_2} \tag{B.3.12}$$

Using these parameters, we obtain more symmetric and simpler form of the solution. We introduce also, instead of e_1 and e_2, two other parameters q_1 and q_2 such that:

$$\begin{aligned}q_1 &= e_1 - \gamma,\\ q_2 &= e_2 + \gamma,\end{aligned} \qquad \gamma = \frac{m_2q_1 - m_1q_2}{\ell}, \qquad \begin{aligned}\sigma_1^2 &= m_1^2 + \gamma^2 - q_1^2,\\ \sigma_2^2 &= m_2^2 + \gamma^2 - q_2^2.\end{aligned} \tag{B.3.13}$$

Finally, after some long calculations and consideration of various possible forms of our solution, its rather short form was found. In this form, all components of the solution are presented as functions of six parameters with only one constraint

$$\{m_1, m_2, q_1, q_2, \ell, \gamma\} \qquad \Big\| \qquad \gamma\ell = m_2q_1 - m_1q_2 \tag{B.3.14}$$

which leaves only five parameters to be independent.

B.4. 5-parametric solution for interacting Reissner-Nordström sources

For the monodromy data (B.3.1),(B.3.2), the corresponding solution is static and its metric and electromagnetic potential in cylindrical Weyl coordinates take the forms

$$\begin{aligned} ds^2 &= Hdt^2 - f(d\rho^2 + dz^2) - \frac{\rho^2}{H}d\varphi^2, \\ A_t &= \Phi, \qquad A_\rho = A_z = A_\varphi = 0, \end{aligned} \tag{B.4.1}$$

where H, f and Φ are real functions of ρ and z. The calculations described in previous subsections lead to the following structure of the functions H, Φ and f:

$$H = \frac{\mathcal{D}^2 - \mathcal{G}^2 + \mathcal{F}^2}{(\mathcal{D}+\mathcal{G})^2}, \qquad \Phi = \frac{\mathcal{F}}{\mathcal{D}+\mathcal{G}}, \qquad f = \frac{f_0(\mathcal{D}+\mathcal{G})^2}{(x_1^2 - \sigma_1^2 y_1^2)(x_2^2 - \sigma_2^2 y_2^2)} \tag{B.4.2}$$

where $\mathcal{D}$, $\mathcal{G}$, $\mathcal{F}$ are polynomial functions of bipolar coordinates with rather simple coefficients depending on the parameters of the solution:

$$\begin{aligned} \mathcal{D} &= x_1x_2 - \gamma^2 y_1 y_2 \\ &\quad + \delta\left[x_1^2 + x_2^2 - \sigma_1^2 y_1^2 - \sigma_2^2 y_2^2 + 2(m_1m_2 - q_1q_2)y_1y_2\right] \\ \mathcal{G} &= m_1x_2 + m_2x_1 + \gamma(q_1y_1 + q_2y_2) \\ &\quad + 2\delta\left[m_1x_1 + m_2x_2 + y_1(q_2\gamma - m_1\ell) + y_2(q_1\gamma + m_2\ell)\right] \\ \mathcal{F} &= q_1x_2 + q_2x_1 + \gamma(m_1y_1 + m_2y_2) \\ &\quad + 2\delta\left[q_1x_1 + q_2x_2 + y_1(m_2\gamma - q_1\ell) + y_2(m_1\gamma + q_2\ell)\right] \end{aligned} \tag{B.4.3}$$

In (B.4.2), f_0 is an arbitrary constant which should be chosen so that $f \to 1$ at spatial infinity and the parameter δ in (B.4.3) is determined by the expression:

$$f_0 = \frac{1}{(1+2\delta)^2}, \qquad \delta = \frac{m_1m_2 - q_1q_2}{\ell^2 - m_1^2 - m_2^2 + q_1^2 + q_2^2} \tag{B.4.4}$$

In accordance with (B.3.13) and (B.3.14), these equations give us an explicit expression of the solution in terms of five free real parameters m_1, m_2, e_1, e_2 and $\ell = z_2 - z_1$. We consider this solution as depending on six parameters m_1, m_2, q_1, q_2, ℓ and γ restricted by the only one constraint (B.3.14) and with the expressions (B.3.13) for σ_1^2, σ_2^2:

$$\gamma\ell = m_2q_1 - m_1q_2, \qquad \sigma_1^2 = m_1^2 + \gamma^2 - q_1^2, \qquad \sigma_2^2 = m_2^2 + \gamma^2 - q_2^2. \tag{B.4.5}$$

This solution is asymptotically flat. As it will be explained in the next section, the parameters m_1 and m_2 characterize the individual masses of the sources and the charges of these sources are $e_1 = q_1 + \gamma$ and $e_2 = q_2 - \gamma$ respectively, while the total mass $m = m_1 + m_2$ and the total charge $e = e_1 + e_2$.

B.5. Physical parameters of the sources

For electrovacuum space-time which admits a time-like Killing vector field $\xi = \partial_t$ we can write (following, for example, (36)) the "dynamical" part of the Einstein-Maxwell equations in the Kinnersley-like self-dual form

$$\nabla_k \overset{+}{\mathcal{H}}{}^{ik} = 0, \qquad \nabla_k \overset{+}{\mathcal{F}}{}^{ik} = 0 \tag{B.5.1}$$

where the bivectors $\mathcal{H}_{ik}$ and $\mathcal{F}_{ik}$ are defined as follows ($\gamma = c = 1$):

$$\begin{aligned} &\overset{+}{\mathcal{H}}_{ik} \equiv \overset{+}{\mathcal{K}}_{ik} - 2\overline{\Phi}\overset{+}{\mathcal{F}}_{kl}, \qquad &&\overset{+}{\mathcal{K}}_{ik} = K_{ik} + \frac{i}{2}\varepsilon_{iklm}K^{lm}, \qquad &&K_{ik} = \partial_i\xi_k - \partial_k\xi_i, \\ & &&\overset{+}{\mathcal{F}}_{ik} = F_{ik} + \frac{i}{2}\varepsilon_{iklm}F^{lm}, &&F_{ik} = \partial_i A_k - \partial_k A_i \end{aligned} \tag{B.5.2}$$

and, due to (B.5.1), these self-dual bivectors possess complex vector potentials, one of which determines the complex scalar function Φ which enters the expressions (B.5.2):

$$\overset{+}{\mathcal{H}}_{ik} = \partial_i\mathcal{H}_k - \partial_k\mathcal{H}_i \qquad \overset{+}{\mathcal{F}}_{ik} = \partial_i\Phi_k - \partial_k\Phi_i \qquad \Phi \equiv \xi^k\Phi_k \tag{B.5.3}$$

