A sequence of four works involving exact solutions of the Einstein-Maxwell equations

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Abstract

The thesis is divided into four chapters, which are based on a sequence of four independent works. Each of them involves an exact solution of the Einstein-Maxwell equations; they are concatenated in a logical way.

The first chapter is devoted to a special 2-soliton solution, constructed by a Schwarzschild metric “dressed” with a Kerr-Newman soliton. We drowned the electric force lines, and a special configuration is considered with a negligible conical singularity.

The second chapter is dedicated to the analysis of the Alekseev-Belinski solution, which is again a special case of a 2-soliton solution. We considered the different equilibrium configurations, stressing the presence of non-Newtonian cases: also two opposite-signed charges can be in equilibrium, due to the repulsive nature of the naked singularity.

The third chapter is devoted to construct a membrane-model of naked singularity which avoids the central singularity of the source and allows an external region with “repulsive-gravity”. We found that a radially-stable configuration with a radius smaller than $Q^2/M$ is indeed possible. This gives a more sensible physical meaning also to the Alekseev-Belinski solution.

Using the same technique of the previous work, in the last chapter we consider the motion, and in particular the intersection, of charged shells. This is a generalization of an article by Barkov-Belinski-Bisnovatji-Kogan to the electric case. We give the energy-exchange formula, finding that the ejection mechanism can be magnified in presence of charges.
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Introduction

In this thesis I will discuss a sequence of four works which use in different ways exact solutions of the Einstein-Maxwell field equations. They are four independent works, indeed each one is self-consistent, but they constitute also a “suite” because one calls the other, as it will be clear in the next.

The original material of this thesis has been published during the three years of my IRAP Ph.D. fellowship in the Refs: [1, 2, 3, 4], and in the related proceedings [5, 6, 7]. When I was not the only author, we worked in parallel, confronting the results, and discussing what to exclude, what to include, and how to explain it — therefore it is not possible to define flatly the personal contributions.

Now, let us give just few methodological remarks related to our work. The first question one may ask is “Why to study exact solutions of the EM equations?”. In theoretical physics there are in general two different way to proceed: one is to consider a real problem, which in general will be very complicate, and by making simplifications, approximations and also numerical tests, try to make some prediction. This is indeed a very important way to proceed in physics and it has carried to very important results — e.g. Enrico Fermi can be regarded as a master in this way of working.

The other way is to proceed more theoretically. When one has a very well established theory (as Electromagnetism and General Relativity are, at least for internal coherence) then one can take it as “true” and analyze what
it exactly predicts in some particular cases. This is what one does studying an exact solution in General Relativity. Then one can also construct mathematical toy models, which can be considered as an idealization of some real situation, and to study in deep them, trying to extract from them some aspects which eventually will describe some relevant feature of a real situation. This is, e.g., what we did in the work on the charged membrane. The advantage in this case is that the toy models are simple to study — for instance, the equations of a charged membrane are simple enough to be solved exactly.

There is also another aspect that has to be underlined, and it is the point of view to adopt studying a mathematical model or an exact solution. As Feynman said: <<The physicist is always interested in the special case; he is never interested in the general case. He is talking about something; he is not talking abstractly about anything. He wants to discuss the gravity law in three dimensions; he never wants the arbitrary force case in n dimensions. So a certain amount of reducing is necessary, because the mathematicians have prepared these things for a wide range of problems. This is very useful, and later on it always turns out that the poor physicist has to come back and say, ‘Excuse me, what you wanted to tell me about four dimensions...?’>> [8].

Our point of view was the one of physicists and not of mathematicians. E.g. we were not interested to discuss the general two-soliton solution with 12 parameters, four of which are unphysical, but to consider the case in which there are only the physical ones. And we did not focus on the mathematical
aspects of the derivations (except the essential ones), while we used the references to address the reader interested in deeper technical details. For the same reason we were not interested in other unphysical strangeness which are commonly encountered in exact solutions of EM equations, as branch points which lead to “other universes”.

In this sense the first work [1, 5] can be considered as an unsuccessful attempt, or, less severely, just a first step toward a really physically-well-behaved solution. Originally the aim was to construct a two-soliton solution representing a Schwarzschild black hole near a Kerr-Newman source in equilibrium without conical singularity, neither other unphysical features like NUT parameter or magnetic charge. In order to do that we dressed the Schwarzschild solution with one soliton using the Inverse Scattering Method (ISM) [9, 10, 11, 12]. However at the end we found that the conical singularity can not to be avoided, and all the unpleasant parameters reappear due to a non-linear effect which is explained in Chapter 1. Anyway, we found a configuration with the Kerr-Newman source much smaller than the Schwarzschild black hole, in which all the unphysical parameters can be putted to zero and the conic singularity is negligible; then we drowned the electric force lines of this configuration finding a result similar to the Hanni-Ruffini [13] one. We stressed also that this plots can give a more easy-to-look understanding of these complicate solutions, more than the twenty-line-long expressions of the metric components and of the electromagnetic potentials. Then, the fact that the conic singularity was negligible in this configuration was a further
clue, beyond the ones already present in literature [14, 15, 16], that such equilibrium could be described by an adequate exact solution.

That solution was indeed achieved (in a little bit simpler case) few months later by Alekseev and Belinski [17, 18, 19]. They finally achieved the task, which was pursued by many researcher in the last 20 years, to find a double Reissner-Nordstrom (RN) solution without conic singularity. They achieved this result by using a different technique which is called Integral Equation Method (IEM) [20, 21]. This method allows a more convenient (with respect to the ISM) parametrization, in particular it allows to construct spinningless sources, i.e. RN instead of Kerr-Newman. Using this new solution, Paolino and I [2] analyzed the different configurations permitted by it; many of these configurations are forbidden by Newtonian physics, and we found that the equilibrium is possible also with opposite charged sources. Finally, with the same technique already used in [1], we plotted the related electric force lines. Furthermore we investigated the stability of the solution with respect to quasi-statical perturbations of the distance between the two bodies [4]. The Alekseev-Belinski solution (AB) results to be stable in the most of the cases, in agreement with what one could expect from the Bonnor analysis on the test particle on a RN [14].

The ensuing work (Chapter III) is about a membrane model (i.e. a thin shell with tension) for a naked singularity [3]. The link with the AB solution is very natural, and it is due to the fact that, there, the equilibrium is allowed only by the presence of a naked singularity and to its repulsive region near
the center. Therefore, in order to give a more sensible physical meaning to this solution there was the need to see if it was possible to construct a model, at least a toy model, for a RN naked source with a radius smaller than $Q^2/M$, i.e. with the repulsive zone outside the matter. We found that this is indeed possible, using a charged membrane with a dark-matter-like equation of state, with $\epsilon = -p$. Obviously this is a very simple model, and it does not resolve all the problems concerning the naked singularity. e.g. how to arrive to such a configuration. Anyway it gives a hint that such a configuration is physically possible — and we know that something like that could exist because the electron has the parameters corresponding to a “naked singularity”. Furthermore the configuration we studied is stable with respect to radial perturbations (in our analysis we restricted only to the realm of classical mechanics, since Einstein equations works only there).

The last work, Ref.[4], is again about a shells-made model, but in this case, instead of a static membrane, we considered the motion of two intersecting shells with positive tangential pressure and charge; for completeness we considered also the presence of a central charge singularity. The motion of each shell is independent from the other until they intersect — indeed until the intersection the outer shell feels the inside shell as a simple RN source. The motion of a single charged shell was known from the Chase’s paper [22], however we re-derived the same formulas using a more-familiar-for-physicist method, which uses the $\delta$-shaped sources. Then we obtained the exchanging energy formula between the two shells due to their intersection. Finally
we described the ejection mechanism, for which one of two bounded shells can acquire enough energy to be kicked out to infinity. This mechanism is present also in the neutral case (see ref. [23], of which our article is a direct generalization), but the effect is magnified by the presence of the charge, if the shells have opposite-signed charge.

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Chapter 1

A peculiar 2-soliton solution
This chapter is based on Ref.[1].

1.1 Introduction

From the ’80s there was a certain number of papers which studied the general 2-soliton solution (with twelve parameters) on the Minkowski background, (see Refs. [21]–[24]). However a physical detailed study of that family of solutions — which means the identification of the physical parameters, the asymptotic behaviour at infinity, the electrical force lines, and above all the regularity conditions on the axis — was still lacking in literature.

The present work can be considered a first step in that direction, above all respect to the study of the electric field which we investigated through the plotting of the force lines, following Ref.[13]. In order to find this solution of the Einstein-Maxwell equations we have used the Inverse Scattering Method (ISM) applied to the EM equations ([10, 11],[12]; and [9] for a self-consistent exposition) which allows to treat the problem in exact form. Although with that method it is possible to describe $N$ Kerr-Newman\(\footnote{More precisely Kerr-Newman-NUT-g sources, indeed they have also a magnetic charge and the NUT parameter.}^1\) sources aligned on the axis (in the general case of $N$-solitons), the most physically relevant configuration is described by the 2-soliton solutions, which should contain as a particular case also the equilibrium configuration of two Reissner-Nordstrom sources.

In particular we considered the most simple case of 2-soliton solution: it
is the case in which one adds one soliton on the Schwarzschild background. That solution is really a new solution which is not trivially included in the already studied family of 2-soliton solutions constructed adding two solitons on the flat background, because the solutions found in that way describe two naked singularities and nobody has yet showed how to reach the case with horizon, nor whether it is possible at all to reach it—maybe it is possible, but in every case it needs some complex and highly non-trivial transformation.

We found this new solution taking the spectral matrix associated to the Minkowski background, dressing it with one soliton and analytically prolonging it to the Schwarzschild case by putting to zero all the parameters but the Schwarzschild mass $m$. That procedure is equivalent to resolve the differential system associated to the Schwarzschild background, but in that way it is possible to avoid the integration and to proceed simply algebraically; the so-found spectral matrix allows to find the n-soliton solution on the Schwarzschild background.

The physical situation which we were interested in is the equilibrium condition of a Schwarzschild black hole near a Kerr-Newman (KN) singularity. However, unfortunately we found that the anomaly region between the two sources (which consists in a conic singularity and a ‘tube singularity’, i.e. $g_{\varphi\varphi}$ negative on the axis) is unavoidable, no matter the values of the parameters are[25]. That fact hampers an easy physical interpretation of the solution; usually people interpret this as a “strut”, or a “string” [26], between the two bodies.
Finally we plotted the force lines of the electric field for different values of the parameters. In both the general case and in the case in which the KN particle is much smaller respect to the black hole, we discovered that in spite of the naive interpretation suggested by the mathematical construction, the black hole of the Schwarzschild background acquires a charge. This is due to the fact that the non-linear superposition of the two solitons mixes and changes the physical interpretation of the mathematical parameters (parameter-mixing phenomenon) —e.g. if $e_1$ and $e_2$ are respectively the charges of the two solitons when they are very far each others, then, when they are nearby, the physical charges will be in general some different constants, say $Q_1$ and $Q_2$, which will be some complicated functions if expressed in terms of the old parameters. The expected meaning of the parameters is recovered only in the far distanced limit —it makes exception the mass parameter $m$ of the Schwarzschild black hole which maintains its original meaning also nearby.

1.2 Summary of the ISM procedure

The soliton method[9] (which is another name for the ISM), admits us to find solutions of the Einstein-Maxwell equations in the form:

$$ds^2 = g_{ab}(\rho, z)dx^a dx^b + f(\rho, z)(d\rho^2 + dz^2)$$ (1.1)
where the indexes \( a, b = 1, 2 \) and correspond, respectively, to the coordinates \( t \) and \( \varphi \). \(^2\) In order to make the integrable Ansatz compatible with the metric (1.1), one should assume the following structure for the electromagnetic potentials:

\[
A_\mu = 0, \quad A_a = A_a(\rho, z)
\]

where \( \mu = 3, 4 \) (i.e. it refers to \( \rho \) and \( z \)). Obviously these solutions are stationary and cylindrical-symmetric. We followed the procedure explained in a self-consistent way in the Ref.[9] (chap. 3), then for brevity we will not report all the passages, but simply the main steps and the final results.

### 1.2.1 One soliton on the flat background

It is well known in literature [21] that adding one soliton to the flat background, one finds the Kerr-Newman-NUT-g (KNNg) solution.

However what we need in the following is only the \( \phi^{(0)} \) spectral matrix which resolves the Einstein-Maxwell problem associated to the Minkowski solution:

\[
\phi^{(0)} = \begin{pmatrix}
\frac{1}{\Gamma} & 0 & 0 \\
\frac{\lambda}{\Gamma} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where

\[
\Gamma \equiv \sqrt{\lambda^2 + \rho^2}, \quad \lambda \equiv z - w.
\]

The complex parameter \( w \) has to take the value \( w_1 \equiv z_1 + i\sigma_1 \), which is the

\(^2\)The original method is more general and enables also to find non-stationary solutions in which \( t \) and \( \rho \) are exchanged.
pole of the dressing matrix $\chi$, 

$$\chi = I + \sum_{k=1}^{n} \frac{R_k}{w - w_k}.$$ 

It is in general a complex constant, however we can take it pure imaginary, say 

$$w_1 = i\sigma_1$$ \hspace{1cm} (1.5) 

because of the z-translational invariance of the Minkowski background (to put $z_1 = 0$ means to put the singularity on the origin). The solution has five constant, that are $\sigma_1$ and the four introduced with the constant vector $l^{(1)}$ (defined in [9], formula (3.66)). It is convenient to use the following parametrization: $l^{(1)} = (1, \frac{m-ib}{a-\sigma_1}, 2\frac{e+ia}{a-\sigma_1})$, and, without loss of generality, 

$$\sigma_1^2 = -m^2 + b^2 + a^2 + e^2 + g^2,$$ \hspace{1cm} (1.6) 

where now $m, a, e, b$ and $g$ are the five independent parameters. With that choice, the final form of the metric, after the simple rotation 

$$dt' = dt + \text{const.}d\varphi,$$ \hspace{1cm} (1.7) 

is the KNNg one in the standard coordinates. Therefore one can recognize $m$ as the mass, $a$ the angular momentum and $e$ the charge, $b$ the NUT parameter and $g$ the magnetic charge.

We put now all the constants to zero but the mass $m$, in order to have the Schwarzschild solution, which reads in cylindrical coordinates as:

$$ds^2 = - \left(1 - \frac{2m}{r}\right)dt^2 + r^2\sin^2\theta d\varphi^2 + \frac{r^2}{(r - m)^2 - m^2\cos^2\theta}(d\rho^2 + dz^2).$$ \hspace{1cm} (1.8)
Note that $\rho$ and $z$ are related to the usual Schwarzschild ones by

\[
\begin{cases}
\rho = \sqrt{r^2 - 2mr \sin \theta} \\
z = (r - m) \cos \theta
\end{cases}
\]  

(1.9)

That metric will be used as the new background in the construction of the new solution (and obviously the background electromagnetic field is everywhere absent).

It is to emphasize that although it appears so natural, at that stage, to put to zero $b$ and $g$ (because they are not physical), that choice is not troubleless in the 2-soliton solution, as we explain in the final remarks, for the mixing-parameter phenomenon.

### 1.2.2 Two-solitons solution: two different ways

The soliton method has the peculiarity that starting from a background solution, and having found the $\phi^{(0)}$ matrix, which resolve the spectral system associated to that Einstein-Maxwell problem, one can construct the whole class of the $n$-soliton solutions associated to that background, in a purely algebraic way. This is possible dressing the $\phi^{(0)}$ matrix by

\[
\phi^{(n)} = \chi \phi^{(0)}.
\]  

(1.10)

Then, $\phi^{(n)}$ still satisfy the spectral equation associated to the Einstein-Maxwell problem, admitting to find the $n$-soliton solution.

Let us return to our case. Roughly speaking one can follow two different ways in order to have a 2-soliton solution on the flat background:
1. Start from the flat background and add two solitons.

2. Start from the flat background, and add one soliton; then restart the same procedure taking the one-soliton solution as the new background and add to it another soliton.

In general, if one considers only naked singularities, the two procedures are completely equivalent. However, since we want to add one soliton to the Schwarzschild background, we have a not trivial problem because the first procedure gives a solution with two naked KNNg (indeed, if one tries to put values that give horizons, then $\sigma_1$ becomes imaginary and thus the metric complex; in the 1-soliton case we have not this problem because the rotation (1.7) “miraculously” eliminates the terms which contains $\sigma_1$ linearly).

Thus we followed the second way apart a little device. Instead of resolve two spectral problems by integrations, we used, for the second one, the dressing procedure. In order to do that we used the following trick: we adapt the $\phi(0)$ matrix to the Schwarzschild case with an analytical continuation. Schematically the procedure is the following:

$$
\phi(0) \rightarrow \phi^{(KN)} = \chi \phi^{(0)} \sigma_1 \rightarrow \text{im} \phi^{(S)}, \quad (1.11)
$$

In other words we take the spectral matrix of the flat background $\phi^{(0)}$, we dress it with one soliton finding the spectral matrix $\phi^{(KN)}$ for the (naked) KN background, then we prolong it to the Schwarzschild case by putting the pole pure real:

$$
i\sigma_1 = m, \quad m \in \mathbb{R} \quad (1.12)
$$
(i.e. taking also \( a = 0, \, e = 0 \)). This is what we mean for ‘analytical continuation’\(^3\).

The validity of that trick can be simply checked showing with an explicit calculation that the spectral problem is identically satisfied by the \( \phi^{(S)} \) found. Using the same normalization of (1.3), we have:

\[
\phi^{(S)} = \begin{pmatrix} \frac{1}{\sqrt{w^2 - m^2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{w^2 - m^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{i}{r} \left[ w - m \cos \theta + \frac{m \lambda}{r} \right] & -\frac{i m}{r} & 0 \\ \frac{i}{r} \left[ m r \sin^2 \theta + (w + m \cos \theta) \lambda \right] & w + m \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

(1.13)

where \( \lambda = z - m \).

Now the parameter \( w \) has to take the value of the second pole (on the Schwarzschild background),

\[
w_2 = z_2 + i \sigma_2,
\]

(1.14)

where \( z_2 \) and \( \sigma_2 \) are two new real independent constants.

1.2.3 The final solution: \( g_{ab}, \, A_a \) and \( f \)

In order to give the explicit form of the solution one should follow the steps summarized in [9] (cap. 3, pagg. 80-81). All these steps are merely algebraical and we report here only the final results. Therefore, the explicit form of the metric tensor is:

\(^3\)It is important to emphasize that one can do that continuation only after having calculated all the quantities used in the construction of the KN solution (and in particular after all the complex conjugations of the previous procedure). Indeed the soliton method for the coupled Einstein-Maxwell equation works only with complex poles (or, at least, with pure imaginary ones).
\[ g_{tt} = -\left(1 - \frac{2m}{r}\right) - \frac{4|l_2^{(2)}|^2}{|w_2 - m^2|^2|\Gamma_2|^2} \left| \vec{U} - iml_2^{(2)} \bar{\Gamma}_2 \right|^2 \\
- \frac{8}{|w_2 - m^2|r^2} \text{Im} \left\{ \frac{1}{\bar{\Gamma}_2} \left[ -l^{(2)}_2 \left| U \right|^2 + im \left[ (r - m)\bar{w}_2 - m^2 \cos \theta \right] \bar{\Gamma}_2 \right] \right\} + i m|l_2^{(2)}|^2 \left[ (r - m)w_2 - m^2 \cos \theta \right] \bar{\Gamma}_2 + (1.15) \\
+i m|l_2^{(2)}|^2 \left[ (r - m)w_2 - m^2 \cos \theta \right] \bar{\Gamma}_2 + m^2 l_2^{(2)} |\Gamma_2|^2 \right\} \}

\[ g_{\varphi \varphi} = \frac{i4|l_2^{(2)}|^2}{|w_2 - m^2|^2|\Gamma_2|^2} \left\{ \frac{[(r-m)\bar{w}_2 - m^2 \cos \theta] - iml_2^{(2)} \bar{\Gamma}_2}{r} \left[ l_2^{(2)} (w_2 + m \cos \theta) \Gamma_2 - iV \right] \right\} + \\
- \frac{i4}{|w_2 - m^2|r^2} \left\{ -l_2^{(2)} \left[ (r - m)\bar{w}_2 - m^2 \cos \theta \right] \bar{V}_2 + m \left[ (m + w_2 \cos \theta) - (w_2 + m \cos \theta)^2 \right] \bar{\Gamma}_2 + \\
i m|l_2^{(2)}|^2 \left[ (r - m)\bar{w}_2 - m^2 \cos \theta \right] \bar{\Gamma}_2 + \\
i m|l_2^{(2)}|^2 \left[ (r - m)\bar{w}_2 - m^2 \cos \theta \right] \bar{\Gamma}_2 + m^2 l_2^{(2)} |\Gamma_2|^2 \right\} \} 

(1.16)

\[ g_{\varphi \varphi} = r^2 \sin^2 \theta + \\
+ \frac{4|l_2^{(2)}|^2}{|w_2 - m^2|^2|\Gamma_2|^2} \left[ i \left[ (m + \bar{w}_2 \cos \theta) - (\bar{w}_2 + m \cos \theta)^2 \right] + l_2^{(2)} \left[ (w_2 + m \cos \theta) \right] \bar{\Gamma}_2 \right]^2 + \\
- \frac{8}{|w_2 - m^2|} \text{Im} \left\{ \frac{1}{\bar{\Gamma}_2} \left[ -l_2^{(2)} \left[ (m + w_2 \cos \theta) - (w_2 + m \cos \theta)^2 \right] \right]^2 + \\
+ [(m + \bar{w}_2 \cos \theta) - (\bar{w}_2 + m \cos \theta)^2] (w_2 + m \cos \theta) \bar{\Gamma}_2 + \\
i |l_2^{(2)}|^2 \left[ (m + w_2 \cos \theta) - (w_2 + m \cos \theta)^2 \right] \left[ (\bar{w}_2 + m \cos \theta) \bar{\Gamma}_2 + l_2^{(2)} (w_2 + m \cos \theta) \bar{\Gamma}_2 \right] \right\} 

(1.17) \]
(the bar means complex conjugation). Only in order to compact the expressions we have used the three symbols

\[
\begin{align*}
U &= (r - m)w_2 - m^2 \cos \theta \\
V &= (m + w_2 \cos \theta)r - (w_2 + m \cos \theta)^2 \\
Z &= (w_2 + m \cos \theta).
\end{align*}
\]

(1.18)

The quantities \( l_2^{(2)} \) and \( l_3^{(2)} \) are two complex constants that can be defined in terms of \( \sigma_2, z_2, m \) and of the new real constants \( \alpha, \beta, l, k \) as:

\[
l_2^{(2)} = \alpha + i\beta
\]

(1.19)

\[
l_3^{(2)} = \frac{l + ik}{\sqrt{w_2^2 - m^2}}.
\]

(1.20)

We also remember that these three components of the metric are not independent, indeed by construction they satisfy the relation

\[
g_{tt}g_{\phi\phi} - (g_{t\phi})^2 = -\rho^2.
\]

(1.21)

Obviously, the metric component \( g_{t\phi} \) is real, although it is not immediately evident from the expression (16).

Then, we have the electromagnetic potential:

\[
\begin{align*}
A_t &= 2\text{Im}\left\{ \frac{l_2^{(2)}}{\sqrt{w_2^2 - m^2}} \left[ (r - m)\overline{w}_2 - m^2 \cos \theta \right] - il_2^{(2)}\overline{\Gamma}_2 \right\} \\
A_{\phi} &= \\
&= 2\text{Im}\left\{ \frac{l_2^{(2)}}{\sqrt{w_2^2 - m^2}} \left[ i [(m + \overline{w}_2 \cos \theta)r - (\overline{w}_2 + m \cos \theta)^2] + l_2^{(2)} [(\overline{w}_2 + m \cos \theta)]\overline{\Gamma}_2 \right] \right\}
\end{align*}
\]

(1.22)

The quantity \( T_2 \) is defined as

\[
T_2 = \frac{2}{|w_2^2 - m^2|^2\sigma_2 r \overline{\Gamma}_2} \left\{ 2\overline{\Gamma}_2 \text{Im}(U\overline{V}) + \Gamma_2 (\overline{U}Z - m\overline{V}) + \\
+ |l_2^{(2)}|^2 \overline{\Gamma}_2 (mV - U\overline{Z}) + 2ml_2^{(2)} |\overline{\Gamma}_2|^2 \text{Im}(Z) + (l^2 + k^2) r \overline{\Gamma}_2 \right\}
\]

(1.23)
Finally we have to give the conformal metric factor $f$:

$$f = C_2 |T_2|^2 \frac{r^2}{(r - m)^2 - m^2 \cos^2 \theta}$$

(1.24)

where $C_2$ is an arbitrary constant which we can take as

$$C_2 = \frac{\sigma_2^2 |w_2^2 - m^2|^2}{4K^2},$$

(1.25)

in order to have the asymptotically flat solution at large distances, i.e. $\lim_{r \to \infty} f = 1$.

The largest part of the difficulties to give a compact form of the 2-soliton solution comes from the irrational nature of the function $\Gamma_2^4$. Nevertheless, there exist three particular ‘surfaces’ on which $\Gamma_2$ is rational; they are

$$\theta = 0$$

(1.26)

$$\theta = \pi$$

(1.27)

$$r = 2m$$

(1.28)

The case $\theta = \pi$ can be reproduced by the case $\theta = 0$ simply inverting the sign of $z_2$.

The main features are that considering $\theta = 0$ one sees that there are no divergence on the axis$^{5}$, and that the north and south poles of the Schwarzschild horizon remain unperturbed in $r = 2m$; then studying $r = 2m$ it comes out

$^{4}$On the other hand $\Gamma_1 = (r - m) - im \cos \theta$ has been rationalized thanks to the peculiar choice of the coordinates $(r, \theta)$.

$^{5}$The Schwarzschild singularity is on $r = 0$ but $\theta = \pi/2$; so physically the singularity is on the axis, but mathematically we do not see it if we define the axis as the set of points for which $\theta = 0$ or $\theta = \pi$. 

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that the new horizon is larger than Schwarzschild horizon, and that \( g_{r\varphi} \) does not vanishes also if \( A_{\text{tot}} = 0 \), i.e. the static case cannot be reached, at least in that simple way.

It comes out that in order to have the standard form of the Minkowski vacuum in the far observer limit, we need the simple rotation

\[
\begin{align*}
\frac{dt'}{dt} &= 1 + R \frac{d\varphi}{d\varphi'}, \\
\frac{d\varphi'}{d\varphi} &= 1,
\end{align*}
\]

which is analogous to the 1.7 used to find KN in the standard coordinates (but in general with a different constant \( R \)).

1.3 Physical interpretation of the new solution

1.3.1 Two particular limits with \( z_2 \to \infty \)

The physical interpretation of the new solution can be easily showed by taking the following two limits:

1. Considering the limit in which

\[
\begin{align*}
z_2 &\to \infty, \\
|z - z_2| &< \infty, \\
\rho &< \infty
\end{align*}
\]

one finds the KN solution; that indeed correspond to put the charge very far from the Schwarzschild black hole (\( z_2 \to \infty \)), and to put the observer near the charge (\( \rho < \infty \) and \( |z - z_2| < \infty \)).
2. On the other hand, taking the limit in which

\[
\begin{aligned}
    z_2 &\to \infty \\
    r &< \infty
\end{aligned}
\]  \quad (1.31)

one finds the Schwarzschild solution with mass \( m \).

At the same time it is now clear that \( z_2 \) is a measure of the distance between the two bodies. However, strictly speaking, it would be not correct to consider the exact solution as a rotating charged particle near a Schwarzschild black hole. Indeed when we take the limit \( z_2 \to \infty \), also the values of the physical parameters change (since they depend on \( z_2 \), see Appendix, 1.55), therefore when \( z_2 \) is finite the interpretation of the parameters would be different because of the non-linear interaction between the two solitons (the interaction is linear only in first approximation when \( z_2 \to \infty \)).

It is also worth noting to say that the imaginary part of the pole \( \sigma_2 \) is linked to the physical constants by

\[
\sigma_2^2 = -(M_{\text{tot}} - m)^2 + A_2^2 + Q_{\text{tot}}^2, \tag{1.32}
\]

where \((M_{\text{tot}} - m), A_2\) and \(Q_{\text{tot}}\) are respectively mass, angular momentum of the KN particle, and total charge\(^6\), as we will see in the next section; then, as in the one-soliton solution, the charged particle is a \textit{naked} KN singularity, otherwise \( \sigma_2 \) would be imaginary and it will give imaginary terms in the metric tensor.

\(^6\)The total charge coincides with the one of the KN particle only in the \( z_2 \to \infty \) limit.
1.3.2 Far-observer approximation; the rotation

Thus, the present solution has seven integration constants: \( m, z_2, \sigma_2, \alpha, \beta, l, k \). Only the first two, have a direct physical meaning, then in order to identify the other physical constants one can proceed, as usual, expanding \( g_{ab} \) and the potential \( A_a \) in power of \( \frac{1}{r} \).

After the mentioned rotation (see 1.29), we have the following approximation form of the metric tensor

\[
\begin{align*}
    g_{tt} &= -1 + \frac{2M_{tot}}{r} - \frac{Q_{tot}^2}{r^2} - 2\frac{M_{tot} - m}{r^2} \left[ M_{tot} - z_2 \cos \theta \right] + O \left( \frac{1}{r^3} \right) \\
    g_{t\varphi} &= -2\frac{M_{tot} - m}{r} A_{tot} \sin^2 \theta + O \left( \frac{1}{r^2} \right) \\
    g_{\varphi\varphi} &= \left[ 1 + 2\frac{M_{tot} - m}{r} + \left[ 2M_{tot}^2 - Q_{tot}^2 - 2(M_{tot} - m)z_2 \cos \theta \right] \frac{2}{r^2} \right] r^2 \sin^2 \theta + O \left( \frac{1}{r} \right)
\end{align*}
\]

(1.33)

and for the potential:

\[
\begin{align*}
    A_t &= \frac{Q_{tot}}{r} - \frac{M_{tot} - m}{r^2} Q_{tot} + \\
    &\quad + 2\frac{\sigma_2}{K} \left[ z_2 \left( z_2^2 + \sigma_2^2 - m^2 + 2m\alpha \sigma_2 + m\beta z_2 \right) - m\beta (\sigma_2^2 + m^2) \right] \frac{2Q_{tot} \cos \theta}{\sigma_2 - A_2} + O \left( \frac{1}{r^3} \right) \\
    A_\varphi &= \frac{Q_{tot}}{r} \left[ R + \frac{(M_{tot} - m)^2 + (\sigma_2 - A_K)^2 \cos^2 \theta}{(\sigma_2 - A_K)^2} \right] \left. \right. + \frac{2\sigma_2}{K} \left[ k \left( z_2^2 - \sigma_2^2 + m^2 \right) - 2l z_2 \sigma_2 \right] \left. \right. \sin^2 \theta + \\
    &\quad + O \left( \frac{1}{r^2} \right).
\end{align*}
\]

(1.34)

Finally we have for the conformal factor:

\[
f = 1 - \frac{2M_{tot}}{r} + O \left( \frac{1}{r^2} \right), \tag{1.35}
\]

The relation between the physical and the mathematical parameters are given in the Appendix. Here and in the following we have already putted to zero the total magnetic charge and the NUT-parameter because they give a non-asymptotic flat solution. In general it cannot be excluded the presence of
two magnetic charges (which in the sum give zero), however we do not care of that fact since it should affect only the anomaly region; furthermore in the following we will put our attention only on the electric field.

It is worth noting that the mass of the black hole coincides exactly with the Schwarzschild mass $m$, indeed using the Komar integral definition [27, 26], the mass $m_H$ inside the horizon $H$ is:

$$m_H = \frac{1}{4\pi} \int_H (\xi^1;i - \xi^{i1};i) \sqrt{-g} d^3x =$$

$$= \frac{1}{4\pi} \int_H \left[ \left( \frac{f^i}{f} \right)_{,\rho} + \left[ \frac{\rho f_z}{f} \right]_{,z} \right] d\varphi d\rho dz$$

$$= \frac{1}{4\pi} \int_{\partial H} (1 + \rho \frac{f_z}{f}) d\varphi dz = m,$$  \quad (1.36)

where $\xi^i = (1, 0, 0, 0)$, and we have used the fact that the horizon lies on the axis ($\rho = 0$), and that the extreme of $H$ remain unperturbed notwithstanding the presence of the second source (i.e. $H = \rho = 0, z \in [-m, m], \varphi \in [0, 2\pi]$). That result holds for each metric of the form (1.1) which has an horizon $H$.

Therefore the mass of the second source has to be $m_2 = M_{tot} - m$, and has a much more complex expression in terms of the mathematical parameters (see the Appendix).

A final remark: in the expansion of $g_{t\varphi}$ (the second eqn. of (33)) we have defined $A_{tot}$ in such a way that it is multiplied for the factor $m_2$—i.e. the mass of the rotating charge—in analogy with the Kerr case. However in the Kerr case there was no doubt that the rotation was due to the only mass...
present, while now we have two bodies which can rotate. So, in order to stress that fact, we introduce a new constant,

\[ A_2 \equiv \frac{2\sigma_2}{K} |X|^2, \tag{1.37} \]

which is the angular momentum of the charged particle (where \( X = w_2 + i\bar{l}_2 \), and \( K \) is a combination of the other constants, see the Appendix). Then we can express \( A_{tot} \) as

\[ A_{tot} = A_2 - \frac{m}{m_2} R, \tag{1.38} \]

thus, in a more clear way one can write

\[ g_{t\phi} = 2(mR - m_2 A_2) \frac{\sin^2 \theta}{r} + O \left( \frac{1}{r^2} \right). \]

Therefore the rotational constant \( R \) can be interpreted as the angular momentum of the black hole. It is worth noting that one can put the angular momentum of the particle, or the total angular momentum to zero (in that case the particle and the black hole are contra-rotating); but it is not possible to put both the momenta to zero. Therefore \( A_{tot} = 0 \) means that the two bodies are contra-rotating, which is different from the static case.

The fact that also the Schwarzschild black hole acquire an angular momentum is another example of the parameter-mixing phenomenon.
1.4 Some analytical features of the 2-soliton solution

1.4.1 The polydromic function $\Gamma_2$ and the cut-line

In the 2-soliton solution it is present the polydromic function\(^7\)

$$\Gamma_2 = \sqrt{\rho^2 - \sigma_2^2 + (z - z_2)^2 - i2\sigma_2(z - z_2)} \quad (1.39)$$

That function has a branch point in

$$\begin{cases} 
\rho = \sigma_2 \\
\rho = z_2 \end{cases} \quad (1.40)$$

so, in order to define $\Gamma_2$, it is better to use a different notation,

$$\Gamma_2 = \sqrt{u + iv} = \sqrt{u^2 + v^2}e^{\text{arg}(u + iv)} \quad (1.41)$$

where $u$ and $v$ are the real functions depending by $r, \theta$ which one can deduce from 1.39. In all the calculation we have taken the cut along the semi-line\(^8\)

$$\begin{cases} 
u \leq 0 \\
v = 0 \end{cases} \quad (1.42)$$

which is the same to define $-\pi \leq \text{arg}(u + iv) \leq \pi$ or, alternatively, $\pi \leq \text{arg}(u + iv) \leq 3\pi$. With that choice the cut-line reaches the axis on $z = z_2$.

It is worth noting that the cut-line defined in that way reduces to a simple point on the axis if one takes the limit $z_2 \to \infty$ (indeed $\Gamma_2 \approx z_2$).

