On the reconstruction of relative motion of components of a binary star in gravitational field of supermassive black hole from its redshift

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Observational evidences have shown that a supermassive black hole with a mass $M \approx 10^6 M_\odot$ exists in the center of our galaxy (Sgr A*).

The results of the observations of the stars in the innermost arcsecond (the so-called S-stars) of our galaxy give as possibilities for testing theories of gravity (the pericenter of the star S2 lies within the 0,01″). The investigation of the redshift of received light can be used for this purpose.

1″ correspond to 0,04 pc $\approx 400000 \cdot M$. 
Introduction

- The astrophysical observations give also evidences for the existence of the X-ray pulsars in the closed vicinity of the Galactic Center. This gives us possibilities to study gravity in the strong field regime.
- In the case of pulsars the pulse arrival time analysis can be used for the purpose of testing theories of gravity. Mathematically this equivalent to studying time evolution of redshift of their light $z$.

Figure. Chandra image of the Galactic Center region. The outer and inner yellow circles correspond to 1 pc and 0.2 pc from the Galactic Center.

The purpose of the present research is to work out method for the solution of inverse problem: unique determination of parameters of the motion of a binary star in external gravitational field using observational redshift data (for usual stars) or pulsar timing data (for pulsars).
\[ R_{ij} - \frac{1}{2} R g_{ij} = \frac{8\pi G}{c^4} T_{ij} \Rightarrow \]

\[ T^{ji}_{\ ;i} = 0 \quad \text{— equations of motion.} \]
Firstly, consider a point-like source in Kerr space-time and calculate the redshift \( z_0(\tau) \), where \( \tau \) is the proper time of the source. The isotropic and timelike geodesics can be found from the system of equations (with corresponding functions \( R(r) \) and \( \Theta(\cos \theta) \)):

\[
\frac{dr}{d\tau_M} = \pm \sqrt{R(r)} ;
\]
\[
\frac{d\cos \theta}{d\tau_M} = \pm \sqrt{\Theta(\cos \theta)} ;
\]
\[
\frac{d\phi}{d\tau_M} = \Phi(r, \cos \theta) ;
\]
\[
\frac{dt}{d\tau_M} = T(r, \cos \theta).
\]

Where \( \{r, \theta, \phi, t\} \) are the Boyer-Lindquist coordinates, \( \tau_M = \int \frac{d\tau}{r^2 + a^2 \cos^2 \theta} \), where \( \tau \) is the proper time of the source (for timelike geodesic), or an affine parameter (for isotropic geodesic) is the Mino time. The obtained system of equations can be solved analytically in terms of elliptic integrals.
Time of arrival of the pulses and the redshift of the received light

\[ z = \frac{\delta \lambda}{\lambda} = \frac{(k^i u_i)_s}{(k^i u_i)_o} - 1. \]

\[ t_{TOA}^{(N)} = t_{TOA}^{(N-1)} + T_p(z + 1) = t_{TOA}^{(N-1)} + T_p \frac{(k^i u_i)_s}{(k^i u_i)_o}. \]

Here \( T_p \) — is the pulsar period in the reference frame of the pulsar, \( z \) is the redshift, \( t_{TOA}^{(j)} \) — is the time of arrival of the \( j \) th pulse.
Comoving reference frame

\[ X^{(\alpha)} = h(\alpha)_{\beta} \eta^i \sigma. \]

\[ g_{ij} = \eta(i)(j) + \varepsilon(i)(j) + \phi(i)(j). \]

where \( \eta(i)(j) = \text{diag}(1, 1, 1, -1) \) and

\[ \varepsilon_{(\alpha)(\beta)} = -\frac{1}{3} R^{(\alpha)(\mu)(\beta)(\nu)} X^{(\mu)} X^{(\nu)} + O(\varrho^3), \]

\[ \varepsilon_{(4)(4)} = -\left(2\Theta + 2\zeta + \zeta^2 - \frac{1}{c^2} K(\alpha) K(\alpha)\right) + O(\varrho^3). \]

\[ K(\alpha) = \varepsilon_{(\alpha)(\kappa)(\tau)} X^{(\tau)} \omega^{(\kappa)}, \quad \zeta = \frac{1}{c^2} \mathcal{W}(\alpha) X^{(\alpha)}, \]

\[ \Theta(\alpha) = \frac{2}{3} R^{(\alpha)(\mu)(\nu)} X^{(\mu)} X^{(\nu)} + O(\varrho^3), \]

\[ \Theta = \frac{1}{2} R^{(4)(\mu)(4)(\nu)} X^{(\mu)} X^{(\nu)} + O(\varrho^3). \]
The equations of motion

Equations of motion of the center of mass

\[
(m_1 + m_2) \frac{dV^{(\kappa)}}{dT} = (m_1 + m_2)(2\varepsilon^{(\kappa)}_{(\mu)(\nu)} V^{(\mu)} \omega^{(\nu)} - 2cR^{(\kappa)}_{(\nu)(\mu)(4)} X^{(\mu)} V^{(\nu)} + 2D^{(\kappa)}_{(\nu)} X^{(\nu)} - W^{(\kappa)}) - 2mcR^{(\kappa)}_{(\nu)(\mu)(4)} x^{(\mu)} v^{(\nu)}.
\]

