Creation and Evolution of Cosmological Horizon in FLRW Universe with Wormhole

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Outline

• Motivation
• Wormhole cosmological models
• Wormhole cosmology & apparent horizons
• Past light cone & particle horizon
• Summary
Motivation

• Black hole cosmological models are active in accelerating universe.
• More general & practical solutions are required.
• Finding exact solutions is a kind of unification of local and global physics.
• Wormhole cosmology plays the same role as black hole cosmology in many reasons.
• Cosmological wormhole can be one of the important points relating to issues on dark energy and gravitational wave sources in accelerating universe.
Creation of wormhole

• Quantum fluctuation & quantum foam
• Topology change
Models

• **Black Hole Cosmology**
  - Kottler (1918): SdS
  - McVittie (1933)
  - Sultana-Dyer (2005)
  - Faraoni-Jacques (2007)

• **Wormhole Cosmology**
  - Hochberg, Kepart (1993)
  - Roman (1993)
  - SWK (1996)
  - Mirza, Eshaghi, Dehdashti (2006)
Black hole cosmology solutions

• McVittie

\[ ds^2 = - \left(1 - \frac{m}{2u} \right)^2 \left(1 + \frac{m}{2u} \right)^4 \left(dt^2 + e^{\beta(t)}(1 + m/2u)4(dr^2 + r^2d\Omega^2)\right) \]

\[ u = re^{\beta/2} \]

\[ m = 0 \rightarrow \text{FLRW universe, } \beta = \text{const.} \rightarrow \text{Schwarzschild} \]

• Schwarzschild-de Sitter

\[ ds^2 = - \left(1 - \frac{2M}{r} + \Lambda r^2 \right) dt^2 + \left(1 - \frac{2M}{r} + \Lambda r^2 \right)^{-1} dr^2 + r^2 d\Omega^2 \]

Two horizons: cosmological horizon + black hole horizon
# Wormhole cosmological models

<table>
<thead>
<tr>
<th>Author</th>
<th>Year</th>
<th>Cosmology</th>
<th>Wormhole</th>
<th>Main Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hochberg, Kepart</td>
<td>1993</td>
<td>FRW</td>
<td>Visser-type</td>
<td>Horizon problem</td>
</tr>
<tr>
<td>Roman</td>
<td>1993</td>
<td>Inflation</td>
<td>MT-wormhole</td>
<td>Wormhole throat expands with $a(t)$</td>
</tr>
<tr>
<td>SWK</td>
<td>1996</td>
<td>FRW</td>
<td>MT-wormhole</td>
<td>Wormhole throat expands with $a(t)$</td>
</tr>
<tr>
<td>Li-Xin Li</td>
<td>2001</td>
<td>Two open universes</td>
<td>Visser-type</td>
<td>Many solutions</td>
</tr>
<tr>
<td>González-Diaz</td>
<td>2003</td>
<td>Acceleration with quintessence</td>
<td>MT-wormhole</td>
<td>Expands with phantom energy Avoid big rip through a short-cut</td>
</tr>
<tr>
<td>Mirza, Eshaghi, Dehdashti</td>
<td>2006</td>
<td>FRW</td>
<td>MT-wormhole</td>
<td>Positive $\rho$ for asymptotically de Sitter spacetime</td>
</tr>
<tr>
<td>Bochiccio &amp; Faraoni</td>
<td>2010</td>
<td>Lemaitre-Tolman-Bondi</td>
<td>shell</td>
<td>$t \rightarrow \infty$, shell becomes comoving with dust-dominated cosmic substratum.</td>
</tr>
<tr>
<td>Ebrahimi &amp; Riazi</td>
<td>2010</td>
<td>BD cosmology</td>
<td>MT-wormhole</td>
<td>N+1 dim solution</td>
</tr>
</tbody>
</table>
\[ ds^2 = e^{2\Phi} dt^2 - \frac{1}{1 - b(r)/r} dr^2 - r^2 d\Omega^2, \quad (MT, 1988) \]

1. \[ ds^2 = e^{2\Phi} dt^2 - e^{Ht} \left( \frac{dr^2}{1 - b(r)/r} + r^2 d\Omega^2 \right) \quad (Roman, 1993) \]