\(^7\)In Greek “poly-” = “multi-”, and “dromos” = “course, place where to run” —which means that the function can be defined on more domains. This is a better expression than the more popular “multivalued” (because a “function” —by definition— can never be “multivalued”).

\(^8\)For simplicity, thanks to the axial symmetry, we refer to the 2-dimensional $z - \rho$-semiplane. Thus, e.g. the ‘cut-line’ is not a line but a surface, and so on.
This is maybe the best reason to adopt such a definition (furthermore, it is mathematically the simplest).

However, in general the choice of a particular cut-line should depend by the boundary condition of the problem. (The only constrain is that every closed line around the charge must cross it).

On the other hand, in order to have a continue solution even on the cut-line, one should define $\Gamma_2$ on the extended complex plane, say

$$\Gamma_2 = \begin{cases} 
\sqrt{u^2 + v^2}e^{\operatorname{arg}(u + iv)} & \text{if } (u, v) \in R_1 \\
-\sqrt{u^2 + v^2}e^{\operatorname{arg}(u + iv)} & \text{if } (u, v) \in R_2
\end{cases}, \quad (1.43)$$

where $R_1$ and $R_2$ are the two Riemann sheets defined by

$$R_1 = \{(u, v) : \pi < \operatorname{arg}(u + iv) \leq \pi\}$$
$$R_2 = \{(u, v) : \pi < \operatorname{arg}(u + iv) \leq 3\pi\} \quad (1.44)$$

From a physical point of view, an observer who comes from $R_1$ (which corresponds to the solution with an asymptotically flat spacetime) would be in the domain $R_2$ after having crossed the cut-line. Then, he can return to $R_1$ simply making another turn around the branch point\(^9\). As it is typical for the ring singularity of the KN kind, the branch-point (i.e. (1.40)) coincide with the charge point.

However a lot of strangeness would appear if the observer could go in the domain $R_2$, for example in that domain the solution has unphysical quantities as $M_{\text{tot}} < 0$ at $r \to \infty$.

\(^9\)A very similar situation, in which it is present a ring singularity which makes as branch point, is described in [24] referring to a family of two-solitons solutions in the non-stationary case.
Thus we have only two possibilities:

1. We can define $\Gamma_2$ on the extended complex plane; then the solution is regular also on the cut-line, but crossing the ring singularity the observer enter in another universe\textsuperscript{10} which has unphysical quantities.

2. We can restrict $\Gamma_2$ to the first Riemann sheet $R_1$; then the solution has a discontinuity on the cut-line, but it is univocally defined for every $(r, \theta)$ (i.e. we have no tunnel to the other universe).

We chose the second possibility as the only physically meaningful. One can interpret the discontinuity on the cut-line as due to the presence of some singular field of matter.

Naturally, using the soliton method we assumed that the only matter was given by the electromagnetic field, however, in a certain region it is possible to match the 2-soliton solution with another solution which takes in consideration another stress-energy tensor. In that picture the 2-soliton metric will hold only outside that region.

\textbf{1.4.2 The region between the two centers; the ‘strut’}

It is a fact that two masses, in absence of some further force, cannot be exactly at rest. In spite of this, we have an exact stationary solution, which describes, as we have showed in the two limits above, a rotating charge

\textsuperscript{10}A similar strangeness would come also in the Kerr metric, if one continue the solution to negative $r$ (see the discussion in [28]).
near a (quasi-)Schwarzschild black hole\textsuperscript{11}. Thus, it is not unnatural to expect some mathematical strangeness in this region that physically represent ‘something’, maybe a stripes of matter or a strut, which prevent the charge from falling down.

Even if the axis is regular in the usual sense, i.e. the $g_{ab}$ tensor and the $A_a$ potential, on the axis do not diverge, it is present a different kind of singularity consisting in the fact that

$$\begin{cases}
g_{\varphi t}|_{\rho=0} = 0 & \text{if } z < 0 \text{ or } z > z_2 \\
g_{\varphi t}|_{\rho=0} = a_0 g_{tt} & \text{if } 0 < z < z_2
\end{cases} \quad (1.45)$$

and that

$$\begin{cases}
f g_{tt}|_{\rho=0} = -1 & \text{if } z < 0 \text{ or } z > z_2 \\
f g_{tt}|_{\rho=0} = -1 + a_1 & \text{if } 0 < z < z_2
\end{cases} \quad (1.46)$$

The behavior at infinity is the correct one, while the jump on the KN ring, which is directly due to the branch point of the $\Gamma_2$ function, give place to two distinguished pathologies in $g_{\varphi\varphi}$ which cannot be eliminated by any change of coordinates. The implications on $g_{\varphi\varphi}$ can be seen using the determinant relation 1.21.

The first anomaly, i.e. $a_0 \neq 0$ in (1.45), implies that on the axis $g_{\varphi\varphi}$ has the opposite sign that it has at infinity (where it is positive) —we called it the ‘tube-syngularity’ since there exist a surface around the axis topologically equivalent to an overturned cone (with the base on the ring and the vertex

\textsuperscript{11}As we will show in the next section, in general the equilibrium is not given by an electric repulsive force, e.g. in the last cases that we have considered the (quasi-)Schwarzschild black hole has a charge of opposite sign respect to the KN and thus the electric force is even attractive.
on the Schwarzschild horizon), on which $g_{\varphi \varphi}$ vanishes, and inside of which $\varphi$ becomes time-like.

The second one is usually called the ‘conic singularity’ (see Ref.[29] for the rigorous relation between the value of the angle deficit and the effective energy-momentum tensor of these struts and strings); indeed the fact that $a_1 \neq 0$ in (1.46), implies that the circumference of a small circle of radius $\rho$ centered on the axis, is not $2\pi \rho$ (even if $a_0 = 0$). This means that the region near that part of the axis would be not homeomorphic to the Minkowski spacetime.

Thus, in order to have a physically well-behaved solution, we should impose the two equilibrium conditions,

$$
\begin{align*}
    a_0(m, M_{\text{tot}}, A_{\text{tot}}, Q_{\text{tot}}, z_2) &= 0 \\
    a_1(m, M_{\text{tot}}, A_{\text{tot}}, Q_{\text{tot}}, z_2) &= 0
\end{align*}
$$

(1.47)

Unfortunately, as it is put out in Ref.[25] (also in the more general case of the RN background, instead of Schwarzschild), they cannot be satisfied by any $z_2$ real distance, for any choice of the other parameters.

From a physical point of view this is interpreted as the presence of a ‘strut’ which should represent the presence of some matter field that prevents the charge to fall in the Schwarzschild black hole\(^\text{12}\).

\(^\text{12}\)Another (rather curious) way to remove the $g_{\varphi \varphi}$-anomaly could consist in putting the cut-line exactly on $g_{\varphi \varphi} = 0$. In that different definition of the Riemann sheets the conic singularity will be relegated to the other universe; however it would remain the discontinuity of the metric on this surface.
1.5 Force lines

The soliton method starts from the charge-free Maxwell equations (i.e. $j^\mu = 0$). This means that the charge is localized at most in some particular points in which the electromagnetic field diverges. As we have said, in our solution that point coincides exactly with the branch-point 1.40 (i.e. the point in which vanishes $\Gamma_2$). This is a typical feature of the $n$-soliton solutions, as well the fact that the charge has a ring structure\(^{13}\) (because of the cylindrical symmetry and the fact that $\sigma_2^2 > 0$). In spite of this, both $g_{ab}$ and $A_a$ are finite there. Nevertheless, everyone of them has a cusp, and thus all theirs derivatives diverge. Further, on that point $f \rightarrow \infty$ because it is proportional to $\frac{1}{|\Gamma_2|^2}$.

One can see the charge point and the behavior of the electric field plotting the lines of force. We define the force lines, in the same way as was made by Hanni and Ruffini in [13]: it is the locus of points with a given value of the flux $\Phi$, which is defined as

$$
\Phi \equiv \int_C F^* = 2 \int_C \left\{ F_{23}^* d\tau d\varphi + F_{24}^* d\theta d\varphi \right\} = 8\pi Q_{tot},
$$

where $Q_{tot} = \text{total electric charge inside } C$ and $F_{ij}^* = \sqrt{-g} \varepsilon_{ijkl} F^{kl}$. Then, at any given point, the slope of the the line of constant flux is given by

$$
\frac{dr}{d\theta} = -\frac{\partial \Phi}{\partial \theta} / \frac{\partial \Phi}{\partial r}
$$

obviously the flux $\Phi$ is now considered as a function of $r, \theta$ because now the integral (1.48) is taken over a piece of a spherical surface (and $r, \theta$ are the

\(^{13}\)In the $n$-soliton case one has $n$ different $\Gamma_i$, and so $n$ rings.
From a mathematical point of view it is the same to resolve the differential system
\[\begin{align*}
\frac{dr}{d\lambda} &= \sqrt{-g} F^{13} = \frac{1}{\sin \theta} \left( g_{\varphi \varphi} \frac{\partial A_t}{\partial r} - g_{t \varphi} \frac{\partial A_{\varphi}}{\partial r} \right) \\
\frac{d\theta}{d\lambda} &= \sqrt{-g} F^{14} = \frac{1}{(r^2 - 2m r) \sin \theta} \left( g_{\varphi \varphi} \frac{\partial A_t}{\partial \theta} - g_{t \varphi} \frac{\partial A_{\varphi}}{\partial \theta} \right) \quad (1.50)
\end{align*}\]

remembering that \( F^{13} = E^r \) and \( F^{14} = E^\theta \) it is clear the meaning of the solutions \((r(\lambda), \theta(\lambda))\): they are the lines which are, at each point, tangent to the electrical field.

The physical interpretation (Christodoulou-Ruffini, quoted in [13]) is the following: a force line is a line tangent to the direction of the electric force measured by a free-falling test charge momentarily at rest, with initial 4-velocity
\[u^t = (\sqrt{g^{tt}}, 0, 0, 0).\quad (1.51)\]

Note that such interpretation is valid only for \( g^{tt} > 0 \), for this reason we have not plotted the lines inside the horizon.

In the following plots we used geometrical units \((G = c = 1)\), in which the unitary length is given by the Schwarzschild mass \(m = 1\). The lines are plotted in cylindrical coordinates\(^{14}\),
\[\begin{align*}
\rho' &= r \sin \theta \\
z' &= r \cos \theta
\end{align*}\quad (1.52)

where \(r\) and \(\theta\) are the usual Schwarzschild coordinates.
1.5.1 General case

Preliminary we can look at the general case, in which all the parameters are of a similar order of magnitude, say:

\[
\begin{align*}
    m &= 1 \\
    m_2 &= 3 \\
    A_2 &= 2 \\
    Q_{tot} &= 4 \\
    A_{tot} &\cong 0.717 \\
    z_2 &= 10 
\end{align*}
\] (1.53)

(note that the independent constants are only five, indeed \(A_{tot}\) is calculated after having fixed the other values using the relation 1.55 given in the Appendix).

From the point of view of the electric flux we have three distinguished regions: a region in which the force lines come from the KN singularity and end in the strut; a region which includes the lines outcoming from the KN singularity and arriving at infinity; finally the region of the lines which go from the black hole to infinity. We printed in bold the electric lines which separate these three regions.

Since near the strut the solution has not a physically meaningful interpretation, we did not plot in fig. 1.5.1 the lines that lay inside the anomaly region and also the ones which go inside and than escape to infinity.

It is indeed to emphasize that only outside these regions that the solution can give a serious physical description of a realistic situation.

\[\text{\textsuperscript{14}They are not the ones defined into 1.9; however they coincide when } r \to \infty\]
The presence of the separatrix line which divides the black hole from the
KN singularity means that the two objects have a charge of the same sign.

As one expects, the behavior far from the charges becomes very speedily
the one of the Coulomb field.

The anomaly region can be shrunk to the axis until to become approx-
imatively a segment by taking small values of charge and mass in the KN
singularity, as we will see in the next section.

1.5.2 A case close to the Hanni-Ruffini one

Hanni and Ruffini[13] considered a non-rotating test charge momentarily at
rest in the Schwarzschild metric. Thus we can approximate that situation
putting the energy of the charge to be very small respect to the mass of
the black hole. However the 2-soliton solution cannot approach \textit{exactly}
the Hanni-Ruffini case for three different reasons. First of all, obviously, because
it is present the strut; in second place because one cannot put \textit{both}
the angular momenta to zero, and, also if the particle is non-rotating, the total
angular momentum does not vanish. And finally because, as we will see now,
also the black hole acquires a charge. However, we can consider a similar
case using the following values:

\[
\begin{align*}
    m &= 1 \\
    m_2 &= 0.01 \\
    A_2 &= 0 \\
    Q_{tot} &= 0.02 \\
    z_2 &= 3
\end{align*}
\] (1.54)
Thus the charge is non-rotating (in the sense that $A_2 = 0$) and has a small mass (and energy) with respect to the Schwarzschild black hole. An important fact is that with this values the angle deficit of the conic singularity is practically negligible (of the order of $1 - a_1 \approx 10^{-3}$).

Instead, the total angular momentum is $A_{tot} \simeq 2.31$, however $g_{t\varphi}$ is quite small because it is proportional also to $m_2$. The resulting line of force are given in the figure 1.2.

Also in that case it exist a separatrix line. Now the separatrix line is nothing else but a force line which has the initial condition very close to the south pole of the horizon (i.e. $r = 2m, \theta = \pi$), however we marked that line because it has the important physical meaning that encloses the horizon, and thus —for the Cauchy theorem— all the lines which lay inside that region cannot go to infinity and have to fall on the horizon. It means that the flux of the electric field on the horizon is surely negative, i.e. the black hole has a net negative charge.

In the case 1.54, using a numerical integration over the horizon surface, one find that the mixing-parameter phenomenon is very strong, indeed the charge is $Q_{blackhole} = -0.009998$, which is practically the half of the total. Now the Schwarzschild horizon $g_{tt} = 0$ is with a very good approximation a sphere of radius $r = 2m$, since the perturbation of the KN particle is very small.

It is worth noting that in that case the KN ring, and thus the cut-line,
has a very small radius and it is practically point-like. The $g_{\varphi\varphi}$-anomaly, i.e. the strut, even if it is still present, now it is very close to the axis and it does not touch the electric lines of our plots. Further, since there are no force lines which end on it, we can deduce that it is neutral.

Plotting the force lines at different values of the distance $z_2$ one can see a smooth transition of the electric field to that of a RN black hole, which is very similar to what was found in the similar Hanni-Ruffini case in Ref. [13].

Finally we want to stress that since in the two cases 1.53 and 1.54 the charge of the black hole has a different sign then it should be exist also the case in which the black hole is neutral —indeed the transition between the two cases has to be continuous at the changing of $m_2$, $A_2$ and $Q_{tot}$.

1.6 Conclusions I

We briefly reassume the main characteristic of the 2-soliton solution:

1. It is a stationary axially symmetric and asymptotically flat solution of the coupled Einstein -Maxwell equations with five physical parameters.

2. In the limit in which $z_2 \to \infty$ (i.e. $z_2 >> m$) it reduces to a Schwarzschild black hole with a (large distanced) naked KN singularity linked by a ‘strut’ (which becomes thinner and thinner as $z_2 \to \infty$, but never vanishes).

3. The charge singularity has a ‘ring’ structure (as in the KN case) with a discontinuity inside the ring (in the the plane $\rho - z$ it corresponds to
a cut-line which links the singularity to the axis).

4. In general, the solution is stationary but not static, also if the total angular momentum constant $A_{tot}$ is put to zero.

5. The metric tensor has a doubly anomalous behavior near the axis in the region between the two centers (inversion of the sign in $g_{zz}$ and the conic singularity); this is what we called the ‘strut’.

6. The Schwarzschild horizon is smoothly perturbed by the presence of the other singularity: it becomes a little bit larger, while the north and sud poles remain unmodified.

7. It is possible to find configurations in which all the physical anomalies are negligible (see Sec.1.5.2).

Maybe, someone could say that it was clear from the beginning that some anomaly would appear in the solution in order to ensure the dynamical equilibrium between the charge and the black hole. Nevertheless the existence of an equilibrium point in the geodesics of a neutral test particle on a Reissner-Nordstrom naked singularity leaves opened the possibility to have such a well-behaved exact solution[17]. Indeed in Sec.1.5.2 we showed a case in which all the physical anomalies are negligible. The equilibrium in that case is due to a peculiar geometrical effect of the naked singularity.
The parameter-mixing phenomenon

The ISM, as actually is formulated, should admit—we cannot yet show it, but we believe that it is the case—a solution without the anomaly region and the conic singularity by choosing a certain peculiar value of $z_2$, which should be determined as a function of the other constants. However, we argue that this condition could be satisfied only in the most general 2-soliton solution, which has twelve mathematical parameters. Indeed also the four of these parameters which are not physical in the limit $z_2 \to \infty$, i.e. the NUT-parameter and the magnetic charge of each soliton (when the solitons are alone), play an important role and cannot be putted to zero from the beginning since they would acquire a different meaning in the entangled case (parameter-mixing). These four unphysical degree of freedom of the solution will be removed by two “equilibrium conditions”, i.e. no-tube and no-conic conditions, and by imposing that total NUT parameter and both the magnetic charge must be zero (these are indeed five condition, but one of them should be used to fix $z_2$). That picture can also explain why we found a charge inside the ‘Schwarzschild’ black hole: the charge parameter of the background solution is no more the physical charge when one adds the second soliton; in order to have a neutral black hole one should put to zero a different quantity which would be a complicated mix of the mathematical constants.
1.7 Appendix: Relations between the physical constants and the mathematical ones

The seven real constants which appear in the 2-soliton solution are $\alpha, \beta, l, k, \sigma_2, z_2$ and $m$. They are linked to the two complex components $l^{(2)}_2$ and $l^{(2)}_3$ by the definitions 1.14, 1.19, 1.20 and 1.32.

Apart $m$ and $z_2$ which have a direct physical meaning, the other physical parameters are:

\[
\begin{align*}
M_{\text{tot}} &= m + m_2 \\
m_2 &= \frac{2\sigma_2}{K} \left[ (-1 + \alpha^2 + \beta^2) m \sigma_2 + \alpha (z_2^2 + \sigma_2^2 - m^2) \right] \\
A_2 &= \sigma_2 - \frac{2\sigma_2}{K} \left[ z_2^2 + \sigma_2^2 + m^2 (\alpha^2 + \beta^2) + 2m (\alpha \sigma_2 + \beta z_2) \right] \\
A_{\text{tot}} &= A_2 - \frac{m R}{m_2} \\
Q_{\text{tot}} &= \frac{2\sigma_2}{K} \left[ l(z_2 + m \beta) + k(\sigma_2 - m \alpha) \right] \\
B &= \frac{4\sigma_2}{K} \left[ (1 + \alpha^2 + \beta^2) m z_2 + \beta (z_2^2 + \sigma_2^2 + m^2) \right] \\
G_{\text{tot}} &= \frac{2\sigma_2}{K} \left[ l(\sigma_2 + m \alpha) - k(z_2 + m \beta) \right]
\end{align*}
\]

which are, respectively, total mass, mass and angular momentum of the KN particle, total angular momentum, total charge, NUT-parameter, and total magnetic charge. We have used the auxiliary constant

\[
K = (1 - \alpha^2 - \beta^2)(z_2^2 + \sigma_2^2 - m^2) + 4 \alpha m \sigma_2 + \frac{l^2 + k^2}{4}.
\]

The constant $R$ used in $A_{\text{tot}}$ and in the rotation of the coordinates is

\[
R = \frac{2\sigma_2}{K} \left[ (1 + \alpha^2 + \beta^2)(z_2^2 + \sigma_2^2 + m^2) + 4 \beta m z_2 \right] .
\]

In order to have a physical solution, as it was explained in the text, one has to impose $B = 0$ and $G_{\text{tot}} = 0$. Considering these constraints, the inverse
relations are:

$$\alpha = \frac{m \sigma_2 (\sigma_2 - A_2)^2 - m^2 \sigma_2 (\sigma_2 - A_2) (|w_2|^2 - m^2)}{m^2 m_2^2 - 2 m m_2 \sigma_2 (\sigma_2 - A_2) + (\sigma_2 - A_2)^2 |w_2|^2}$$

$$\beta = - \frac{m^2 m_2^2 - 2 m m_2 \sigma_2 (\sigma_2 - A_2) + (\sigma_2 - A_2)^2 |w_2|^2}{m m_2 (m_2^2 + (\sigma_2 - A_2)^2)}$$

$$l = \frac{2 Q_{tot}}{\sigma_2 - A_2} \left( z_2 - \frac{m^2 m_2^2 - 2 m m_2 \sigma_2 (\sigma_2 - A_2) + (\sigma_2 - A_2)^2 |w_2|^2}{m m_2 (m_2^2 + (\sigma_2 - A_2)^2)} z_2 \right)$$

$$k = \frac{2 Q_{tot}}{\sigma_2 - A_2} \left( \sigma_2 + \frac{m^2 \sigma_2 (\sigma_2 - A_2)^2 - m^2 \sigma_2 (\sigma_2 - A_2) (|w_2|^2 - m^2)}{m^2 m_2^2 - 2 m m_2 \sigma_2 (\sigma_2 - A_2) + (\sigma_2 - A_2)^2 |w_2|^2} \right)$$
Figure 1.1: Force lines of the electric field in the general case 1.53. The horizontal segment which joins the charge to the axis is not a force line but the cut-line of the KN ring; there the force lines also are discontinue. The circle of radius $r = 2m$ is the Schwarzschild horizon. The point-like line marks the $g_{\varphi \rho} = 0$ surface; inside of that region the metrics has an unphysical behaviour. We printed thicker the two force lines which make as separatrices between the three different regions.
Figure 1.2: The force lines in the case of a small charge at $r = 4m$. The bold line ends into $r = 2m$, $\theta = \pi$, then it is a separatrix, because the lines inside that region are entrapped and fall in the horizon, instead the others escape to infinity. The circle is the Schwarzschild surface. The dotted line indicate the ‘strut’, which is now very close to the axis, and it does not touch the force line of our plot.
Figure 1.3: The force lines in the case of a small charge at $r = 3m$ in (a) and $r = 2.2m$ in (b). From a qualitative point of view the behavior of the force lines is the same of 1.2, however now the separatrices lines are more close to the horizon.
Chapter 2

Analysis of the Alekseev-Belinski solution
This chapter is based on Refs.[2, 6, 7].

2.1 Two Reissner-Nordstrom sources in equilibrium

In the previous chapter we have studied a special case of 2-soliton solution. Now we pass to another 2-sources solution, which has been recently found by Alekseev and Belinski[17] (in the following denoted with AB). That is a very important result because it has solved the long standing problem of the static equilibrium of two charged masses in the context of GR, since it is conic-singularity-free. This solution — differently from the one treated in the previous chapter — is static and not just stationary (i.e. $g_{t\varphi} = 0$), and the sources are two RN singularities.

Let us now spend few words about such an equilibrium in General Relativity.

While in the Newtonian theory the equilibrium condition is simply $m_1m_2 = e_1e_2$, in the relativistic regime the problem is much more complicated because one has to solve the full system of the Einstein-Maxwell equations, and find a static solution with two sources. Furthermore, as we have seen in general this solution will present a conic singularity on the symmetry axis; to find the equilibrium condition is equivalent to require the absence of any conic singularity, i.e. the axis has to be locally Minkowskian — physically that means that there must be neither “struts” nor “strings” which prevent the two bodies to fall or run away each other.
The key point to understand the main differences between classic and relativistic regime is the repulsive nature of gravity in GR near a naked singularity. This can be seen just by looking at the RN metric

\[ g_{tt} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{2.1} \]

where gravity is repulsive for \( r < \frac{Q^2}{M} \): it is for that reason that the equilibrium is allowed only at certain distances. Indeed, e.g. if one considers the geodesic of a neutral particle on that background, it is easy to find a (stable) equilibrium precisely at

\[ r_c = \frac{Q^2}{M}. \tag{2.2} \]

For charged particles an equilibrium is also possible at a fixed distance[14]; in these cases it can be both stable or unstable, according to the choice of the parameters.

In the AB-solution the Newtonian equilibrium condition is restored taking the limit of large distance between the two singularities.

The important achievement of the AB-solution is the extreme compactness of all the formulas, despite of complexity of calculations by which it was found[18]. They get the wanted task using the IEM, which presents some advantages with respect to the ISM — using which, as we have seen, there are also unphysical parameters (NUT parameter, magnetic charge) and the rotation which are not easy to be eliminated.

As they showed, the equilibrium is possible (apart from the well-known Majumdar-Papapetrou case where the charge of each source is equal to its
mass) only for a naked singularity near a black hole (b.h.). We excluded from our analysis the b.h.-b.h. and naked-naked configurations since the former give not separable objects, while the latter do not exist at all in the equilibrium state.

This chapter is organized as follows: we give a brief historical review of the works in literature (Sec.2.2) (this section can be skipped by the ones interested only to the physical contents); for the easy of the reader we also add the reproduction of the AB solution in Sec.2.3; we give some details clarifying the use of the coordinates systems involved (Sec.2.4); finally we graph the plots of the electric force lines in the various qualitatively different cases (Sec. 2.5). More precisely, in this last section, we consider at the beginning the general case with two charges, firstly with $e_1e_2 > 0$ and then $e_1e_2 < 0$; and finally that in which only one object (the naked singularity) is charged. The last particular case of the solution in different form was presented in Ref. [19]. Of each case we present also the limit in which one source has a much smaller mass and charge than the other.

In particular we consider the limit case of a small charged particle near a Schwarzschild black hole, finding electric force lines plots congruent with the Hanni-Ruffini[13] ones.

Finally in Sec. 2.6 we consider the stability of the solution with respect to distance displacement of the two sources.
2.2 Some Historical Remarks

The problem of the equilibrium of two charged masses and their resulting gravitational and electric fields has a long history in GR literature (see table 2.1). It is possible to distinguish two different kind of results: approximate results, and exact solutions.

In the contest of the approximate results, the first to be mentioned is the one of Copson [30], who gave in 1927 the electric potential of a test charge on the Schwarzschild background (therefore it was neglected the backreaction of the particle on the metric tensor). That work was important because it gave the potential in a closed analytic form, however that result was not completely correct because it implicated that the black hole would have an induced charge: the correct potential was given by Linet [31] only in 1976—the electric potential of the AB-solution indeed reduces to that form in the limit in which the naked singularity source can be considered as a test particle.

In 1973 Hanni and Ruffini [13] gave for the first time the plots of the electric force lines\(^1\), again for a test particle near a Schwarzschild black hole (but they used a multipole expansion of the electric potential).

Later a certain number of papers have been published in which different authors (using exact solutions, PN and PPN approaches) arrived to different conclusions about the possibility/impossibility of an equilibrium config-

\(^1\)We follow this work for the construction of the plots of the present solution.
uration, however no final statements were achieved because of the use of supplementary hypothesis or for the incompleteness of the analysis.

In 1993 the already mentioned article of Bonnor [14] gave an important hint to clarify the problem: studying the equilibrium configurations in the test particle limit, namely a test charge on the RN background, he pointed out that equilibrium configurations were possible when the ratio $e/m$ was less than unity for the background and more than unity for the particle, or vice versa; he showed also that equilibrium was possible for charges of opposite signs too. It is worth noting that the Alekseev-Belinski solution confirms practically word-by-word (from a qualitative point of view) that picture.

Then in 1997 Perry and Cooperstock [15] found three numerical example showing that the equilibrium was possible for naked-b.h. configurations using an exact solution.

Finally it is to mention the Bini-Geralico-Ruffini articles [32, 33], in which the authors found, using the Zerilli perturbative approach, the correction to the test particle approximation, considering the back-reaction of the particle to the background until the first order. Surprisingly they found that the Bonnor condition remain unchanged also considering these corrections.

For what concerns the exact solutions history, the first two important articles were the ones of Majumdar and Papapetrou [34, 35], which exhibited the fields of an arbitrary number of sources in reciprocal equilibrium, each one with $m_i = e_i$. 

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For many years that was the only exact result known, the next step was made by Belinski and Zakharov [10, 11] in 1978 with the foundation of the Inverse Scattering Method in General Relativity (purely gravitational), which was then extended also to the Einstein-Maxwell equations by Alekseev [12] (see Ref. [9] for a self-consistent review). This method allows to find stationary, axially symmetric solutions with an arbitrary number of sources. From this time in principle the solution of our problem was available. However, practically, the constraints necessary to eliminate the rotation, the conic singularity and the unphysical parameters (NUT parameter, magnetic charge) were too complicate to be handled analytically.

The next step was made by Ernst and Hauser [36, 37], Sibgatullin [38] and Alekseev [20], who developed different integral equation methods for constructing of solutions of Einstein-Maxwell equations. (The first method of such kind for pure gravity was already formulated in Ref. [10]). The method of Ref. [20] was used by Alekseev and Belinski to find the present solution [17] (see also Ref. [18]), the important achievement of which is the extreme simplicity of the formulas and of the equilibrium condition.

### 2.3 Summary of the Alekseev-Belinski formulas

The following (2.3)–(2.11) formulas are the reproduction of formulas (1)-(10) of Ref. [17].

The solution, which can be interpreted as the non-linear superposition of
<table>
<thead>
<tr>
<th>Table 2.1: Some historical remarks.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Perturbation Methods</strong></td>
</tr>
<tr>
<td><strong>Copson</strong> (1927)</td>
</tr>
<tr>
<td>Electric field of a test charge</td>
</tr>
<tr>
<td>near a Schwarzschild b.h.</td>
</tr>
<tr>
<td><strong>Hanni – Ruffini</strong> (1973)</td>
</tr>
<tr>
<td>Electric force lines of a test</td>
</tr>
<tr>
<td>charge near a Schwarzschild b.h.</td>
</tr>
<tr>
<td><strong>Linet</strong> (1976)</td>
</tr>
<tr>
<td>A correction of Copson solution</td>
</tr>
<tr>
<td><strong>Belinski – Zakharov</strong> (1978)</td>
</tr>
<tr>
<td><strong>Vacuum Solitons</strong></td>
</tr>
<tr>
<td>and <em>Alekseev</em> (1980)</td>
</tr>
<tr>
<td><strong>Electrovacuum Solitons</strong></td>
</tr>
<tr>
<td>Solutions of <em>Hauser – Ernst</em></td>
</tr>
<tr>
<td>and <em>Sibgatulling</em> (1984)</td>
</tr>
<tr>
<td>by IEM for rational axis data,</td>
</tr>
<tr>
<td>and of <em>Alekseev</em> (1985)</td>
</tr>
<tr>
<td>by IEM for rational monodromy data</td>
</tr>
<tr>
<td><strong>Integral Equation Method</strong></td>
</tr>
<tr>
<td><strong>Bonnor</strong> (1993)</td>
</tr>
<tr>
<td>Equilibrium of a test particle</td>
</tr>
<tr>
<td>on RN background</td>
</tr>
<tr>
<td><strong>Perry – Cooperstock</strong> (1997)</td>
</tr>
<tr>
<td>Equilibrium is possible</td>
</tr>
<tr>
<td>(3 numerical examples)</td>
</tr>
<tr>
<td><strong>Bini – Geralico – Ruffini</strong></td>
</tr>
<tr>
<td>Equilibrium of a test charge on RN</td>
</tr>
<tr>
<td>with back-reaction until first</td>
</tr>
<tr>
<td>order</td>
</tr>
</tbody>
</table>

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two RN source at a fixed distance on the z-axis, is of the form

\[ ds^2 = Hdt^2 - \frac{\rho^2}{H}d\phi^2 - f(d\rho^2 + dz^2) \]  
\[ A_t = \Phi, \quad A_\phi = A_\rho = A_z = 0 \]

where \( H, f \) and \( \Phi \) are real function of \( \rho \) and \( z \) only. In what follows \( m_1, m_2 \) and \( e_1, e_2 \) are the physical masses and charges of each source respectively\(^2\); the masses include also the interaction energy therefore \( M_{\text{tot}} = m_1 + m_2 \), and \( Q_{\text{tot}} = e_1 + e_2 \). It is convenient to use the spheroidal coordinates \((r_1, \theta_1)\) and \((r_2, \theta_2)\) which are linked to the Weyl coordinates \((\rho, z)\) by:

\[
\begin{align*}
\rho &= \sqrt{(r_1 - m_1)^2 - \sigma_1^2 \sin^2 \theta_1} \\
z &= z_1 + (r_1 - m_1) \cos \theta_1
\end{align*}
\]
\[
\begin{align*}
\rho &= \sqrt{(r_2 - m_2)^2 - \sigma_2^2 \sin^2 \theta_2} \\
z &= z_2 + (r_2 - m_2) \cos \theta_2
\end{align*}
\]

By definition \( l \equiv z_2 - z_1 \) is the distance, expressed in the Weyl coordinate \( z \), between the two objects. Then, the explicit solution is:

\[ H = \frac{[\frac{(r_1 - m_1)^2 - \sigma_1^2 - \gamma^2 \sin^2 \theta_2]}{D^2}][\frac{(r_2 - m_2)^2 - \sigma_2^2 + \gamma^2 \sin^2 \theta_1]}{D^2}][\frac{(r_1 - m_1)^2 - \sigma_1^2 \cos^2 \theta_1]}{D^2}][\frac{(r_2 - m_2)^2 - \sigma_2^2 \cos^2 \theta_2]}{D^2}] \]  
\[ \Phi = \frac{[(e_1 - \gamma)(r_2 - m_2) + (e_2 + \gamma)(r_1 - m_1) + \gamma(m_1 \cos \theta_1 + m_2 \cos \theta_2)]}{D} \]  
\[ f = \frac{D^2}{[(r_1 - m_1)^2 - \sigma_1^2 \cos^2 \theta_1]}[\frac{(r_2 - m_2)^2 - \sigma_2^2 \cos^2 \theta_2]}{D^2}] \]

where

\[ D = r_1r_2 - (e_1 - \gamma - \gamma \cos \theta_2)(e_2 + \gamma - \gamma \cos \theta_1), \]  

\(^2\)The expressions were found with the help of the Gauss theorem.
while $\gamma$, $\sigma_1$ and $\sigma_2$ are defined by:

$$
\gamma = (m_2 e_1 - m_1 e_2) (l + m_1 + m_2)^{-1},
$$

(2.10)

$$
\sigma_1^2 = m_1^2 - e_1^2 + 2 e_1 \gamma, \quad \sigma_2^2 = m_2^2 - e_2^2 - 2 e_2 \gamma.
$$

It is easy to see that $(fH)_{\rho=0} = 1$ on the whole axis, i.e. automatically there is no conic singularity. The above formulas give the solution satisfying the Einstein-Maxwell system only under the equilibrium condition

$$
m_1 m_2 = (e_1 - \gamma)(e_2 + \gamma).
$$

(2.11)

Each of the parameters $\sigma_1$ and $\sigma_2$ can be either real (in the case of a black hole) or imaginary (for a naked singularity); however in the following it will be always

$$
\sigma_1^2 > 0, \quad \sigma_2^2 < 0, \quad \text{and} \quad \sigma_1 > 0
$$

(2.12)

i.e. the first source is “dressed” and the second is “naked”. Since we want to deal only with separable objects, we require also the non-overcrossing condition

$$
l - \sigma_1 > 0
$$

(2.13)

(it means that the naked singularity must be outside the horizon). Using (2.11), the distance $l$ can be written as a function of the other parameters by the very simple formula:

$$
l = -m_1 - m_2 + \frac{m_1 e_2 - m_2 e_1}{2(m_1 m_2 - e_1 e_2)} \left[ (e_2 - e_1) \pm \sqrt{(e_1 + e_2)^2 - 4m_1 m_2} \right]
$$

(2.14)

we always choose the sign in front of the root in (2.14) in order to satisfy the non-overcrossing condition (2.13). From (2.14) it is clear that the parameters
must satisfy the restriction

\((e_2 + e_1)^2 > 4m_1m_2\). \hspace{1cm} (2.15)

### 2.4 Some further Details of the Solution

The solution has a very simple form, the only price to pay is just the simultaneous use of two pairs of coordinates. Obviously for practical purposes, as for the electric lines plot, one needs the use of only one system—in our case we choose \((r_1, \theta_1)\), the one related to the black hole (which is centered on the origin, since we took \(z_1 = 0\) for simplicity, and consequently \(z_2 = l\)). The linking relations are:

\[
\begin{cases}
  r_2 - m_2 = \frac{1}{\sqrt{2}} \sqrt{b^2 + \sqrt{b^4 - 4\sigma_2^2(z-z_2)^2}} \\
  \cos \theta_2 = (z-z_2)(r_2 - m_2)^{-1}
\end{cases}, \hspace{1cm} (2.16)
\]

where \(b^2 \equiv \rho^2 + \sigma_2^2 + (z-z_2)^2\), while \(\rho\) and \(z\) have to be expressed using the first couple of (2.5). We take the plus sign of the roots in the first of (2.16) since \(r_1\) and \(r_2\) must coincide at infinity.