The "dynamical" equations (B.5.1) allow us to construct the "conserved" quantities—the additive integral values which characterize the sources and which can be calculated as the integrals over the spherical-like 2-surfaces surrounding different parts of the sources on the space-like hypersurfaces slicing the space-time region outside the sources. For stationary axisymmetric spacetime outside the field sources we consider the space-like hypersurfaces $\Sigma_t\ :\ t = const$ where t is the Killing parameter. Contraction the equations (B.5.1) with the gradient $\partial_i t$, we obtain

$$\overset{(3)}{\nabla}_\delta(N_i\overset{+}{\mathcal{H}}{}^{i\delta}) = 0, \qquad \overset{(3)}{\nabla}_\delta(N_i\overset{+}{\mathcal{F}}{}^{i\delta}) = 0 \tag{B.5.4}$$

where $\delta = 1, 2, 3$; $\overset{(3)}{\nabla}$ is a covariant derivative with respect to a three-dimensional metric on the hypersurfaces Σ_t and N_i is a unit time-like future-directed normal to these hypersurfaces. Integrating the above equations on Σ_t over a three-dimensional region between a sphere of a large radius B_∞ located in the asymptotically Minkowski region and by a closed surface B_s surrounding the sources, and applying the Gauss theorem, we obtain

$$\int_{B_s} (\overset{+}{\mathcal{H}}_{ik} N^i n^k) d^2\sigma = \int_{B_\infty} (\overset{+}{\mathcal{H}}_{ik} N^i n^k) d^2\sigma, \quad \int_{B_s} (\overset{+}{\mathcal{F}}_{ik} N^i n^k) d^2\sigma = \int_{B_\infty} (\overset{+}{\mathcal{F}}_{ik} N^i n^k) d^2\sigma \tag{B.5.5}$$

where n_k is a unit vector tangent to the hypersurface Σ_t and representing the outward normal to the corresponding boundary B_s or B_∞ and $d^2\sigma$ is the area element on these boundaries. At first, we calculate the integrals over B_∞.

For any stationary axisymmetric asymptotically flat electrovacuum solution of Einstein-Maxwell equations, the metric and electromagnetic potential of the form (B.1.4) at spatial infinity admit the expansions (the NUT parameter, the total magnetic charge and the additive constants in A_t and A_φ are assumed to vanish):

$$\begin{aligned} g_{tt} &= 1 - \frac{2m}{r} + O(\frac{1}{r^2}) & f &= 1 + \frac{2m}{r} + O(\frac{1}{r^2}) \\ g_{t\varphi} &= \frac{2am}{r}\sin^2\theta + O(\frac{1}{r^2}) & A_t &= \frac{e}{r} + O(\frac{1}{r}) \\ g_{\varphi\varphi} &= -r^2\sin^2\theta - 2mr\sin^2\theta + O(r^0) & A_\varphi &= O(\frac{1}{r}) \end{aligned}$$

where $\rho = r\sin\theta, z = z_* + r\cos\theta$ and the constants m, a and e mean the total mass, total angular momentum per unit mass and the total electric charge respectively. For the complex vector potentials $\mathcal{H}_i$ and Φ_i introduced in (B.5.3) for $r \to \infty$ in the coordinates $\{t, \rho, z, \varphi\}$ we obtain the expansions

$$\begin{aligned} \mathcal{H}_i &= \{-\frac{2m}{r} + O(\frac{1}{r^2}),\ 0,\ 0,\ -2im\cos\theta + O(\frac{1}{r})\} \\ \Phi_i &= \{\frac{e}{r} + O(\frac{1}{r^2}),\ 0,\ 0,\ ie\cos\theta + O(\frac{1}{r})\} \end{aligned} \tag{B.5.6}$$

The components of the vector N^i in these coordinates take the form

$$N^i = \{1 + \frac{m}{r} + O(\frac{1}{r^2}),\ 0,\ 0,\ \frac{2am}{r^3} + O(\frac{1}{r^4})\}$$

For a sphere of a large radius, the spatial unit normal vector in the leading order is $n^i\partial_i = \partial/\partial r$ and in the limit $r \to \infty$ for the integrals (B.5.5) over B_∞ we obtain

$$\int_{B_\infty} (\overset{+}{\mathcal{H}}_{ik} N^i n^k) d^2\sigma = -8\pi m, \qquad \int_{B_\infty} (\overset{+}{\mathcal{F}}_{ik} N^i n^k) d^2\sigma = 4\pi e$$

where m and e are the total mass and charge of the field configuration. This allows us to express (in accordance with (B.5.5)) the total mass and charge of a system of sources in terms of the integrals over the surface B_s surrounding the sources:

$$m = -\frac{1}{8\pi} \int\limits_{B_s} (\overset{+}{\mathcal{H}}_{ik} N^i n^k) d^2\sigma, \qquad e = \frac{1}{4\pi} \int\limits_{B_s} (\overset{+}{\mathcal{F}}_{ik} N^i n^k) d^2\sigma \tag{B.5.7}$$