2. \[ ds^2 = e^{2\Phi} dt^2 - a(t) \left( \frac{dr^2}{1 - kr^2 - \frac{b(r)}{r}} + r^2 d\Omega^2 \right) \quad (SWK, 1996) \]

- Isotropic form of MT wormhole for \( b(r) = b_0^2/r^2 \)

\[ ds^2 = e^{2\Phi} dt^2 - a(t) \left( 1 + \frac{b_0^2}{4r^2} \right)^2 (dr^2 + r^2 d\Omega^2) \]

\[ ds^2 = e^{2\Phi} dt^2 - a(t) \left( \frac{1}{(1 + kr^2/4)^2} + \frac{b_0^2}{4r^2} \right)^2 (dr^2 + r^2 d\Omega^2) \]

(Mirza, Eshaghi, Dehdashti, 2006)
Isotropic Wormhole solution 1

- Morris-Thorne type wormhole

\[ ds^2 = e^{2\Phi} dt^2 - \frac{1}{1 - b(r)/r} dr^2 - r^2 d\Omega^2, \]

- Isotropic form of the solution

\[ ds^2 = A^2 \tilde{d}t^2 - B^2 (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2). \]

\[ A(\tilde{t}, \tilde{r}) = e^{\Phi(r)}, \tilde{t} = t, B\tilde{r} = r \quad B = re^{-\int \frac{dr}{\sqrt{r^2 - b(r)r}}}. \]

- For simple case, \( A = 1, b = \frac{b_0^2}{r} \quad (r > b_0) \)

\[ ds^2 = d\tilde{t}^2 - \left(1 + \frac{b_0^2}{4\tilde{r}^2}\right)^2 (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2) \quad \left( \tilde{r} > \frac{b_0}{2} \right) \]
Isotropic Wormhole solution 2

Matter solutions

\[
\frac{b'}{8\pi r^2} = \rho_w = -\frac{b_0^2}{8\pi r^4} \\
\frac{b}{8\pi r^3} = \tau_w = \frac{b_0^2}{8\pi r^4} \\
\frac{b - b'r}{16\pi r^3} = P_w = \frac{b_0^2}{8\pi r^4}
\]

\[
T_{\mu \nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} T_{\alpha \beta}
\]

\[
G^{0}_0 = -\frac{4^4 b_0^2 \tilde{r}^4}{(b_0^2 + 4\tilde{r}^2)^4} = 8\pi \rho_w \\
G^{1}_1 = \frac{4^4 b_0^2 \tilde{r}^4}{(b_0^2 + 4\tilde{r}^2)^4} = 8\pi \tau_w \\
G^{2}_2 = -\frac{4^4 b_0^2 \tilde{r}^4}{(b_0^2 + 4\tilde{r}^2)^4} = 8\pi P_w \\
G^{3}_3 = -\frac{4^4 b_0^2 \tilde{r}^4}{(b_0^2 + 4\tilde{r}^2)^4} = 8\pi P_w.
\]

\(\rho_w, \tau_w, P_w\) are wormhole energy density, tension, and pressure.
Wormhole solution embedded in FLRW universe 1

- Isotropic form of FRLW universe ($\tilde{r} \rightarrow r$, when there is no confusion)

$$ds^2 = dt^2 - \frac{a^2(t)}{(1 + kr^2)^2}(dr^2 + r^2d\Omega^2)$$

$a(t)$ is the scale factor and $k = 1/4\mathcal{L}$ is the curvature.

- Let’s start from the metric form for wormhole in FRLW universe

$$ds^2 = e^{\zeta(r,t)}dt^2 - e^{\nu(r,t)}(dr^2 + r^2d\Omega^2)$$

- Assume the matter distribution as

- $\rho_c c^2 \sim 9 \times 10^{-10} \text{J/m}^3$ over the universe

- $\rho_w c^2 \sim 6 \times 10^6 \left(\frac{10 \text{m}}{r}\right)^4 \left(\frac{b_0}{10 \text{m}}\right)^2 \text{J/m}^3$ near throat

$$a(t)\rho(r,t) = a(t)\rho_c(t) + \rho_w(r)$$

$$a(t)p_1(r,t) = a(t)p_{1c}(t) + p_{1w}(r)$$

$$a(t)p_2(r,t) = a(t)p_{2c}(t) + p_{2w}(r)$$

$$a(t)p_3(r,t) = a(t)p_{3c}(t) + p_{3w}(r)$$
Wormhole solution embedded in FLRW universe 2