The peculiarity of the coordinates used needs a clarification in order to understand the physical property of the solution, first of all where the “true” divergences are and what happens on the horizons.

**Using** \((r_1, \theta_1)\)

These coordinates are centered on the black hole and can be considered as the natural generalization of the Schwarzschild ones. For the peculiar
choice of the \((r_1, \theta_1)\)-coordinates, the horizon remains a perfect circle (it can be seen also analytically that \(H\) vanishes at \(r_h = m_1 \pm \sigma_1\) as for the single RN black hole). However the spherical symmetry is only apparent, indeed the invariants have a \(\theta_1\)-dependence and vary on the horizon. In this frame is not possible to reach the inside of the spheroid \(r_2 < m_2\) (we called the surface \(r_2 = m_2\) the ‘critical spheroid’, as in [17]), therefore the second source (the naked RN centered in \(z = z_2\)), appears squeezed “inside” a horizontal segment that cuts the vertical axis: this happens because the naked singularity lies inside the region not covered by \((r_1, \theta_1)\).

**Using \((r_2, \theta_2)\)**

Conversely, if one would to use \((r_2, \theta_2)\), the ‘critical spheroid’ of the naked RN will appear as a sphere of coordinates \(r_2 = m_2\), while the black hole horizon as a segment squeezed on the axis: in this case it is the ‘critical spheroid’ of the first source, i.e. \(r_1 < m_1\), that cannot be reached. Again, that has nothing to do with physics but just with the choice of the coordinate system).

<table>
<thead>
<tr>
<th>Physical description</th>
<th>Location</th>
</tr>
</thead>
</table>
| Horizon                               | \(\{ \rho = 0, \ z_1 - \sigma_1 \leq z \leq z_1 + \sigma_1 \} \)  
|                                       | or equivalently \(\{ r_1 = m_1 + \sigma_1, \ \forall \theta_1 \} \) |
| Critical spheroid of the naked singularity | \(\{ 0 \leq \rho \leq \text{Im}[\sigma_2], \ z = z_2 \} \)  
|                                       | or equivalently \(\{ r_2 = m_2, \ \forall \theta_2 \} \) |

Table 2.2: The two peculiar regions in Weyl and in the spheroidal coordinates.

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Table 2.3: Characteristic points of the first source. Note the degeneracy of the Weyl coordinates. For the numerical evaluation we used \( m_1 = 1, e_1 = 0.7, m_2 = 0.3, e_2 = 0.44, l = 5 \) (the same used for fig.1). The central singularity of the b.h. is splitted in two points: 

\[
r_1^{(I)} = \frac{m_1 + m_2 + l - \sqrt{(m_1 + m_2 + l)^2 - 4e_1e_2}}{2}, \quad r_1^{(II)} = \frac{m_1 - m_2 - l + \sqrt{(m_1 - m_2 - l)^2 - 4e_1e_2}}{2}.
\]

In Table 2.4 we localize the two peculiar regions (the horizon and the critcs spheroid), using Weyl coordinates, with the respective translations in \((r_1, \theta_1)\) or \((r_2, \theta_2)\); while in Tables 2.4-2.4 we give a detailed description of the relevant physical quantities in the notable points of these two zones. It is also to note the “degeneracy” of the Weyl coordinates: to the same point in \((\rho, z)\) it can corresponds different values of the spheroidal coordinates.
<table>
<thead>
<tr>
<th>Description</th>
<th>((r_2, \theta_2))</th>
<th>((\rho, z))</th>
<th>H</th>
<th>(\Phi)</th>
<th>f</th>
<th>(F^{ij}F_{ij})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naked sing.</td>
<td></td>
<td>(0, l)</td>
<td>+\infty</td>
<td>-\infty</td>
<td>0</td>
<td>-\infty</td>
</tr>
<tr>
<td>Crossing of the cut</td>
<td></td>
<td>(0, l)</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
</tr>
<tr>
<td>i.e. (down border)</td>
<td></td>
<td>(0, l)</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
</tr>
<tr>
<td>Extremes of the cut</td>
<td></td>
<td>(\text{Im}[\sigma_2], l)</td>
<td>fin.</td>
<td>fin.</td>
<td>+\infty</td>
<td>fin</td>
</tr>
</tbody>
</table>

Table 2.4: Characteristic points of the naked source: the first three points correspond both to \((\rho = 0, z = z_2 = 0)\). The same numerical values of the Tab. 2.4 are used.

**The electromagnetic invariant**

In order to understand where the charges are located it is useful to consider the electromagnetic invariant \(\mathcal{F} = F^{ij}F_{ij}/2\). For the solution (2.3) it has the form:

\[
\mathcal{F} = -\frac{f H [r_2 - m]^2 - \sigma_1^2 (\partial_{r_1} \Phi)^2 + (\partial_{\theta_1} \Phi)^2}{f H [(r_1 - m_1)^2 - \sigma_1^2 \cos^2 \theta_1]}.
\] (2.17)

It can be seen numerically (see tables 2.4, 2.4) that it diverges inside the horizon and *inside* the critical spheroid of the naked RN \(^3\).

It is also worth noting that on the critical spheroid, although in the \((r_1, \theta_1)\) representation it is a line, the up- and down-limit of \(\mathcal{F}\) do not coincide, since they correspond to different points of the physical space-time. Then,

\(^3\)The spheroid, i.e. the line \(\{0 < \rho < \text{Im}(\sigma_1), \ z = z_2\}\), seems apparently regular in \((r_1, \theta_1)\) coordinates just because its interior can be reached only using \((r_2, \theta_2)\)
looking at $F$ it is possible to see that no real discontinuity exists on the horizon, indeed it diverges only on the central singularities. Finally, the other invariant, $\epsilon^{ijkl}F_{ij}F_{kl} = EB$, is identically zero.

2.5 Plots of the Force Lines

In the plots, what we called “second source” (i.e. the naked RN) is always up, while the “first source” (i.e. the black hole) is always down and centered on the origin. For the physical meaning of the force lines see Part I, 1.5.

The lines are plotted in $(x, y)$ Cartesian coordinates defined as

\[
\begin{align*}
  x &= r_1 \sin \theta_1 \\
  y &= r_1 \cos \theta_1
\end{align*}
\]  

(2.18)

(they coincide with $(\rho, z)$ defined in (2.5) when $r_1 \to \infty$).

In the plots we have used geometrical units ($G = c = 1$), in which the unitary length is given by the Schwarzschild mass $m_1 = 1$.

The graphical Faraday criterium is used, namely we plotted the electric force lines such that

\[
\frac{\text{Number of lines from the first source}}{\text{Number of lines from the second source}} \simeq \frac{e_1}{e_2}.
\]

The separatrix

In general, when there are two charges, the electric force diagram will present a separatrix, which is a force line which reach asymptotically a saddle point of the potential and separates the lines of the two charges in the case they have the same sign, or —in the case of opposite sign charge— it delimits the
region in which the lines flow from one to the other source. We marked these separatrix lines in bold; may be it is worth to mention that on the saddle point they have an invariant definition since on that point $\mathcal{F} = 0$.

**Inside the horizon**

In the following plots the force lines are graphed only outside the horizon since there it is no more possible to consider a static observer, i.e. the physical interpretation given in Sec. 1.5 does not hold because (1.51) becomes imaginary. However, when the separatrix starts from the inside of the horizon, the study of that region is important to understand the difference between cases with the same or opposite charges. Therefore, in the case of fig. 2.3, in which the saddle point is inside the horizon, we calculated the point where the separatrix touches the horizon, and we plotted the diagram just from there. (This was possible because mathematically the eqn. (1.50) is well defined also inside the horizon).

In the following three sub-sections we analyze the three qualitatively different sub-cases: $e_1e_2 > 0$ (2.5.1), $e_1e_2 < 0$ (2.5.2), and finally $e_1 = 0$ (2.5.3).
2.5.1 Two charges of equal sign \((e_1 e_2 > 0)\)

General case: two comparable RN sources

Let us consider the case in which the two RN sources have charges and masses of comparable dimensions

\[
m_1 \approx m_2 , \quad e_1 \approx e_2
\]

\[
m_1^2 > e_1^2 \quad m_2^2 < e_2^2 \quad (2.19)
\]

\[
e_1 e_2 > 0.
\]

This is the closest case to the classical picture, indeed here the equilibrium is mainly due to the classical balance of the electrostatic force and gravitational field. The resulting plot is given in fig.(2.1).

The qualitative behavior of the force lines does not change with the changing of the distance \(l\).

Small\(^4\) charge (naked) near a RN black hole

The equilibrium configurations of this case (see fig. 2), with

\[
m_1 \gg m_2 , \quad |e_1| >> |e_2|
\]

\[
m_1^2 > e_1^2 \quad m_2^2 < e_2^2 \quad (2.20)
\]

\[
e_1 e_2 > 0,
\]

have been studied in the test particle approximation first in Ref. [14], and recently in Ref. [32]-[33], where they took in account also the back-reaction of the test particle.

\(^4\)Here and in the following we say ‘small’ charge and not ‘test’ charge because the exact nature of the solution automatically takes in account all the back-reaction terms even if they can be very small (while the ‘test’ limit is in general referred as the one in which all those terms are completely neglected).
Figure 2.1: Force lines in the general case (2.19), when the two RN have charges of the same sign. Note that the critical spheroid in that coordinate representation (2.18) is an horizontal segment. The bold line is the separatrix. The circle on the bottom is the external horizon of the first source. Parameters used: $m_1 = 1, e_1 = 0.7, m_2 = 0.3, e_2 = 0.44, l = 5$.

**Small charge (with horizon) near a naked RN**

This case does not exist for $e_1 e_2 > 0$.

**2.5.2 Two charges of opposite sign ($e_1 e_2 < 0$)**

Although it is easy to show that in the previous cases with $e_1 e_2 > 0$ the implications

$$m_1^2 > e_1^2 \Rightarrow \sigma_1^2 > 0$$

$$m_2^2 < e_2^2 \Rightarrow \sigma_2^2 < 0,$$

(2.21)
Figure 2.2: Force lines of a small charge near a RN with horizon, case (2.20). Parameters used: $m_1 = 1$, $e_1 = 0.1$, $m_2 = 10^{-3}$, $e_2 = 1.3 \cdot 10^{-2}$, $l = 2.5$). The bold line is the separatrix.

are always true, it is not so if $e_1 e_2 < 0$. However in the following we considered two cases in which (2.21) holds.

**Two comparable RN sources**

This case, with

$$m_1 \approx m_2 \quad e_1 \approx -e_2$$

$$m_1^2 > e_1^2 \quad m_2^2 < e_2^2$$

(2.22)

$$e_1 e_2 < 0,$$

is the case in which the relativistic effects are much evident since here also the electric force is attractive (see fig. (2.3)): in this case the equilibrium is due to the repulsive nature of the naked singularity.
Figure 2.3: Force lines in the general case (2.22), with charges of the opposite sign. Parameters used: $m_1 = 1$, $e_1 = 0.05$, $m_2 = 0.3$, $e_2 = -1.66$, $l = 5$. The bold line is the separatrix, which now encircles also the central singularity of the b.h.: inside that region the lines go from one charge to the other. Outside that region the lines go from $e_2$ to infinity (some of them pass also through the horizon).
Small charge near a RN

It is also possible to find values that corresponds to a small charge with horizon near a naked RN:

\[ m_1 \ll m_2, \quad |e_1| \ll |e_2|, \]
\[ m_1^2 > e_1^2, \quad m_2^2 < e_2^2, \]
\[ e_1 e_2 < 0. \]  

(2.23)

However in this case it would be useless to plot the force lines because the electric field is trivially Coulombian (the first source is weakly interacting both gravitationally and electrically).

The inverse case, namely a small charge \textit{naked} near a RN \textit{with horizon}, does not exist for particles lying outside the horizon (i.e. requiring \( l > \sigma_1 \)), as noted by Bonnor [14].

2.5.3 Cases with only one charge

In the following we will consider the cases with a naked singularity near a neutral black hole; they are qualitatively different from the previous ones since now there is no separatrix and the electric flux over the horizon surface is zero.

In the particular case in which the first source is neutral (i.e. \( e_1 = 0 \)), the equilibrium distance is even simpler,

\[ l = -m_1 - m_2 + \frac{e_2^2}{2m_2} \left( 1 \pm \sqrt{1 - 2m_1 \left( \frac{e_2^2}{2m_2} \right)^{-1}} \right), \]  

(2.24)
Figure 2.4: Force lines for the values (2.25). The blank circle of radius $2m_1$ is the Schwarzschild horizon. Parameters used: $m_1 = 1$, $m_2 = 0.3$, $e_2 = 1.5$, $l = 5$.

which can be always satisfied for sufficiently large values of the charge parameter $e_2$.

**RN near a Schwarzschild black hole (comparable masses)**

Thanks to the exact nature of the solution, it is very interesting also the case in which the RN source has comparable mass with the Schwarzschild black hole, say

$$m_1 \approx m_2, \quad e_1 = 0,$$

$$\sigma_1 = m_1, \quad m_2^2 < e_2^2,$$  \hspace{1cm} (2.25)

indeed this case cannot be achieved by a perturbative approach, see fig. 4.

It is possible to see that the electric lines are just slightly deformed by the
gravitational field.

**Small charge near a Schwarzschild black hole**

We can also consider the small-charge limit,

\[ m_1 >> m_2, e_2, e_1 = 0, \]

\[ \sigma_1 = m_1, \quad m_2^2 < e_2^2, \quad (2.26) \]

i.e. the second source is a small RN naked singularity. That is the only case in which we have a good comparing in literature, since it is the only case already studied (as we know) by using the force lines plots [13], although by a perturbative approach. Strictly speaking the Hanni-Ruffini case refers to a slightly different situation, since they considered a particle *momentarily* at rest in the Schwarzschild metric, while the AB solution is exactly static\(^5\). However the present solution confirms very nearly their multipole expansion, since we find that the plots are in practice coincident. In order to have the best possible comparing we considered the same distances between the charge and the horizon (figg. 5-7). Since now \( l \) is not an independent parameter we fixed the masses values \( m_1 = 1 \) and \( m_2 = 10^{-4} \), then varying the distance we found (using (2.14)) the relative parameter \( e_2 \). The test particle is at \( z = l \), or equivalently at \( r_1 = l + m_1 \). (Just to clarify the link with [13]'s notations: their \( r \) is our \( r_1 \), and their \( M \) is our \( m_1 \)).

From the flux equation (1.49), considering that now \( \sigma_1 = m_1 \), it is easy to see that the corrections to the Hanni-Ruffini approximation are limited only

\(^5\)From another point of view, Hanni-Ruffini do not use (2.24) to determine the fourth parameter (because in their approximation the fourth parameter, say \( m_2 \), is considered arbitrarily small, therefore it is not present at all)
Figure 2.5: Force lines for the values (2.26), with $l = 3m_1$, i.e. in the spheroidal coordinates the particle is in $r_1 = 4m_1$. The circle of radius $2m_1$ is the Schwarzschild horizon. The plots are practically identical to the ones found by Hanni and Ruffini.
Figure 2.6: Now the distance is $l = 2m_1$, or equivalently the charge is in $r_1 = 3m_1$.

Figure 2.7: Now the distance is $l = 1.2m_1$, or equivalently the charge is in $r_1 = 2.2m_1$. 
on the exact form of the $A_t$ potential, since to use the Schwarzschild metric or the functions $H$ and $f$ given in (2.6) and (2.8) does not change the force lines.

2.6 Stability analysis

In the following we will study the stability of the AB solution in a very restrictive sense, i.e. with respect to spatial displacements of the two sources. Indeed there can be a lot of different perturbations, as rotational ones, anyway we think that these are some of the most physically significant. Our analysis make no use of the usual perturbative methods (i.e. to put a perturbation in the Einstein-Maxwell equations and see how they evolve in time), we use instead the properties of the conical singularity, following the Sokolov-Starobinski definition.

If the two sources are displaced at a different distance, say $l = l_0 + x$, then one can have still a static solution but in that case it appears a conic singularity between the two bodies[18], namely on $\{\rho = 0, z \in [z_1, z_2]\}$ (let us suppose that source-1 is in $z_1 = 0$ and source-2 in $z_2 = l$). This is interpreted as a strut or a string to which it can be associated a force. It is called “strut” if it has pressure, “string” if it has tension; since the difference is only a sign, in the following we will call them simply “strut”.

Now, we assume that in the reality there will be no struts if the two sources will be displaced from the equilibrium position, but that the two sources will oscillate near the distance $l_0$ (stable configuration), or go far away or collapse.
Then we assume that the force exercised by the two bodies one to the other will be precisely the opposite of $F_{\text{Strut}}$, say

$$F_{\text{Bodies}} = -F_{\text{Strut}} ; \quad (2.27)$$

indeed the eventual presence of a strut with such a force would balance the repulsion/attraction of the bodies, keeping the system exactly in “equilibrium”, with $F_{\text{tot}} = F_{\text{Bodies}} + F_{\text{Strut}} = 0$. Therefore once that one has found the $F_{\text{Bodies}}$ the analysis of the equilibrium follows the usual procedure of classical mechanics.

### 2.6.1 Force of the strut.

To calculate the force of the strut we need to calculate the energy-momentum tensor $T^j_i$ on this segment. Since we know the metric (from the AB solution in his general 5-parameters formulation[18]) it is convenient to define $T^j_i$ by the Einstein equations

$$8\pi T^j_i = R^j_i - 1/2 \delta^j_i R.$$

If the parameter $l \neq l_0$, then the AB metric has a conic singularity; that means that if we expand the solution near the axis, it has the form:

$$ds^2 = a^2 \rho^2 d\varphi^2 + d\rho^2, \quad t, \rho, z = \text{const.}, \quad \rho \approx 0, \quad 0 < z < l, \quad (2.28)$$

where $a$ is a constant (in general $a \neq 1$ between the two sources) given by

$$a = \frac{1}{(\sqrt{fH})_{\rho=0}} = \frac{1 + 2\delta}{1 - 2\delta} \quad (2.29)$$
with
\[\delta = \frac{m_1 m_2 - (e_1 - \gamma)(e_2 + \gamma)}{(l_0 + x)^2 - m_1^2 - m_2^2 + (e_1 - \gamma)^2 + (e_2 + \gamma)^2}\] (2.30)
\[\gamma = \frac{m_2 e_1 - m_1 e_2}{l_0 + x + m_1 + m_2}\] (2.31)
\[l_0 = -m_1 - m_2 + \frac{m_1 e_2 - m_2 e_1}{2(m_1 m_2 - e_1 e_2)} \left[ (e_2 - e_1) \pm \sqrt{(e_1 + e_2)^2 - 4m_1 m_2} \right].\] (2.32)

Then a direct calculation of \( R^j_i \) gives \( R^j_i = 0, R = 0 \), and thus \( T^j_i = 0 \).
However, we can introduce a distribution-like source using the Gauss-Bonnet theorem as in Ref.[29]:
\[\int_S K d\sigma = 2\pi - \int_{\partial S} k_g ds ,\] (2.33)
where
\[K\] is the Gaussian curvature, it is the half of the Ricci scalar, \( K = R/2 \); \( k_g \) is the geodesic curvature:
\[k_g = \epsilon_{ij} \left( \frac{d^2 x^i}{ds^2} \frac{dx^j}{ds} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} \right) \left( g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \right)^{-1/2}\] (2.34)
If \( S \) is a (small) disk of radius \( \epsilon \) around the surface, we obtain:
\[R = \pi \frac{1 - a}{a} \delta(\rho), \quad \int_{0}^{\epsilon} \int_{0}^{2\pi} \rho \delta(\rho) d\rho d\phi \equiv 1\] (2.35)
the result is independent of the radius of the disk.

Then, using \( R^z_z = R/2 \) and the Einstein equations we find:
\[T^z_z = \frac{1 - a}{4a} \delta(\rho),\] (2.36)
using which we find the expression of the force:
\[F_{Strut} = -\int_{0}^{\epsilon} \int_{0}^{2\pi} T^z_z \sqrt{g_{\rho\rho} g_{\phi\phi}} d\rho d\phi\]
\[= \frac{1}{2} (a - 1).\] (2.37)
2.6.2 The different cases in the AB solution

In the AB solution, for \( l = l_0 + x \), the Eqn.(2.37) becomes[39]:

\[
F_{\text{Strut}} = \frac{\delta}{1 - 2\delta} = \frac{m_1 m_2 - (e_1 - \gamma)(e_2 + \gamma)}{l^2 - (m_1 + m_2)^2 + (e_1 + e_2)^2}
\] (2.38)

at \( x = 0 \) one has \( l = l_0 \) and \( F_{\text{Strut}} = 0 \).

Therefore the stability can be deduced from the sign of the derivative of the force w.r.t. \( x \); obviously we have to evaluate this quantity on the equilibrium point \( x = 0 \):

\[
(\partial_x F_B)_{x=0} = \frac{m_2 e_1 - m_1 e_2}{(l_0 + m_1 + m_2)^2} \left[ \frac{2\gamma_0 - e_1 + e_2}{l_0^2 - m_1^2 - m_2^2 + (e_1 - \gamma_0)^2 + (e_2 + \gamma_0)^2} \right];
\] (2.39)

where \( \gamma_0 \) is \( \gamma \) evaluated on \( x = 0 \); the stability condition is:

\[
(\partial_x F_B)_{x=0} < 0 .
\] (2.40)

The previous formula can be simplified without loss of generality using the following considerations\(^6\):

1. Using the arbitrariness of the electric charge’s sign definition:

\[
e_2 > 0;
\] (2.41)

2. Since we are considering a black hole and a naked singularity:

\[
\frac{e_1}{m_1} < 1 < \frac{e_2}{m_2} ;
\] (2.42)

\(^6\)Obviously a necessary condition for the stability is also that the equilibrium exists.
3. Separability requirement:

\[ l_0 > \sigma_1; \quad (2.43) \]

4. Finally, the existence of a real \( l_0 \) needs[2]:

\[ (e_2 + e_1)^2 > 4m_1m_2. \quad (2.44) \]

Thus, from conditions (2.41) and (2.42) we find that

\[ \gamma_0 < 0 \quad \text{always}, \quad (2.45) \]

Then, from Eq.(2.43) it is easy to show that the denominator in Eq.(2.40) is always positive, i.e.

\[ l_0^2 - m_1^2 - m_2^2 + (e_1 - \gamma_0)^2 + (e_2 + \gamma_0)^2 = l_0^2 - \sigma_1^2 - \sigma_2^2 + 2\gamma_0^2 > 0 \quad (2.46) \]

(indeed \( \sigma_2^2 < 0 \) because source-two is naked). Thus, stability condition (2.40) can be reduced to the following one:

\[ 2\gamma_0 - e_1 + e_2 > 0. \quad (2.47) \]

If the previous inequality is not fulfilled, it means that the configuration is unstable. Inequality (2.47) can be rewritten as:

\[ (m_2 - m_1)(e_1 + e_2) + (e_2 - e_1)l_0 > 0; \quad (2.48) \]

for commodity in the following we define the quantity

\[ X \equiv (m_2 - m_1)(e_1 + e_2) + (e_2 - e_1)l_0, \]

which is anyway an irrational 4-parameters quantity.
(A) Equal signed charges \((e_1 > 0, e_2 > 0)\)

This is the presumably-only case in which we found also unstable equilibria.

**Sub-case (A.1) \(m_1 < m_2\). (BH smaller than NS)**

If \(m_2 > m_1\) then, necessarily from (2.42), \(e_2 > e_1\); consequently \(X > 0\) is always satisfied and the equilibrium is always stable.

**Sub-case (A.2) \(m_2 < m_1\). (BH larger than NS)**

**Sub-sub-case (A.2.1) \(m_2 < m_1\) and \(e_1 < e_2\).** Numerically we found only stable equilibrium (when it exist).

**Sub-sub-case (A.2.2) \(m_2 < m_1\) and \(e_2 < e_1\).** In this case \(X\) is always negative and thus the equilibrium unstable. A particular situation of this sub-case is the small-particle limit (i.e. \(m_2 = e_2, e_2 \to 0\), with \(\alpha < 1\) constant)

That agrees with the instability found by Bonnor[14].

(B) Opposite signed charges \((e_1 < 0, e_2 > 0)\)

In this case we suspect that the equilibrium is always stable (however there is one sub-case in which we was not able to demonstrate it analytically).

Since \(X\) now is

\[
X = (m_2 - m_1)(e_2 - |e_1|) + (e_2 + |e_1|)l_0, \quad (2.49)
\]

then we can consider the two different sub-cases: \(m_2 > m_1\) and \(m_1 > m_2\).
Sub-case (B.1) \( m_2 > m_1 \). (BH smaller than NS)

If \( m_2 > m_1 \), then from condition (2.42) we have necessarily \( e_2 > e_1 \), which implies that \( X \) is always positive.

Sub-case (B.2) \( m_1 > m_2 \). (BH larger than NS)

Otherwise, if \( m_1 > m_2 \), then we need to consider the two different sub-sub-cases: \( |e_1| \gtrless e_2 \).

Sub-sub-case (B.2.1) \( m_1 > m_2, \ |e_1| > e_2 \). If \( |e_1| > e_2 \) then one can see at first sight from (2.49) that \( X \) is always positive.

Sub-sub-case (B.2.1) \( m_1 > m_2, \ |e_1| < e_2 \). If \( |e_1| < e_2 \) we are not able to demonstrate that \( X \) is always positive, but we can say that at least for enough large values of \( e_2 \) this is true, because

\[
\lim_{e_2 \to \infty} l_0 \approx \frac{m_1}{|e_1|} e_2, \quad (2.50)
\]

and thus \( X \to \frac{m_1}{|e_1|} e_2^2 \to +\infty \).

(C) One charge only \( (e_1 = 0) \)

This case is always stable. Indeed, considering \( e_1 = 0 \), the stability condition (2.48) becomes:

\[
l_0 + m_2 > m_1. \quad (2.51)
\]
Then, considering the separability condition, which is now:

\[ l_0 > \sigma_1 = m_1, \]  

(2.52)

it is immediate to see that (2.51) is always true.

### 2.7 Conclusions II

The main result of our analysis is that the exact solution seems to confirm quite strictly the test-charge approximation on the RN background (see, e.g. Ref. [14]), which seems to give a good test of the exact picture.

**Size of the naked singularity** Sometimes in literature has been guessed (Ref. [40], cap.15; Ref. [41]) that \( e^2 / 2m \) should be considered as a ‘critical radius’ of the naked singularity inside of which the RN solution has no physical meaning since it should be matched with a more realistic matter field tensor, in order to avoid the well known problems of a pointlike source, as the divergence of the electric energy.

If the quantity \( e_2^2 / 2m_2 \) can be roughly considered as the physical size of the RN charge, then from formula (2.24) it is easy to see that the equilibrium configurations exist only for \( e_2^2 / 2m_2 \) larger than the Schwarzschild radius \((2m_1)\). That seems to suggest that a real ‘small’ charge limit cannot be achieved, in the sense that the particle can be ‘small’ only gravitationally (and electrically), but *not geometrically* because it would have a size larger than the Schwarzschild horizon.
However in the next chapter we will present a finite-size model of RN source with a radius smaller than $e_2^2/2m_2$.

**Coordinate dependence of the plots** Any plot of the force lines change drastically for different choices of the coordinates. However, what is interesting is to compare different situation by using the same coordinate representation, e.g. as we did for the Hanni-Ruffini case.

**Stability**

1. The most of the cases are stable.

2. The only unstable case we found is the one in which the naked RN has a smaller mass than the RN black hole (with equal-signed charge). That agrees with Bonnor limit.

3. The one-charge case is always stable.

4. This criterion of stability, although very peculiar, can be considered at least as a *necessary* condition for the stability in general sense. It becomes also *sufficient* in the limit in which one source is much smaller than the other.

5. Considering the values of the parameters used in the previous plots one finds that they are all stable, except the case with a small charge (with horizon) near the naked RN.
Chapter 3

A membrane model for the Reissner-Nordstrom singularity
This chapter is based on Ref.[3].

3.1 Introduction

One of the interesting effects of relativistic gravity which has no analogue in the Newtonian theory is the presence of gravitational repulsive forces. The classical example is the Reissner-Nordstrom (RN) field in the region close enough to the central singularity. Indeed, in the RN metric

\[-ds^2 = -f c^2 dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\]  

where

\[f = 1 - \frac{2kM}{c^2 r} + \frac{kQ^2}{c^4 r^2},\]  

the radial motion of a test neutral particle follows the equation:

\[\frac{d^2 r}{ds^2} = -\frac{1}{2} \frac{df}{dr} = \frac{k}{c^4 r^2} \left( \frac{Q^2}{r} - Mc^2 \right)\]  

from where one can see the appearance of repulsive force in the region of small \(r\). In this zone the gradient of the gravitational potential \(f(r)\) is negative and the gravitational force in Eq.(3.3) is directed toward the outside of the central source.

For the RN naked singularity case \((Q^2 > kM^2)\), in which we are interested in the present study, the potential \(f(r)\) is everywhere positive and has a minimum at the point \(r = Q^2/Mc^2\). Therefore at this point a neutral particle can stay at rest in the state of stable equilibrium (the detailed study can be found in [42, 14]).
It is an interesting and nontrivial fact that the same sort of stationary equilibrium state due to the repulsive gravity exists also as an exact asymptotically flat two-body solution of the Einstein Maxwell equations which describes a Schwarzschild black hole situated at rest in the field of a RN naked singularity without any strut or string between these two objects [17, 19]. However, solutions of this kind have the feature that the object creating the repelling region has naked singularity and this last property has no clear physical interpretation. Consequently the pertinent question is whether the repelling phenomenon around a charged source arises only due to the presence of the naked singularity or it can be also a feature of physically reasonable structure of the space-time and matter.

By other words the question is whether or not it is possible to construct a regular material source which can block the central singularity and join the external repulsive region in a proper way. Then we are interested to construct a body with the following properties:

1. inside the body there are no singularities;

2. outside the body there is the RN field (3.1)-(3.2), corresponding to the case $Q^2 > kM^2$;

3. the radius of the body is less than $Q^2/Mc^2$, so between the surface of the body and the sphere $r = Q^2/Mc^2$ arises the repulsive region;

4. such stationary state of the body is stable with respect to collapse or expansion.
In Ref. [3] we have proposed a new model for such body in the form of spherically symmetric thin membrane with positive tension. We assert that there exists a physically acceptable range of parameters for which all the above four conditions (1)-(4) can be satisfied. We illustrate this conclusion by the especially transparent case of a Nambu-Goto membrane with equation of state $\epsilon = \tau$.

Then the existence of everywhere-regular material sources possessing RN “antigravity” properties in the vicinity of their surfaces attribute to this phenomenon and to the RN naked singularity solution more sensible physical status.

It is necessary to mention that at least two exact solutions of Einstein-Maxwell equations representing a compact continuous spherically symmetric distribution of charged matter under the tension producing the gravitationally repulsive forces inside the matter as well as in some region outside of it already exist in the literature. These are solutions constructed in Ref. [43] and Ref. [44]. A more detailed study of these two results can be found in Ref. [45]. An interesting possibility to have a gravitationally repulsive core of electrically neutral but viscous matter has been communicated in Ref. [46].

It is worth to remark that the first (to our knowledge) mentioning of the gravitational repulsive force due to the presence of electric field was made already in 1937 in the Ref. [47] in connection to the nonlinear model of electrodynamics of Born-Infield type. One of the first paper where a repulsive phenomenon in the framework of the conventional Einstein-Maxwell theory
has been mentioned is Ref.[48]. The general investigation of the different aspects of this phenomenon apart from the already mentioned references [42]-[48] can be found also in the more detailed works [49, 50, 51, 52]. Some part of these papers is dedicated to a possibility of construction a classical model for electron. This is doubtful enterprise, however, because the intrinsic structure of electron is a matter out of classical physics. Nonetheless the mathematical results obtained are useful and can be applied to the physically sensible situations, e.g. for construction the models of macroscopical objects.

3.2 Equation of motion of a membrane with empty space inside

The equation of motion for the most general case of a thin charged spherically symmetric fluid shell with tangential pressure moving in the RN field have been derived 38 years ago by J.E. Chase[22]. The corresponding dynamics for a charged elastic membrane with tension follows from his equation simply by the change of the sign of the pressure. We derived, however, the membrane’s dynamics again using a different approach.

Chase used the geometrical method which have been applied to the description of singular surfaces in relativistic gravity in [53] and have been elaborated in [48, 54] for some special cases of charged shells. An essential development of the Israel approach in application to the cosmological domain walls can be found in the series of works of V.Berezin, V.Kuzmin and I. Tkachev, see Ref.[55] and references therein. Our treatment follows the
method more habitual for physicists which have been used in [23], where the motion of a neutral fluid shell in a Schwarzschild field was derived by the direct integration of the Einstein equations with appropriate δ-shaped source. Now we generalized this approach for the charged membrane and charged central source.

Of course, the membrane's equation of motion that we obtained coincides with that of Chase. Nonetheless the different approach to the same problem often has a methodological value and gives new details. We hope that our case makes no exception, then for an interested reader we put the main steps of our derivation in Appendix (where we considered a general case with central source).

In this section we study only the particular solution in which there is no central body, that is inside the membrane we have flat space-time.

Although the basic formulas of this section follow from the Appendix under restriction $M_{in} = Q_{in} = 0$ the exposition we give here is more or less self-consistent. Only the definitions of 4-dimensional membrane's energy density and tension need some clarification which can be found in Appendix.

For the thin spherically symmetric membrane with empty space inside and with radius which depends on time the metrics inside, outside and on membrane are:
\[- (ds^2)_{in} = - \Gamma^2(t)c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.4) \]
\[- (ds^2)_{out} = - f(r)c^2dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.5) \]
\[- (ds^2)_{on} = - c^2d\eta^2 + r_0^2(\eta)(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.6) \]

In the interval (3.6) $\eta$ is the proper time of the membrane. The factor $\Gamma^2$ in (3.4) is necessary to ensure the continuity of the global time coordinate $t$ through the membrane. The metric coefficient $f(r)$ in the region outside the membrane is given by Eq.(3.2).