Now we transform the integrands in (B.5.7) using the self-duality of $\overset{+}{\mathcal{H}}$ and $\overset{+}{\mathcal{F}}$:

$$\begin{array}{l} \int\limits_{B_s} \overset{+}{\mathcal{H}}{}^{ij} N_i n_j d^2\sigma = i \int\limits_{B_s} \overset{+}{\mathcal{H}}_{ij} k^i l^j d^2\sigma \\ \int\limits_{B_s} \overset{+}{\mathcal{F}}{}^{ij} N_i n_j d^2\sigma = i \int\limits_{B_s} \overset{+}{\mathcal{F}}_{ij} k^i l^j d^2\sigma \end{array} \quad \Bigg\| \quad \begin{array}{l} \varepsilon^{ijkl} N_k n_l = k^i l^j - k^j l^i \\ \varepsilon^{ijkl} N_k n_l k_i l_j = 1 \end{array} \tag{B.5.8}$$

Here we introduced two spatial unit vectors k^i and l^i which are orthogonal to each other and tangent to 2-surface B_s. These vectors determine the orientation of the 2-surface B_s so that the spatial basis $\{n^i, k^j, l^k\}$ is positively oriented. Choosing the 2-surface B_s to be axially symmetric, we specify the choice of the vectors k^i and l^j in the tangent space of B_s so that l^j would be the rotational Killing vector $\partial/\partial_\varphi$ and k^i is tangent to a curve $\mathcal{L}$ on B_s with $\varphi = const$:

$$k^i = \{0, \frac{d\rho}{d\ell}, \frac{dz}{d\ell}, 0\} \qquad\qquad l^i = \frac{1}{\sqrt{-g_{\varphi\varphi}}}\{0, 0, 0, 1\}$$

where the parameter ℓ is the length on the curve $\mathcal{L}$ and the direction on $\mathcal{L}$ is chosen so that it goes from some point on the positive part of the z-axis to some point on its negative part (provided the point $\rho = z = 0$ is located somewhere inside B_s). The element of the area on B_s is $\sqrt{-g_{\varphi\varphi}} d\ell d\varphi$. After the integration over φ we observe that the integrands of the remaining contour integrals over $\mathcal{L}$ are the differentials of the φ-components of the potentials of self-dual bivectors $\overset{+}{\mathcal{H}}$ and $\overset{+}{\mathcal{F}}$:

$$\begin{array}{l} m = -\frac{i}{4} \int\limits_{\mathcal{L}} (\partial_\mu \mathcal{H}_\varphi) dx^\mu = \frac{i}{4} \left[(\mathcal{H}_\varphi)_+ - (\mathcal{H}_\varphi)_- \right] \\ e = \frac{i}{2} \int\limits_{\mathcal{B}L} (\partial_\mu \Phi_\varphi) dx^\mu = -\frac{i}{2} \left[(\Phi_\varphi)_+ - (\Phi_\varphi)_- \right] \end{array} \tag{B.5.9}$$

where $(\ldots)_+$ means the value of a potential at the beginning of $\mathcal{L}$, i.e. on the positive part of the z-axis, and $(\ldots)_-$—its value on the negative part of this axis.

It is important to note that $\mathcal{H}_\varphi$ and Φ_φ are constant along the regular parts of the symmetry axis outside the sources and therefore, the integrals (B.5.9) do not change if we deform the surface of integration B_s surrounding the sources in the domain of regularity of the solution.

Now we consider the case, in which the source of the field consists of two parts (black holes, naked singularities or extended bodies) separated by some segment of the symmetry axis. In this case, we can deform the surface B_s surrounding the sources into two spheres each surrounding one of these parts of the sources and a thin tube surrounding the intermediate part of the axis between the sources. If $\mathcal{H}_\varphi$ and Φ_φ are constant on the intermediate part of the axis, the integral over the thin tube vanish and we obtain that the integrals for the total mass and the total charge are expressed as the sums of the integrals of the same type over the spheres surrounding each of the sources. In particular, for the sources consisting of two parts, as it is in our solution for two Reissner-Nordström sources, we obtain

$$m = m_1 + m_2, \qquad e = e_1 + e_2$$

where m_1, m_2 and e_1, e_2 can be interpreted respectively as the masses (energies) and charges of the corresponding parts of the source. To conclude this section, we mention that rather tedious calculations show that the parameters m_1, m_2 and e_1, e_2 in our electrostatic solution presented in the previous section and describing the field of two Reissner-Nordström sources, coincide with the values of the integrals (B.5.9) calculated for each part of the sources and therefore, these parameters can be interpreted as the additive masses and charges of these interacting sources.

B.6. Equilibrium of two Reissner-Nordström sources

The 5-parametric solution (B.4.1)-(B.4.5) presented above is asymptotically flat and the space-time geometry is regular far enough from the sources. However, for an arbitrary choice of parameters m_1, m_2, q_1, q_2 and ℓ, this solution may have some physically not reasonable singularities ("struts") on the part of the axis of symmetry between the sources. Like at the vertex of the cone, at these points, the local Euclidean properties of space-time may be violated so that on the space-time sections $t = const$, $z = const$ the ratio of the length $\mathcal{L}$ of a small circle, surrounding the axis of symmetry $\rho = 0$ and contracting to the point z of this axis, to its radius $\mathcal{R}$ multiplied by 2π is not equal to a unit. The constraint imposed on the parameters which provide the absence of such conical singularities plays the role of equilibrium condition because it allows to select a physically acceptable solution in which these singularities are absent and the sources are in the equilibrium because of the balance between their gravitational and electromagnetic interactions.

As it follows from elementary considerations, the limiting value of the square of the ratio $(2\pi\mathcal{R})/\mathcal{L}$ for $\mathcal{R} \to 0$ is equal to the value of the product fH at the point z of the axis. However, the field equations imply that on the regular parts of the axis of symmetry the product $f(\rho = 0, z)H(\rho = 0, z)$ does not depend on z and therefore, it is a constant. However, the values of this constant can be different on different parts of the axis of symmetry separated by the sources, and the condition $fH = 1$ of the absence of conical singularities should be satisfied at each of these parts of the axis of symmetry.