- Einstein field equation

\[
G^0_0 = -\frac{1}{4r} \left\{ \left[ (8\nu' + 4\nu''r + \nu'^2r) e^{-\nu} - 3\nu^2 r \right] e^{-\zeta} \right\}
\]

\[
G^0_1 = \frac{1}{2} (-2\nu' + \nu'\zeta') e^{-\zeta}
\]

\[
G^1_1 = -\frac{1}{2r} \left\{ \left[ r(-2\nu' + (-\frac{3}{2} \nu + \zeta')\nu) e^{-\nu} + 2\nu' + 2\zeta' + \zeta'\nu' + \nu' + \nu' + \frac{1}{2} r\nu'^2 \right] e^{-\nu} \right\}
\]

\[
G^1_0 = -\frac{1}{2} (-2\nu' + \nu'\zeta') e^{-\nu}
\]

\[
G^2_2 = G^3_3 = \frac{1}{4r} \left\{ [-2\zeta' - 2\nu' - 2\nu''r - 2\zeta''r - \zeta'^2 r] e^{-\nu} - 2r(-2\nu' - \frac{3}{2} \nu^2 + \zeta'\nu) e^{-\zeta} \right\}
\]

For the case of ultra-static observer, \( e^\zeta = 1 \)

- Exact solution is

\[
G^1_0 = 0 \quad \nu' = 0
\]

\[
\nu(r, t) = \alpha(t) + \beta(r) \quad \text{or} \quad e^{\nu(t, r)} = e^{\alpha(t)} e^{\beta(r)}
\]
Wormhole solution embedded in FLRW universe 3

• Solution for spatial part
  – Boundary conditions

\[ e^\beta(r) = \begin{cases} 
(1 + \frac{b_0^2}{4r^2})^2 & (k = 0 \text{ or } r \to b_0/2) \\
(1 + kr^2)^{-2} & (b_0 = 0 \text{ or } r \to \infty) 
\end{cases} \]

When we compare \( G_1 \) and \( G_2 \):

\[ 2\kappa(p_2 - p_1) = [\nu'' - \frac{1}{r} \nu' - \frac{1}{2}(\nu')^2]e^{-\nu} \]

\[ [\beta'' - \frac{1}{r}\beta' - \frac{1}{2}(\beta')^2]e^{-\beta} = 2\kappa a(t)(p_2 - p_1) = 2\kappa(p_{2w} - p_{1w}) \]

– Inhomogeneous differential equation

\[ \beta = \beta_c + \beta_w \]

\[ \beta_c = -2 \ln(kr^2 + 1) \]

\[ \beta_w = 2 \ln \left( 1 + \frac{b_0^2}{4r^2} \right) \]

\[ e^\nu(r,t) = e^{\alpha(t)+\beta(r)} = \frac{e^{\alpha(t)}}{(kr^2+1)^2} \left[ 1 + \frac{b_0^2}{4r^2} \right]^2 \]
Wormhole solution embedded in FLRW universe 4

• Final metric

\[ ds^2 = dt^2 - \frac{a^2(t)}{(kr^2 + 1)^2} \left( 1 + \frac{b_0^2}{4r^2} \right)^2 (dr^2 + r^2d\Omega^2) \]

• Discussions
  – Compare with the previous solution (Mirza, Eshaghi, Dehdashti, 2006)
  – Generalization from wormhole to cosmological wormhole

\[ \left( 1 + \frac{b_0^2}{4r^2} \right) \rightarrow w(t_1, r_1) \left( 1 + \frac{q(t_1)}{4r^2} \right) \]

\[ dl^2 = -dv^2 + a^2(v) \frac{1}{(1 + kx^2/4)^{m/2}} \left( \frac{b_m}{\sqrt{m}} \right)^{4/m} (dx^2 + x^2d\Omega^2) \]