Matching conditions for the intervals (3.4)-(3.6) through the membrane’s surface are:

\[ [ (ds^2)_{in} ]_{r=r_0(\eta)} = [ (ds^2)_{out} ]_{r=r_0(\eta)} = (ds^2)_{on} \quad (3.7) \]

If the equation of motion of the membrane $r = r_0(\eta)$ is known, then from these conditions the connection $t(\eta)$ between global and proper times and factor $\Gamma(t)$ follow easily:

\[ \Gamma(t) = \frac{f(r_0)\sqrt{1+c^{-2}(r_0,\eta)^2}}{\sqrt{f(r_0) + c^{-2}(r_0,\eta)^2}} \quad (3.8) \]
\[ \frac{dt}{d\eta} = \frac{\sqrt{f(r_0) + c^{-2}(r_0,\eta)^2}}{f(r_0)} \quad (3.9) \]

The differential equation for the function $r_0(\eta)$ follows from Einstein-Maxwell equations with energy-momentum tensor and charge current concentrated on the surface of the membrane. It is:

\[ Me^2 = m(r_0)c^2\sqrt{1 + \left(\frac{dr_0}{c d\eta}\right)^2} + \frac{Q^2}{2r_0} - \frac{k m^2(r_0)}{2r_0} \quad (3.10) \]
Here $m(r_0) > 0$ is the effective rest mass of the membrane in the radially comoving frame. This quantity includes the membrane’s rest mass as well as all kinds of interaction mass-energies between membrane’s constituents, that is those intrinsic energies which are responsible for the tension. The constants $Q$ and $M$ are the total charge of the membrane and total relativistic mass of the system. These are the same constants which appeared earlier in Eq.(3.2). The membrane’s energy density $\epsilon$ and tension $\tau$ are (see Appendix for a further clarification):

$$\epsilon = \epsilon_0(r_0)\delta[r - r_0(\eta)] \quad \tau = \tau_0(r_0)\delta[r - r_0(\eta)]$$ \hspace{1cm} (3.11)

where

$$\epsilon_0 = \frac{m(r_0)c^2}{8\pi r_0^2} \left[ \frac{1}{\sqrt{1 + c^{-2}(r_0,\eta)^2}} + \frac{f(r_0)}{\sqrt{f(r_0) + c^{-2}(r_0,\eta)^2}} \right]$$ \hspace{1cm} (3.12)

$$\tau_0(r_0) = \frac{dm(r_0)}{dr_0} \frac{r_0\epsilon_0(r_0)}{2m(r_0)}$$ \hspace{1cm} (3.13)

The electromagnetic potentials have the form $A_r = A_\theta = A_\phi = 0$, $A_t = A_t(t, r)$ and for the electric field strength $\partial A_t/\partial r$ the solution is

$$\frac{\partial A_t}{\partial r} = \begin{cases} \frac{Q}{r^2} & \text{for } r > r_0(\eta) \\ 0 & \text{for } r < r_0(\eta) \end{cases}$$ \hspace{1cm} (3.14)

The formulas (3.4)-(3.14) give the complete solution of the problem for the case of empty space inside the membrane.

Finally we would like to stress the following important point. As follows from discussion in Appendix, the signs of the square roots $\sqrt{1 + c^{-2}(r_0,\eta)^2}$ and $\sqrt{f(r_0) + c^{-2}(r_0,\eta)^2}$ coincide with the signs of the time component $u^0$.
of the 4-velocity of the membrane evaluated from inside and outside of the
membrane respectively. The component $u^0$ is a continuous quantity by def-
inition and can not change the sign when passing through the membrane’s
surface. Besides, for macroscopical objects we are interested in in this anal-
ysis $u^0$ should be positive. Consequently the both aforementioned square
roots should be positive. From another side it is easy to show that equation
(3.10) can be written also in the following equivalent form

$$Mc^2 = mc^2 \sqrt{f(r_0) + \left(\frac{d r_0}{c d\eta}\right)^2 + \frac{Q^2}{2r_0} + \frac{k m^2}{2r_0}}$$

Then from this expression and from (3.10) follows that both square roots will
be positive if and only if

$$Mc^2 - \frac{Q^2}{2r_0} - \frac{k m^2}{2r_0} > 0$$

This is unavoidable constraint which must be adopted as additional condition
for any physically realizable solution of the equation of motion (3.10) in
classical macroscopical realm.

### 3.3 Nambu-Goto membrane with “antigravity” effect

To proceed further we must specify the function $m(r_0)$, which is equivalent
to specifying an equation of state, as can be seen from (3.13).

Let us analyze the membrane with equation of state $\epsilon = \tau$. This model
can be interpreted as “bare” Nambu-Goto charged membrane[56, 57], or as
Zeldovich-Kobzarev-Okun charged domain wall[58]. It follows from (3.13) that for such type of membrane we have:

\[ m = \sigma r_0^2 \]  

(3.17)

where \( \sigma \) is an arbitrary constant. In this and next section we consider only the case of positive constants \( \sigma \) and \( M \):

\[ \sigma > 0, \quad M > 0. \]  

(3.18)

The sign of \( Q \) is of no matter since the charge appear everywhere in square. Now we write the equation of motion (3.10) in the following form:

\[ 4 \left( \frac{d r_0}{c d \eta} \right)^2 - \left( \frac{k \sigma r_0}{c^2} + \frac{2 M}{\sigma^2} - \frac{Q^2}{c^2 \sigma^3} \right)^2 = -4. \]  

(3.19)

Formally this can be considered as the equation of motion of a non-relativistic particle with the “mass” equal to 8 moving in the potential \( U(r_0) \),

\[ U(r_0) = -\left( \frac{k \sigma r_0}{c^2} + \frac{2 M}{\sigma^2} - \frac{Q^2}{c^2 \sigma^3} \right)^2 \]  

(3.20)

and under that condition that particle is forced to live on the “total energy” level equal to minus four.

For the existence of the stable stationary state we are interested in, the following conditions should hold:

1. The gravitational field in the exterior region should correspond to the super-extreme RN metric:

\[ Q^2 > kM^2. \]  

(3.21)
2. The potential \( U(r_0) \) should have a local minimum at some value \( r_0 = R_{\text{min}} \). The form (3.20) of \( U(r_0) \) permit this if and only if

\[
k\sigma^2 Q^6 < (Mc^2)^4.
\]

(3.22)

Under this restriction the potential \( U(r_0) \) has three extrema, two maxima at points \( r_0 = R_{\text{max}}^{(1)} \) and \( r_0 = R_{\text{max}}^{(2)} \) and a minimum which is located between them: \( R_{\text{max}}^{(1)} < R_{\text{min}} < R_{\text{max}}^{(2)} \). We show the shape of the potential \( U(r_0) \) for this case in Fig.1.

The equation \( U(r_0) = 0 \) has only one real root and this is also the first local maximum \( R_{\text{max}}^{(1)} \). The minimum and the second maximum are coming as two other roots of the equation \( \frac{dU}{dr_0} = 0 \).

The equation for \( R_{\text{min}} \) is:

\[
k\sigma^2 R_{\text{min}}^4 - 4Mc^2 R_{\text{min}} + 3Q^2 = 0.
\]

(3.23)

This fourth order equation has only two real solutions and \( R_{\text{min}} \) is the smaller one.

3. For the stationary position of the membrane at the minimum of the potential we must ensure the relation \( U(R_{\text{min}}) = -4 \) which is:

\[
\frac{k\sigma}{c^3} R_{\text{min}} + \frac{2M}{\sigma} R_{\text{min}}^{-2} - \frac{Q^2}{c^2 \sigma} R_{\text{min}}^{-3} = 2
\]

(3.24)

(the minus two in the r.h.s. of (3.24) would be incompatible with Eq.(3.23) under condition (3.18)).
Figure 3.1: The membrane's motion can be described as the motion of a non-relativistic point particle in the potential $U(r_0)$. 
4. To have repulsive region it is necessary for the membrane’s radius $R_{\text{min}}$ to be less than the minimum of the gravitational potential $f(r)$, that is less than the quantity $Q^2/Mc^2$. In this case outside of the membrane surface in the region $R_{\text{min}} < r < Q^2/Mc^2$ we have the repulsive effect. Then we demand:

$$R_{\text{min}} < \frac{Q^2}{Mc^2}.$$  \hspace{1cm} (3.25)

5. Also the additional constraint (3.16) should be satisfied. This means that for our stationary solution we have to satisfy the inequality:

$$Mc^2 - \frac{Q^2}{2R_{\text{min}}} - \frac{k\sigma^2}{2}R_{\text{min}}^3 > 0.$$  \hspace{1cm} (3.26)

6. We have also another condition: that the electric field nearby the membrane should be not too large, otherwise the stability of the model would be destroyed by the strong macroscopical consequences of quantum effects, e.g. by the intensive electron-positron pair creation. This condition (which was suggested by J.A. Wheeler long time ago, see the reference with this Wheeler’s proposal in the paper of Bekenstein[59]) is:

$$\frac{Q}{R_{\text{min}}^2} \ll \mathcal{E}_{\text{cr}},$$

$$\mathcal{E}_{\text{cr}} = \frac{m_e^2c^3}{e_e\hbar},$$  \hspace{1cm} (3.27)

where $m_e$ and $e_e$ are the electron's mass and charge). $\mathcal{E}_{\text{cr}}$ is the well known critical electric field above which the intensive process of pair creation starts.
To satisfy these six conditions we have to find a physically acceptable domain in the space of the four parameters $M$, $Q$, $\sigma$ and $R_{\text{min}}$. The point is that such domain indeed exists and it is wide enough. If we introduce the dimensionless radius of the stationary membrane $x$ as

$$\frac{k\sigma}{c^2} R_{\text{min}} = x,$$

then one can check directly that the first five of the above formulated conditions will be satisfied under the following three constraints:

$$x < 1$$

$$M = \frac{c^4}{k^2 \sigma^2} (3x^2 - 2x^3)$$

$$Q^2 = \frac{c^8}{k^3 \sigma^2} (4x^3 - 3x^4)$$

The last two of these relations are just the equations (3.23) and (3.24) but written in the form resolved with respect to $M$ and $Q^2$.

The formulas (3.29)-(3.31) shows that for the first five conditions it is convenient to take $x < 1$ and $\sigma$ as independent parameters, and then to calculate the mass and charge necessary to obtain the model we need.

As for the last constraint (3.27) it gives some restriction also for parameter $\sigma$:

$$k\sigma^2 << \frac{x}{4 - 3x} \mathcal{E}_{\text{cr}}^2.$$

The energy density $\epsilon$ for the stationary state at $r_0 = R_{\text{min}}$, expressed in terms of parameters $x$ and $\sigma$, is:

$$\epsilon = \frac{\sigma c^2}{8\pi} (1 + \sqrt{x^2 - 2x + 1}) \delta(r - R_{\text{min}}).$$
3.4 Conclusions III

1. We showed that exists a possibility to have a spherically charged membrane in stable stationary state producing RN repulsive gravitational force outside its surface and having flat space inside. To construct such model one should take a pair of constants $0 < x < 1$ and $\sigma > 0$ satisfying the inequality (3.32) and calculate from (3.28) and (3.30)-(3.31) the membrane’s radius $R_{\text{min}}$, total mass $M$ and charge $Q$.

2. The equation of motion (3.10) can be used also for the description of the oscillation of the membrane in the potential well ABC (see fig.1) above the equilibrium point C. If we slightly increase the total membrane’s energy $Mc^2$ then the potential $U(r_0)$ around its minimum (i.e. the point C and its vicinity) will be shifted slightly down but the level ”minus four” in Eq.(3.20) on which the system lives will remain at the same position. Then the membrane will oscillate between the new shifted walls AC and CB.

3. It is easy to see that in the general dynamical state the membrane can live only inside the potential well ABC. All regions outside ABC are forbidden. In the region to the right from the point $R_{\text{max}}^{(2)}$ and above the potential $U(r_0)$ any location of the membrane is impossible due to the fact that inequality (3.16) is violated there.

This means that a membrane of considered type in principle can not have the radius (no matter in which state) greater than $R_{\text{max}}^{(2)}$. In turn for $R_{\text{max}}^{(2)}$ it is easy to obtain from the potential (20) the upper limit $R_{\text{max}}^{(2)} < \frac{e^2}{k\sigma} \left( \frac{4k^2\sigma M}{c^4} \right)^{1/3}$.
The same violation of the inequality (3.16) take place in the domain between \( R_{\text{max}}^{(1)} \) and \( R_{\text{max}}^{(2)} \) and above the segment AB. The motion in the region to the left from the point \( R_{\text{max}}^{(1)} \) and above the curve \( U(r_0) \) is forbidden again due to the same violation of the condition (3.16). This means that a membrane of considered type in principle can not have the radius less than \( R_{\text{max}}^{(1)} \). In particular there is no way for a membrane with positive effective rest mass \( m \) to collapse to the point \( r_0 = 0 \) leaving outside the field corresponding to the RN naked singularity solution. This conclusion is in agreement with the main result of the paper \[60\].

4. Although we claimed that the stationary state of a membrane constructed is stable this stability should be understood in a very restrict sense, that is as stability in the framework of the dynamics described by the equation (3.10). We do not know what will happen to our membrane after the whole set of arbitrary perturbations will be given.

5. In general the arbitrary perturbations will change also the equation of state. We investigated a membrane with equation of state \( \epsilon = \tau \). However this case can be considered only as “bare” Nambu-Goto membrane, by other words as a toy model. In the papers \[56, 57, 61, 62, 63, 64\] it was shown that arbitrary perturbations essentially renormalize the form of the equation of state of the strings and membranes. Moreover for the membranes \[57\] (differently from the strings) the fixed points of the renormalization group for the transverse and longitudinal perturbations does not coincide, which means that for the general “wiggly” membrane there is no equation of state
of the type $\epsilon = \epsilon(\tau)$ at all.

6. We also would like to stress that for appearance of repulsive force the presence of electric field is of no principal necessity. For example the repulsive gravitational forces arise also in neutral viscous fluid [46] and in the course of interaction between electrically neutral topological gravitational solitons [65].

7. From the conditions (21)-(26) also follows that in addition to the inequality (25) the radius $R_{\text{min}}$ of the shell in the stable stationary state cannot be less than $\frac{Q^2}{2M^2}$. A simple analysis shows that there is no way for $R_{\text{min}}$ to be arbitrarily small keeping some finite non-zero value for $M$ and $Q$.

### 3.5 Appendix

For the spherically symmetric case the metric\(^1\) is:

$$-(ds)^2_{\text{in}} = g_{00}c^2dt^2 + g_{11}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.34}$$

where $g_{00}$ and $g_{11}$ depend only on $t$, $r$ and the standard notation for the coordinates is:

$$(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi) . \tag{3.35}$$

The Electromagnetic tensor $F_{ik}$ has the form:

$$F_{ik} = A_{k,i} - A_{i,k} \tag{3.36}$$

---

\(^1\)We use the notations in which the interval is written as $-ds^2 = g_{ik}dx^idx^k$ and metric signature is $(-, +, +, +)$, i.e. the time-time component $g_{00}$ is negative. The norm of a time-like vector is negative. The Roman indices take values 0, 1, 2, 3. The Newtonian constant is denoted by $k$. The simple partial derivatives we designated by a comma, while covariant derivatives by semicolon.
and the Einstein-Maxwell equations are:

\[ R^k_i - \frac{1}{2} R \delta^k_i = \frac{8\pi k}{c^4} T^k_i \quad (3.37) \]

\[ (F^{ik})_{;k} = \frac{4\pi}{c} \rho u^i \quad (3.38) \]

The energy-momentum tensor for a spherical charged membrane with energy density \( \epsilon \) and tangential tension \( \tau \) is:

\[ T^k_i = \epsilon u^i u^k - (\delta^2_i \delta^k_2 + \delta^3_i \delta^k_3) \tau + \frac{1}{4\pi} (F_{it} F^{kt} - \frac{1}{4} \delta^k_i F_{lm} F^{lm}) \quad (3.39) \]

and for the membrane’s 4-velocity \( u^i \) we have:

\[ u^0 = u^0(t, r), \quad u^1 = u^1(t, r), \quad u^2 = u^3 = 0, \quad u^i u_i = -1. \quad (3.40) \]

The main step is to define the 4-invariant charge and energy densities \( \rho \) and \( \epsilon \). After that, the tension \( \tau \) follows automatically from the Einstein-Maxwell equations and from the equation of state. To construct \( \rho \) and \( \epsilon \) we apply the Landau-Lifschitz procedure[66].

The charge \( dq \) in the 3-volume element \( dV = \sqrt{g_{11}g_{22}g_{33}} \, dx^1dx^2dx^3 \) is a 4-invariant quantity by definition (although \( dV \) is not a 4-scalar). The three-dimensional charge density \( \rho^{(3)} \) can be introduced by the relation \( dq = \rho^{(3)} dV \). Consequently, for the spherically symmetric membrane case it is:

\[ \rho^{(3)} = \frac{Q \delta(r - r_0)}{4\pi r^2 \sqrt{g_{11}}}, \quad (3.41) \]

where \( Q \) is the electric charge of the membrane and \( r_0 \) is the membrane’s radius. Indeed it is easy to check that \( Q = \int \rho^{(3)} dV \) as it should be\(^2\).

\(^2\)The \( \delta \)-function in curved metric (3.34) is defined by the usual relation \( \int \delta(r - r_0) dr = 1 \). Such \( \delta \)-function has dimension \( cm^{-1} \).
Since $\rho^{(3)}dV$ is a 4-scalar the quantities $\rho^{(3)}dVdx^i$ represent a 4-vector.

With the use of the previous formula we obtain:

$$c\rho^{(3)}dVdx^i = \frac{cQ\delta(r - r_0)}{4\pi r^2 u^0 \sqrt{-g_{00}g_{11}}} u^i \sqrt{-g} d^4 x , \quad (3.42)$$

where $g$ is the 4-metric’s determinant. The last formula shows that the factor in front of $u^i \sqrt{-g} d^4 x$ is a 4-scalar. This scalar is nothing else but the 4-invariant charge density $\rho$ which appeared in the Maxwell equation (3.38):

$$\rho = \frac{cQ\delta[r - r_0(t)]}{4\pi r^2 u^0 \sqrt{-g_{00}g_{11}}} . \quad (3.43)$$

For the electric current $j^k$ we have $j^k = \rho u^k$.

The 4-scalar energy density $\epsilon$ which figure in the energy-momentum 4-tensor (3.39) can be constructed exactly in the same way if we observe that the rest energy of the matter in a 3-volume element $dV$ (i.e. the sum of the all kinds of the internal energies of this element in the reference system in which this element is at rest) is a 4-invariant quantity by definition. Then we can introduce the 3-dimensional rest energy density (the direct analogue of the previous charge density $\rho^{(3)}$) which under integration over 3-volume gives the total rest energy $mc^2$ of the membrane. Then $mc^2$ is the sum of the all kinds of internal energies of the membrane in the radially comoving system in which membrane is at rest. In this way we obtain:

$$\epsilon = \frac{mc^2\delta[r - r_0(t)]}{4\pi r^2 u^0 \sqrt{-g_{00}g_{11}}} . \quad (3.44)$$

Clearly the effective rest mass $m$ of the membrane in the presence of a tension depends on the membrane radius $r_0(t)$.
In the case of spherical symmetry the electromagnetic potentials \( A_i \) can be taken in the form:

\[
A_0 = A_0(t, r), \quad A_1 = A_2 = A_3 = 0,
\]

(3.45)

which gives only one nonvanishing component for the electromagnetic tensor \( F_{ik} \), namely \( F_{10} \) (and its antisymmetric partner \( F_{01} \)):

\[
F_{10} = A_{0,1}.
\]

(3.46)

Now, we enter with definitions (3.34)-(3.36) and (3.39)-(3.46) into the Einstein-Maxwell equations (3.37)-(3.38) to calculate the solution. These calculations need special care since we are dealing with distributions in application to the non-linear theory. In general this is not a trivial task (see e.g. [67, 68, 69]), however, for particular case of spherical symmetry everything is tractable and can be done easily thanks to the specially simple structure of the field equations. The resulting solution contains four arbitrary constants of integration \( M_{\text{in}}, Q_{\text{in}} \) and \( M_{\text{out}}, Q_{\text{out}} \) which have an obvious interpretation as mass and charge of a central RN source and the total mass and charge of the whole system (the central body together with the membrane) respectively. The membrane’s charge \( Q \) is simply the difference of \( Q_{\text{out}} \) and \( Q_{\text{in}} \):

\[
Q = Q_{\text{out}} - Q_{\text{in}}.
\]

(3.47)

To represent the solution in compact form we use the proper time \( \eta \) of the membrane, denoting the membrane’s equation of motion as \( r = r_0(\eta) \), and
introducing the following notations:

\[
\phi_{in}(r) = 1 - \frac{2kM_{in}}{c^2 r} + \frac{kQ_{in}^2}{c^4 r^2}
\]

\[
\phi_{out}(r) = 1 - \frac{2kM_{out}}{c^2 r} + \frac{kQ_{out}^2}{c^4 r^2}
\]

\[
S_{in}(\eta) = \sqrt{\phi_{in}(r_0) + c^{-2}(r_0,\eta)^2}
\]

\[
S_{out}(\eta) = \sqrt{\phi_{out}(r_0) + c^{-2}(r_0,\eta)^2}
\]

(3.48)

(3.49)

We consider the global time \( t \) in (3.34) as continuous quantity when passing through the membrane. Then the intervals inside, outside and on the membrane are:

\[
-(ds^2)_{in} = -\Gamma(t)\phi_{in}(r)c^2 dt^2 + \frac{dr^2}{\phi_{in}(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

(3.50)

\[
-(ds^2)_{out} = -\phi_{out}(r)c^2 dt^2 + \frac{dr^2}{\phi_{out}(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

(3.51)

\[
-(ds^2)_{on} = -c^2 d\eta^2 + r_0^2(\eta)(d\theta^2 + \sin^2 \theta d\phi^2)
\]

(3.52)

The matching conditions for these intervals through the membrane are:

\[
[(ds^2)_{in}]_{r=r_0(\eta)} = [(ds^2)_{out}]_{r=r_0(\eta)} = (ds^2)_{on}
\]

(3.53)

Using the relations (3.53), the factor \( \Gamma(t) \) in (3.50) and the connection \( t(\eta) \) between global and proper times can be expressed through the membrane’s radius \( r_0(\eta) \):

\[
\frac{dt}{d\eta} = \frac{S_{out}}{\phi_{out}(r_0)}
\]

(3.54)

\[
\Gamma(t) = \frac{\phi_{out}(r_0)S_{in}}{\phi_{in}(r_0)S_{out}}
\]

(3.55)
Namely the continuity conditions (3.53) and continuous character of the time variable $t$ are responsible for the appearance of the term $\Gamma^2(t)$ in $g_{00}$ in Eq.(3.54). Since this term depends only on time, it can be easily removed by passing to the internal time variable $t_{in}$ by the transformation

$$\Gamma dt = dt_{in},$$

(3.56)

which can be found with the help of (3.54) after the function $r_0(\eta)$ became known. In terms of the variables $(t_{in}, r)$ also the internal metric (3.50) takes the standard RN form.

As it was already mentioned, the membrane’s effective rest mass $m$ which appeared in the energy density (3.44) depends on the membrane radius. The concrete form of the function $m(r_0)$ is not known in advance and its specification is equivalent to the specification of the equation of state. For an arbitrary $m(r_0)$ the Einstein-Maxwell equations (3.37)-(3.38) give the following equation of motion for the membrane:

$$M_{out}c^2 - M_{in}c^2 = \frac{1}{2}(S_{in} + S_{out})mc^2 + \frac{QQ_{in}}{r_0} + \frac{Q^2}{2r_0},$$

(3.57)

together with the condition that both square roots $S_{in}$ and $S_{out}$ defined by (3.49), should have the same sign. The provenance of this condition is due to the fact that the signs of $S_{in}$ and $S_{out}$ are nothing else but the signs of the time-component of $u^0$ of the membrane’s 4-velocity when it is seen from the inside ($r \rightarrow r_0 - 0$) and outside ($r \rightarrow r_0 + 0$) of the membrane surface respectively. In our approach (with continuous coordinates $t, r$) we can consider the
4-velocity $u^i$ as a field continuous through the surface of the membrane. We can define $u^i$ everywhere in space-time simply by smooth parallel transport from the membrane’s surface, no matter that the membrane is concentrated only at the points $r = r_0$. This concentration is ensured not by $u^i$ but due to the $\delta$-functions in the densities $\rho$ and $\epsilon$. Since $u^0$ can not change sign passing through the membrane, $S_{in}$ and $S_{out}$ should have the same sign.

Of course, we need to know the fields $u^0$ and $u^1$ only on the membrane, and there they are:

$$u^0 = t_{,\eta} ; \quad u^1 = c^{-1}r_{0,\eta}$$

(3.58)

It is easy to check that the matching conditions (3.54) and (3.55) are nothing else but the demand that the normalization constraint $u^iu_i = -1$ should hold independently from which side we approach the surface of the membrane.

It is worth to be remarked that the Einstein-Maxwell equations also demand for the trajectory $r_0(\eta)$ the second order (in time) differential equation of motion. However, this last one represents simply the result of the differentiation in time of the first order equation (3.57). Then this second-order equation we can forget safely.

The resulting expressions for the energy density and tension are:

$$\epsilon = \frac{mc^2}{8\pi r_0^2} \left[ \frac{\phi_{in}(r_0)}{S_{in}} + \frac{\phi_{out}(r_0)}{S_{out}} \right] \delta[r - r_0(\eta)]$$

(3.59)

$$\tau = \frac{r_0}{2m} \frac{dm}{dr_0} \epsilon.$$  

(3.60)
The electric field $F_{10}$ outside the membrane is:

$$F_{10} = \frac{Q_{\text{out}}}{r^2}, \quad r > r_0.$$  \hspace{1cm} (3.61)

Inside the membrane we have:

$$F_{10} = \frac{Q_{\text{in}}}{r^2} \frac{dt_{\text{in}}}{dt}, \quad r < r_0,$$  \hspace{1cm} (3.62)

where the factor $\frac{dt_{\text{in}}}{dt}$ depends only on time and can be calculated from the relations (3.55) and (3.56). The origin of this factor is due to the fact that we use the time $t$ as continuous global time including the region inside the membrane. If we describe the internal metric in terms of internal time $t_{\text{in}}$ the field strength $F_{10}$ would be simply $Q_{\text{in}}/r^2$.

The formulas (3.48)-(3.52), (3.54), (3.55) and (3.57)-(3.62) provide the complete solution of the problem. It is worth explaining briefly the main steps of our integration procedure that we applied to the Einstein-Maxwell equations.

As in any spherically symmetric problem it is convenient to use, instead of the full original Einstein equations (3.37), only its ($^0_0$), ($^1_1$) and ($^1_0$) components, and the hydrodynamical equations $T^k_{i;k} = 0$. All the remaining components of equations (3.37) after that will be satisfied identically either due to the Bianchi identities or due to the symmetry of the problem. Then the solution for $g_{11}$ together with the basic eq.(3.57) follows from ($^0_0$) and ($^1_0$) components of Einstein equations (3.37), and after that the solution for $g_{00}$ follows from the difference of the ($^0_0$) and ($^1_1$) components of (3.37). The solution for the electric...
field $F_{10}$ is the result of the Maxwell equations (3.38). The hydrodynamical equations $T_{i;k}^k = 0$ give only two relations. The first one simply express the tension $\tau$ in terms of other quantities and this is the formula (3.60). The second one results in the already mentioned second order differential equation for $r_0(\eta)$ which represents the differentiation in time of the first order equation (3.57). Then this second order equation is of no importance.

We remark also that the procedure described above need a caution because the symbolic function are involved. Nevertheless everything going well under the following three standard operation rules with such functions:

1. $\frac{d}{dx} \theta(x) = \delta(x)$ ,

2. $F(x) \delta(x) = \frac{1}{2} [F(-0) + F(+0)] \delta(x)$ ,

3. $\frac{d}{dx} \theta^2(x) = 2 \theta(x) \delta(x) = \delta(x)$ .

(To call the third rule as the standard one is a little exaggeration; however it works well and final results indeed coincide with those obtained in literature by different approaches). Originally we obtained the solution in global form using the step function $\theta(x)$ and only after that we represented the results separately in the regions $r > r_0$ and $r < r_0$. However, since $\theta(x)$ is defined also at the point $x = 0$ [$\theta(0) = 1/2$], we found by the way the values for the metric and electric field also at the points of the membrane’s surface. Such
global form is:

$$\frac{1}{g_{11}} = 1 - \frac{2kM_{in}}{c^2r} - \frac{2k(M_{out} - M_{in})}{c^2r}\theta[r - r_0(\eta)] + \frac{k}{c^4r^2}\{Q_{in} + Q\theta[r - r_0(\eta)]\}^2$$

(3.63)

$$\frac{1}{\sqrt{-g_{00}g_{11}}} = \frac{1}{\Gamma} + \left(1 - \frac{1}{\Gamma}\right)\theta[r - r_0(\eta)]$$

(3.64)

$$F_{10} = \frac{\sqrt{-g_{00}g_{11}}}{r^2}\{Q_{in} + Q\theta[r - r_0(\eta)]\},$$

(3.65)

to which should be added the equation (3.57). This equation arise as self-consistency condition for the $^{(0)}_0$ and $^{(1)}_0$ components of Einstein equations, which can be verified by the direct substitution into these components of the above global expressions together with eqs. (3.58)-(3.60).

Finally it should be mentioned that the membrane’s equation of motion (3.57) can be written in the following two equivalent forms:

$$mc^2S_{in} = M_{out}c^2 - M_{in}c^2 - \frac{Q_{in}Q}{r_0} - \frac{Q^2}{2r_0} + \frac{k m^2}{2r_0}$$

(3.66)

$$mc^2S_{out} = M_{out}c^2 - M_{in}c^2 - \frac{Q_{in}Q}{r_0} - \frac{Q^2}{2r_0} - \frac{k m^2}{2r_0}.\) (3.67)

Each of these two equations is equivalent to (3.57) which can be checked easily by simple algebraic manipulations. For practical calculations we can use only one of these equations, however, in addition it is necessary to ensure the same sign for both quantities $S_{in}$ and $S_{out}$. (For a membrane with empty space inside they both should be positive). More convenient is relation (3.66)
which we write as

\[ M_{\text{out}} c^2 = M_{\text{in}} c^2 + mc^2 \sqrt{\phi_{\text{in}}(r_0) + c^{-2}(r_0,\eta)^2} \]

\[ + \frac{Q_{\text{in}}Q}{r_0} + \frac{Q^2}{2r_0} - \frac{k m^2}{2r_0} \]  

(3.68)

This is the equation obtained by Chase[22] with the aid of a different derivation procedure which makes use of Gauss-Codazzi equations (see Israel[53]).

Eqn.(3.68) is interesting because in spite of the fact that \( m \) depends on time (or on \( r_0 \)) this equation looks like an usual integral of motion, that is as if \( m \) was a constant. Relation (3.68) expresses the conservation of the total energy \( M_{\text{out}} c^2 \) of the system which is the sum of the five familiar constituents: 1) the rest energy of the central body, 2) the kinetic energy of the membrane together with its gravitational potential energy in the gravitational field of the central body, 3) the electric interaction energy between membrane and central source, 4) the positive electric self-interaction energy of the membrane, and 5) the negative gravitational self-interaction energy of the membrane.
Chapter 4

Intersections of self-gravitating charged shells
This chapter is based on Ref.[4].

4.1 Introduction

The mathematical model that we analyze now describes the dynamic evolution of two spherical shells of charged matter which freely move outside the field of a central Reissner-Nordstrom (RN) source. Microscopically these shells are assumed to be composed by charged particles which move on elliptical orbits with a collective variable radius. The angular motion, distributed uniformly and isotropically on the shell surfaces, is mathematically described by a tangential-pressure term in the energy momentum tensor of the Einstein equations. The definition of the shell implies that all the particles have the same following three ratios: energy/mass, angular momentum/mass, and charge/mass. Indeed, since the equations of motion for any singled-out particle “a” are

\[
\frac{dt_a}{ds} = \frac{1}{-ma^2 c^2 g_{tt}(r_a)} (E_a + e_a A_0(r_a)) \\
\left(\frac{dr_a}{ds}\right)^2 = \frac{1}{m_a^2 c^4} (E_a + e_a A_0(r_a))^2 \left(\frac{1}{-g_{tt}(r_a)g_{rr}(r_a)}\right) - \left(\frac{l_a^2}{m_a^2 c^2 r^2} + 1\right) \frac{1}{g_{rr}(r_a)} \\
\left(\frac{d\theta_a}{ds}\right)^2 = \frac{l_a^2}{m_a^2 c^2 r^4} - \frac{k_a^2}{m_a^2 c^2 r^4 \sin^2 \theta_a} \\
\frac{d\phi_a}{ds} = \frac{k_a}{m_a c r^2 \sin^2 \theta_a}
\]

\((g_{tt} \text{ and } g_{rr} \text{ are the components of a spherical symmetric metric and } A_0 \text{ its electric potential; } k_a \text{ and } l_a \text{ are arbitrary constants}), \text{ it is easy to see that the}}

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radial motion for all particles is the same if

$$\frac{E_a}{m_a} = \text{const.}, \quad \frac{e_a}{m_a} = \text{const.}, \quad \frac{|l_a|}{m_a} = \text{const.}, \quad \forall a, \quad (4.5)$$

where each \textit{const.} does not depend on the index \(a\). Therefore, if at the beginning the particles are on the same radius \(r_a = R_0\), then the shell will evolve “coherently”, i.e. all particles will evolve with the same radius.

Now the problem we are interested in is to find the exchange of energy between the two shells after the intersection. Indeed the motion of the shells before and after the intersection can be easily deduced from the equation of motion for just one shell, which equation has been found many years ago by Chase[22] with a geometrical method first used by Israel [53].

What we achieve in the present study is the determination of the constant parameters after the intersection knowing just the parameters before the intersection. Actually the unknown parameter is only one, \(m_{21}\), which is the Schwarzschild mass parameter measured by an observer between the shells after the intersection. This parameter is strictly related to the energy transfer which takes place in the crossing, and it is found imposing a proper continuity condition on the shells velocities.

In the model we assume that the emission of electromagnetic waves is negligible, and that there are no other interactions between the two shells apart the gravitational and electrostatic ones. In particular the shells, during the intersection, are assumed to be “transparent” each other (i.e. no scattering processes).
The chapter is divided as follows: in Sec.2 we preliminarily discuss the one-shell case; in Sec.3, which is the central part of this article, we find the unknown parameter $m_{21}$; then, Secs.4-7 are devoted to some applications: post-Newtonian approximation, zero effective masses case (i.e. ultra-relativistic case), test-shell case, and finally the ejection mechanism.

In this analysis we deal only with the mathematical aspects of the problem; some astrophysical applications of charged shells in the field of a RN black hole have been considered in Ref.CR

4.2 A gravitating charged shell with tangential pressure

The motion of a thin charged dust-shell with a central RN singularity was firstly studied by De La Cruz and Israel[48], while the case with tangential pressure was achieved by Chase[22] in 1970. All these authors used the extrinsic curvature tensor and the Gauss-Codazzi equations. However we followed a different way, indeed the same solution can be found also by using $\delta$ and $\theta$ distributions and then by direct integration of the Einstein-Maxwell equations (see Ref.BBB and the appendix in Ref.BPP). This method has the advantage of a clearer physical interpretation, and it is also straightforward in the calculations; however in the following we will give only the main passages.