In our solution with two separated sources, there exists three parts of the axis of symmetry where the condition of the absence of conical points should be considered. There are two semi-infinite parts of the axis outside the sources. On the negative part L_-, we have $-\infty < z < z_1 - \sigma_1$ (if the first source is a black hole) or $-\infty < z < z_1$ (if the first source is a naked singularity) and on this part $x_1 = z_1 - z$, $x_2 = z_2 - z$, $y_1 = -1$, $y_2 = -1$. On the positive part L_+ of the axis, $z_2 + \sigma_2 < z < \infty$ (if the second source is a black hole) or $z_2 < z < \infty$ (if the second source is a naked singularity) and we have there $x_1 = z - z_1$, $x_2 = z - z_2$, $y_1 = 1$, $y_2 = 1$. It is easy to see that on these parts of the axis the condition $fH = 1$ is satisfied for any choice of free parameters of our solution. However, it is not the case on the part L_0 of the axis between the sources. On L_0 we have $z_1 + Re(\sigma_1) < z < z_2 - Re(\sigma_2)$ and $x_1 = z - z_1$, $x_2 = z_1 + \ell - z$, $y_1 = 1$, $y_2 = -1$ and the product fH also takes there some constant value whose equality to a unit give us the equilibrium condition:

$$m_1 m_2 - q_1 q_2 = 0 \tag{B.6.1}$$

It is interesting to note that this equilibrium condition looks just like the Newtonian condition of equilibrium of two charged point-like masses, but in the case of General Relativity this condition relates the masses of the Reissner-Nordström sources not with their charges e_1 and e_2, but with the parameters q_1 and q_2 whose expressions in terms of masses m_1, m_2 and charges e_1, e_2 depend also on the z-distance ℓ separating the sources.

The equilibrium condition (B.6.1) allows also to simplify the solution (B.4.1)-(B.4.5) and leads to the 4-parametric family which describes the superposed field of two Reissner-Nordström sources in equilibrium. This solution has been presented in our previous short paper (34). Here we persent it in a bit different form using m_1, m_2, q_1, q_2, γ as the basic set of parameters, such that the first four of them should satisfy the equilibrium condition (B.6.1). For this solution the metric functions have the same expressions (B.4.2) where the polynomials $\mathcal{D}$, $\mathcal{G}$ and $\mathcal{F}$ have the expressions

$$\left.\begin{array}{l} \mathcal{D} = x_1 x_2 - \gamma^2 y_1 y_2 \\ \mathcal{G} = m_1 x_2 + m_2 x_1 + \gamma(q_1 y_1 + q_2 y_2) \\ \mathcal{F} = q_1 x_2 + q_2 x_1 + \gamma(m_1 y_1 + m_2 y_2) \end{array}\right\| \quad m_1 m_2 = q_1 q_2$$

and $f_0 = 1$. The other physical parameters—the z-distance ℓ separating the sources and their charges are determined by the expressions

$$\ell \equiv z_2 - z_1 = (m_2 q_1 - m_1 q_2)/\gamma, \qquad e_1 = q_1 + \gamma, \quad e_2 = q_2 - \gamma$$

The independent parameters m_1, m_2, q_1, q_2, γ of this solution should be chosen so that the sources would be separated actually by some positive distance, i.e. $\ell > Re(\sigma_1) + Re(\sigma_2)$.

This concludes our description in this paper of some fragments of the monodromy transform approach and of the procedure for solution of the corresponding singular integral equations for stationary axisymmetric electrovacuum fields with simple rational monodromy data, which leads to the construction of the 5-parametric family of solutions for the field of two interacting Reissner-Nordström sources, and of the derivation from this solution of the 4-parametric family of solutions for the fields of equilibrium configurations of these sources. Some properties of these equilibrium field configurations have been discussed in (34), however, a more detail analysis of physical and geometrical properties of these configurations, such as the structure of the superposed fields, influence on the geometry of horizons and on the space-time geometry inside the horizon of the external gravitational and electromagnetic fields created by another source, tidal influence of these fields on the structure of naked singularities, stability of equilibrium and probably, some others, are expected to be the subject of our next publications.

C. A membrane model of the Reissner-Nordstrom singularity with repulsive gravity

C.1. Introduction

One of the interesting effects of relativistic gravity which has no analogue in the Newtonian theory is the presence of gravitational repulsive forces. The classical example is the Reissner-Nordstrom (RN) field in the region close enough to the central singularity. Indeed, in the RN metric

$$-ds^2 = -f\, c^2 dt^2 + f^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{C.1.1}$$

where

$$f = 1 - \frac{2kM}{c^2 r} + \frac{kQ^2}{c^4 r^2}\,, \tag{C.1.2}$$

the radial motion of a test neutral particle follows the equation:

$$\frac{d^2 r}{ds^2} = -\frac{1}{2}\frac{df}{dr} = \frac{k}{c^4 r^2}\left(\frac{Q^2}{r} - Mc^2\right) \tag{C.1.3}$$

from where one can see the appearance of repulsive force in the region of small r. In this zone the gradient of the gravitational potential $f(r)$ is negative and the gravitational force in Eq.(C.1.3) is directed toward the outside of the central source.

For the RN naked singularity case ($Q^2 > kM^2$), in which we are interested in the present paper, the potential $f(r)$ is everywhere positive and has a minimum at the point $r = Q^2/Mc^2$. Therefore at this point a neutral particle can stay at rest in the state of stable equilibrium (the detailed study can be found in (46; 47)).

It is an interesting and nontrivial fact that the same sort of stationary equilibrium state due to the repulsive gravity exists also as an exact asymptotically flat two-body solution of the Einstein Maxwell equations which describes a Schwarzschild black hole situated at rest in the field of a RN naked singularity without any strut or string between these two objects (48; 49). However, solutions of this kind have the feature that the object creating the repelling region has naked singularity and this last property has no clear physical interpretation. Consequently the pertinent question is whether the repelling phenomenon around a charged source arises only due to the presence of the naked singularity or it can be also a feature of physically reasonable structure of the space-time and matter.