  – Coupling term & Interaction: \[ \frac{1}{(1+kr^2)^2} \leftrightarrow \frac{b_0^2}{4r^2} \]

  – Inverse transformation
Apparent horizons 1

• With new coordinate

\[ R \equiv a \left( \frac{1 + \frac{b_0^2}{4r^2}}{1 + kr^2} \right) r = a(t) A(r) \]

\[ ds^2 = - \left( 1 - \frac{R^2 H^2}{r^2 J^2} \right) F^2 dT^2 + \frac{1}{r^2 J^2} \left( 1 - \frac{R^2 H^2}{r^2 J^2} \right)^{-1} dR^2 + R^2 d\Omega^2 \]

\[ \Delta \]

Case 1: \( b(t) < \frac{R_H}{2} \)
Two horizons
Case 2: \( b(t) = \frac{R_H}{2} \)
1 horizon
Case 3: \( b(t) > \frac{R_H}{2} \)
no horizon
Apparent horizons 2

- $k = 0$,

$$R = ar \left(1 + \frac{b_0^2}{4r^2}\right) = ar + \frac{ab_0^2}{4r}$$

$$r = \frac{R}{2a} \pm \sqrt{\left(\frac{R}{2a}\right)^2 - \left(\frac{b_0^2}{4}\right)}$$

$$R_\pm = \frac{1}{\sqrt{2H}} \left[1 \pm \sqrt{1 - (2b(t)H)^2}\right]^{1/2} < R_H = \frac{1}{H}$$

$$R_0 < R_- < R_+ < R_H$$

Misner-Sharp-Hermandez mass:

$$M_{\text{MSH}} = \frac{H^2}{2(1 - b^2/r^2)} R^3 = \frac{4\pi}{3} R^3 \rho_c \left[1 - (b/R)^2\right]^{-1}$$
Apparent horizon 3

In terms of matter distribution

\[ b_0^2 = \frac{1}{8\pi} \frac{1}{|\rho_w(r)|} \bigg|_{r=b_0/2} = \frac{1}{8\pi} \frac{1}{|\rho^p_w|} \]

\[ a^2 H^2 = \frac{8\pi}{3} \rho_0 \frac{2}{3(1+\omega)} c^\frac{1+3\omega}{3(1+\omega)} \]

\[ 1 - 4b_0^2a^2H^2 \simeq 1 - \frac{4}{3} \left( \frac{\rho_0}{|\rho^p_w|} \right) \geq 0 \]

\[ \rho^p_w \text{ is the peak of } \rho_w(r). \]

When \( H \) is very large or \( \rho_0 > \frac{3}{4} \rho^p_w \), no horizon appears.
Hawking Temperature 1

\[ ds^2 = - \left( 1 - \frac{R^2/(R_+^2 + R_-^2)}{1 - b^2/R^2} \right) dt^2 - \frac{2HR}{1 - b^2/R^2} dt dR + \frac{1}{1 - b^2/R^2} dR^2 + R^2 d\Omega^2 \]

- Kodama vector

\[ K^a = -\varepsilon^{ab} \partial_b R = \sqrt{1 - \frac{b^2}{R^2}} \left( \frac{\partial}{\partial t} \right)^a \]

- Hamilton-Jacobi equation

\[ \omega = -K^a \partial_a S = -\sqrt{1 - \frac{b^2}{R^2}} \partial_t S, \quad k_R = \left( \frac{\partial}{\partial R} \right)^a \partial_a S = \partial_R S \]

\[ \text{Im } S = \text{Im } \int -HR \pm \sqrt{H^2 R^2 + \lambda \left[ 1 - \frac{m^2}{\omega^2} (1 - b^2/R^2) \right]} \frac{R^2(R_+^2 + R_-^2)\omega}{\sqrt{1 - b^2/R^2(R_+^2 - R^2)(R^2 - R_-^2)}} \]

\[ = \pi R_+ \omega. \]

\[ T = \frac{1}{2\pi R_+} \]
Hawking temperature 2

- Radial null geodesic

\[ \dot{R} = HR \pm \sqrt{(HR)^2 + (1 - b^2/R^2 - R^2/(R_+^2 + R_-^2))}, \]