Let there be a central body of mass $m_{in}$ and charge $e_{in}$ and let a spherical massive charged shell with charge $e$ move outside this body. It is clear in advance that the field internal to the shell will be RN, while externally we
will have again a RN metric but with different mass and charge parameters $m_{out}$ and $e_{out} = e_{in} + e$. Using the coordinates $x^0 = ct$ and $r$, which are continuous when passing through the shell, we can write the intervals inside, outside, and on the shell as

$$-(ds)_{in}^2 = -e^{T(t)} f_{in}(r)c^2 dt^2 + f_{in}^{-1}(r) dr^2 + r^2 d\Omega^2$$ (4.6)

$$-(ds)_{out}^2 = -f_{out}(r)c^2 dt^2 + f_{out}^{-1}(r) dr^2 + r^2 d\Omega^2$$ (4.7)

$$-(ds)_{on}^2 = -c^2 d\tau^2 + r_0(\tau)^2 d\Omega^2$$ (4.8)

where we denoted

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

and

$$\phi_{in} = 1 - 2 \frac{G m_{in}}{c^2 r} + \frac{G e_{in}^2}{c^4 r^2}, \quad f_{out} = 1 - 2 \frac{G m_{out}}{c^2 r} + \frac{G (e_{in} + e)^2}{c^4 r^2}. \quad (4.9)$$

In the interval (4.8), $\tau$ is the proper time of the shell. The “dilaton” factor $e^{T(t)}$ in (4.6) is required to ensure the continuity of the time coordinate $t$ through the shell. If the equation of motion for the shell is

$$r = R_0(t), \quad (4.10)$$

then joining the angular part of the three intervals (4.6)-(4.8), one has

$$r_0(\tau) = R_0[t(\tau)], \quad (4.11)$$

where the function $t(\tau)$ describes the relationship between the global time and the proper time of the shell. Joining the radial-time parts of the intervals
(4.6)-(4.7) on the shell requires that the following relations hold:

\[ \phi_{in}(r_0) \left( \frac{dt}{d\tau} \right)^2 e^{T(t)} - \phi_{in}^{-1}(r_0) \left( \frac{dr_0}{cd\tau} \right)^2 = 1, \quad (4.12) \]

\[ f_{out}(r_0) \left( \frac{dt}{d\tau} \right)^2 - f_{out}^{-1}(r_0) \left( \frac{dr_0}{cd\tau} \right)^2 = 1. \quad (4.13) \]

If the equation of motion for the shell — i.e. the function \( r_0(\tau) \) — is known, then the function \( t(\tau) \) follows from (4.13) and consequently \( T(t) \) can be deduced by (4.12). Thus the problem consist only in determining \( r_0(\tau) \), which can be done by direct integration of the Einstein-Maxwell equations

\[ \begin{cases} R_i^k - \frac{1}{2} R g_i^k = \frac{8\pi G}{c^4} T_i^k \\ (\sqrt{-g} F^{ik})_{,k} = \sqrt{-g} \frac{4\pi}{c^3} \rho u^i \end{cases} \quad (4.14) \]

with the energy-momentum tensor given by:

\[ T_i^k = \epsilon u_i u^k + (\delta^2 \delta_2^k + \delta^3 \delta_3^k)p + T^{(el)}_i^k \quad (4.15) \]

\[ T^{(el)}_i^k = \frac{1}{4\pi} (F_{il} F_l^k - \frac{1}{4} \delta_i^k F_{lm} F^{lm}) \quad (4.16) \]

Here on we employ the following notations:

\[-ds^2 = g_{ik} dx^i dx^k, \quad g_{ik} \text{ has signature } (-, +, +, +)\]

\[ x^k = (ct, r, \theta, \varphi) \quad i, j, k... = 0, 1, 2, 3\]

\[ p \equiv p(R_0) = p_\theta = p_\varphi = \text{tangential pressure} \quad (p_r = 0)\]

\[ F_{ik} = A_{k,i} - A_{i,k}\]

The above equations are to be solved for the metric

\[-ds^2 = g_{00}(t, r)c^2 dt^2 + g_{11}(t, r)dr^2 + r^2 d\Omega^2; \quad (4.17)\]
and for the potential

\[ A_0 = A_0(t, r), \quad A_1 = A_2 = A_3 = 0, \quad (4.18) \]

As follows from the Landau-Lifshitz approach LL (see Ref.BBB) the energy distribution of the shell is

\[ \epsilon = \frac{M(t)c^2\delta[r - R_0(t)]}{4\pi r^2 u^0 \sqrt{-g_{00}g_{11}}}, \quad (4.19) \]

while its charge density is

\[ \rho = \frac{c e\delta[r - R_0(t)]}{4\pi r^2 u^0 \sqrt{-g_{00}g_{11}}}, \quad (4.20) \]

where \( \delta \) is the standard \( \delta \)-function. In the absence of tangential pressure \( p \), the quantity \( M \) in Eqn.(4.19) would be a constant, but in presence of pressure, \( Mc^2 \) includes the rest energy along with the energy (in the radially comoving frame) of the tangential motions of the particles that produce this pressure.

It can be checked that the Einstein part of (4.14) actually lead to the solution (4.6)-(4.8) with, in addition, the “joint condition”

\[ \sqrt{\phi_{in}(r_0)} + \left( \frac{dr_0}{c d\tau} \right)^2 + \sqrt{f_{out}(r_0)} + \left( \frac{dr_0}{c d\tau} \right)^2 = 2 \frac{(m_{out} - M_{in})}{\mu(\tau)} - \frac{e^2}{\mu(\tau)c^2 r_0}, \quad (4.21) \]

where we denoted

\[ \mu(\tau) = M[t(\tau)], \quad (4.22) \]

while

\[ m_{out} - M_{in} = \frac{E}{c^2} \quad (4.23) \]
is a constant which can be interpreted as the total amount of energy of the shell. Then, from the Maxwell side of (4.14) the only non-vanishing component of the electric field is

\[ F_{01} = -\frac{\sqrt{-g_{00}g_{11}}}{r^2} \{ e_{in} + e\theta[r - R_0(t)] \} \]  

(\theta(x) is the standard step function). Finally, the equations \( T_{ik} = 0 \) can be reduced to the only one relation:

\[ p = -\frac{dM}{dt} \frac{c^2\delta[r - R_0]}{8\pi ru^1\sqrt{-g_{00}g_{11}}} \]  

We will not treat here the steady case (i.e. \( r_0 = \text{const} \)) which should be treated separately; thus in the following we will assume always \( r_0 \neq \text{const} \).

The joint condition (4.21) can be written in several different forms: two of them, which will be useful in the following, are

\[ \sqrt{\varphi_{in}(r_0) + \left( \frac{d r_0}{c d\tau} \right)^2} = \frac{(m_{out} - M_{in})}{\mu(\tau)} + \frac{G\mu^2(\tau) - e^2 - 2ee_{in}}{2\mu(\tau)c^2r_0} \]  

and

\[ \sqrt{f_{out}(r_0) + \left( \frac{d r_0}{c d\tau} \right)^2} = \frac{(m_{out} - M_{in})}{\mu(\tau)} - \frac{G\mu^2(\tau) + e^2 + 2ee_{in}}{2\mu(\tau)c^2r_0}. \]  

As in Ref. BBB, all the radicals encountered here are taken positive, since for astrophysical considerations only these cases are meaningful. To proceed further, we must specify the equation of state, i.e. the function \( \mu(\tau) \). Here we consider a particle-made shell, therefore the sum of kinetic and rest energy of all the particles is

\[ M c^2 = \sum_a \left( m_a c^2 \sqrt{1 + \frac{p_a^2}{m_a^2 c^2}} \right), \]
where $p_a$ is the tangential momentum of each particle (the electric interaction between the particles is already taken into account by the self-energy term of, e.g., (4.26), thus one has not to include it in $M$ too). From the definition of the shell (see Introduction) it follows:

$$\frac{p_a^2}{m_a^2} = \frac{l_a^2}{m_a^2 R_0^2} = \text{const} \frac{1}{R_0^2},$$ (4.29)

the square root in (4.28) does not depend on the index $a$; then defining

$$\sum_a m_a c^2 = m c^2, \quad \sum_a |l_a| = L,$$

formula (4.28) can be re-written (remembering definition (4.22) too) as

$$\mu(\tau) = \sqrt{m^2 + \frac{L^2}{c^2 r_0^2(\tau)}}.$$

(4.30)

Thus, now, one can determine the function $r_0(\tau)$ from equation (4.21) [or from one of the equivalent forms (4.26)-(4.27)] if the initial radius of the shell and the six arbitrary constants $M_{in}, m_{out}, m, e_{in}, e$ and $L$ are specified. Accordingly with (4.19), (4.25), (4.22) and (4.30), the equation of state that relates the shell energy density $\epsilon$ to the tangential pressure $p$ is

$$p = \frac{\epsilon L^2}{2 m^2 c^2 R_0^2} \left( 1 + \frac{L^2}{m^2 c^2 R_0^2} \right)^{-1},$$

(4.31)

as in the uncharged case, i.e. the presence of the charges do not modify the relation between energy density and pressure (indeed the presence of the charge is hidden in the equation of motion). Note that when the shell expands to infinity ($R_0 \to \infty$) the angular momentum becomes irrelevant and the equation of state tends to the dust case $p \ll \epsilon$.  

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Figure 4.1: The four region in which it is divided the spacetime; the two lines represent the trajectories of shell-1 and shell-2.

4.3 The shells intersection

Let us now consider the case of two shells which move in the field of a central charged mass. The generalization from the previous (single-shell) case is straightforward if the shells do not intersect: indeed the outer shell do not affect the motion of the inner one, while the inner one appears from outside just as a RN metric. Therefore the principal aim of this section is to consider the intersection eventuality and to predict the motion of the two shells after the crossing, having specified the initial conditions before the crossing. After the intersection one has a new unknown constant that has to be found by imposing opportune joining conditions as now we are going to explain (the analysis follows step by step the Ref.BBB’s one).
Let us previously analyze the space-time in the \((t, r)\) coordinates (which are continuous through the shells). We define the point \(O \equiv (t_*, r_*)\) as the intersection point; then the space-time is divided in four regions (see Fig.1):

\[
\begin{align*}
COB & \quad (r > R_1, r > R_2), \\
COA & \quad (R_1 < r < R_2), \\
AOD & \quad (r < R_1, r < R_2), \\
BOD & \quad (R_2 < r < R_1).
\end{align*}
\]

Correspondingly to these regions we have the metric in form (4.13) but with different coefficients \(g_{00}\) and \(g_{11}\):

\[
\begin{align*}
g_{00}^{(COB)} &= -f_{out}(r), \\
g_{11}^{(COB)} &= f_{out}^{-1}(r) \\
g_{00}^{(COA)} &= -e^{T_1(t_*)} f_{12}(r), \\
g_{11}^{(COA)} &= f_{12}^{-1}(r) \\
g_{00}^{(AOD)} &= -e^{T_0(t_*)} f_{in}(r), \\
g_{11}^{(AOD)} &= f_{in}^{-1}(r) \\
g_{00}^{(BOD)} &= -e^{T_2(t_*)} f_{21}(r), \\
g_{11}^{(BOD)} &= f_{21}^{-1}(r)
\end{align*}
\]

The dilaton factor \(T(t)\) allows to cover all the space-time with only one \(t\)-coordinate; here, \(\phi_{in}\) and \(f_{out}\) are the same as those in (4.9) while \(f_{12}\) and \(f_{21}\) are given by similar expressions:

\[
\begin{align*}
f_{12} &= 1 - \frac{2Gm_{12}}{c^2 r} + \frac{G(e_{in} + e_1)^2}{c^4 r^2} \\
f_{21} &= 1 - \frac{2Gm_{21}}{c^2 r} + \frac{G(e_{in} + e_2)^2}{c^4 r^2}
\end{align*}
\]

As we said, the parameters \(M_{in}, m_{12}, m_{out}, e_{in}, e_1, e_2\) are assumed to be specified at the beginning, while \(m_{21}\) is the actual unknown constant which has yet to be determined from the joining conditions on \((t_*, r_*)\).
Before the intersection

Let us write the equation of motion for the two shells before the intersection
(shell-1 inner and shell-2 outer). This can be made easily adapting the (4.27)
and (4.26) to the present case:

\[
\sqrt{f_{12}(r_1)} + \left(\frac{d r_1}{c d \tau_1}\right)^2 = \frac{(m_{12} - M_{in})}{M_1} - \frac{GM_1^2 + e_1^2 + 2e_{in}e_1}{2M_1c^2r_1}
\]  
(4.39)

for shell 1, while for shell 2

\[
\sqrt{f_{12}(r_2)} + \left(\frac{d r_2}{c d \tau_2}\right)^2 = \frac{(m_{12} - M_{in})}{M_2} + \frac{GM_2^2 - e_2^2 - 2(e_{in} + e_1)e_2}{2M_2c^2r_2}
\]  
(4.40)

with

\[
M_1 = \sqrt{m_1^2 + \frac{L_1^2}{c^2 r_1^2}}, \quad M_2 = \sqrt{m_2^2 + \frac{L_2^2}{c^2 r_2^2}}.
\]  
(4.41)

Here, \(\tau_1\) and \(\tau_2\) are the proper times of the first and second shells respectively,
while \(r_1(\tau_1) = R_1[t(\tau_1)]\) and \(r_2(\tau_2) = R_2[t(\tau_2)]\). Now we have to impose the
joining conditions for the intervals on both the shells. For the first shell (on
curve AO) one has:

\[
e^{T_1(t)}f_{12}(r_1) \left(\frac{d t}{d \tau_1}\right)^2 - f_{12}^{-1}(r_1) \left(\frac{d r_1}{c d \tau_1}\right)^2 = 1
\]  
(4.42)

\[
e^{T_0(t)}\phi_{in}(r_1) \left(\frac{d t}{d \tau_1}\right)^2 - \phi_{in}^{-1}(r_1) \left(\frac{d r_1}{c d \tau_1}\right)^2 = 1;
\]  
(4.43)

while for the second shell:

\[
f_{out}(r_2) \left(\frac{d t}{d \tau_2}\right)^2 - f_{out}^{-1}(r_2) \left(\frac{d r_2}{c d \tau_2}\right)^2 = 1
\]  
(4.44)

\[
e^{T_1(t)}f_{12}(r_2) \left(\frac{d t}{d \tau_2}\right)^2 - f_{12}^{-1}(r_2) \left(\frac{d r_2}{c d \tau_2}\right)^2 = 1.
\]  
(4.45)
If all free parameters and initial data to Eqs.(4.39)-(4.41) were specified and if the functions \( r_1(\tau_1) \) and \( r_2(\tau_2) \) were derived, then their substitution in (4.42)-(4.45) gives the functions \( \tau_1(t) \), \( \tau_2(t) \) and \( T_1(t) \), \( T_0(t) \), which is enough for determining the motion of the shells before the intersection. Therefore the intersection point \((t_*, r_*)\) can be found by solving the system

\[
\begin{align*}
  r_* &= r_1(\tau_1(t_*)) \\
  r_* &= r_2(\tau_2(t_*)),
\end{align*}
\]

which we assume that has a solution.

**After the intersection**

The equation of motion for the shells after the intersection time \( t_* \) can be constructed in the same way again by turning to Eqns.(4.26) and (4.27), and introducing the new parameter \( m_{21} \) which characterize the “Schwarschild mass” seen by an observer in the region \( BOD \). We use Eq.(4.26) for (now outer) shell 1 and Eq.(4.27) for (now inner) shell 2:

\[
\sqrt{f_{21}(r_1) + \left( \frac{dr_1}{cd\tau_1} \right)^2} = (m_{out} - m_{21}) \frac{GM_1^2 - e_1^2 - 2e_1(e_{in} + e_2)}{2M_1 c^2 r_1}, \quad (4.47)
\]

\[
\sqrt{f_{21}(r_2) + \left( \frac{dr_2}{cd\tau_2} \right)^2} = \frac{(m_{21} - M_{in})}{M_2} - \frac{GM_2^2 + e_2^2 + 2e_2 e_{in}}{2M_2 c^2 r_2}. \quad (4.48)
\]

Naturally, \( M_1(r_1) \) and \( M_2(r_2) \) are given by the same expression of (4.41) but now they have to be calculated on \( r_1(\tau_1) \) and \( r_2(\tau_2) \) after the intersection.

Joining the intervals on the first shell (on curve \( OB \)) yields

\[
f_{out}(r_1) \left( \frac{dt}{d\tau_1} \right)^2 - f_{out}^{-1}(r_1) \left( \frac{dr_1}{cd\tau_1} \right)^2 = 1 \quad (4.49)
\]
\[ e^{T_2(t)} f_{21}(r_1) \left( \frac{dt}{dr_1} \right)^2 - f_{21}^{-1}(r_1) \left( \frac{dr_1}{cd\tau_1} \right)^2 = 1. \quad (4.50) \]

Then, joining the second shell (on curve OB) we obtain:

\[ e^{T_2(t)} f_{21}(r_2) \left( \frac{dt}{dr_2} \right)^2 - f_{21}^{-1}(r_2) \left( \frac{dr_2}{cd\tau_2} \right)^2 = 1 \quad (4.51) \]

\[ e^{T_0(t)} \phi_{in}(r_2) \left( \frac{dt}{dr_2} \right)^2 - \phi_{in}^{-1}(r_2) \left( \frac{dr_2}{cd\tau_2} \right)^2 = 1. \quad (4.52) \]

Since the initial data to Eqs. (4.47) and (4.48) have already been specified (from the previous evolution), then the evolution of the shells after the intersection would be determined from Eqs. (4.47)-(4.52) if parameter \( m_{21} \) were known. Thus we need an additional physical condition from which we could determine \( m_{21} \).

This condition follows from the fact that the Christoffel symbols (i.e. the accelerations) of the shells have only finite discontinuities (finite jumps), therefore the relative velocity of the shells must remain continuous through the crossing point.

In the presence of two shells, we can construct one more invariant than in the single shell case (where only \( u_i u^i = -1 \) was possible): the scalar product between the two 4-velocities of the shells. We can also avoid to apply the parallel transport if we evaluate the 4-velocities on the intersection point \((t_*, r_*)\). The continuity condition can be found imposing that the scalar product has to have the same value when evaluated in both the two limits \( t \to t_*^- \) and \( t \to t_*^+ \).
Determination of $Q$.

Let us start determining the quantity

$$Q \equiv \left\{ g_{00}^{(COA)} u^0_{AO} u^0_{CO} + g_{11}^{(COA)} u^1_{AO} u^1_{CO} \right\}_{t=t^*, r=r_1=r_2=r_*},$$

(4.53)

which is the scalar product of the two 4-velocities evaluated in the intersection point from the region $AOC$ (along the curves $AO$ and $CO$). Written explicitly, the unit tangent vector to trajectory $AO$ is

$$u^i_{AO} = (u^0_{AO}, u^1_{AO}, u^2_{AO}, u^3_{AO})$$

$$= \left( \frac{dt}{d\tau_1}, \frac{dr_1}{c d\tau_1}, 0, 0 \right)_{t \leq t^*},$$

(4.54)

while for the trajectory $CO$ we have

$$u^i_{CO} = (u^0_{CO}, u^1_{CO}, u^2_{CO}, u^3_{CO})$$

$$= \left( \frac{dt}{d\tau_2}, \frac{dr_2}{c d\tau_2}, 0, 0 \right)_{t \leq t^*}. $$

(4.55)

The fact that these are actually unit vectors follows from the joining equations (4.42) and (4.45).

The components of the vector (4.54) can be easily expressed from Eqs.(4.39) and (4.42) as

$$\left( \frac{dt}{d\tau_1} \right)_{t \leq t^*} = \frac{e^{-T_1(t)/2}}{M_1(r_1) f_{12}(r_1)} \left( m_{12} - M_m - \frac{GM_1^2(r_1) + e_1^2 + 2e_1 e_m}{2c^2 r_1} \right)$$

(4.56)
\[
\left( \frac{dr_1}{cd\tau_1} \right)_{t \leq t^*} = \frac{\delta_1}{M_1(r_1)f_{12}(r_1)} \sqrt{\left( m_{12} - M_{in} - \frac{GM_1^2(r_1) + e_1^2 + 2e_1e_{in}}{2c^2r_1} \right)^2 - M_1^2(r_1)f_{12}(r_1)}
\]

where
\[
\delta_1 = \text{sgn} \left( \frac{dr_1}{cd\tau_1} \right)_{t \leq t^*}.
\]

Analogously, for the components of vector (4.55), we obtain the following expressions from Eqs.(4.40) and (4.45):

\[
\left( \frac{dt}{d\tau_2} \right)_{t \leq t^*} = \frac{e^{-\tau_1(t)/2}}{M_2(r_2)f_{12}(r_2)} \left( m_{out} - m_{12} + \frac{GM_2^2(r_2) - e_2^2 - 2e_2(e_{in} + e_1)}{2c^2r_2} \right)
\]

\[
\left( \frac{dr_2}{cd\tau_2} \right)_{t \leq t^*} = \frac{\delta_2}{M_2(r_2)f_{12}(r_2)} \cdot \sqrt{\left( m_{out} - m_{12} + \frac{GM_2^2(r_2) - e_2^2 - 2e_2(e_{in} + e_1)}{2c^2r_2} \right)^2 - M_2^2(r_2)f_{12}(r_2)}
\]

\[
\delta_2 = \text{sgn} \left( \frac{dr_2}{cd\tau_2} \right)_{t \leq t^*}.
\]
Thus, from the preceding results, we obtain:

\[
Q = -\frac{1}{M_1 M_2 f_{12}} \cdot 
\left\{ \left( m_{12} - M_{in} - \frac{GM_1^2 + \epsilon_1^2 + 2 \epsilon_1 \epsilon_{in}}{2c^2 r_*} \right) \left( m_{out} - m_{12} + \frac{GM_2^2 - \epsilon_2^2 - 2 \epsilon_2 (\epsilon_{in} + \epsilon_1)}{2c^2 r_*} \right) + 
\right.
\]

\[
- \delta_1 \delta_2 \sqrt{ \left( m_{12} - M_{in} - \frac{GM_1^2 + \epsilon_1^2 + 2 \epsilon_1 \epsilon_{in}}{2c^2 r_*} \right)^2 - M_1^2 f_{12}}
\]

\[
\left. \sqrt{ \left( m_{out} - m_{12} + \frac{GM_2^2 - \epsilon_2^2 - 2 \epsilon_2 (\epsilon_{in} + \epsilon_1)}{2c^2 r_*} \right)^2 - M_2^2 f_{12}} \right\};
\]

(4.62)

here and in the following we omit the coordinate dependence of \( f_a, M_a \) etc., implicitly assuming that they have to be evaluated on \((t_*, r_*)\) where not differently indicated.

**Determination of \( Q' \).**

It is possible to apply the same procedure to the region \( BOD \) (i.e. after the intersection time), finding the quantity

\[
Q' \equiv \{ g_{00}^{(BOD)} u_{OB}^0 u_{OD}^0 + g_{11}^{(BOD)} u_{OB}^1 u_{OD}^1 \}_{t=t^*, r=r_1=r_2=r_*}.
\]

(4.63)

Now the unit tangent vectors to trajectories \( OB \) and \( OD \) are\(^1\):

\[
u_{OB}^i = (u_{OB}^0, u_{OB}^1, u_{OB}^2, u_{OB}^3)
= \left( \frac{dt}{d\tau_1}, \frac{dr_1}{d\tau_1}, 0, 0 \right)_{t \geq t_*},
\]

(4.64)

\(^1\)Obviously, when we say \( t \geq t_* \), we tacitly assume before a (possible) second intersection.
and

\[ u_{iOD} = (u^0_{OD}, u^1_{OD}, u^2_{OD}, u^3_{OD}) = \left( \frac{dt}{d\tau_2}, \frac{dr_2}{d\tau_2}, 0, 0 \right)_{t \geq t_\star}; \quad (4.65) \]

from the joining conditions (4.50) and (4.51) it is possible to see that these are actually unit vectors. The components of \( u_{iOB} \) can be deduced from Eqs.(4.47) and (4.50), while the components of \( u_{iOD} \) from Eqs.(4.48) and (4.51). Then, using the metric in the region \( BOD \), it is possible to calculate the scalar product

\[ Q' = \frac{-1}{M_1 M_2 f_21} \cdot \{ \left( m_{out} - m_{21} + \frac{GM_1^2 - \varepsilon_1^2 - 2\varepsilon_1 (e_{in} + e_2)}{2c^2 r_s} \right) \left( m_{21} - M_{in} - \frac{GM_2^2 + \varepsilon_2^2 + 2\varepsilon_2 e_{in}}{2c^2 r_s} \right) + \\
\quad \delta'_1 \delta'_2 \sqrt{\left( m_{out} - m_{21} + \frac{GM_1^2 - \varepsilon_1^2 - 2\varepsilon_1 (e_{in} + e_2)}{2c^2 r_s} \right)^2 - M_1^2 f_{21}} - \\
\quad \sqrt{\left( m_{21} - M_{in} - \frac{GM_2^2 + \varepsilon_2^2 + 2\varepsilon_2 e_{in}}{2c^2 r_s} \right)^2 - M_2^2 f_{21}} \} , \quad (4.66) \]

where \( \delta'_1 \) and \( \delta'_2 \) have been defined as in (4.58) and (4.61), but for \( t \geq t_\star \). We introduced these symbols only for generality, but actually we are interested only in the case with\(^2\)

\[ \delta'_1 = \delta_1, \quad \delta'_2 = \delta_2 . \quad (4.67) \]

\(^2\)This is the only possible case if one excludes \( v_1(t^\star) = v_2(t^\star) = 0 \), because there are non discontinuities in the velocities.
The necessary continuity requirement is thus

\[ Q = Q', \quad (4.68) \]

then, since \( r_* \) is assumed to be known, this equation allows to find \( m_{21} \).

**Physical meaning of Q and Q'**.

Using standard definition for the shell velocities before the intersection one has

\[
\left( \frac{v_1}{c} \right)^2 = \frac{g_{11}^{(COA)}(r_1)}{-g_{00}^{(COA)}(r_1)} \left( \frac{dr_1}{cdt} \right)^2, \tag{4.69}
\]

\[
\left( \frac{v_2}{c} \right)^2 = \frac{g_{11}^{(COA)}(r_2)}{-g_{00}^{(COA)}(r_2)} \left( \frac{dr_2}{cdt} \right)^2, \tag{4.70}
\]

and similarly for the velocities after the intersection,

\[
\left( \frac{v_1'}{c} \right)^2 = \frac{g_{11}^{(BOD)}(r_1)}{-g_{00}^{(BOD)}(r_1)} \left( \frac{dr_1}{cdt} \right)^2, \tag{4.71}
\]

\[
\left( \frac{v_2'}{c} \right)^2 = \frac{g_{11}^{(BOD)}(r_2)}{-g_{00}^{(BOD)}(r_2)} \left( \frac{dr_2}{cdt} \right)^2. \tag{4.72}
\]

Then it is easy to obtain from the definitions (4.53) and (4.63), that\(^3\)

\[
Q = \left\{ \frac{v_1v_2/c^2 - 1}{\sqrt{1-v_1^2/c^2}\sqrt{1-v_2^2/c^2}} \right\}_{t=t_*,r_1=r_2=r_*} \tag{4.73}
\]

and

\[
Q' = \left\{ \frac{v_1'v_2'/c^2 - 1}{\sqrt{1-(v_1')^2/c^2}\sqrt{1-(v_2')^2/c^2}} \right\}_{t=t_*,r_1=r_2=r_*}. \tag{4.74}
\]

\(^3\)It is also worth noting that \( \sqrt{Q^2 - 1}/Q = -|v_1/c - v_2/c|/(1 - v_1v_2/c^2) \), which is the relative velocity definition of two “particles” in relativistic mechanics.
Determination of $P$ and $P'$.

First of all it is convenient to introduce new symbols to simplify the expressions of $Q$ and $Q'$. With

\[
q_1 \equiv -\frac{GM_1^2 + e_1^2 + 2e_1e_{in}}{2c^2r_*},
\]
\[
q_2 \equiv \frac{GM_2^2 - e_2^2 - 2e_2(e_{in} + e_1)}{2c^2r_*},
\]

and

\[
q'_1 \equiv \frac{GM_1^2 - e_1^2 - 2(e_{in} + e_2)}{2c^2r_*},
\]
\[
q'_2 \equiv -\frac{GM_2^2 + e_2^2 + 2e_2e_{in}}{2c^2r_*},
\]

then $Q$ and $Q'$ can be re-written as

\[
Q = \frac{-1}{M_1M_2f_{12}},
\]
\[
\cdot \left\{ (m_{12} - M_{in} + q_1)(m_{out} - m_{12} + q_2) + \right.
\]
\[
\left. -\delta_1\delta_2 \sqrt{(m_{12} - M_{in} + q_1)^2 - M_1^2f_{12}} \right\}^{(4.75)}
\]
\[
\sqrt{(m_{out} - m_{12} + q_2)^2 - M_2^2f_{12}}
\]
and

\[ Q' = \frac{-1}{M_1 M_2 f_{21}} \cdot \left\{ \left( m_{\text{out}} - m_{21} + q'_1 \right) \left( m_{21} - M_{\text{in}} + q'_2 \right) + \right. \]
\[ \left. -\delta'_1 \delta'_2 \sqrt{\left( m_{\text{out}} - m_{21} + q'_1 \right)^2 - M_{\text{f}_{12}}} \right\}, \]  

(4.76)

Now, in principle is possible to find \( m_{21} \) by squaring and solving \( Q = Q' \) (which is a quartic equation). However the procedure is cumbersome and moreover it is not possible with Eq.(4.68) alone to determine the sign of the roots. Fortunately, as in the non-charged case, it is possible to follow another easier way. Indeed, it is possible to introduce two other invariants, say \( P \) and \( P' \), similar to \( Q \) and \( Q' \), which are constructed using the scalar products of the 4-velocities of the two shell, but now taking the limit to \( (t_*, r_*) \) from the \( AOD \) and \( COB \) regions respectively. More explicitly, we define

\[ P \equiv \left\{ g_{00}^{(AOD)} u^{0}_{AO} u^{0}_{OD} + g_{11}^{(AOD)} u^{1}_{AO} u^{1}_{OD} \right\}_{t'=t, r'=r_1=r_2=r_*}, \]  

(4.77)

and

\[ P' \equiv \left\{ g_{00}^{(COB)} u^{0}_{CO} u^{0}_{OB} + g_{11}^{(COB)} u^{1}_{CO} u^{1}_{OB} \right\}_{t'=t, r'=r_1=r_2=r_*}. \]  

(4.78)

Then, the same continuity requirement of Eq.(4.68) implies that it must hold also that

\[ Q = P, \quad P = P'. \]  

(4.79)

Following the same method used to find \( Q \) and \( Q' \), after some calculations,
one arrives to

\[ P = \frac{-1}{M_1 M_2 \phi_{in}} \cdot \left\{ (m_{12} - M_{in} + p_1) (m_{21} - M_{in} + p_2) + \delta_1 \delta'_2 \sqrt{(m_{12} - M_{in} + p_1)^2 - M_1^2 \phi_{in}} \right\} \]

\[ \sqrt{(m_{21} - M_{in} + p_2)^2 - M_2^2 \phi_{in}} \}

(4.80)

and

\[ P' = \frac{-1}{M_1 M_2 \phi_{in}} \cdot \left\{ (m_{out} - m_{21} + p'_1) (m_{out} - m_{12} + p'_2) + \delta'_1 \delta_2 \sqrt{(m_{out} - m_{21} + p'_1)^2 - M_1^2 f_{out}} \right\} \]

\[ \sqrt{(m_{out} - m_{12} + p'_2)^2 - M_2^2 f_{out}} \}

(4.81)

where we have denoted

\[ p_1 \equiv \frac{GM_1^2 - e_1^2 - 2e_1e_{in}}{2c^2 r_*} \]

\[ p_2 \equiv \frac{GM_2^2 - e_2^2 - 2e_2e_{in}}{2c^2 r_*} \]

and

\[ p'_1 \equiv -\frac{GM_1^2 + e_1^2 + 2e_1(e_{in} + e_2)}{2c^2 r_*} \]

\[ p'_2 \equiv -\frac{GM_2^2 + e_2^2 + 2e_2(e_{in} + e_1)}{2c^2 r_*} . \]
Determination of $m_{21}$; the energy transfer

Thus the complete set of continuity conditions at the point of intersection can be written as

\[ Q = Q', \quad Q = P, \quad Q = P'. \]  \hspace{1cm} (4.82)

It turns out that this three quartic equations for the unknown parameter $m_{21}$ have only one common root. It is possible to find the solution using hyperbolic functions (see Appendix). The final result is remarkably simple:

\[ m_{21} = M_{in} + m_{out} - m_{12} - \frac{e_1 e_2}{c^2 r_s} - \frac{G M_1 M_2}{c^2 r_s} Q , \]  \hspace{1cm} (4.83)

or equivalently, in terms of $f_{21}$:

\[ f_{21} = \phi_{in} + f_{out} - f_{12} + 2 \frac{G^2 M_1 M_2}{c^4 r_s^2} Q . \]  \hspace{1cm} (4.84)

It can be easily seen from Eqn.(4.83) that the charge $e_{in}$ of the central singularity does not affect the result (but it affects the equation of the motion of the shells and thus $Q$). Formula (4.83) solves the problem of determining the mass parameter $m_{21}$ from the quantities specified at the evolutionary stage before intersection. It is then possible to determine the energy transfer between the shells. Indeed the energy of shell 1 and 2 before the intersection are, respectively

\[ E_1 = (m_{12} - M_{in}) c^2 , \quad E_2 = (m_{out} - m_{12}) c^2 , \]  \hspace{1cm} (4.85)

while, after the intersection

\[ E_1' = (m_{out} - m_{21}) c^2 , \quad E_2' = (m_{21} - M_{in}) c^2 . \]  \hspace{1cm} (4.86)
The conservation of total energy is automatically ensured by the above formulas, indeed
\[ E_1 + E_2 = E_1' + E_2' . \]  
(4.87)

Then it is natural to define the exchange energy as
\[ \Delta E = E_2' - E_2 = -(E_1' - E_1) . \]  
(4.88)

Then, from Eqn.(4.83) and the above definitions, it follows that
\[ \Delta E = -\frac{e_1 e_2}{r_*} - \frac{G M_1 M_2}{r_*} Q . \]  
(4.89)

It is also useful (especially for the Newtonian approximation) to use Eqn.(4.73) and re-express \( \Delta E \) as:
\[ \Delta E = -\frac{e_1 e_2}{r_*} - \frac{G M_1 M_2}{r_*} \left\{ \frac{v_1 v_2/c^2 - 1}{\sqrt{1 - v_1^2/c^2} \sqrt{1 - v_2^2/c^2}} \right\} r=r_* . \]  
(4.90)

4.4 Post-Newtonian approximation

For slow velocities of the shells it is interesting to consider the Post-Newtonian limit of Eqn.(4.90):

\[ \Delta E = \frac{G m_1 m_2 - e_1 e_2}{r_*} + \frac{1}{2c^2} \left\{ \frac{G m_1 m_2}{r_*} [v_1(r_*) - v_2(r_*)]^2 + \frac{G m_2 L_1^2}{m_1 r_*^3} + \frac{G m_1 L_2^2}{m_2 r_*^3} \right\} + o \left( \frac{1}{c^4} \right) . \]  
(4.91)

It is worth noting that only the zeroth order in \( 1/c^2 \) changes with respect to the uncharged case (because of the Coulomb term \(-e_1 e_2/r_*\)), while all the
other orders remain unchanged, being of kinetic origin; \( m_1 \) and \( m_2 \) are the rest masses of the shells, indeed we have used for the masses \( M_1 \) and \( M_2 \) the definitions (4.41).