By other words the question is whether or not it is possible to construct a regular material source which can block the central singularity and join the external repulsive region in a proper way. Then we are interested to construct a body with the following properties:

1. inside the body there are no singularities;
2. outside the body there is the RN field (C.1.1)-(C.1.2), corresponding to the case $Q^2 > kM^2$;
3. the radius of the body is less than Q^2/Mc^2, so between the surface of the body and the sphere $r = Q^2/Mc^2$ arises the repulsive region;
4. such stationary state of the body is stable with respect to collapse or expansion.

In this paper we propose a new model for such body in the form of spherically symmetric thin membrane with positive tension. We assert that there exists a physically acceptable range of parameters for which all the above four conditions (1)-(4) can be satisfied. We illustrate this conclusion by the especially transparent case of a Nambu-Goto membrane with equation of state $\epsilon = \tau$.

Then the existence of everywhere-regular material sources possessing RN "antigravity" properties in the vicinity of their surfaces attribute to this phenomenon and to the RN naked singularity solution more sensible physical status.

It is necessary to mention that at least two exact solutions of Einstein-Maxwell equations representing a compact continuous spherically symmetric distribution of charged matter under the tension producing the gravitationally repulsive forces inside the matter as well as in some region outside of it already exist in the literature. These are solutions constructed in Ref.(50) and Ref.(51). A more detailed study of these two results can be found in Ref.(52). An interesting possibility to have a gravitationally repulsive core of electrically neutral but viscous matter has been communicated in Ref.(53).

It is worth to remark that the first (to our knowledge) mentioning of the gravitational repulsive force due to the presence of electric field was made already in 1937 in the Ref.(54) in connection to the nonlinear model of electrodynamics of Born-Infield type. One of the first paper where a repulsive phenomenon in the framework of the conventional Einstein-Maxwell theory has been mentioned is Ref.(55). The general investigation of the different aspects of this phenomenon apart from the already mentioned references (46)-(55) can be found also in the more detailed works (56; 57; 58; 59). Some part of these papers is dedicated to a possibility of construction a classical model for electron. This is doubtful enterprise, however, because the intrinsic structure of electron is a matter out of classical physics. Nonetheless the mathematical results obtained are useful and can be applied to the physically sensible situations, e.g. for construction the models of macroscopical objects.

C.2. Equation of motion of a membrane with empty space inside

The equation of motion for the most general case of a thin charged spherically symmetric fluid shell with tangential pressure moving in the RN field have been derived 38 years ago by J.E. Chase(60). The corresponding dynamics for a charged elastic membrane with tension follows from his equation simply by the change of the sign of the pressure. We derived, however, the membrane's dynamics again using a different approach.

Chase used the geometrical method which have been applied to the description of singular surfaces in relativistic gravity in (61) and have been elaborated in (55; 62) for some special cases of charged shells. An essential development of the Israel approach in application to the cosmological domain walls can be found in the series of works of V.Berezin, V.Kuzmin and I. Tkachev, see Ref.(63) and references therein. Our treatment follows the method more habitual for physicists which have been used in (64), where the motion of a neutral fluid shell in a Schwarzschild field was derived by the direct integration of the Einstein equations with appropriate δ-shaped source. Now we generalized this approach for the charged membrane and charged central source.

In this section we study only the particular solution in which there is no central body, that is inside the membrane we have flat space-time.

The exposition we give here is more or less self-consistent, we reserve to give more details on the procedure used on a forthcoming paper.

For the thin spherically symmetric membrane with empty space inside and with radius which depends on time the metrics inside, outside and on membrane are:

$$-(ds^2)_{in} = -\Gamma^2(t)c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{C.2.1}$$

$$-(ds^2)_{out} = -f(r)c^2dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{C.2.2}$$

$$-(ds^2)_{on} = -c^2d\eta^2 + r_0^2(\eta)(d\theta^2 + \sin^2\theta d\phi^2) \tag{C.2.3}$$

In the interval (C.2.3) η is the proper time of the membrane. The factor Γ^2 in (C.2.1) is necessary to ensure the continuity of the global time coordinate t through the membrane. The metric coefficient $f(r)$ in the region outside the membrane is given by Eq.(C.1.2).

Matching conditions for the intervals (C.2.1)-(C.2.3) through the membrane's surface are:

$$[(ds^2)_{in}]_{r=r_0(\eta)} = [(ds^2)_{out}]_{r=r_0(\eta)} = (ds^2)_{on} \tag{C.2.4}$$

If the equation of motion of the membrane $r = r_0(\eta)$ is known, then from these conditions the connection $t(\eta)$ between global and proper times and factor $\Gamma(t)$ follow easily:

$$\Gamma(t) = \frac{f(r_0)\sqrt{1 + c^{-2}(r_{0,\eta})^2}}{\sqrt{f(r_0) + c^{-2}(r_{0,\eta})^2}} \tag{C.2.5}$$

$$\frac{d\,t}{d\eta} = \frac{\sqrt{f(r_0) + c^{-2}(r_{0,\eta})^2}}{f(r_0)} \tag{C.2.6}$$

The differential equation for the function $r_0(\eta)$ follows from Einstein-Maxwell equations with energy-momentum tensor and charge current concentrated on the surface of the membrane. It is:

$$Mc^2 = \mu(r_0)c^2\sqrt{1 + \left(\frac{d\,r_0}{c\,d\eta}\right)^2} + \frac{Q^2}{2r_0} - \frac{k\mu^2(r_0)}{2r_0} \tag{C.2.7}$$