\[ \text{Im } S = \text{Im} \int_{R_i}^{R_f} p_R dR = \text{Im} \int_{R_i}^{R_f} dp'_R dR, \]

\[ \text{Im } S = \text{Im} \int_{R_i}^{R_f} dR \int d\hat{H} \frac{1}{\hat{R}} \]

\[ = \text{Im} \int_{R_i}^{R_f} dR \frac{\omega}{\hat{R} \sqrt{1 - b^2/R^2}} \]

\[ = -\omega \text{Im} \int_{R_i}^{R_f} \frac{dR}{\sqrt{1 - b^2/R^2}(\sqrt{(HR)^2 + (1 - b^2/R^2 - R^2/(R_+^2 + R_-^2))} - HR)} \]

\[ = \pi R_+ \omega. \]

\[ T = \frac{\omega}{2 \text{Im } S} = \frac{1}{2\pi R_+} \]
Wormhole Cosmology

- Friedmann equation (It does not change.)
\[ H^2 = \left(\frac{\ddot{a}}{a}\right)^2 = \frac{8\pi \rho_c}{3} \]

- Energy conservation, equation of state (parameter \( \omega \))
\[
a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3\omega)} \equiv \left(\frac{t}{t_0}\right)^{\frac{q}{3}}, \quad \omega \neq -1, \text{ present time } t_0
\]

\[
\frac{dR_H}{dt} = \frac{1}{q} = 1 + \xi, \quad \xi = -\ddot{a}a/\dot{a}^2 \text{ deceleration parameter}
\]

- Apparent horizons are
\[
R_{\pm} = \frac{t}{\sqrt{2|q|}} \left[ 1 \pm \sqrt{1 - (2q \mathcal{b}_r)^2 \left(\frac{t}{t_0}\right)^{2(q-1)}}\right]^{1/2}
\]
\[
\mathcal{b}_r = \frac{b_0}{t_0} \quad \text{relative wormhole size, } t_0: \text{ present time, } a(t_0) = 1
\]
Creation & evolution

- Creation at

\[ t_c = (2qb_r)^{(1-q)} t_0 \] coincidence time

\[ R_c = \frac{(2qb_r)^{1/(1-q)}}{\sqrt{2q^2}} t_0 = \frac{t_c}{\sqrt{2q^2}} \]

- Wormhole shrinks, has minimum value, and expands

\[ t_{min} = [\sqrt{q}(1+q)b_r]^{(1-q)} t_0 \] minimum size time of \( R_- \)

\[ \frac{t_{min}}{t_c} = \left( \frac{(1+q)^2}{4q} \right)^{1/2(1-q)} > 1 \]

\[ R_{\text{min}} = \frac{1}{\sqrt{q(1+q)}} \frac{t_{min}}{t_0} \]
**Matter-dominated universe**

\[ q = \frac{2}{3}, \quad a(t) = \left(\frac{t}{t_0}\right)^{2/3} \]

\[ t_{mc} = \left(\frac{4b_r}{3}\right)^3 t_0, \quad b(t) = b_0 \left(\frac{t}{t_0}\right)^{2/3} \]

\[ R_c = \frac{16}{9} \sqrt{2} b_r^3 t_0, \quad R_{min} = \frac{50}{81} \sqrt{15} b_r^3 \]

\[ \lim_{t \to \infty} \frac{dR_+}{dt} = \pm \infty \]

\[ \lim_{t \to t_c} \frac{dR_+}{dt} = \frac{1}{|q|} \]

\[ \lim_{t \to \infty} \frac{dR_-}{dt} = b_0 q \left(\frac{t}{t_0}\right)^{(q-1)} = b_0 \dot{a} \]
$\Lambda$-dominated universe

\[ a(t) = a_0 e^{H_0 t}, \quad H = H_0 = \text{const.} \]

\[ t_{\Lambda c} = t_0 - \frac{\ln(2H_0b_0)}{H_0}, \quad b(t) = b_0 a_0 e^{H_0 t}, \quad R_c = \frac{1}{\sqrt{2}H_0} \]
Matter & $\Lambda$-dominated universe I