It can be also useful to re-express all the quantities in a Newtonian language and consider only the zeroth order in \( 1/c^2 \), e.g. we can expand the energy as

\[
E = mc^2 + \mathcal{E} + o\left(\frac{1}{c^2}\right),
\]

(4.92)

where \( m \) and \( \mathcal{E} \) do not depend on \( c \). Therefore, similarly, we can define at the first order in \( 1/c^2 \)

\[
\begin{align*}
    m_{12} - M_{in} &= m_1 + \frac{\mathcal{E}_1}{c^2}, & m_{out} - m_{12} &= m_2 + \frac{\mathcal{E}_2}{c^2}, \\
    m_{out} - m_{21} &= m_1 + \frac{\mathcal{E}'_1}{c^2}, & m_{21} - M_{in} &= m_1 + \frac{\mathcal{E}'_2}{c^2}.
\end{align*}
\]

(4.93)

(4.94)

Then it follows also that the energy conservation law takes the form

\[
\mathcal{E}_1 + \mathcal{E}_2 = \mathcal{E}'_1 + \mathcal{E}'_2,
\]

(4.95)

and Eqn.(4.88) becomes

\[
\begin{align*}
    \mathcal{E}'_1 &= \mathcal{E}_1 - \Delta \mathcal{E}, & \mathcal{E}'_2 &= \mathcal{E}_2 + \Delta \mathcal{E},
\end{align*}
\]

(4.96)

where \( \Delta \mathcal{E} = (\Delta E)_{c \to \infty} \). Thus from the above formulas and definitions it is clear that

\[
\Delta \mathcal{E} = \frac{Gm_1m_2 - e_1e_2}{r_s}.
\]

(4.97)
4.5 Pressureless shells with zero effective masses

\( L_1 = L_2 = 0 \) \text{ and } \( M_1 = M_2 = 0 \)

It is interesting also to consider the case in which the motion of the particles of the shells is only radial (i.e. \( L_1 = L_2 = 0 \)) and the rest masses are negligible with respect to the kinetic energies and to the charges —indeed this is the case for two shells composed by (ultra)relativistic electrons and positrons. In this case the effective masses can be replaced by

\[ M_1 = M_2 = \lambda, \quad (4.98) \]

where \( \lambda \) is a parameter arbitrary small. From Eqn.(4.89), with \( Q \) expressed by formula (4.62), it is easy to find that the energy transfer in this case is

\[ \Delta E = -\frac{e_1 e_2}{r_s} + \frac{c^4 r_s}{2G f_{12}} (\phi_{in} - f_{12}) (f_{12} - f_{out}) + o(\lambda^2), \quad (4.99) \]

having assumed that the shells have opposite-directed velocities, i.e.

\[ \delta_1 \delta_2 = -1. \quad (4.100) \]

Otherwise, if the shells goes in the same direction, i.e.

\[ \delta_1 \delta_2 = 1, \quad (4.101) \]

then Eqn.(4.89) becomes simply

\[ \Delta E = -\frac{e_1 e_2}{r_s} + o(\lambda^2); \quad (4.102) \]

obviously the previous formulas make sense only if \( r_s \) exists. We want to underline the presence of the term \( o(\lambda^2) \), because, strictly speaking, a charge
cannot have zero rest mass, therefore we are in the case of just small effective masses. As expected, in the case of vanishing charges ($e_1 = e_2 = 0$), Eqn.(4.102) gives zero at $\lambda = 0$ because this is the case of two photon-shells which go in the same direction and therefore cannot never intersect.

4.6 The intersection of a test shell with a gravitating one

One-shell case

Let us consider firstly the case of a test shell on the RN field. This limit has the only aim to show that the shell’s equation of motion (4.26) actually reduce to the simple test-particle case; the limit can be obtained by putting

$$m \to \lambda m, \quad e \to \lambda e, \quad L \to \lambda L, \quad (m_{\text{out}} - M_{\text{in}})c^2 \to \lambda E$$

(4.103)

with $\lambda \to 0$. Then, considering also (4.30), we find that Eqn.(4.26) becomes

$$E = \mu c^2 \sqrt{\phi_{\text{in}}(r_0)} + \left(\frac{dr_0}{cdt}\right)^2 + \frac{ee_{\text{in}}}{r_0} - \lambda \frac{G\mu^2 - e^2}{2r_0},$$

(4.104)

now, putting $\lambda = 0$ the self-energy term is killed; then re-writing Eqn.(4.104) using the more familiar Schwarzschild time $t$ [and notation (4.11)],

$$E = c^2 \sqrt{m^2 + \frac{L^2}{c^2 R_0^2(t)}} \sqrt{\frac{\phi_{\text{in}}^3(R_0)}{\phi_{\text{in}}^2(R_0) - \left(\frac{dR_0}{cdt}\right)^2} + \frac{ee_{\text{in}}}{R_0} + o(\lambda)},$$

(4.105)

it is easy to recognize that Eqn.(4.105) coincides with the first integral of motion of a test-charge particle on the Reissner-Nordstrom background, where $E$ is the conserved energy of the particle, $m$ the rest mass, $e$ the charge and $L$ the angular momentum.

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Two-shell case, with one test-shell

Now we can deal with the more interesting two-shell case, in which shell-2 is considered “test”. To gain this limit we have to put

\[ m_2 \to \lambda m_2 , \quad e_2 \to \lambda e_2 , \quad L_2 \to \lambda L_2 , \quad (4.106) \]

\[ (m_{\text{out}} - m_{12})c^2 \to \lambda E_2 , \quad (m_{21} - M_{\text{in}})c^2 \to \lambda E'_2 . \]

Then, using Eqn.(4.83) with \( Q \) given by formula (4.62), one obtains

\[ \Delta E = -\frac{\epsilon_1 \epsilon_2}{r_*} + \frac{1}{r_* f_{12}} \cdot \frac{\left\{ \left( E_1 - \frac{GM_1^2 + e_1^2 + 2 e_1 e_{\text{in}}}{2c^2 r_*} \right) \left( E_2 - \frac{e_2(e_{\text{in}} + e_1)}{c^2 r_*} + \lambda \frac{GM_2^2 - e_2^2}{2c^2 r_*} \right) + \right.}{\left. -\delta_1 \delta_2 \sqrt{\left( E_1 - \frac{GM_1^2 + e_1^2 + 2 e_1 e_{\text{in}}}{2c^2 r_*} \right)^2 - M_1^2 f_{12}} \right\} . \]

(4.107)

Thus, only the self-energy terms of shell-2 are killed by \( \lambda = 0 \).

Now, it is worth noting the following fact: shell-1 does not have any discontinuity when it intersect the shell-2 (this is natural because shell-2 is “test” and does not affect the metric), on the other hand shell-2 undergoes a discontinuity in the metric when it cross shell-1 and consequently it has an actual discontinuity in the velocity. It is easy to calculate this gap; indeed using the definition (4.70) of velocity \( v_2 \) [with the time \( \frac{dT}{dr_2} \) given by the joint condition (4.45)], with metric coefficient (4.34), and with the help the first integral of motion (4.40), one finds

\[ v_2^2(r_2) = 1 - f_{\text{out}}(r_2) \left( \frac{E_2}{M_2(r_2)} - \frac{e_2(e_1 + e_{\text{in}})^2}{M_2(r_2)^2} \right)^2 + o(\lambda) , \quad t \leq t_* , \quad (4.108) \]
where we have used $f_{12} = f_{\text{out}} + o(\lambda)$; in the same way, using (4.72), (4.51), (4.36), and (4.48), the velocity $v'_2$ (after the intersection) is

$$[v'_2(r_2)]^2 = 1 - \phi_{\text{in}}(r_2) \left( \frac{E'_2}{M_2(r_2)} - \frac{e_2(e_1 + e_{\text{in}})^2}{M_2^2(r_2)r_2} \right)^2 + o(\lambda), \quad t \geq t_*, \quad (4.109)$$

where $E'_2$ can be expressed in function of $E_2$ with the help of (4.107). From the previous formulas it is clear that in general

$$v'_2(r_*) - v_2(r_*) \neq 0. \quad (4.110)$$

### 4.7 Shell ejection

The exchange in energy of the shells during the intersection makes possible that one initially bounded shell can acquire enough energy to escape to infinity.

The shell ejection mechanism can take place also in the Newtonian regime. In this case, from Eqs.(4.96)-(4.97) it results that

$$E'_1 = E_1 - \frac{Gm_1 m_2 - e_1 e_2}{r'_*} , \quad E'_2 = E_2 + \frac{Gm_1 m_2 - e_1 e_2}{r'_*} , \quad (4.111)$$

and then, after the first intersection

$$E''_1 = E'_1 + \frac{Gm_1 m_2 - e_1 e_2}{r''_*} = E_1 - (Gm_1 m_2 - e_1 e_2) \left( \frac{1}{r'_*} - \frac{1}{r''_*} \right) ,$$

$$E''_2 = E'_2 - \frac{Gm_1 m_2 - e_1 e_2}{r''_*} = E_2 + (Gm_1 m_2 - e_1 e_2) \left( \frac{1}{r'_*} - \frac{1}{r''_*} \right) , \quad (4.112)$$

where we have denoted the radius of the first and second intersection with $r'_*$ and $r''_*$ respectively. In the following we will consider only the case

$$Gm_1 m_2 - e_1 e_2 > 0 , \quad (4.113)$$

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this is e.g. the case in which the two shells have opposite charges. Thus, also in the case \( E_1, E_2 < 0 \), if

\[
r''_s > r'_s ,
\]

(4.114)

and if the initial condition were in such a way that \( r'_s \) is enough small and \( r''_s \) not too much close to \( r'_s \), then it is possible to have \( E''_2 > 0 \), i.e. the ejection of the second shell.

Let us now assume that \( r''_s > r'_s \), and consider a “semi-relativistic” case in which at the first intersection we use the full relativistic formulas\(^4\),

\[
\begin{align*}
E'_1 &= E_1 - \frac{M_1(r'_s)M_2(r'_s)}{r'_s}(-Q) + \frac{e_1e_2}{r'_s} , \\
E'_2 &= E_2 + \frac{M_1(r'_s)M_2(r'_s)}{r'_s}(-Q) - \frac{e_1e_2}{r'_s} ,
\end{align*}
\]

(4.115)

while at the second intersection we use the Newtonian approximation,

\[
\begin{align*}
E''_1 &= E'_1 + \frac{Gm_1m_2 - e_1e_2}{r''_s} \\
&= E_1 - \left[ \frac{M_1(r'_s)M_2(r'_s)(-Q) - e_1e_2}{r'_s} - \frac{Gm_1m_2 - e_1e_2}{r''_s} \right] ,
\end{align*}
\]

(4.116)

\[
\begin{align*}
E''_2 &= E'_2 - \frac{Gm_1m_2 - e_1e_2}{r''_s} \\
&= E_2 + \left[ \frac{M_1(r'_s)M_2(r'_s)(-Q) - e_1e_2}{r'_s} - \frac{Gm_1m_2 - e_1e_2}{r''_s} \right] .
\end{align*}
\]

This approximation is always justified if the radius of the second intersection \( r''_s \) is enough large. Now, it is remarkable that whatever the value of \( r'_s \) is,

\(^4\)Remember that \(-Q = 1 + o(1/c^2)\).
the first term in the square brackets in Eqn.(4.116) satisfies the inequality
\[
\frac{M_1(r'_s)M_2(r'_s)(-Q) - e_1e_2}{r'_s} > \frac{Gm_1m_2 - e_1e_2}{r'_s}.
\] (4.117)
Comparing the expressions (4.116), (4.117) and (4.112) it is possible to see that in the relativistic regime the shell ejection possibility is even greater than in the Newtonian case. Furthermore, it is worth noting that the presence of the charge do not change qualitatively the pure gravitational analysis, but just magnifies the ejection effect.

### 4.7.1 \( Gm_1m_2 - e_1e_2 < 0 \) case

Let us consider also briefly the case in which the shells are equal-signed charged and the repulsion overcome the gravity attraction, i.e. \( Gm_1m_2 - e_1e_2 < 0 \). In this case the ejection can happen only after an odd number of intersections.

E.g. after three intersections, from the previous formulas we have, in the Newtonian approximation:
\[
E'''_1 = E_1 - (Gm_1m_2 - e_1e_2)\left(\frac{1}{r_*'} - \frac{1}{r_*''} + \frac{1}{r_*'''}\right).
\] (4.118)
Obviously this formula has a meaning only if
\[
\frac{1}{r_*'} < \frac{1}{r_*''} - \frac{1}{r_*'''};
\] (4.119)
otherwise the ejection happens at the first intersection (and then there would not be other crossings, and no \( r_*'' \), \( r_*''' \)), or never more; if Eqn.(4.119) is true, then it means that the barycenter of the two shells is falling into the center singularity.
4.8 Conclusions IV

We have found the energy exchange between two charged crossing shells (formula (4.90)). Then we have studied special cases of physical interest in which the formulas simplify: the non relativistic case, the massless shells, the test shell, and finally the ejection mechanism in a semi-Newtonian regime: we found that the ejection mechanism is more efficient in the charged case than in the neutral one if the charges have opposite sign (because the energy transfer is larger due to the Coulomb interaction).
Chapter 5

Appendix: Articles
GRAVITATIONAL FIELD AND ELECTRIC FORCE 
LINES OF A NEW 2-SOLITON SOLUTION

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A new exact solution of the coupled Einstein–Maxwell equations is given and studied. It is found using the soliton method, adding one soliton to the Schwarzschild background. The solution is stationary and axial-symmetric, and has five physical parameters. The physical interpretation we give is that it describes a Kerr–Newman (KN) naked singularity linked by a “strut” to a charged black hole. Indeed, on the axis, between the two bodies an unavoidable anomaly region is present \((g_{\phi\phi} < 0\) and a conic singularity). The solution is stationary also in the case with zero total angular momentum. Finally, we give the force lines of the electrical field in a general case, and in the case in which the KN singularity has a much smaller mass than the nearby black hole; we also considered the behavior at different distances of the charge. In spite of the naive interpretation suggested by the mathematical construction of the solution, what we expected to be a “Schwarzschild” black hole appears to be charged and rotating; we interpret this fact as a parameter-mixing phenomenon.

Keywords: Einstein–Maxwell equations; soliton solutions; electric force lines.

1. Introduction

The problem of two charged bodies in reciprocal equilibrium, in the context of General Relativity, has a long history in the literature and was treated by different methods which were both approximated and exact (see, for example, Refs. 1–10). Only recently was it found\(^\text{10}\) that an equilibrium condition at a certain distance is possible also for two non-extreme Reissner–Nordström sources in the case in which one is naked and the other is with a horizon. However, in our work we consider a different case which includes rotating sources, since we have a stationary but in general not static (i.e. \(g_{t\phi} \neq 0\)) solution.

In the present work, we have used the Inverse Scattering Method (ISM) applied to the Einstein–Maxwell equations (see Refs. 11, 12 and 13 for a self-consistent exposition) which allows one to treat the problem in an exact form. Although with
that method it is possible to describe $N$ Kerr–Newman sources\(^a\) aligned on the axis (in the general case of $N$-solitons), the most physically relevant configuration is described by the 2-soliton solutions, which should contain as a particular case also the equilibrium configuration of two Reissner–Nordström sources.

From the 1980s there was a certain number of papers which studied the general 2-soliton solution (with twelve parameters) on the Minkowski background.\(^{14–16}\) However, a physical detailed study of that family of solutions — which means the identification of the physical parameters, the asymptotic behavior at infinity, the electrical force lines, and above all the regularity conditions on the axis — is still lacking in the literature.

The present work aims to be a first step in that direction, particularly with respect to the study of the electric field which we investigate through the plotting of the force lines, following Ref. 2. In particular, we consider the most simple case of 2-soliton solution, where one adds one soliton on the Schwarzschild background. That solution is really a new solution which is not trivially included in the already studied family of 2-soliton solutions constructed adding two solitons on the flat background, because the solutions found in that way describe two naked singularities and nobody has yet shown how to reach the case with a horizon, nor whether it is possible to reach it at all — maybe it is possible, but in every case it requires some complex and highly non-trivial transformation.

We find that new solution, taking the spectral matrix associated to the Minkowski background, dressing it with one soliton, and analytically extending it to the Schwarzschild case by setting to zero all the parameters but the Schwarzschild mass $m$. That procedure is equivalent to resolving the differential system associated with the Schwarzschild background, but in that way it is possible to avoid the integration and to proceed simply algebraically; this spectral matrix allows one to find the $n$-soliton solution on the Schwarzschild background.

The physical situation which we are interested in is the equilibrium condition of a Schwarzschild black hole near a Kerr–Newman (KN) singularity. However, unfortunately we found that the anomaly region between the two sources (which consists of a conic singularity and a “tube singularity,” i.e. $g_{\phi\phi}$ negative on the axis) is unavoidable, no matter what the values of the parameters are.\(^{21}\) That fact hampers an easy physical interpretation of the solution; usually this is interpreted as a “strut,” or a “string”\(^{20}\) between the two bodies.

Finally, we plot the force lines of the electric field for different values of the parameters. In both the general case and in the case in which the KN particle is much smaller with respect to the black hole, we discover that in spite of the naive interpretation suggested by the mathematical construction, the black hole of the Schwarzschild background acquires a charge. This is because the non-linear superposition of the two solitons mixes and changes the physical interpretation of

\(^a\)More precisely Kerr–Newman-NUT-g sources; they also have a magnetic charge and the NUT parameter.
the mathematical parameters (parameter-mixing phenomenon), e.g. if \( e_1 \) and \( e_2 \) are, respectively, the charges of the two solitons when they are very far from each other, then when they are nearby, the physical charges will be in general some different constants, say \( Q_1 \) and \( Q_2 \), which will be some complicated functions if expressed in terms of the previous parameters. The expected meaning of the parameters is recovered only in the far distance limit — an exception is the mass parameter \( m \), which maintains its original meaning also nearby.

2. Summary of the Procedure

The soliton method\(^{13}\) (which is another name for the ISM) allows us to find solutions of the Einstein–Maxwell equations in the form:

\[
ds^2 = g_{ab}(\rho, z)dx^a dx^b + f(\rho, z)(d\rho^2 + dz^2),
\]

where the indexes \( a, b = 1, 2 \) corresponding, respectively, to the coordinates \( t \) and \( \varphi \).\(^{b}\) In order to make the integrable ansatz compatible with the metric (1), one should assume the following structure for the electromagnetic potentials:

\[
A_\mu = 0, \quad A_a = A_a(\rho, z),
\]

where \( \mu = 3, 4 \) (i.e. it refers to \( \rho \) and \( z \)). Obviously, these solutions are stationary and cylindrical-symmetric. We followed the procedure explained in a self-consistent way in Ref. 13 (Chap. 3). For brevity, we will not report all the details, but simply the main steps and the final results.

2.1. One soliton on the flat background

As is well known in the literature,\(^{14}\) adding one soliton to the flat background, one finds the Kerr–Newman-NUT-g (KNNg) solution. However, what we need in the following is only the \( \phi^{(0)} \) spectral matrix which resolves the Einstein–Maxwell problem associated with the Minkowski solution:

\[
\phi^{(0)} = \begin{pmatrix}
1 & 0 & 0 \\
\frac{i \lambda}{\Gamma} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where

\[
\Gamma \equiv \sqrt{\lambda^2 + \rho^2}, \quad \lambda \equiv z - w.
\]

\(^{b}\)The original method is more general and also enables us to find non-stationary solutions in which \( t \) and \( \rho \) are exchanged.
The complex parameter \( w \) has to take the value \( w_1 \equiv z_1 + i\sigma_1 \), which is the pole of the dressing matrix \( \chi \),

\[
\chi = I + \sum_{k=1}^{n} \frac{R_k}{w - w_k}.
\]

This is, in general, a complex constant, but we can make it purely imaginary, say,

\[
w_1 = i\sigma_1,
\]

(5)

because of the \( z \)-translational invariance of the Minkowski background (putting \( z_1 = 0 \) means putting the singularity at the origin). The solution has five constants, which are \( \sigma_1 \) and the four introduced with the constant vector \( l^{(1)} \) [defined in Ref. 13, formula (3.66)]. It is convenient to use the following parametrization:

\[
l^{(1)} = (1, \frac{m - ib}{a - \sigma_1}, \frac{e + ia}{a - \sigma_1}),
\]

and, without loss of generality,

\[
\sigma_1^2 = -m^2 + b^2 + a^2 + e^2 + g^2,
\]

(6)

where now \( m, a, e, b \) and \( g \) are the five independent parameters. With that choice, the final form of the metric, after the simple rotation,

\[
dt' = dt + \text{const.} \times d\varphi,
\]

(7)

is the KNNg one in the standard coordinates. Therefore, one can recognize \( m \) as the mass, \( a \) the angular momentum, \( e \) the charge, \( b \) the NUT parameter, and \( g \) the magnetic charge.

We now set all the constants to zero but the mass \( m \), in order to have the Schwarzschild solution, which reads in cylindrical coordinates

\[
ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + r^2 \sin^2 \theta d\varphi^2 + \frac{r^2}{(r - m)^2 - m^2 \cos^2 \theta} (d\rho^2 + dz^2).
\]

(8)

Note that \( \rho \) and \( z \) are related to the usual Schwarzschild ones by

\[
\begin{cases}
\rho = \sqrt{r^2 - 2mr \sin \theta}, \\
z = (r - m) \cos \theta.
\end{cases}
\]

(9)

This metric will be used as the new background in the construction of the new solution (and obviously the background electromagnetic field is absent everywhere).

One should emphasize that although it appears so natural at this stage to set \( b \) and \( g \) to zero (because they are not physical), this would not be without problems in the 2-soliton solution, as we argue in the final remarks, for the mixing-parameter phenomenon.

2.2. Two-solitons solution: Two different ways

The soliton method has the peculiarity that starting from a background solution, and having found the \( \phi^{(0)} \) matrix, which resolves the spectral system associated with the Einstein–Maxwell problem, one can construct the whole class of \( n \)-soliton
solutions associated with that background, in a purely algebraic way. This is possible by dressing the $\phi^{(0)}$ matrix:

$$
\phi^{(n)} = \chi \phi^{(0)}.
$$

Then, $\phi^{(n)}$ still satisfies the spectral equation associated with the Einstein–Maxwell problem, allowing one to find the $n$-soliton solution.

Let us return to our case. Roughly there are two different ways of finding a 2-soliton solution on the flat background:

(i) Start from the flat background and add two solitons.
(ii) Start from the flat background, and add one soliton. Then, restart the same procedure taking the one-soliton solution as the new background and add another soliton to it.

In general, if one considers only naked singularities, the two procedures are completely equivalent. However, since we want to add one soliton to the Schwarzschild background, we have a non-trivial problem because the first procedure gives a solution with two naked KN$\sigma_1$ (indeed, if one tries to put values that give horizons, then $\sigma_1$ becomes imaginary and thus the metric complex; in the 1-soliton case, we do not have this problem because the rotation (7) “miraculously” eliminates the terms that contains $\sigma_1$ linearly).

Thus, we took the second approach with the exception of a small device. Instead of resolving two spectral problems by integrations for the second we used, the dressing procedure. In order to do that, we used the following trick: we adapt the $\phi^{(0)}$ matrix to the Schwarzschild case with an analytical continuation. Schematically, the procedure is the following:

$$
\phi^{(0)} \rightarrow \phi^{(KN)} = \chi \phi^{\sigma_1 \rightarrow -im} \phi^{(S)}.
$$

In other words, we take the spectral matrix of the flat background $\phi^{(0)}$, dress it with one soliton by finding the spectral matrix $\phi^{(KN)}$ for the (naked) KN background, and extend it to the Schwarzschild case by making the pole pure real:

$$
i \sigma_1 = m, \quad m \in \mathbb{R}
$$

(i.e. taking also $a = 0$, $e = 0$). This is what we mean by “analytical continuation.”

The validity of this trick can be simply checked by showing with an explicit calculation that the spectral problem is identically satisfied by the $\phi^{(S)}$ found.

---

Footnote:

It is important to emphasize that one can do that continuation only after having calculated all the quantities used in the construction of the KN solution (and in particular after all the complex conjugations of the previous procedure). Indeed, the soliton method for the coupled Einstein–Maxwell equation works only with complex poles (or, at least, with pure imaginary ones).
Using the same normalization as in Ref. 13, we have

\[
\phi(S) = \begin{pmatrix}
\frac{1}{\sqrt{w^2 - m^2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{w^2 - m^2}} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\Gamma} \left[ w - m \cos \theta + \frac{m \lambda}{r} \right] & -\frac{m}{r} & 0 \\
\frac{i}{\Gamma} \left[ m r \sin^2 \theta + \left( w + m \cos \theta \right) \lambda \right] & w + m \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(13)

where \( \lambda = z - m \).

Now the parameter \( w \) has to take the value of the second pole (on the Schwarzschild background):

\[ w_2 = z_2 + i\sigma_2, \]

where \( z_2 \) and \( \sigma_2 \) are two new real independent constants.

2.3. The final solution: \( g_{ab}, A_a \) and \( f \)

In order to give the explicit form of the solution, one should follow the steps summarized in Ref. 13 (Chap. 3, pp. 80–81). All these steps are merely algebraical and we report here only the final results. Therefore, the explicit form of the metric tensor is

\[
g_{tt} = -\left( 1 - \frac{2m}{r} \right) - \frac{4|l_3(2)|^2}{|w_2^2 - m^2|^2 |T_2 \Gamma_2|^2 |r^2|} |U - iml_2(2)\Gamma_2|^2
\]

\[
- \frac{8}{|w_2^2 - m^2|r^2} \text{Im} \left\{ \frac{1}{T_2 \Gamma_2} [\overline{l_2(2)} |U|^2 + im\overline{U} \Gamma_2 \right\}
\]

\[
+ im|l_2(2)|^2 U \overline{\Gamma_2} + m^2|l_2(2)|^2 |\Gamma_2|^2 \right\}
\]

(15)

\[
g_{t\phi} = \frac{i4|l_3(2)|^2}{|w_2^2 - m^2|^2 |T_2 \Gamma_2|^2 r^2} \left\{ \frac{\overline{U} - iml_2(2) \Gamma_2 \overline{l_2(2)} \Gamma_2 - iV}{r} \right\}
\]

\[
- \frac{i4}{|w_2^2 - m^2|r^2} \left\{ \overline{U} \overline{Z} \Gamma_2 + |l_2(2)|^2 m \overline{\Gamma_2}^2 - iml_2(2)|\Gamma_2|^2 Z \right\}
\]

\[
- \frac{i4}{|w_2^2 - m^2|r^2} \left\{ -|l_2(2)|^2 \overline{U} Z \Gamma_2 - m \overline{V} \Gamma_2 + iml_2(2) \overline{\Gamma_2} |\Gamma_2|^2 Z \right\}
\]

(16)
Gravitational Field and Electric Force Lines of a New 2-Soliton Solution

\[ g_{\varphi \varphi} = r^2 \sin^2 \theta + \frac{4 |l_3^{(2)}|^2}{|w_2^2 - m^2|^2 |T_2 \Gamma_2|^2} V + (l_2^{(2)} \Gamma_2)^2 \]

\[- \frac{8}{|w_2^2 - m^2|} \text{Im} \left\{ \frac{1}{T_2 \Gamma_2} [-l_2^{(2)} |V|^2 \right. \]

\[ + \sqrt{Z \Gamma_2} + il_2^{(2)} |2V \Gamma_2 + l_2^{(2)} |Z \Gamma_2|^2] \right\} \]  

(17)

(the bar denotes complex conjugation). In order to make the expressions, more compact we have used the three symbols

\[ \begin{cases} U = (r - m)w_2 - m^2 \cos \theta, \\
V = (m + w_2 \cos \theta)r - (w_2 + m \cos \theta)^2, \\
Z = (w_2 + m \cos \theta) \end{cases} \]  

(18)

The quantities \( l_2^{(2)} \) and \( l_3^{(2)} \) are two complex constants that can be defined in terms of \( \sigma_2, z_2, m \) and of the new real constants \( \alpha, \beta, l, k \) as:

\[ l_2^{(2)} = \alpha + i\beta, \]

\[ l_3^{(2)} = \frac{l + ik}{\sqrt{w_2^2 - m^2}}. \]  

(19)

(20)

We also remember that these three components of the metric are not independent. Indeed, by construction, they satisfy the relation

\[ g_{tt} g_{\varphi \varphi} - (g_{t \varphi})^2 = -\rho^2. \]  

(21)

Obviously, the metric component, \( g_{t \varphi} \) is real, although it is not immediately evident from expression (16).

Then, we have the electromagnetic potential:

\[ \begin{cases} A_t = 2 \text{Im} \left\{ \frac{l_3^{(2)}}{\sqrt{w_2^2 - m^2} r T_2 \Gamma_2} [\bar{U} - i ml_2^{(2)} \Gamma_2] \right\}, \\
A_\varphi = 2 \text{Im} \left\{ \frac{l_3^{(2)}}{\sqrt{w_2^2 - m^2} r T_2 \Gamma_2} [i \bar{V} + l_2^{(2)} \bar{Z} \Gamma_2] \right\}. \]  

(22)

The quantity \( T_2 \) is defined as

\[ T_2 = \frac{2}{w_2^2 - m^2} (2 l_2^{(2)} \text{Im}(U \bar{V}) + \Gamma_2 (\bar{U}Z - m \bar{V}) \)

\[ + |l_2^{(2)}|^2 T_2 (mV - U \bar{Z}) + 2ml_2^{(2)} |\Gamma_2|^2 \text{Im}(Z) + (l^2 + k^2) r \Gamma_2). \]  

(23)
Finally, we give the conformal metric factor $f$:

$$f = C_2 |T_2|^2 \frac{r^2}{(r - m)^2 - m^2 \cos^2 \theta}, \tag{24}$$

where $C_2$ is an arbitrary constant, which we can take as

$$C_2 = \frac{\sigma_2^2 |w_2^2 - m^2|^2}{4K^2}, \tag{25}$$

in order to have the asymptotically flat solution at large distances, i.e. $\lim_{r \to \infty} f = 1$.

The largest part of the problem of giving a compact form of the 2-soliton solution comes from the irrational nature of the function $\Gamma_2$. Nevertheless, there exist three particular “surfaces” on which $\Gamma_2$ is rational; they are

$$\theta = 0, \tag{26}$$
$$\theta = \pi, \tag{27}$$
$$r = 2m. \tag{28}$$

The case of $\theta = \pi$ can be reproduced by the case of $\theta = 0$ simply by inverting the sign of $z_2$. The main features are that considering $\theta = 0$ one sees that there is no divergence on the axis, and that the north and south poles of the Schwarzschild horizon remain unperturbed in $r = 2m$. Studying $r = 2m$, it transpires that the new horizon is larger than the Schwarzschild horizon, and that $g_{t\phi}$ does not vanish also if $A_{\text{tot}} = 0$, i.e. the static case cannot be reached, at least in that simple way.

It transpires that in order to have the standard form of the Minkowski vacuum in the far observer limit, we need the simple rotation

$$\begin{cases}
  dt' = dt + Rd\varphi, \\
  d\varphi' = d\varphi,
\end{cases} \tag{29}$$

which is analogous to that in Eq. (7), used to find KN in the standard coordinates (but in general with a different constant $R$).

3. Physical Interpretation of the New Solution

3.1. Two particular limits with $z_2 \to \infty$

The physical interpretation of the new solution can easily be shown by taking the following two limits:

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On the other hand, $\Gamma_1 = (r - m) - im \cos \theta$ has been rationalized thanks to the peculiar choice of the coordinates $(r, \theta)$.

The Schwarzschild singularity is on $r = 0$ but $\theta = \pi/2$; so, physically the singularity is on the axis, but mathematically we do not see it if we define the axis as the set of points for which $\theta = 0$ or $\theta = \pi$. 

---
(i) Considering the limit in which

\[
\begin{aligned}
& z_2 \to \infty, \\
& |z - z_2| < \infty, \\
& \rho < \infty,
\end{aligned}
\]  

one finds the KN solution, which indeed corresponds to putting the charge very far from the Schwarzschild black hole \((z_2 \to \infty)\), and putting the observer near the charge \((\rho < \infty \text{ and } |z - z_2| < \infty)\).

(ii) On the other hand, taking the limit in which

\[
\begin{aligned}
& z_2 \to \infty, \\
& r < \infty,
\end{aligned}
\]  

one finds the Schwarzschild solution with mass \(m\).

At the same time, it is now clear that \(z_2\) is a measure of the distance between the two bodies. However, strictly speaking, it would be not correct to consider the exact solution as a rotating charged particle near a Schwarzschild black hole. Indeed, when we take the limit \(z_2 \to \infty\), the values of the physical parameters also change (since they depend on \(z_2\); see Appendix A; Eq. (54)), therefore when \(z_2\) is finite the interpretation of the parameters would be different because of the non-linear interaction between the two solitons (the interaction is linear only in the first approximation when \(z_2 \to \infty\)).

It is also worth noting that the imaginary part of the pole \(\sigma_2\) is linked to the physical constants by

\[
\sigma_2^2 = -(M_{\text{tot}} - m)^2 + A_2^2 + Q_{\text{tot}}^2,
\]

where \((M_{\text{tot}} - m), A_2\) and \(Q_{\text{tot}}\) are, respectively, mass, angular momentum of the KN particle, and total charge, as we will see in the next section. Then, as in the one-soliton solution, the charged particle is a \textit{naked} KN singularity; otherwise, \(\sigma_2\) would be imaginary and it would give imaginary terms in the metric tensor.

\subsection{Far-observer approximation; the rotation}

Thus, the present solution has seven integration constants: \(m, z_2, \sigma_2, \alpha, \beta, l, k\). Only the first two have a direct physical meaning. In order to identify the other

\footnote{The total charge coincides with the one of the KN particles only in the \(z_2 \to \infty\) limit.}
physical constants, one can proceed, as usual, by expanding $g_{ab}$ and the potential $A_a$ in powers of $(\frac{1}{r})$.