Here $\mu(r_0) > 0$ is the effective rest mass of the membrane in the radially comoving frame. This quantity includes the membrane's rest mass as well as all kinds of interaction mass-energies between membrane's constituents, that is those intrinsic energies which are responsible for the tension. The constants Q and M are the total charge of the membrane and total relativistic mass of the system. These are the same constants which appeared earlier in Eq.(C.1.2). The membrane's energy density ϵ and tension τ are :

$$\epsilon = \epsilon_0(r_0)\delta[r - r_0(\eta)] \quad \tau = \tau_0(r_0)\delta[r - r_0(\eta)] \tag{C.2.8}$$

where

$$\epsilon_0 = \frac{\mu(r_0)c^2}{8\pi r_0^2}\left[\frac{1}{\sqrt{1 + c^{-2}(r_{0,\eta})^2}} + \frac{f(r_0)}{\sqrt{f(r_0) + c^{-2}(r_{0,\eta})^2}}\right] \tag{C.2.9}$$

$$\tau_0(r_0) = \frac{d\mu(r_0)}{dr_0}\frac{r_0\epsilon_0(r_0)}{2\mu(r_0)} \tag{C.2.10}$$

The electromagnetic potentials have the form $A_r = A_\theta = A_\phi = 0$, $A_t =$

$A_t(t,r)$ and for the electric field strength $\partial A_t/\partial r$ the solution is

$$\frac{\partial A_t}{\partial r} = \begin{cases} \frac{Q}{r^2} & \text{for} \quad r > r_0(\eta) \\ \\ 0 & \text{for} \quad r < r_0(\eta) \end{cases} \tag{C.2.11}$$

The formulas (C.2.1)-(C.2.11) give the complete solution of the problem for the case of empty space inside the membrane.

Finally we would like to stress the following important point. The signs of the square roots $\sqrt{1+c^{-2}(r_{0,\eta})^2}$ and $\sqrt{f(r_0)+c^{-2}(r_{0,\eta})^2}$ coincide with the signs of the time component u^0 of the 4-velocity of the membrane evaluated from inside and outside of the membrane respectively. The component u^0 is a continuous quantity by definition and can not change the sign when passing through the membrane's surface. Besides, for macroscopical objects we are interested in in this paper u^0 should be positive. Consequently the both aforementioned square roots should be positive. From another side it is easy to show that equation (C.2.7) can be written also in the following equivalent form

$$Mc^2 = \mu c^2 \sqrt{f(r_0) + \left(\frac{d\,r_0}{c\,d\eta}\right)^2} + \frac{Q^2}{2r_0} + \frac{k\mu^2}{2r_0} \tag{C.2.12}$$

Then from this expression and from (C.2.7) follows that both square roots will be positive if and only if

$$Mc^2 - \frac{Q^2}{2r_0} - \frac{k\mu^2}{2r_0} > 0 \tag{C.2.13}$$

This is unavoidable constraint which must be adopted as additional condition for any physically realizable solution of the equation of motion (C.2.7) in classical macroscopical realm.

C.3. Nambu-Goto membrane with "antigravity" effect

To proceed further we must specify the function $\mu(r_0)$, which is equivalent to specifying an equation of state, as can be seen from (C.2.10).

Let us analyze the membrane with equation of state $\epsilon = \tau$. This model can be interpreted as "bare" Nambu-Goto charged membrane(65; 66), or as Zeldovich-Kobzarev-Okun charged domain wall(67). It follows from (C.2.10) that for such type of membrane we have:

$$\mu = \gamma r_0^2 \tag{C.3.1}$$

where γ is an arbitrary constant. In this and next section we consider only the case of positive constants γ and M:

$$\gamma > 0\,, \quad M > 0\,. \tag{C.3.2}$$

The sign of Q is of no matter since the charge appear everywhere in square. Now we write the equation of motion (C.2.7) in the following form:

$$4\left(\frac{d\,r_0}{c\,d\eta}\right)^2 - \left(\frac{k\gamma r_0}{c^2} + \frac{2M}{\gamma r_0^2} - \frac{Q^2}{c^2\gamma r_0^3}\right)^2 = -4\,. \tag{C.3.3}$$

Formally this can be considered as the equation of motion of a non-relativistic particle with the "mass" equal to 8 moving in the potential $U(r_0)$,

$$U(r_0) = -\left(\frac{k\gamma r_0}{c^2} + \frac{2M}{\gamma r_0^2} - \frac{Q^2}{c^2\gamma r_0^3}\right)^2 \tag{C.3.4}$$

and under that condition that particle is forced to live on the "total energy" level equal to minus four.

For the existence of the stable stationary state we are interested in, the following conditions should hold:

1. The gravitational field in the exterior region should correspond to the super-extreme RN metric:

$$Q^2 > kM^2. \tag{C.3.5}$$

2. The potential $U(r_0)$ should have a local minimum at some value $r_0 = R_{min}$. The form (C.3.4) of $U(r_0)$ permit this if and only if

$$k\gamma^2 Q^6 < (Mc^2)^4. \tag{C.3.6}$$

Under this restriction the potential $U(r_0)$ has three extrema, two maxima at points $r_0 = R_{max}^{(1)}$ and $r_0 = R_{max}^{(2)}$ and a minimum which is located between them: $R_{max}^{(1)} < R_{min} < R_{max}^{(2)}$. We show the shape of the potential $U(r_0)$ for this case in Fig.1.

The equation $U(r_0) = 0$ has only one real root and this is also the first local maximum $R_{max}^{(1)}$. The minimum and the second maximum are coming as two other roots of the equation $\frac{dU}{dr_0} = 0$.