- Friedmann eq. \( \frac{H^2}{H_0^2} = \frac{\Omega_m}{a^3} + \Omega_{\Lambda_0} \)
- Redshift \( \frac{a_0}{a} = (1 + z) \)
  \[ H(z) = H_0 \sqrt{\Omega_m (1 + z)^3 + \Omega_{\Lambda_0}} \]
- Coincidence time is the solution to the equation
  \[ \Omega_m (1 + z)^3 - \left( \frac{1}{2b_0 H_0} \right)^2 (1 + z)^2 + \Omega_{\Lambda_0} = 0 \]
- In the limit of \( 1 + z \to \infty \) (near origin)
  \[ (1 + z) \approx \left( \frac{1}{2b_0 H_0} \right)^2 \frac{1}{\Omega_m} \Rightarrow t_{mc} = \left( \frac{4b_r}{3} \right)^3 t_0 \]
- In the limit of \( 1 + z \to 0 \)
  \[ (1 + z) \approx (2b_0 H_0) \sqrt{\Omega_{\Lambda_0}} \Rightarrow t_{\Lambda c} = t_0 - \ln(2H_0t_0b_r) / H_0 \]
Matter & $\Lambda$-dominated universe II

\[
H_0 = \frac{1}{14} \\
\Omega_{m0} = 0.3 \\
\Omega_{\Lambda0} = 0.7
\]

2 critical times
Near $z = -1$ and $\infty$
Particle horizon

• The maximum distance from which particles could have traveled to the observer at the present time

• Proper distance

\[ d_p = a(t) \int_0^r \left( 1 + \frac{b_0^2}{4r^2} \right) dr = a(t)r \left( 1 - \frac{b_0^2}{4r^2} \right) \equiv a(t)r_* \]

• Null radial geodesic

\[ \int \frac{dt}{a} = \int_0^r \left( 1 + \frac{b_0^2}{4r^2} \right) dr = r_* \quad \text{Proper distance} \]

\[ \frac{dt}{a} = - \frac{dz}{H(z)} \]
------ case without wormhole
_____ case with wormhole
Past light cone

\[ \tau = \int \frac{dt}{a} \]

\[ r \left(1 - \frac{b_0^2}{4r^2}\right) \]

Without wormhole

With wormhole

past light cone in the universe without wormhole

past light cone in wormhole universe
$\Lambda$CDM model without wormhole
Penrose diagram \((k = \omega = 0)\)
Horizon problem

FIG. 1. A spacetime diagram representation of a Friedman cosmology with flat \((E^3)\) spatial sections, with the past light cone for an observer at \(A\). Radiation received at \(A\) from opposite directions was emitted from \(B\) and \(C\). A wormhole connects events in the past light cones of the otherwise spatially separated points \(B\) and \(C\). A photon is shown going down the wormhole in \(B\)’s past and exiting the wormhole in \(C\)’s past.

(Hochberg & Kepart, 1993)
Summary & future works

- Reviews on wormhole-cosmology & new exact solution
- Time-dependence of the horizons for cosmological matter distribution
  - Apparent horizon existence
    - Matter-dominated universe
      - Appear at early coincidence times
    - Lambda-dominated universe
      - Disappear at late coincidence times
    - Lambda-CDM universe
      - two coincidence times (at early times and late times)
  - Particle horizon: hide the region near origin
  - Light cone: narrower than the case of no wormhole
  - Light cone from other universe & same universe
  - Horizon problem
- In future,
  - the more general solution will be studied, such as model with accretion or curvature, etc.
• Discussion
  – Stability
  – Matter interaction or mixing?
  – Why no horizon before \( t_c \) for matter-dominated universe?
    • Near throat highly concentrated: \( \rho_c \ll \rho_w \)
      \[
      1 - 4b_0^2a^2H^2 \approx 1 - \frac{4}{3} \left( \frac{\rho_{c0}}{|\rho_{w}^p|} \right) \geq 0
      \]
    • Before \( t_c \), \( \rho_{c0} \) dominates \( |\rho_{w}^p| \) near throat
  • Perturbation of background (in preparation)
    – Radial inflow through non-zero \( G_0^1 \) by background perturbation