After the mentioned rotation (29), we have the following approximation form of the metric tensor

$$
\begin{align*}
g_{tt} &= -1 + \frac{2M_{\text{tot}}}{r} - \frac{Q_{\text{tot}}^2}{r^2} - 2\frac{M_{\text{tot}} - m}{r^2} [M_{\text{tot}} - z_2 \cos \theta] + O \left( \frac{1}{r^3} \right), \\
g_{t\phi} &= -2\frac{M_{\text{tot}} - m}{r} A_{\text{tot}} \sin^2 \theta + O \left( \frac{1}{r^2} \right), \\
g_{\phi\phi} &= \left\{ 1 + 2\frac{M_{\text{tot}} - m}{r} + \left[ 2M_{\text{tot}}^2 - Q_{\text{tot}}^2 - 2(M_{\text{tot}} - m)z_2 \cos \theta \right] \frac{2}{r^2} \right\} \\
&\quad \times r^2 \sin^2 \theta + O \left( \frac{1}{r}\right),
\end{align*}
$$

and for the potential:

$$
\begin{align*}
A_t &= \frac{Q_{\text{tot}}}{r} - \frac{M_{\text{tot}} - m}{r^2} Q_{\text{tot}} \\
&\quad + \frac{2\sigma_2}{K} \left[ z_2 (z_2^2 + \sigma_2^2 - m^2 + 2m\alpha \sigma_2 + m\beta z_2) - m\beta (\sigma_2^2 + m^2) \right] \\
&\quad \times \frac{2Q_{\text{tot}}}{\sigma_2 - A_2} \cos \theta + O \left( \frac{1}{r^3} \right) \\
A_\phi &= \frac{Q_{\text{tot}}}{r} \left\{ R + \frac{(M_{\text{tot}} - m)^2 + (\sigma_2 - A_K)^2 \cos^2 \theta}{(\sigma_2 - A_K)} \right\} \\
&\quad + \frac{2\sigma_2}{K} \left[ k(z_2^2 - \sigma_2^2 + m^2) - 2lz_2 \sigma_2 \right] \frac{\sin^2 \theta}{r} + O \left( \frac{1}{r^2} \right).
\end{align*}
$$

Finally, we have for the conformal factor:

$$
f = 1 - \frac{2M_{\text{tot}}}{r} + O \left( \frac{1}{r^2} \right).
$$

The relation between the physical and the mathematical parameters are given in Appendix A. Here and in the following, we have already set to zero the total magnetic charge and the NUT-parameter because they give a non-asymptotic flat solution. In general, the presence of two magnetic charges cannot be excluded (which in sum give zero); however, we do not concern ourselves with that fact since it should only affect the anomaly region. Furthermore, in the following we will pay attention only to the electric field.
It is worth noting that the mass of the black hole coincides exactly with the Schwarzschild mass \( m \). Indeed, using the Komar integral definition,\(^{18,20}\) the mass \( m_H \) inside the horizon \( H \) is

\[
m_H = \frac{1}{4\pi} \int_H \left( \xi^1 \xi^1 - \xi^1 i_1 \right) \sqrt{-g} d^3x
\]

\[
= \frac{1}{4\pi} \int_H \left[ \frac{(f \rho)}{f} \right]_{,\rho} + \left[ \frac{\rho f z}{f} \right]_{,z} d\varphi d\rho dz
\]

\[
= \frac{1}{4\pi} \int_{\partial H} \left( 1 + \frac{\rho f \rho}{f} \right) d\varphi dz = m,
\]

where \( \xi^i = (1, 0, 0, 0) \), and we have used the fact that the horizon lies on the axis \( (\rho = 0) \), and that the limits of \( H \) remain unperturbed notwithstanding the presence of the second source (i.e. \( H = \{ \rho = 0, z \in [-m, m], \varphi \in [0, 2\pi] \} \)). That result holds for each metric of the form (1) which has an horizon \( H \).

Therefore, the mass of the second source has to be \( m_2 = M_{\text{tot}} - m \), and has a much more complex expression in terms of the mathematical parameters (see Appendix A).

A final remark: in the expansion of \( g_{t\varphi} \) [the second equation of (33)] we have defined \( A_{\text{tot}} \) in such a way that it is multiplied by the factor \( m_2 \) — i.e. the mass of the rotating charge — by analogy with the Kerr case. However, in the Kerr case, there was no doubt that the rotation was due only to the mass present, while now we have two bodies which can rotate. So, in order to stress that fact, we introduce a new constant:

\[
A_2 \equiv \sigma_2 - \frac{2\sigma_2}{K} |X|^2,
\]

which is the angular momentum of the charged particle (where \( X = w_2 + i\bar{l}_2 \), and \( K \) is a combination of the other constants; see Appendix A). Then, we can express \( A_{\text{tot}} \) as

\[
A_{\text{tot}} = A_2 - \frac{m}{m_2} R.
\]

Thus, in a clearer way one can write \( g_{t\varphi} = 2(mR - m_2 A_2) \frac{\text{d}^2\theta}{\text{d}x^2} + O \left( \frac{1}{r^2} \right) \). Therefore, the rotational constant \( R \) can be interpreted as the angular momentum of the black hole. It is worth noting that one can put the angular momentum of the particle, or the total angular momentum, to zero (in that case, the particle and the black hole are contra-rotating), but it is not possible to put both the momenta to zero. Therefore, \( A_{\text{tot}} = 0 \) means that the two bodies are contra-rotating, which is different from the static case.

The fact that the Schwarzschild black hole also acquires an angular momentum is another example of the parameter-mixing phenomenon.
4. Some Analytical Features of the 2-Soliton Solution

4.1. The polydromic function \( \Gamma_2 \) and the cut-line

In the 2-soliton solution, the polydromic function is present\(^8\)

\[
\Gamma_2 = \sqrt{\rho^2 - \sigma_2^2 + (z - z_2)^2} - i2\sigma_2(z - z_2). \tag{39}
\]

This function has a branch point in

\[
\begin{align*}
\rho = \sigma_2, \\
z = z_2,
\end{align*} \tag{40}
\]

so, in order to define \( \Gamma_2 \), it is better to use a different notation:

\[
\Gamma_2 = \sqrt{u + iv} = \sqrt{u^2 + v^2}e^{i\arg(u + iv)}, \tag{41}
\]

where \( u \) and \( v \) are the real functions depending on \( r, \theta \), which one can deduce from (39). In all the calculation, we have taken the cut along the semi-line\(^b\)

\[
\begin{align*}
u \leq 0, \\
v = 0,
\end{align*} \tag{42}
\]

which is the same as defining \(-\pi \leq \arg(u + iv) \leq \pi\) or, alternatively, \(\pi \leq \arg(u + iv) \leq 3\pi\). With that choice, the cut-line reaches the axis on \( z = z_2 \).

It is worth noting that the cut-line defined in that way reduces to a simple point on the axis if one takes the limit \( z_2 \to \infty \) (indeed, \( \Gamma_2 \approx z_2 \)). This is perhaps the best reason to adopt such a definition (furthermore, it is mathematically the simplest).

However, in general, the choice of a particular cut-line should depend on the boundary condition of the problem. (The only constraint is that every closed line around the charge must cross it.)

On the other hand, in order to have a continuous solution even on the cut-line, one should define \( \Gamma_2 \) on the extended complex plane, say

\[
\Gamma_2 = \begin{cases} 
\sqrt{u^2 + v^2}e^{i\arg(u + iv)} & \text{if } (u, v) \in R_1, \\
-\sqrt{u^2 + v^2}e^{i\arg(u + iv)} & \text{if } (u, v) \in R_2,
\end{cases} \tag{43}
\]

where \( R_1 \) and \( R_2 \) are the two Riemann sheets defined by

\[
\begin{align*}
R_1 &= \{(u, v) : -\pi < \arg(u + iv) \leq \pi\}, \\
R_2 &= \{(u, v) : \pi < \arg(u + iv) \leq 3\pi\}. \tag{44}
\end{align*}
\]

From a physical point of view, an observer who comes from \( R_1 \) (which corresponds to the solution with an asymptotically flat space–time) would be in the domain \( R_2 \)

\(^8\)In Greek “poly-” = multi-, and “dromos” = racetrack — which means that the function can be defined on more domains. This is a better expression than the more popular “multivalued” (because a “function” — by definition — can never be “multivalued”).

\(^b\)For simplicity, thanks to the axial symmetry, we refer to the two-dimensional \( z-\rho \)-semiplane. Thus, e.g. the “cut-line” is not a line but a surface, and so on.
after having crossed the cut-line. Then, he can return to $R_1$ simply making another turn around the branch point.\textsuperscript{1} As is typical for the ring singularity of the KN kind, the branch-point [i.e. (40)] coincides with the charge point. However, a lot of strangeness would appear if the observer could go in the domain $R_2$ (for example, in that domain the solution has unphysical quantities such as $M_{tot} < 0$ at $r \to \infty$).

Thus, we have only two possibilities:

(i) We can define $\Gamma_2$ on the extended complex plane; then the solution is also regular on the cut-line. However, crossing the ring singularity, the observer enters another universe\textsuperscript{3} which has unphysical quantities.

(ii) We can restrict $\Gamma_2$ to the first Riemann sheet $R_1$. Then the solution has a discontinuity on the cut-line, but it is univocally defined for every $(r, \theta)$ (i.e. we have no tunnel to the other universe).

We chose the second possibility as the only physically meaningful one. One can interpret the discontinuity on the cut-line as due to the presence of some singular field of matter. Naturally, using the soliton method we assumed that the only matter was given by the electromagnetic field. However, in a certain region, it is possible to match the 2-soliton solution with another solution which takes into consideration another stress–energy tensor. In that picture, the 2-soliton metric will hold only outside that region.

4.2. The region between the two centers; the “strut”

It is a fact that two masses, in the absence of some further force, cannot be exactly at rest. In spite of this, we have an exact stationary solution, which describes, as we have shown in the two limits above, a rotating charge near a (quasi-)Schwarzschild black hole.\textsuperscript{k} Thus, it is not unnatural to expect some mathematical strangeness in this region that physically represents “something,” maybe stripes of matter or a strut, which prevents the charge from falling down.

Even if the axis is regular in the usual sense, i.e. the $g_{ab}$ tensor and the $A_a$ potential on the axis do not diverge, there exists a different kind of singularity where

\[
\begin{align*}
g_{\varphi t}|_{\rho=0} &= 0 & \text{if } z < 0 \text{ or } z > z_2, \\
g_{\varphi t}|_{\rho=0} &= a_0g_{tt} & \text{if } 0 < z < z_2,
\end{align*}
\]  

\textsuperscript{1}A very similar situation, in which a ring singularity is present that makes a branch point, is described in Ref. 16 referring to a family of two-solitons solutions in the non-stationary case.

\textsuperscript{3}A similar strangeness would also come in the Kerr metric, if one continues the solution to negative $r$ (see the discussion in Ref. 19).

\textsuperscript{k}As we will show in the next section, in general, the equilibrium is not given by an electric repulsive force, e.g. in the last cases that we have considered, the (quasi-)Schwarzschild black hole has a charge of opposite sign with respect to the KN and thus the electric force is even attractive.
and

\[
\begin{cases}
fg_{tt}|_{\rho=0} = -1 & \text{if } z < 0 \text{ or } z > z_2, \\
fg_{tt}|_{\rho=0} = -1 + a_1 & \text{if } 0 < z < z_2.
\end{cases}
\] (46)

The behavior at infinity is the correct one, while the jump on the KN ring, which is directly due to the branch point of the $\Gamma_2$ function, gives rise to two distinguished paths in $g_{\varphi\varphi}$, which cannot be eliminated by any change of coordinates. The implications on $g_{\varphi\varphi}$ can be seen using the determinant relation (21).

The first anomaly, i.e. $a_0 \neq 0$ in (45), implies that on the axis $g_{\varphi\varphi}$ has the opposite sign to what it has at infinity (where it is positive) — we call it the “tubesingularity” since there exists a surface around the axis topologically equivalent to an overturned cone (with the base on the ring and the vertex on the Schwarzschild horizon), on which $g_{\varphi\varphi}$ vanishes, and inside of which $\varphi$ becomes time-like.

The second one is usually called the “conic singularity.” Indeed, the fact that $a_1 \neq 0$ in (46) implies that the circumference of a small circle of radius $\rho$ centered on the axis is not $2\pi \rho$ (even if $a_0$ vanished). This means that the region near that part of the axis would not be homeomorphic to the Minkowski space–time.

Thus, in order to have a physically well-behaved solution, we should impose the two equilibrium conditions:

\[
\begin{cases}
a_0(m, M_{\text{tot}}, A_{\text{tot}}, Q_{\text{tot}}, z_2) = 0, \\
a_1(m, M_{\text{tot}}, A_{\text{tot}}, Q_{\text{tot}}, z_2) = 0.
\end{cases}
\] (47)

Unfortunately, as is shown in Ref. 21 (also in the more general case of the Reissner–Nordström background, instead of Schwarzschild), they cannot be satisfied by any $z_2$ real distance, for any choice of the other parameters.

From a physical point of view, this is interpreted as the presence of a “strut”, which should represent the presence of some matter field that prevents the charge to from falling into the Schwarzschild black hole.\(^1\)

5. Force Lines

The soliton method starts from the charge-free Maxwell equations (i.e. $j^\mu = 0$). This means that the charge is localized at most at some particular point at which the electromagnetic field diverges. As we have said, in our solution that point coincides exactly with the branch-point (40) (i.e. the point at which vanishes $\Gamma_2$). This is a typical feature of $n$-soliton solutions, as well as the fact that the charge has a ring structure\(^m\) (because of the cylindrical symmetry and the fact that $\sigma_2^2 > 0$). In spite of this, both $g_{ab}$ and $A_a$ are finite there. Nevertheless, all of them have a cusp,

\(^1\)Another (rather curious) way to remove the $g_{\varphi\varphi}$-anomaly could consist of putting the cut-line exactly on $g_{\varphi\varphi} = 0$. In that different definition of the Riemann sheets, the conic singularity will be relegated to the other universe; however, it would remain the discontinuity of the metric on this surface.

\(^m\)In the $n$-soliton case, one has $n$ different $\Gamma_i$, and thus $n$ rings.
and thus all their derivatives diverge. Further, at that point \( f \to \infty \) because it is proportional to \( \frac{1}{r^2} \).

One can see the charge point and the behavior of the electric field by plotting the lines of force. We define the force lines in the same way as Hanni and Ruffini in Ref. 2: the locus of points with a given value of the flux \( \Phi \), which is defined as

\[
\Phi \equiv \oint_C F^* = 2 \int_C \{ F^*_{23} dr d\varphi + F^*_{24} d\theta d\varphi \} = 4\pi Q_{\text{tot}},
\]

where \( Q_{\text{tot}} = \text{total electric charge inside } C \) and \( F^*_{ij} = \sqrt{-g} \varepsilon_{ijkl} F^{kl} \). Then, at any given point, the slope of the line of constant flux is given by

\[
\frac{dr}{d\theta} = -\frac{\partial \Phi}{\partial \theta} \left/ \frac{\partial \Phi}{\partial r} \right..
\]

Obviously, the flux \( \Phi \) is now considered as a function of \( r, \theta \) because now the integral (48) is taken over a piece of a spherical surface (and \( r, \theta \) are the limits of the integration). From a mathematical point of view, it is the same as resolving the differential system

\[
\begin{align*}
\frac{dr}{d\lambda} &= \sqrt{-g} F^{13} = \frac{1}{\sin \theta} \left( g_{\varphi \varphi} \frac{\partial A_t}{\partial r} - g_{\nu \varphi} \frac{\partial A_{\varphi}}{\partial r} \right), \\
\frac{d\theta}{d\lambda} &= \sqrt{-g} F^{14} = \frac{1}{(r^2 - 2mr) \sin \theta} \left( g_{\varphi \varphi} \frac{\partial A_t}{\partial \theta} - g_{\nu \varphi} \frac{\partial A_{\varphi}}{\partial \theta} \right)
\end{align*}
\]

Remembering that \( F^{13} = E_r \) and \( F^{14} = E^\theta \). The meaning of the solutions is clear \((r(\lambda), \theta(\lambda))\): they are the lines which are, at each point, tangential to the electrical field.

In the following plots we used geometrical units \((G = c = 1)\), where the unitary length is given by the Schwarzschild mass \( m = 1 \). The lines are plotted in cylindrical coordinates\(^n\):

\[
\begin{align*}
\rho' &= r \sin \theta, \\
z' &= r \cos \theta,
\end{align*}
\]

where \( r \) and \( \theta \) are the usual Schwarzschild coordinates.

### 5.1. General case

Initially, we look at the general case, in which all the parameters are of a similar order of magnitude, say:

\[
\begin{align*}
m &= 1, \\
m_2 &= 3, \\
A_2 &= 2, \\
Q_{\text{tot}} &= 4, \\
A_{\text{tot}} &\cong 0.717, \\
z_2 &= 10,
\end{align*}
\]

\(^n\)They are not the ones defined in (9); however, they coincide as \( r \to \infty \).
(note that there are only independent constants; $A_{\text{tot}}$ is calculated after having fixed the other values using the relation (54) given in Appendix A).

From the point of view of the electric flux, we have three separate regions: a region in which the force lines come from the KN singularity and end in the strut; a region which includes the lines coming from the KN singularity and arriving at infinity; finally, the region of the lines which go from the black hole to infinity. We show with thick lines the electric lines which separate these three regions.

Since near the strut the solution does not have physically meaningful interpretation, we do not plot in Fig. 1 the lines that lie inside the anomaly region and also the ones which go inside and then escape to infinity. In the same way, we do inside the horizon, because it is not easy to give a physical interpretation of the electric field for the time-like behavior of the $r$ coordinate. This emphasizes that only outside these regions can the solution give a serious physical description of a realistic situation.

The presence of the separatrix line which divides the black hole from the KN singularity means that the two objects have a charge of the same sign.

![Diagram of electric field lines](image)

**Fig. 1.** Force lines of the electric field in the general case (52). The horizontal segment which joins the charge to the axis is not a force line but the cut-line of the KN ring; there the force lines also are discontinuous. The circle of radius $= 2m$ is the Schwarzschild horizon. The point-like line marks the $\theta = 0$ surface; inside of that region the metrics have an unphysical behavior. The two force lines, are shown with thick lines, separatrices between the three different regions.
As one expects, the behavior far from the charges becomes very rapidly that of the Coulomb field.

The anomaly region can be shrunk to the axis until it becomes approximatively a segment by taking small values of charge and mass in the KN singularity, as we will see in the next section.

5.2. A case close to the Hanni–Ruffini one

Hanni and Ruffini\(^2\) considered a non-rotating test charge momentarily at rest in the Schwarzschild metric. Thus, we can approximate that situation by making the energy of the charge very small with respect to the mass of the black hole. However, the 2-soliton solution cannot approach the Hanni–Ruffini case exactly for three different reasons. First of all, obviously, because of the presence of the strut. Second, because one cannot put both the angular momenta to zero, and, also if the particle is non-rotating, the total angular momentum does not vanish. Finally, because, as we will see now, the black hole also acquires a charge. However, we can consider a similar case using the following values:

\[
\begin{align*}
    m &= 1, \\
    m_2 &= 0.01, \\
    A_2 &= 0, \\
    Q_{tot} &= 0.02, \\
    \varepsilon_2 &= 3.
\end{align*}
\] (53)

Thus, the charge is non-rotating (in the sense that \(A_2 = 0\)) and has a small energy with respect to the Schwarzschild black hole. The resulting total angular momentum is \(A_{tot} \approx 2.31\). However, \(g_{t\varphi}\) is quite small because it is also proportional to \(m_2\). The resulting lines of force are given in Fig. 2.

Also in that case, there exists a separatrix line. The separatrix line is nothing more than a force line which has the initial condition very close to the south pole of the horizon (i.e. \(r = 2m, \theta = \pi\)). However, we mark that line because it has an important physical meaning that encloses the horizon, and thus — for the Cauchy theorem — all the lines which lie inside that region cannot go to infinity and have to fall on the horizon. This means that the flux of the electric field on the horizon is definitely negative, i.e. the black hole has a net negative charge.

In the case (53), using a numerical integration over the horizon surface, one finds that the mixing-parameter phenomenon is very strong: the charge is \(Q_{blackhole} = -0.009998\), which is practically half of the total. Now, the Schwarzschild horizon \(g_{tt} = 0\) is to a very good approximation a sphere of radius \(r = 2m\), since the perturbation of the KN particle is very small.

It is worth noting that in that case the KN ring, and thus the cut-line, has a very small radius and is practically point-like. The \(g_{\varphi\varphi}\)-anomaly, i.e. the strut, even if it is still present, is now very close to the axis and does not touch the electric lines of our plots. Further, since there are no force lines which end on it, we can deduce that it is neutral.
Fig. 2. The force lines in the case of a small charge at $r = 4m$. The bold line ends at $r = 2m, \theta = \pi$, then it is a separatrix, because the lines inside that region are entrapped and fall in the horizon, instead the others escape to infinity. The circle is the Schwarzschild surface. The dotted line indicates the “strut”, which is now very close to the axis, and it does not touch the force lines of our plot.

Plotting the force lines at different values of the distance $z_2$, one can see a smooth transition of the electric field to that of a Reissner–Nordström black hole, which is very similar to what was found in the similar Hanni–Ruffini case in Ref. 2.

Finally, we want to stress that since in the two cases (52) and (53) the charge of the black hole has a different sign, then the case should exist in which the black hole is neutral — indeed the transition between the two cases has to be continuous at the changing of $m_2, A_2$ and $Q_{\text{tot}}$.

6. Conclusions

We briefly summarize the main characteristics of the 2-soliton solution:

(i) It is a stationary axially symmetric and asymptotically flat solution of the coupled Einstein–Maxwell equations with five physical parameters.
(ii) In the limit in which \( z_2 \to \infty \) (i.e. \( z_2 \gg m \)), it reduces to a Schwarzschild black hole with a (distant) naked KN singularity linked by a “strut” (which becomes thinner and thinner as \( z_2 \to \infty \), but never vanishes).

(iii) The charge singularity has a “ring” structure (as in the KN case) with a discontinuity inside the ring (in the plane \( \rho - z \), it corresponds to a cut-line which links the singularity to the axis).

(iv) In general, the solution is stationary but not static, also if the total angular momentum constant \( A_{\text{tot}} \) is set at zero.

(v) The metric tensor has a doubly anomalous behavior near the axis in the region between the two centers (inversion of the sign in \( g_{\varphi \varphi} \) and the conic singularity); this is what we called the “strut”.

(vi) The Schwarzschild horizon is smoothly perturbed by the presence of the other singularity: it becomes a little bit larger, while the north and sud poles remain unmodified.

(vii) There is also a charge inside the black hole.

Perhaps, it could be said that it was clear from the beginning that some anomaly would appear in the solution in order to ensure the dynamical equilibrium between the charge and the black hole. Nevertheless, the existence of an equilibrium point in the geodesics of a neutral test particle on a Reissner–Nordström naked singularity leaves open the possibility of having such a well-behaved exact solution.\(^{10}\) The equilibrium in that case would be due to a peculiar geometrical effect.
6.1. The parameter-mixing phenomenon

The soliton method, as formulated, should allow — we cannot yet show it, but we believe that it is the case — a solution without the anomaly region and the conic singularity by choosing a certain peculiar value of \( z_2 \), which should be determined as a function of the other constants. However, we argue that this condition could be satisfied only in the most general 2-soliton solution, which has twelve mathematical parameters. Indeed, also the four of these parameters which are not physical in the limit \( z_2 \to \infty \), i.e. the NUT-parameter and the magnetic charge of each soliton (when the solitons are alone), play an important role and cannot be set to zero from the beginning since they would acquire a different meaning in the entangled case (parameter-mixing). These four unphysical degrees of freedom of the solution will be removed by two “equilibrium conditions,” i.e. no-tube and no-conic conditions, and by imposing that both the total NUT parameter and the magnetic charge must be zero (there are indeed five conditions, but one of them should be used to fix \( z_2 \)). This picture can also explain why we found a charge inside the “Schwarzschild” black hole: the charge parameter of the background solution is no longer the physical charge when one adds the second soliton; in order to have a neutral black hole, one should set to zero a different quantity which would be a complicated mix of the mathematical constants.

We are working on such more general solution, but we will talk about it in later communications.

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Appendix A: Relations between the Physical Constants and the Mathematical Ones

The seven real constants which appear in the 2-soliton solution are \( \alpha, \beta, l, k, \sigma_2, z_2 \) and \( m \). They are linked to the two complex components \( l_2^{(2)} \) and \( l_3^{(2)} \) by the definitions (14), (19), (20) and (32).

Apart from \( m \) and \( z_2 \), which have a direct physical meaning, the other physical parameters are

\[
M_{\text{tot}} = m + m_2,
\]

\[
m_2 = \frac{2\sigma_2}{K} \left[ (-1 + \alpha^2 + \beta^2) m\sigma_2 + \alpha(z_2^2 + \sigma_2^2 - m^2) \right],
\]

\[
A_2 = \sigma_2 - \frac{2\sigma_2}{K} \left[ z_2^2 + \sigma_2^2 + m^2(\alpha^2 + \beta^2) + 2m(\alpha\sigma_2 + \beta z_2) \right],
\]
We have used the auxiliary constant total angular momentum, total charge, NUT-parameter, and total magnetic charge. which are, respectively, total mass, mass and angular momentum of the KN particle, relations are

\[ A_{\text{tot}} = A_2 - \frac{m}{m_2} R, \]  
\[ Q_{\text{tot}} = \frac{\sigma_2}{K} [l(z_2 + m\beta) + k(\sigma_2 - m\alpha)], \]
\[ B = \frac{4\sigma_2}{K} [(1 + \alpha^2 + \beta^2)mz_2 + \beta(z_2^2 + \sigma_2^2 + m^2)], \]
\[ G_{\text{tot}} = \frac{\sigma_2}{K} [l(\sigma_2 + m\alpha) - k(z_2 + m\beta)], \]

which are, respectively, total mass, mass and angular momentum of the KN particle, total angular momentum, total charge, NUT-parameter, and total magnetic charge. We have used the auxiliary constant

\[ K = (1 - \alpha^2 - \beta^2)(z_2^2 + \sigma_2^2 - m^2) + 4\alpha m\sigma_2 + \frac{l^2 + k^2}{4}. \]  

The constant \( R \) used in \( A_{\text{tot}} \) and in the rotation of the coordinates is

\[ R = \frac{2\sigma_2}{K} [(1 + \alpha^2 + \beta^2)(z_2^2 + \sigma_2^2 + m^2) + 4\beta mz_2]. \]  

In order to have a physical solution, as explained in the text, one has to impose that \( B = 0 \) and \( G_{\text{tot}} = 0 \). Considering these constraints, the inverse relations are

\[ \alpha = \frac{m\sigma_2((\sigma_2 - A_2)^2 - m_2^2) + m_2(\sigma_2 - A_2)(|w_2|^2 - m_2^2)}{m^2 m_2^2 - 2m m_2 \sigma_2(\sigma_2 - A_2) + (\sigma_2 - A_2)^2 |w_2|^2}, \]
\[ \beta = -\frac{m \sigma_2(z_2^2 + m_2^2)}{m^2 m_2^2 - 2m m_2 \sigma_2(\sigma_2 - A_2) + (\sigma_2 - A_2)^2 |w_2|^2}, \]
\[ l = \frac{2Q_{\text{tot}}}{\sigma_2 - A_2} \left( \frac{z_2 - m_2^2 m_2^2 + (\sigma_2 - A_2)^2 |w_2|^2}{m^2 m_2^2 - 2m m_2 \sigma_2(\sigma_2 - A_2) + (\sigma_2 - A_2)^2 |w_2|^2} \right), \]
\[ k = \frac{2Q_{\text{tot}}}{\sigma_2 - A_2} \left( \frac{\sigma_2 + m_2^2 \sigma_2((\sigma_2 - A_2)^2 - m^2) + m_2 m_2 \sigma_2(\sigma_2 - A_2)(|w_2|^2 - m_2^2)}{m^2 m_2^2 - 2m m_2 \sigma_2(\sigma_2 - A_2) + (\sigma_2 - A_2)^2 |w_2|^2} \right). \]

References
ELECTRIC FORCE LINES OF THE DOUBLE REISSNER–NORDSTROM EXACT SOLUTION

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Recently Alekseev and Belinski have presented a new exact solution to the Einstein–Maxwell equation which describes two Reissner–Nordstrom (RN) sources in reciprocal equilibrium (no struts or strings); one source is a naked singularity, the other is a black hole: this is the only possible configuration for two separable objects, apart from the well-known extreme case ($m_i = e_i$).

In the present paper, after a brief summary of this solution, we study in some detail the coordinate systems used and the main features of the gravitational and electric fields. In particular, we graph the plots of the electric force lines in three qualitatively different situations: equal-sign charges, opposite charges and the case of a naked singularity near a neutral black hole.

Keywords: Electric force lines; Reissner–Nordstrom sources; exact solutions.

1. Introduction

The new solution which has recently been found by Alekseev and Belinski\(^1\) (in the following denoted with AB) has solved the long-standing problem of the static equilibrium of two charged masses in the context of GR.

While in the Newtonian theory the equilibrium condition is simply $m_1 m_2 = e_1 e_2$, in the relativistic regime the problem is much more complicated because one has to solve the full system of the Einstein–Maxwell equations,

\[
\begin{align*}
R_{ij} - \frac{1}{2} R g_{ij} &= 2 \left( F_{ik} F_j{}^k + \frac{1}{2} F_{lm} F^{lm} g_{ij} \right), \\
(\sqrt{-g} F^{ik})_{,k} &= \sqrt{-g} F_{ij},
\end{align*}
\]

(1)
and find a static solution with two sources. Furthermore, this solution will in general present conic singularities at the symmetry axis.\footnote{It is called “conic singularity” because the ratio between a small circumference around the axis and its radius is not \(2\pi\) (as for a circle painted on a cone around its vertex).} To find the equilibrium condition is equivalent to requiring the absence of any conic singularity, i.e. the axis has to be locally Minkowskian — physically this means that there must be neither “struts” nor “strings”\footnote{In the ISM there are also some unphysical parameters (NUT parameter, magnetic charge) and the rotation which are not easy to eliminate.} (see Ref. 2 for the rigorous relation between the value of the angle deficit and the effective energy--momentum tensor of these struts and strings) which prevent the two bodies falling or running away from each other.

The key point in understanding the main differences between the classic and the relativistic regime is the repulsive nature of gravity in GR near a naked singularity. This can be seen just by looking at the RN metric

\[ g_{tt} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (2) \]

where gravity is repulsive for \(r < \frac{Q^2}{M}\): it is for this reason that the equilibrium is allowed only at certain distances. Indeed, for example if one considers the geodesic of a neutral particle on that background, it is easy to find an equilibrium (stable) precisely at

\[ r_c = \frac{Q^2}{M}. \quad (3) \]

For charged particles an equilibrium is also possible at a fixed distance;\footnote{In these cases it can be either stable or unstable, according to the choice of the parameters.} in these cases it can be either stable or unstable, according to the choice of the parameters.

In the AB solution the Newtonian equilibrium condition is restored taking the limit of large distance between the two singularities.

Although in principle such an exact solution could be found already many years ago — by using the inverse scattering method (ISM) or the integral equation method (IEM) — practically nobody was able to eliminate the conic singularity in a reasonably explicit way. Indeed, the important achievement of the AB solution is the extreme compactness of all the formulas, despite the complexity of calculations by which it was found.\footnote{They get the wanted task using the IEM, which presents some advantages over the ISM.} As they showed, the equilibrium is possible, apart from the well-known Majumdar--Papapetrou case, where the charge of each source is equal to its mass, only for a naked singularity near a black hole (BH). We excluded from our analysis the BH–BH and naked–naked configurations, since the objects in the former configuration are not separable (their horizons cross), while the latter do not exist at all in an equilibrium state.

This paper is organized as follows. We give a brief historical review of the works in the literature (Sec. 2) (this section can be skipped by those interested only in...
the physical contents). For the convenience of the reader we add a reproduction of the AB solution in Sec. 3. We give some details clarifying the use of the coordinate systems involved (Sec. 4). Then we recall the definition of the electric field in GR (Sec. 5). Finally, we graph the plots of the electric force lines in the various qualitatively different cases (Sec. 6) — which is the main task of our work. More precisely, in this last section, we consider at the beginning the general case with two charges, first with $e_1e_2 > 0$ and then with $e_1e_2 < 0$; and finally that in which only one object (the naked singularity) is charged. The last particular case of the solution in a different form is presented in Ref. 5. For each case we present also the limit in which one source has a much smaller mass and charge than the other.

In particular, we consider the limit case of a small charged particle near a Schwarzschild black hole, finding electric force line plots congruent with the Hanni–Ruffini\textsuperscript{6} ones.

2. Some Historical Remarks

The problem of the equilibrium of two charged masses and their resulting gravitational and electric fields has a long history in the GR literature (see Table 1). It is possible to distinguish two different kinds of results: approximate results and exact solutions.

In the contest of the approximate results, the first to be mentioned is that of Copson,\textsuperscript{7} who gave in 1927 the electric potential of a test charge on the Schwarzschild background (thus there was neglected the backreaction of the particle on the metric tensor). That work was important because it gave the potential in a closed analytic form. However, that result was not completely correct, because it implied that the black hole would have an induced charge; the correct potential was given by Linet\textsuperscript{8} only in 1976 — the electric potential of the AB solution indeed reduces to that form in the limit in which the naked singularity source can be considered as a test particle.

In 1973 Hanni and Ruffini\textsuperscript{6} gave for the first time the plots of the electric force lines,\textsuperscript{c} again for a test particle near a Schwarzschild black hole (but they used a multipole expansion of the electric potential).

Later a certain number of papers were published in which different authors (using exact solutions, PN and PPN approaches) arrived at different conclusions about the possibility/impossibility of an equilibrium configuration. However, no final statements were achieved because of the use of supplementary hypotheses or the incompleteness of the analysis.

In 1993 the already-mentioned article of Bonnor\textsuperscript{3} gave an important hint to clarify the problem: studying the equilibrium configurations in the test particle limit, namely a test charge on the RN background, he pointed out that equilibrium configurations were possible when the ratio $e/m$ was less than unity for the background and more than unity for the particle, and vice versa; he showed also

\textsuperscript{c}We follow this work for the construction of the plots of the present solution.
Table 1. A timeline of the main papers on the two-body static problem in GR.

<table>
<thead>
<tr>
<th>Perturbation methods</th>
<th>Exact solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copson (1927)</td>
<td>Majumdar–Papapetrou (1947)</td>
</tr>
<tr>
<td><strong>Electric field of a test charge</strong> near a Schwarzschild BH</td>
<td></td>
</tr>
<tr>
<td><strong>Electric force lines of a test charge</strong> near a Schwarzschild BH</td>
<td>and Alekseev (1980)</td>
</tr>
<tr>
<td>Linet (1976)</td>
<td><strong>Vacuum solitons</strong></td>
</tr>
<tr>
<td><strong>A correction of the Copson solution</strong></td>
<td><strong>Electrovacuum Solitons</strong></td>
</tr>
<tr>
<td><strong>Equilibrium of a test particle on the RN background</strong></td>
<td>and Sibgatulling (1984)</td>
</tr>
<tr>
<td>Perry–Cooperstock (1997)</td>
<td>by the IEM for rational axis data,</td>
</tr>
<tr>
<td><strong>Equilibrium is possible</strong> (3 numerical examples)</td>
<td>and of Alekseev (1985)</td>
</tr>
<tr>
<td>Bonnor (1993)</td>
<td>by the IEM for rational monodromy data</td>
</tr>
<tr>
<td><strong>Exact solution for equilibrium (without strut) of two RN sources</strong></td>
<td>Integral equation method</td>
</tr>
</tbody>
</table>

that equilibrium was possible for charges of opposite signs too. It is worth noting that the AB solution confirms practically word for word (from a qualitative point of view) that picture.

Then, in 1997, Perry and Cooperstock\(^9\) found three numerical examples showing that the equilibrium was possible for naked–BH configurations using an exact solution.

Finally, one should mention the Bini–Geralico–Ruffini articles,\(^{10,11}\) in which the authors found, using the Zerilli perturbative approach, the correction to the test particle approximation, considering the backreaction of the particle to the background until the first order. Surprisingly, they found that the Bonnor condition remains unchanged also considering these corrections.

For what concerns the history of the exact solutions, the first two important articles were those of Majumdar and Papapetrou,\(^{12,13}\) which exhibited the fields of an arbitrary number of sources in reciprocal equilibrium, each with \(m_i = e_i\).
For many years that was the only exact result known. The next step was taken by Belinski and Zakharov\textsuperscript{14,15} in 1978 with the foundation of the ISM in general relativity (purely gravitational), which was then extended to the Einstein–Maxwell equations by Alekseev\textsuperscript{16} (see Ref. 17 for a self-consistent review). This method allows one to find stationary, axially symmetric solutions with an arbitrary number of sources. From that time in principle the solution to our problem was available. However, practically, the constraints necessary to eliminate the rotation, the conic singularity and the unphysical parameters (NUT parameter, magnetic charge) were too complicated to be handled analytically.