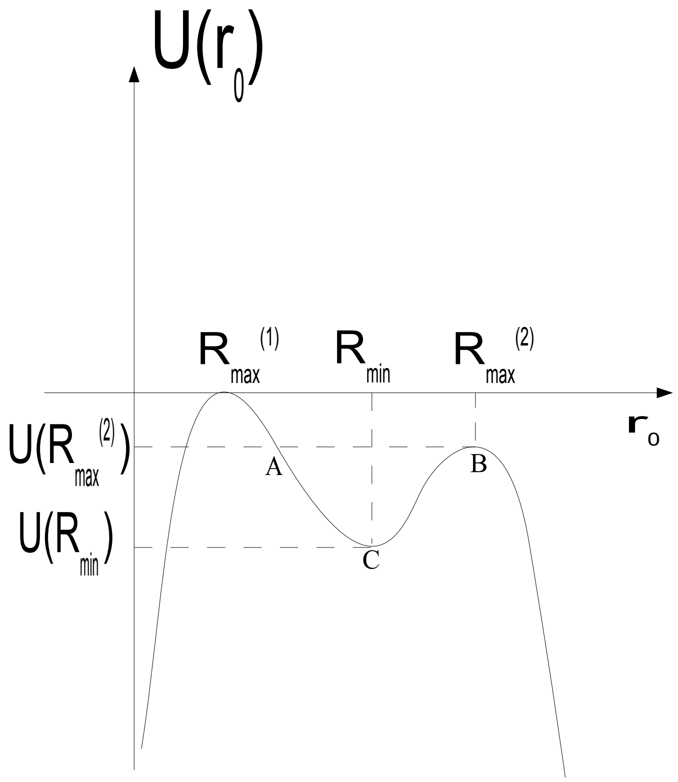

Figure C.1.: The membrane's motion can be described as the motion of a non-relativistic point particle in the potential $U(r_0)$.

The equation for R_{min} is:

$$k\gamma^2 R_{min}^4 - 4Mc^2 R_{min} + 3Q^2 = 0\ . \tag{C.3.7}$$

This fourth order equation has only two real solutions and R_{min} is the smaller one.

3. For the stationary position of the membrane at the minimum of the potential we must ensure the relation $U(R_{min}) = -4$ which is:

$$\frac{k\gamma}{c^2} R_{min} + \frac{2M}{\gamma} R_{min}^{-2} - \frac{Q^2}{c^2\gamma} R_{min}^{-3} = 2 \tag{C.3.8}$$

(the minus two in the r.h.s. of (C.3.8) would be incompatible with Eq.(C.3.7) under condition (C.3.2)).

4. To have repulsive region it is necessary for the membrane's radius R_{min} to be less than the minimum of the gravitational potential $f(r)$, that is less than the quantity Q^2/Mc^2. In this case outside of the membrane surface in the region $R_{min} < r < Q^2/Mc^2$ we have the repulsive effect. Then we demand:

$$R_{min} < \frac{Q^2}{Mc^2}\ . \tag{C.3.9}$$

5. Also the additional constraint (C.2.13) should be satisfied. This means that for our stationary solution we have to satisfy the inequality:

$$Mc^2 - \frac{Q^2}{2R_{min}} - \frac{k\gamma^2}{2} R_{min}^3 > 0\ . \tag{C.3.10}$$

6. We have also another condition: that the electric field nearby the membrane should be not too large, otherwise the stability of the model would be destroyed by the strong macroscopical consequences of quantum effects, e.g. by the intensive electron-positron pair creation. This condition (which was suggested by J.A. Wheeler long time ago(68)) is:

$$\frac{Q}{R_{min}^2} << \mathcal{E}_{cr}\ , \qquad \mathcal{E}_{cr} = \frac{m_e^2 c^3}{e_e \hbar}\ , \tag{C.3.11}$$

where m_e and e_e are the electron's mass and charge). $\mathcal{E}_{cr}$ is the well known critical electric field above which the intensive process of pair creation starts.

To satisfy these six conditions we have to find a physically acceptable domain in the space of the four parameters M, Q, γ and R_{min}. The point is that such domain indeed exists and it is wide enough. If we introduce the dimensionless radius of the stationary membrane x as

$$\frac{k\gamma}{c^2} R_{min} = x\,, \tag{C.3.12}$$

then one can check directly that the first five of the above formulated conditions will be satisfied under the following three constraints:

$$x < 1 \tag{C.3.13}$$

$$M = \frac{c^4}{k^2\gamma}(3x^2 - 2x^3) \tag{C.3.14}$$

$$Q^2 = \frac{c^8}{k^3\gamma^2}(4x^3 - 3x^4) \tag{C.3.15}$$

The last two of these relations are just the equations (C.3.7) and (C.3.8) but written in the form resolved with respect to M and Q^2.

The formulas (C.3.13)-(C.3.15) shows that for the first five conditions it is convenient to take $x < 1$ and γ as independent parameters, and then to calculate the mass and charge necessary to obtain the model we need.

As for the last constraint (C.3.11) it gives some restriction also for parameter γ:

$$k\gamma^2 << \frac{x}{4 - 3x}\mathcal{E}_{cr}^2\,. \tag{C.3.16}$$

The energy density ϵ for the stationary state at $r_0 = R_{min}$, expressed in terms of parameters x and γ, is:

$$\epsilon = \frac{\gamma c^2}{8\pi}(1 + \sqrt{x^2 - 2x + 1})\delta(r - R_{min}). \tag{C.3.17}$$

C.4. Summary

1. We showed that exists a possibility to have a spherically charged membrane in stable stationary state producing RN repulsive gravitational force outside its surface and having flat space inside. To construct such model one should take a pair of constants $0 < x < 1$ and $\gamma > 0$ satisfying the inequality (C.3.16) and calculate from (C.3.12) and (C.3.14)-(C.3.15) the membrane's radius R_{min}, total mass M and charge Q.

2. The equation of motion (C.2.7) can be used also for the description of the oscillation of the membrane in the potential well ABC (see fig.1) above the equilibrium point C. If we slightly increase the total membrane's energy Mc^2

then the potential $U(r_0)$ around its minimum (i.e. the point C and its vicinity) will be shifted slightly down but he level "minus four" in Eq.(C.3.4) on which the system lives will remain at the same position. Then the membrane will oscillate between the new shifted walls AC and CB.

3. It is easy to see that in the general dynamical state the membrane can live only inside the potential well ABC. All regions outside ABC are forbidden. In the region to the right from the point $R_{max}^{(2)}$ and above the potential $U(r_0)$ any location of the membrane is impossible due to the fact that inequality (C.2.13) is violated there.

This means that a membrane of considered type in principle can not have the radius (no matter in which state) greater than $R_{max}^{(2)}$. In turn for $R_{max}^{(2)}$ it is easy to obtain from the potential (20) the upper limit $R_{max}^{(2)} < \frac{c^2}{k\gamma}\left(\frac{4k^2\gamma M}{c^4}\right)^{1/3}$.