(3a)

Equations of the relative motion of the components of the binary

\[
\frac{dv^{(\kappa)}}{dT} = \left( \frac{G(m_1 + m_2)}{r} \right)^{\left(\kappa\right)} - 2\varepsilon^{(\kappa)}_{(\alpha)(\tau)} \omega^{(\alpha)} v^{(\tau)} - \frac{2c(m_2 - m_1)}{(m_1 + m_2)} R^{(\kappa)}_{(\nu)(\mu)(4)} x^{(\mu)} v^{(\nu)} + 2D^{(\kappa)}_{(\nu)} x^{(\mu)} + \frac{2}{3} c (R^{(\alpha)}_{(\mu)(\kappa)(4)} + R^{(\alpha)}_{(\kappa)(\mu)(4)})(X^{(\mu)} v^{(\alpha)} + x^{(\mu)} V^{(\alpha)}).
\]

(3b)

The equations (3a) and (3b) are given in article:

Formula for the calculation of the redshift of light of an extended object

\[ 1 + z = (1 + z_0) \left[ 1 - \frac{d}{d\tau} (n_i \eta^i \varrho) - \frac{1}{2} \frac{d}{d\tau} (n_i ; j \eta^i \eta^j \varrho^2) + \frac{1}{2} (n_i \eta^i)^2 \frac{d}{d\tau} \left( \frac{1}{z_0 + 1} \frac{dz_0}{d\tau} \right) \varrho^2 - \right. \\
\left. \frac{1}{2} R_{lijm} u^i_C u^j_C \eta^l \eta^m \varrho^2 \right] + O(\varrho^3, \varrho^2 \nu, \nu^2). \]  

(4)

Here \( n^i = \frac{k^i}{\omega_0(z_0 + 1)} \) and \( n_i \eta^i \varrho = n(\alpha) X(\alpha) \).

\[ t = \int_0^{\tau_S} (z(\tau') + 1)d\tau'. \]  

(5)

Where

\[ \frac{d\tau}{d\tau_S} = 1 - R_{lijm} u^i_C u^j_C \eta^l \eta^m \varrho^2 + O(\varrho^3, \varrho^2 \nu, \nu^2). \]  

(6)

Numerical results of the calculation of redshift for a model binary star

Figure: The redshift as a function of the proper time of the source. The chosen parameters of the motion are:
- Kerr parameter $a = 0, 1M$
- Angular momentum per unit mass, $L = 4, 9M$
- Carter constant per unit mass, $Q = 3, 0$
- Energy per unit mass, $E = 0, 984$
- Mass of the source, $m_1 = 8, 89 \cdot 10^{-7}M$
- Mass of the companion star, $m_2 = 4, 45 \cdot 10^{-7}M$
- Initial relative distances, $x^i = \{0; 0, 01M; 0\}$
- Initial relative velocities, $v^i / Mc = \{0, 004; 0; 0, 003\}$.
The existing methods for the solution of inverse problem mainly based on a statistical approach. The goal is minimizing the statistic $\chi^2$:

$$\chi^2 = \chi^2_P + \chi^2_N; \quad \text{where}$$

$$\chi^2_P = \sum_{j=1}^{N} \left[ \frac{(\alpha_j - \alpha_{\text{obs},j})^2 + (\beta_j - \beta_{\text{obs},j})^2}{\sigma^2_P} \right]; \quad \chi^2_Z = \sum_{j=1}^{N} \left[ \frac{(z_j - z_{\text{obs},j})^2}{\sigma^2_P} \right],$$

The alternative approach

Using the obtained analytical results it is possible to split redshift into two parts approximately.

\[ c \int_{T_1}^{T_2} \left( \frac{1 + z(\tau_P)}{f(t(\tau_P))} - 1 \right) d\tau_P \approx n(\alpha) X_1(\alpha) \bigg|_{T_1}^{T_2} + c(T_2 - T_1) \cdot O \left( \frac{\rho^2}{M^2}, \frac{v^2}{c^2} \right). \quad (8) \]

The application of the well-known Newtonian methods to the part that connect to relative motion gives possibilities to solve inverse problem for the relative motion of the components.
Our results

**Table:** The model parameters and the reconstructed from the solution of inverse problem parameters. The reconstruction is performed for the motion of the center of mass of the system for the interval of time $60 < \tau < 73$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Model value</th>
<th>Reconstructed value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eccentricity, $e$</td>
<td>0.68</td>
<td>0.58</td>
</tr>
<tr>
<td>Period of relative motion, $T$</td>
<td>20.268 $Mc^{-1}$</td>
<td>20.264 $Mc^{-1}$</td>
</tr>
<tr>
<td>Mass function, $M_2$</td>
<td>0.0062 $M^{1/3}$</td>
<td>0.0066 $M^{1/3}$</td>
</tr>
<tr>
<td>Pericenter longitude, $\omega_r$</td>
<td>1.56 rad</td>
<td>1.32 rad</td>
</tr>
<tr>
<td>Orbital inclination, $i$</td>
<td>1.55 rad</td>
<td>1.31 rad</td>
</tr>
<tr>
<td>Position angle, $\zeta$</td>
<td>1.72 rad</td>
<td>3.05 rad</td>
</tr>
</tbody>
</table>
Thank you for your attention!