The same violation of the inequality (C.2.13) take place in the domain between $R_{max}^{(1)}$ and $R_{max}^{(2)}$ and above the segment AB. The motion in the region to the left from the point $R_{max}^{(1)}$ and above the curve $U(r_0)$ is forbidden again due to the same violation of the condition (C.2.13). This means that a membrane of considered type in principle can not have the radius less than $R_{max}^{(1)}$. In particular there is no way for a membrane with positive effective rest mass μ to collapse to the point $r_0 = 0$ leaving outside the field corresponding to the RN naked singularity solution. This conclusion is in agreement with the main result of the paper (69).

4. Although we claimed that the stationary state of a membrane constructed is stable this stability should be understood in a very restrict sense, that is as stability in the framework of the dynamics described by the equation (C.2.7). We do not know what will happen to our membrane after the whole set of arbitrary perturbations will be given.

5. In general the arbitrary perturbations will change also the equation of state. We investigated a membrane with equation of state $\epsilon = \tau$. However this case can be considered only as "bare" Nambu-Goto membrane, by other words as a toy model. In the papers (65; 66; 70; 71; 72; 73) it was shown that arbitrary perturbations essentially renormalize the form of the equation of state of the strings and membranes. Moreover for the membranes (66) (differently from the strings) the fixed points of the renormalization group for the transverse and longitudinal perturbations does not coincide, which means that for the general "wiggly" membrane there is no equation of state of the type $\epsilon = \epsilon(\tau)$ at all.

6. We also would like to stress that for appearance of repulsive force the presence of electric field is of no principal necessity. For example the repulsive gravitational forces arise also in neutral viscous fluid (53) and in the course of interaction between electrically neutral topological gravitational solitons (74).

7. From the conditions (21)-(26) also follows that in addition to the inequal-

ity (25) the radius R_{min} of the shell in the stable stationary state cannot be less than $\frac{Q^2}{2Mc^2}$. A simple analysis shows that there is no way for R_{min} to be arbitrarily small keeping some finite non-zero value for M and Q.

D. Intersections of self-gravitating charged shells in a Reissner-Nordstrom field

The mathematical model that we have analyzed in the paper submitted to IJMPD, Ref.(75), describes the dynamic evolution of two spherical shells of charged matter which freely move outside the field of a central Reissner-Nordstrom (RN) source. Microscopically these shells are assumed to be composed by charged particles which move on elliptical orbits with a collective variable radius. The angular motion, distributed uniformly and isotropically on the shell surfaces, is mathematically described by a tangential-pressure term in the energy momentum tensor of the Einstein equations. The definition of the shell implies that all the particles have the same following three ratios: energy/mass, angular momentum/mass, and charge/mass. Thus, if at the beginning the particles are on the same radius $r_a = R_0$, then the shell will evolve "coherently", i.e. all particles will evolve with the same radius. This work is a direct generalization to the electric case of Barkov-Belinski-Bisnovati-Kogan paper.

The problem we were interested in was to find the exchange of energy between the two shells after the intersection. Indeed the motion of the shells before and after the intersection can be easily deduced from the equation of motion for just one shell, which equation has been found many years ago by Chase with a geometrical method first used by Israel. All these authors used the extrinsic curvature tensor and the Gauss-Codazzi equations. However we followed a different way, finding the solution by using δ and θ distributions and then by direct integration of the Einstein-Maxwell equations. This method has the advantage of a clearer physical interpretation, and it is also straightforward in the calculations.

What we concretely achieved in this paper is the determination of the constant parameters after the intersection knowing just the parameters before the intersection. Actually the unknown parameter is only one, the Schwarzschild mass parameter measured by an observer between the shells after the intersection. This parameter is strictly related to the energy transfer which takes place in the crossing, and it is found joining in a proper way the intervals inside, outside, and on the shell

$$\begin{aligned}
-(ds)^2_{in} &= -e^{T(t)} f_{in}(r) c^2 dt^2 + f_{in}^{-1}(r) dr^2 + r^2 d\Omega^2 \\
-(ds)^2_{out} &= -f_{out}(r) c^2 dt^2 + f_{out}^{-1}(r) dr^2 + r^2 d\Omega^2 \\
-(ds)^2_{on} &= -c^2 d\tau^2 + r_0(\tau)^2 d\Omega^2
\end{aligned}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

$$f_{in} = 1 - 2\frac{G\, m_{in}}{c^2 r} + \frac{G e_{in}^2}{c^4 r^2}\,, \quad f_{out} = 1 - 2\frac{G\, m_{out}}{c^2 r} + \frac{G(e_{in}+e)^2}{c^4 r^2};$$

and imposing a proper continuity condition on the shells velocities.

In the model we assumed that the emission of electromagnetic waves is negligible, and that there are no other interactions between the two shells apart the gravitational and electrostatic ones. In particular the shells, during the intersection, are assumed to be "transparent" each other (i.e. no scattering processes). In this paper we dealt only with the mathematical aspects of the problem; astrophysical applications will be considered elsewhere.

The main formula we achieved is the equation of the energy exchange between the two charged crossing shells:

$$\Delta E = -\frac{e_1 e_2}{r_*} - \frac{G M_1 M_2}{r_*} \left\{ \frac{v_1 v_2/c^2 - 1}{\sqrt{1 - v_1^2/c^2}\sqrt{1 - v_2^2/c^2}} \right\}_{r=r_*} .$$

Then we have used this formula to study the ejection-mechanism: indeed starting with two bounded shells, it is possible (thanks to the energy-exchange) that one shell is kicked out to infinity. We also considered special cases of physical interest in which the formulas simplify: the non relativistic case, the massless shells, the test shell, and finally the ejection mechanism in a semi-Newtonian regime. We found that the ejection mechanism is more efficient in the charged case than in the neutral one if the charges have opposite sign, because the energy transfer is larger due to the Coulomb interaction.

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