The next step was taken by Ernst and Hauser,\textsuperscript{18,19} Sibgatullin\textsuperscript{20} and Alekseev,\textsuperscript{21} who developed different integral equation methods for constructing solutions to Einstein–Maxwell equations. (The first method of this kind for pure gravity was already formulated in Ref. 14.) The method of Ref. 21 was used by Alekseev and Belinski to find the present solution\textsuperscript{1} (see also Ref. 4), the important achievement of which is the extreme simplicity of the formulas and of the equilibrium condition.

3. Summary of the Alekseev–Belinski Formulas

The formulas (4)–(12) are a reproduction of the formulas (1)–(10) of Ref. 1.

The solution, which can be interpreted as the nonlinear superposition of two RN sources at a fixed distance on the $z$ axis, is of the form

$$ds^2 = H dt^2 - \frac{H}{\rho^2} d\varphi^2 - f(d\rho^2 + dz^2),$$

$$A_t = \Phi, \quad A_\varphi = A_\rho = A_z = 0,$$

where $H$, $f$ and $\Phi$ are real functions of $\rho$ and $z$ only. In what follows, $m_1$, $m_2$ and $e_1$, $e_2$ are the physical masses and charges of each source respectively\textsuperscript{d}; the masses include also the interaction energy, so $M_{\text{tot}} = m_1 + m_2$ and $Q_{\text{tot}} = e_1 + e_2$. It is convenient to use the spheroidal coordinates $(r_1, \theta_1)$ and $(r_2, \theta_2)$, which are linked to the Weyl coordinates $(\rho, z)$ by

$$\begin{cases}
\rho = \sqrt{(r_1 - m_1)^2 - \sigma_1^2 \sin \theta_1}, \\
z = z_1 + (r_1 - m_1) \cos \theta_1,
\end{cases}

\begin{cases}
\rho = \sqrt{(r_2 - m_2)^2 - \sigma_2^2 \sin \theta_2}, \\
z = z_2 + (r_2 - m_2) \cos \theta_2.
\end{cases}$$

By definition $l \equiv z_2 - z_1$ is the distance, expressed in the Weyl coordinate $z$, between the two objects. Then, the explicit solution is

$$H = \frac{[(r_1 - m_1)^2 - \sigma_1^2 + \gamma^2 \sin^2 \theta_1][(r_2 - m_2)^2 - \sigma_2^2 + \gamma^2 \sin^2 \theta_2]}{D^2},$$

$$\Phi = \frac{(e_1 - \gamma)(r_2 - m_2) + (e_2 + \gamma)(r_1 - m_1) + \gamma(m_1 \cos \theta_1 + m_2 \cos \theta_2)}{D},$$

\textsuperscript{d}The expressions were found with the help of the Gauss theorem.
\[ f = \frac{D^2}{[(r_1 - m_1)^2 - \sigma_1^2 \cos^2 \theta_1][(r_2 - m_2)^2 - \sigma_2^2 \cos^2 \theta_2]}, \]  

(9)

where

\[ D = r_1 r_2 - (e_1 - \gamma - \gamma \cos \theta_2)(e_2 + \gamma - \gamma \cos \theta_1), \]  

(10)

while \( \gamma, \sigma_1 \) and \( \sigma_2 \) are defined by

\[ \gamma = (m_2 e_1 - m_1 e_2)(l + m_1 + m_2)^{-1}, \]

\[ \sigma_1^2 = m_1^2 - e_1^2 + 2 e_1 \gamma, \quad \sigma_2^2 = m_2^2 - e_2^2 - 2 e_2 \gamma. \]  

(11)

It is easy to see that \( (fH)_{\rho=0} = 1 \) on the whole axis, i.e. automatically there is no conic singularity. The above formulas give the solution satisfying the Einstein–Maxwell system only under the equilibrium condition

\[ m_1 m_2 = (e_1 - \gamma)(e_2 + \gamma). \]  

(12)

Each of the parameters \( \sigma_1 \) and \( \sigma_2 \) can be either real (in the case of a black hole) or imaginary (for a naked singularity); however, in the following it will always be

\[ \sigma_1^2 > 0, \quad \sigma_2^2 < 0, \quad \text{and} \quad \sigma_1 > 0, \]  

(13)

i.e. the first source is “dressed” and the second is “naked.” Since we want to deal only with separable objects, we require also the non-overcrossing condition

\[ l - \sigma_1 > 0 \]  

(14)

(this means that the naked singularity must be outside the horizon). Using (12), the distance \( l \) can be written as a function of the other parameters by the very simple formula

\[ l = -m_1 - m_2 + \frac{m_1 e_2 - m_2 e_1}{2(m_1 m_2 - e_1 e_2)}[(e_2 - e_1) \pm \sqrt{(e_1 + e_2)^2 - 4 m_1 m_2}]; \]  

(15)

we always choose the sign in front of the root in (15) in order to satisfy the non-overcrossing condition (14). From (15) it is clear that the parameters must satisfy the restriction

\[ (e_2 + e_1)^2 > 4 m_1 m_2. \]  

(16)

4. Some Further Details of the Solution

The solution has a very simple form; the only price to pay is just the simultaneous use of two pairs of coordinates. Obviously, for practical purposes, as for the electric lines plot, one needs the use of only one system — in our case we choose \((r_1, \theta_1)\), the one related to the black hole (which is centered on the origin, since we took \( z_1 = 0 \) for simplicity, and consequently \( z_2 = l \)). The linking relations are

\[
\begin{align*}
  r_2 - m_2 &= \frac{1}{\sqrt{2}} \sqrt{b^2 + \sqrt{b^4 - 4 \sigma_2^2(z - z_2)^2}}, \\
  \cos \theta_2 &= (z - z_2)(r_2 - m_2)^{-1},
\end{align*}
\]  

(17)
where \( b^2 \equiv \rho^2 + \sigma_2^2 + (z - z_2)^2 \), while \( \rho \) and \( z \) have to be expressed using the first couple of (6). We take the plus sign of the roots in the first equation of (17) since \( r_1 \) and \( r_2 \) must coincide at infinity.

The peculiarity of the coordinates used needs clarification in order to understand the physical property of the solution; first of all, where the “true” divergences are and what happens on the horizons.

4.1. Using \((r_1, \theta_1)\)

These coordinates are centered on the black hole and can be considered as the natural generalization of the Schwarzschild ones. For the peculiar choice of the \((r_1, \theta_1)\) coordinates, the horizon remains a perfect circle (it can be seen also analytically that \( H \) vanishes at \( r_h = m_1 \pm \sigma_1 \), as for the single RN black hole). However, the spherical symmetry is only apparent; indeed, the invariants have a \( \theta_1 \) dependence and vary on the horizon. In this frame it is not possible to reach the inside of the spheroid \( r_2 < m_2 \) (we called the surface \( r_2 = m_2 \) the “critical spheroid,” as in Ref. 1), so the second source (the naked RN centered in \( z = z_2 \)) appears squeezed “inside” a horizontal segment that cuts the vertical axis: this happens because the naked singularity lies inside the region not covered by \((r_1, \theta_1)\).

4.2. Using \((r_2, \theta_2)\)

Conversely, if one wishes to use \((r_2, \theta_2)\), the “critical spheroid” of the naked RN will appear as a sphere of coordinates \( r_2 = m_2 \), and the black hole horizon as a segment squeezed on the axis. In this case it is the “critical spheroid” of the first source, i.e. \( r_1 < m_1 \), that cannot be reached. Again, that has nothing to do with physics, but just with the choice of the coordinate system.

In Table 2 we localize the two peculiar regions (the horizon and the critical spheroid), using Weyl coordinates, with the respective translations in \((r_1, \theta_1)\) and \((r_2, \theta_2)\); while in Tables 3 and 4 we give a detailed description of the relevant physical quantities in the notable points of these two zones. One should also note the “degeneracy” of the Weyl coordinates: to the same point in \((\rho, z)\) there can correspond different values of the spheroidal coordinates.

<table>
<thead>
<tr>
<th>Physical description</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon</td>
<td>( { \rho = 0, ; z_1 - \sigma_1 \leq z \leq z_1 + \sigma_1 } ) or, equivalently, ( { r_1 = m_1 + \sigma_1, ; \forall \theta_1 } )</td>
</tr>
<tr>
<td>Critical spheroid of the naked singularity</td>
<td>( { 0 \leq \rho \leq \text{Im}{\sigma_2}, ; z = z_2 } ) or, equivalently, ( { r_2 = m_2, ; \forall \theta_2 } )</td>
</tr>
</tbody>
</table>
Table 3. Characteristic points of the first source. Note the degeneracy of the Weyl coordinates. For the numerical evaluation we used \( m_1 = 1 \), \( e_1 = 0.7 \), \( m_2 = 0.3 \), \( e_2 = 0.44 \), \( l = 5 \) (the same used for Fig. 1). The central singularity of the BH is split into two points:

\[ r_1^{(I)} = \frac{m_1 + m_2 + l - \sqrt{(m_1 + m_2 + l)^2 - 4e_1e_2}}{2} \quad \text{and} \quad r_1^{(II)} = \frac{m_1 + m_2 - l + \sqrt{(m_1 - m_2 - l)^2 - 4e_1e_2}}{2}. \]

<table>
<thead>
<tr>
<th>Description</th>
<th>((r_1, \theta_1))</th>
<th>((\rho, z))</th>
<th>(H)</th>
<th>(\Phi)</th>
<th>(f)</th>
<th>(F^{ij} F_{ij})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two branch points</td>
<td>(r_1 = m_1)</td>
<td>(0, 0)</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = 0, \pi)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equatorial point of the ext. horizon</td>
<td>(r_1 = m_1 + \sigma_1)</td>
<td>(0, 0)</td>
<td>0</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = \pi/2)</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>Equatorial point of the int. horizon</td>
<td>(r_1 = m_1 - \sigma_1)</td>
<td>(0, 0)</td>
<td>0</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = \pi/2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BH singularity I</td>
<td>(r_1^{(CI)} = 0.04974)</td>
<td>(0, r_1 - m_1)</td>
<td>+(\infty)</td>
<td>-(\infty)</td>
<td>0</td>
<td>-(\infty)</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BH singularity II</td>
<td>(r_1^{(CII)} = 0.05938)</td>
<td>(0, -r_1 + m_1)</td>
<td>+(\infty)</td>
<td>-(\infty)</td>
<td>0</td>
<td>-(\infty)</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = \pi)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>North pole of the ext. horizon</td>
<td>(r_1 = m_1 + \sigma_1)</td>
<td>(0, \sigma_1)</td>
<td>0</td>
<td>fin.</td>
<td>+(\infty)</td>
<td>fin.</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>South pole of the int. horizon</td>
<td>(r_1 = m_1 - \sigma_1)</td>
<td>(0, \sigma_1)</td>
<td>0</td>
<td>fin.</td>
<td>+(\infty)</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = \pi)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>North pole of the int. horizon</td>
<td>(r_1 = m_1 - \sigma_1)</td>
<td>(0, -\sigma_1)</td>
<td>0</td>
<td>fin.</td>
<td>+(\infty)</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>South pole of the ext. horizon</td>
<td>(r_1 = m_1 + \sigma_1)</td>
<td>(0, -\sigma_1)</td>
<td>0</td>
<td>fin.</td>
<td>+(\infty)</td>
<td>fin.</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = \pi)</td>
<td></td>
<td></td>
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</tbody>
</table>

Table 4. Characteristic points of the naked source: the first three points correspond to \((\rho = 0, z = z_2 = l)\). The same numerical values of Table 3 are used.

<table>
<thead>
<tr>
<th>Description</th>
<th>((r_2, \theta_2))</th>
<th>((\rho, z))</th>
<th>(H)</th>
<th>(\Phi)</th>
<th>(f)</th>
<th>(F^{ij} F_{ij})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naked singularity</td>
<td>(r_2 = \frac{l + M_{tot} - \sqrt{(l + M_{tot})^2 - 4e_1e_2}}{2})</td>
<td>((0, l))</td>
<td>+(\infty)</td>
<td>-(\infty)</td>
<td>0</td>
<td>-(\infty)</td>
</tr>
<tr>
<td></td>
<td>(\theta_2 = \pi)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Crossing of the cut with the axis (up border)</td>
<td>(r_2 = m_2)</td>
<td>((0, l))</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
</tr>
<tr>
<td></td>
<td>(\theta_2 = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ditto (down border)</td>
<td>(r_2 = m_2)</td>
<td>((0, l))</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
<td>fin.</td>
</tr>
<tr>
<td></td>
<td>(\theta_2 = \pi)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extremes of the cut</td>
<td>(r_2 = m_2)</td>
<td>((\text{Im}[\sigma_2], l))</td>
<td>fin.</td>
<td>fin.</td>
<td>+(\infty)</td>
<td>fin.</td>
</tr>
<tr>
<td></td>
<td>(\theta_2 = \pi/2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.3. The electromagnetic invariant

In order to know where the charges are located, it is useful to consider the electromagnetic invariant \( F = F^{ij} F_{ij} / 2 \). For the solution (4) it has the form

\[
F = -\frac{[\sigma_1^2 \cos^2 \theta_1 + \sigma_2^2 \cos^2 \Phi]}{4H[f\sigma_1^2 \cos^2 \theta_1 + (\partial_{\rho_1} \Phi)^2 (\partial_{\theta_1} \Phi)^2]}.
\] (18)
It can be seen numerically (see Tables 3 and 4) that it diverges inside the horizon and inside the critical spheroid of the naked RN.

It is also worth noting that on the critical spheroid, although in the \((r_1, \theta_1)\) representation it is a line, the up- and the down-limit of \(\mathcal{F}\) do not coincide, since they correspond to different points of the physical space–time.

Looking at \(\mathcal{F}\) it is possible to see that no real discontinuity exists on the horizon; indeed, it diverges only on the central singularities.

The other invariant, \(\epsilon^{ijkl} F_{ij} F_{kl} = \mathbf{E} \mathbf{B}\), is identically zero.

5. Definition of Electric Force Lines

Just to understand better the plots, we will recall the definition of the electrical vector (which is not a trivial choice in GR). Following Ref. 6, we define the electric field as the three nondiagonal timelike components of the controvariant tensor \(F^{ij}\):

\[
E^\alpha = F^{\alpha 0}, \quad \alpha = 1, 2, 3.
\] (19)

That identification is geometrically justified by the Gauss theorem generalized to the curved manifolds:\textsuperscript{22}

\[
4\pi Q = \int_C * \mathbf{F} = \int_C * F_{ij} dx^i \wedge dx^j,
\] (20)

where \(* F_{ij} = 1/2 \epsilon_{ijkl} F^{kl} \sqrt{-g}\) is the dual tensor of \(F^{ij}\). Then it is natural to define the force lines in the usual way, as the trajectories of the dynamical system,

\[
\begin{align*}
\frac{d}{d\lambda} r_1 &= E^{r_1}, \\
\frac{d}{d\lambda} \theta_1 &= E^{\theta_1},
\end{align*}
\] (21)

or, equivalently, by

\[
\frac{d r_1}{d\theta_1} = \frac{E^{r_1}}{E^{\theta_1}}, \quad \frac{E^{r_1}}{E^{\theta_1}} = ((r_1 - m_1)^2 - \sigma_1^2) \frac{\partial_{r_1} \Phi}{\partial_{\theta_1} \Phi}.
\] (22)

Then, from the equation of motion for this problem, restricting to our case, we have

\[
F^{r_1} u^t d\theta_1 - F^{\theta_1} u^t dr_1 = 0,
\] (23)

having used the coordinates \(x^i = (t, \varphi, r_1, \theta_1)\).

The physical interpretation (Christodoulou–Ruffini, quoted in Ref. 6) is the following: a force line is a line tangent to the direction of the electric force measured by a free-falling test charge momentarily at rest, with initial 4-velocity

\[
u^t = (\sqrt{g^{tt}}, 0, 0, 0).
\] (24)

Note that such an interpretation is valid only for \(g^{tt} > 0\); for this reason we have not plotted the lines inside the horizon.

\textsuperscript{e}The spheroid, i.e. the line \(\{0 < \rho < \text{Im}(\sigma_1), \ z = z_2\}\), seems regular in \((r_1, \theta_1)\) coordinates just because its interior can be reached only using \((r_2, \theta_2)\).
In the \((t, \varphi, r_1, \theta_1)\) coordinates the metric (4) becomes
\[
ds^2 = Hdt^2 - \frac{\rho^2}{H}d\varphi^2 - f[ (r_1 - m) + \sigma^2 \cos^2 \theta_1 ] \left[ \frac{dr_1^2}{(r_1 - m)^2 - \sigma^2} + d\theta_1^2 \right],
\]
while the electric potential remains unchanged. Then for the electric field we have
\[
E^\varphi = 0,
\]
\[
E^{r_1} = g^{tt} g^{r_1 r_1} \frac{\partial A_t}{\partial r_1},
\]
\[
E^{\theta_1} = g^{tt} g^{\theta_1 \theta_1} \frac{\partial A_t}{\partial \theta_1}.
\]
Therefore the force lines are given by the solution of
\[
\frac{dr_1}{d\theta_1} = \left( (r_1 - m)^2 - \sigma^2 \right) \frac{\partial r_1 A_t}{\partial \theta_1 A_t}.
\]
It is worth noting that the force lines depend only on the two ratios \(\frac{\partial r_1 A_t}{\partial \theta_1 A_t}\) and \(\frac{g^{tt}}{g^{\theta_1 \theta_1}}\) [indeed, the conformal factor \(f\) and both \(g^{tt}\) and \(g^{\varphi \varphi}\) do not appear in (27)].

6. Plots of the Force Lines

In the plots, what we called the “second source” (i.e. the naked RN) is always up, while the “first source” (i.e. the black hole) is always down and centered on the origin.

The lines are plotted in \((x, y)\) Cartesian coordinates, defined as
\[
\begin{cases}
x = r_1 \sin \theta_1, \\
y = r_1 \cos \theta_1
\end{cases}
\]
[they coincide with \((\rho, z)\), defined in (6), when \(r_1 \to \infty\)].

In the plots we have used geometrical units \((G = c = 1)\), in which the unitary length is given by the Schwarzschild mass \(m_1 = 1\).

The graphical Faraday criterion is used; namely, we have plotted the electric force lines such that
\[
\frac{\text{Number of lines from the first source}}{\text{Number of lines from the second source}} \approx \frac{e_1}{e_2}.
\]

6.1. The separatrix

In general, when there are two charges, the electric force diagram will present a separatrix, which is a force line which reaches asymptotically a saddle point of the potential and separates the lines of the two charges in the case where they have the same sign, or — in the case of opposite sign charge — it delimits the region in which the lines flow from one to the other source. We have marked these separatrix lines in bold; maybe it is worth mentioning that on the saddle point they have an invariant definition, since on that point \(F = 0\).
6.2. Inside the horizon

In the following plots the force lines are graphed only outside the horizon since there it is no more possible to consider a static observer, i.e. the physical interpretation given in Sec. 5 does not hold because (24) becomes imaginary. However, when the separatrix starts from the inside of the horizon, the study of that region is important to understand the difference between cases with the same or opposite charges. Therefore, in the case of Fig. 3, in which the saddle point is inside the horizon, we calculated the point where the separatrix touches the horizon, and we plotted the diagram just from there. [This was possible because mathematically the Eq. (27) is well defined also inside the horizon].

In the following three subsections we analyze the three qualitatively different subcases: \( e_1 e_2 > 0 \) (6.1), \( e_1 e_2 < 0 \) (6.2) and, finally, \( e_1 = 0 \) (6.3).

6.3. Two charges of equal sign \((e_1 e_2 > 0)\)

6.3.1. General case: Two comparable RN sources

Let us consider the case in which the two RN sources have charges and masses of comparable dimensions:

\[
\begin{align*}
  m_1 &\approx m_2, & e_1 &\approx e_2, \\
  m_1^2 &> e_1^2, & m_2^2 &< e_2^2, \\
  e_1 e_2 &> 0.
\end{align*}
\]

(29)

This is the case closest to the classical picture; indeed, here the equilibrium is mainly due to the classical balance of the electrostatic force and gravitational field. The resulting plot is given in Fig. 1.

The qualitative behavior of the force lines does not change with the changing of the distance \( l \).

6.3.2. Small\(^1\) charge (naked) near an RN black hole

The equilibrium configurations of this case (see Fig. 2), with

\[
\begin{align*}
  m_1 &\gg m_2, & |e_1| &\gg |e_2|, \\
  m_1^2 &> e_1^2, & m_2^2 &< e_2^2, \\
  e_1 e_2 &> 0,
\end{align*}
\]

(30)

have been studied in the test particle approximation first in Ref. 3, and recently in Refs. 10 and 11, which also took into account the backreaction of the test particle.

\(^1\)Here and in the following we say “small” charge and not “test” charge because the exact nature of the solution automatically takes into account all the backreaction terms even if they can be very small (while the “test” limit is in general referred as the one in which all those terms are completely neglected).
6.3.3. Small charge (with horizon) near a naked RN

This case does not exist for \( e_1 e_2 > 0 \).

6.4. Two charges of opposite sign (\( e_1 e_2 < 0 \))

Although it is easy to show that in the previous cases with \( e_1 e_2 > 0 \) the implications

\[
\begin{align*}
    m_1^2 &> e_1^2 \implies \sigma_1^2 > 0, \\
    m_2^2 &< e_2^2 \implies \sigma_2^2 < 0
\end{align*}
\]  

are always true, it is not so if \( e_1 e_2 < 0 \). However, in the following, we consider two cases in which (31) holds.

6.4.1. Two comparable RN sources

This case, with

\[
\begin{align*}
    m_1 &\approx m_2, \quad e_1 \approx -e_2, \\
    m_1^2 &> e_1^2, \quad m_2^2 < e_2^2, \\
    e_1 e_2 &< 0,
\end{align*}
\]  

(32)
is the case in which the relativistic effects are very evident since here also the electric force is attractive (see Fig. 3): in this case the equilibrium is due to the repulsive nature of the naked singularity.

6.4.2. Small charge near an RN

It is also possible to find values that correspond to a small charge with horizon near a naked RN:

\[ m_1 \ll m_2, \quad |e_1| \ll |e_2|, \]
\[ m_1^2 > e_1^2, \quad m_2^2 < e_2^2, \]
\[ e_1 e_2 < 0. \]  \hspace{1cm} (33)

However, in this case it would be useless to plot the force lines because the electric field is trivially Coulombian (the first source is weakly interacting, both gravitationally and electrically).

The inverse case, namely a small charge *naked* near an RN *with horizon*, does not exist for particles lying outside the horizon (i.e. requiring \( l > \sigma_1 \)), as noted by Bonnor. \(^3\)
Fig. 3. Force lines in the general case (32), with charges of the opposite sign. Parameters used: $m_1 = 1$, $e_1 = 0.05$, $m_2 = 0.3$, $e_2 = -1.66$, $l = 5$. The bold line is the separatrix, which now encircles also the central singularity of the BH: inside that region the lines go from one charge to the other. Outside that region the lines go from $e_2$ to infinity (some of them also pass through the horizon).

6.5. Cases with only one charge

In the following we will consider the cases with a naked singularity near a neutral black hole; they are qualitatively different from the previous ones since now there is no separatrix and the electric flux over the horizon surface is zero.

In the particular case where the first source is neutral (i.e. $e_1 = 0$), the equilibrium distance is even simpler,

$$ l = -m_1 - m_2 + \frac{e_2^2}{2m_2} \left( 1 \pm \sqrt{1 - 2m_1 \left( \frac{e_2^2}{2m_2} \right)^{-1}} \right), $$  \hspace{1cm} (34)

which can be always satisfied for sufficiently large values of the charge parameter $e_2$.

6.5.1. RN near a Schwarzschild black hole (comparable masses)

Thanks to the exact nature of the solution, it is very interesting to note also the case in which the RN source has a comparable mass to the Schwarzschild black
hole — say,
\[
\begin{align*}
    m_1 & \approx m_2, \quad e_1 = 0, \\
    \sigma_1 &= m_1, \quad m_2^2 < e_2^2.
\end{align*}
\] (35)

Indeed, this case cannot be achieved with a perturbative approach; see Fig. 4. It is possible to see that the electric lines are just slightly deformed by the gravitational field.

6.5.2. Small charge near a Schwarzschild black hole

We can also consider the small-charge limit,
\[
\begin{align*}
    m_1 & \gg m_2, \quad e_2, e_1 = 0, \\
    \sigma_1 &= m_1, \quad m_2^2 < e_2^2,
\end{align*}
\] (36)
i.e. the second source is a small RN naked singularity. That is the only case in which we have a good comparison in the literature, since it is the only case already studied (as we know) by using the force lines plots, although with a perturbative approach. Strictly speaking, the Hanni–Ruffini case refers to a slightly different

![Fig. 4. Force lines for the values (35). The circle of radius 2m1 is the Schwarzschild horizon. Parameters used: m1 = 1, m2 = 0.3, e2 = 1.5, l = 5.](image-url)
situation, since they considered a particle *momentarily* at rest in the Schwarzschild metric, while the AB solution is exactly static. However, the present solution confirms very nearly their multipole expansion, since we find that the plots are in practice coincident. In order to have the best possible comparison, we considered the same distances between the charge and the horizon (Figs. 5–7). Since now $l$ is not an independent parameter, we fixed the masses values $m_1 = 1$ and $m_2 = 10^{-4}$; then, varying the distance, we found [using (15)] the relative parameter $e_2$. The test particle is at $z = l$ or, equivalently, at $r_1 = l + m_1$. (Just to clarify the link with the notations of the authors of Ref. 6: their $r$ is our $r_1$, and their $M$ is our $m_1$.)

From Eq. (22), considering that now $\sigma_1 = m_1$, it is easy to see that the corrections to the Hanni–Ruffini approximation are limited only on the exact form of the $A_t$ potential, since using the Schwarzschild metric or the functions $H$ and $f$ given in (7) and (9) does not change the force lines.

![Figure 5](image-url)

Fig. 5. Force lines for the values (36), with $l = 3m_1$, i.e. in the spheroidal coordinates the particle is in $r_1 = 4m_1$. The circle of radius $2m_1$ is the Schwarzschild horizon. The plots are practically identical to the ones found by Hanni and Ruffini.

*From another point of view, Hanni and Ruffini do not use (34) to determine the fourth parameter (because in their approximation the fourth parameter — say, $m_2$ — is considered arbitrarily small, and therefore it is not present at all).*
7. Final Remarks

The main result of our analysis is that the exact solution seems to confirm quite strictly the test-charge approximation on the RN background (see e.g. Ref. 3), which seems to give a good test of the exact picture.

7.1. Size of the naked singularity

In the literature, it has sometimes been guessed (Ref. 23, Chap. 15; Ref. 24) that $e^2/2m$ should be considered as a “critical radius” of the naked singularity inside of which the RN solution has no physical meaning since it should be matched with a more realistic matter field tensor, in order to avoid the well-known problems of a pointlike source, as the divergence of the electric energy.

If the quantity $e_2^2/2m_2$ can be roughly considered as the physical size of the RN charge, then from the formula (34) it is easy to see that the equilibrium configurations exist only for $e_2^2/2m_2$ larger than the Schwarzschild radius ($2m_1$). This seems to suggest that a real “small” charge limit cannot be achieved, in the sense that the particle can be “small” only gravitationally (and electrically), but not geometrically because it would have a size larger than the Schwarzschild horizon.
Fig. 7. Now the distance is $l = 1.2m_1$ or, equivalently, the charge is in $r_1 = 2.2m_1$.

However, this is just a speculation, since further investigations should be made to model the interior of a realistic RN source and find its minimum radius.

7.2. Coordinate dependence of the plots

Any plot of the force lines changes drastically for different choices of the coordinates. However, what is interesting is to compare different situations by using the same coordinate representation, for example as we did for the Hanni–Ruffini case.

7.3. Stability

If the solution turns out to be unstable, that will mean that it is a completely academic problem, since the equilibrium will be physically not allowed. However, in the geodesic/test particle approximation — which gives the essential features of the problem — the equilibrium is stable, and therefore at least in some range of values it should be the same also in the exact case (indeed, the exact solution smoothly converges to the test particle approximation in the limit $e_i, m_i \to 0$, with $e_i/m_i$ finite, $i = 1$ or 2).
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References

Charged membrane as a source for repulsive gravity

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We demonstrate an alternative (with respect to the ones existing in literature) and more habitual for physicists derivation of exact solution of the Einstein-Maxwell equations for the motion of a charged spherical membrane with tangential tension. We stress that the physically acceptable range of parameters for which the static and stable state of the membrane producing the Reissner-Nordstrom (RN) repulsive gravity effect exists. The concrete realization of such state for the Nambu-Goto membrane is described. The point is that membrane are able to cut out the central naked singularity region and at the same time to join in appropriate way the RN repulsive region.

As result we have a model of an everywhere-regular material source exhibiting a repulsive gravitational force in the vicinity of its surface: this construction gives a more sensible physical status to the RN solution in the naked singularity case.

Keywords: Nambu-Goto membrane; Reissner-Nordstrom; Exact solutions.

1. Introduction

One of the interesting effects of relativistic gravity which has no analogue in the Newtonian theory is the presence of gravitational repulsive forces. The classical example is the Reissner-Nordstrom (RN) field in the region close enough to the central singularity. Indeed, in the RN metric

$$-ds^2 = -f c^2 dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(1)
where
\[ f = 1 - \frac{2kM}{c^2 r} + \frac{kQ^2}{c^4 r^2}, \tag{2} \]
the radial motion of a test neutral particle follows the equation:
\[ \frac{d^2 r}{ds^2} = -\frac{1}{2} \frac{df}{dr} = \frac{k}{c^4 r^2} \left( \frac{Q^2}{r} - M c^2 \right) \tag{3} \]
from where one can see the appearance of repulsive force in the region of small \( r \). In this zone the gradient of the gravitational potential \( f(r) \) is negative and the gravitational force in Eq.(3) is directed toward the outside of the central source.

For the RN naked singularity case \((Q^2 > kM^2)\), in which we are interested in the present paper, the potential \( f(r) \) is everywhere positive and has a minimum at the point \( r = \frac{Q^2}{Mc^2} \). Therefore at this point a neutral particle can stay at rest in the state of stable equilibrium (the detailed study can be found in [1, 2]).

It is an interesting and nontrivial fact that the same sort of stationary equilibrium state due to the repulsive gravity exists also as an exact asymptotically flat two-body solution of the Einstein Maxwell equations which describes a Schwarzschild black hole situated at rest in the field of a RN naked singularity without any strut or string between these two objects [3, 4]. However, solutions of this kind have the feature that the object creating the repelling region has naked singularity and this last property has no clear physical interpretation. Consequently the pertinent question is whether the repelling phenomenon around a charged source arises only due to the presence of the naked singularity or it can be also a feature of physically reasonable structure of the space-time and matter.

By other words the question is whether or not it is possible to construct a regular material source which can block the central singularity and join the external repulsive region in a proper way. Then we are interested to construct a body with the following properties:

(1) inside the body there are no singularities;
(2) outside the body there is the RN field (1)-(2), corresponding to the case \( Q^2 > kM^2 \);
(3) the radius of the body is less than \( \frac{Q^2}{Mc^2} \), so between the surface of the body and the sphere \( r = \frac{Q^2}{Mc^2} \) arises the repulsive region;
(4) such stationary state of the body is stable with respect to collapse or expansion.

In this paper we propose a new model for such body in the form of spherically symmetric thin membrane with positive tension. We assert that there exists a physically acceptable range of parameters for which all the above four conditions (1)-(4) can be satisfied. We illustrate this conclusion by the especially transparent case of a Nambu-Goto membrane with equation of state \( \epsilon = \tau \).

Then the existence of everywhere-regular material sources possessing RN “anti-gravity” properties in the vicinity of their surfaces attribute to this phenomenon and to the RN naked singularity solution more sensible physical status.
It is necessary to mention that at least two exact solutions of Einstein-Maxwell equations representing a compact continuous spherically symmetric distribution of charged matter under the tension producing the gravitationally repulsive forces inside the matter as well as in some region outside of it already exist in the literature. These are solutions constructed in Ref.[5] and Ref.[6]. A more detailed study of these two results can be found in Ref.[7]. An interesting possibility to have a gravitationally repulsive core of electrically neutral but viscous matter has been communicated in Ref.[8].

It is worth to remark that the first (to our knowledge) mentioning of the gravitational repulsive force due to the presence of electric field was made already in 1937 in the Ref.[9] in connection to the nonlinear model of electrodynamics of Born-Infeld type. One of the first paper where a repulsive phenomenon in the framework of the conventional Einstein-Maxwell theory has been mentioned is Ref.[10]. The general investigation of the different aspects of this phenomenon apart from the already mentioned references [1-10] can be found also in the more detailed works [11, 12, 13, 14]. Some part of these papers is dedicated to a possibility of construction a classical model for electron. This is doubtful enterprise, however, because the intrinsic structure of electron is a matter out of classical physics. Nonetheless the mathematical results obtained are useful and can be applied to the physically sensible situations, e.g. for construction the models of macroscopical objects.

2. Equation of motion of a membrane with empty space inside

The equation of motion for the most general case of a thin charged spherically symmetric fluid shell with tangential pressure moving in the RN field have been derived 38 years ago by J.E. Chase\textsuperscript{15}. The corresponding dynamics for a charged elastic membrane with tension follows from his equation simply by the change of the sign of the pressure. We derived, however, the membrane’s dynamics again using a different approach.

Chase used the geometrical method which have been applied to the description of singular surfaces in relativistic gravity in [16] and have been elaborated in [10, 17] for some special cases of charged shells. An essential development of the Israel approach in application to the cosmological domain walls can be found in the series of works of V.Berezin, V.Kuzmin and I. Tkachev, see Ref.[18] and references therein. Our treatment follows the method more habitual for physicists which have been used in [19], where the motion of a neutral fluid shell in a Schwarzschild field was derived by the direct integration of the Einstein equations with appropriate \(\delta\)-shaped source. Now we generalized this approach for the charged membrane and charged central source.

Of course, the membrane’s equation of motion that we obtained coincides with that of Chase. Nonetheless the different approach to the same problem often has a methodological value and gives new details. We hope that our case makes no exception, then for an interested reader we put the main steps of our derivation in
Appendix (where we considered a general case with central source).

In this section we study only the particular solution in which there is no central body, that is inside the membrane we have flat space-time. Although the basic formulas of this section follow from the Appendix under restriction $M_{in} = Q_{in} = 0$ the exposition we give here is more or less self-consistent. Only the definitions of 4-dimensional membrane’s energy density and tension need some clarification which can be found in Appendix.

For the thin spherically symmetric membrane with empty space inside and with radius which depends on time the metrics inside, outside and on membrane are:

\[ - (ds^2)_{in} = - \Gamma^2(t)c^2 dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  \hspace{1cm} (4)

\[ - (ds^2)_{out} = - f(r)c^2 dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  \hspace{1cm} (5)

\[ - (ds^2)_{on} = - c^2 d\eta^2 + r_0^2(\eta)(d\theta^2 + \sin^2 \theta d\phi^2) \]  \hspace{1cm} (6)

In the interval (6) $\eta$ is the proper time of the membrane. The factor $\Gamma^2$ in (4) is necessary to ensure the continuity of the global time coordinate $t$ through the membrane. The metric coefficient $f(r)$ in the region outside the membrane is given by Eq.(2).

Matching conditions for the intervals (4)-(6) through the membrane’s surface are:

\[ [(ds^2)_{in}]_{r=r_0(\eta)} = [(ds^2)_{out}]_{r=r_0(\eta)} = (ds^2)_{on} \]  \hspace{1cm} (7)

If the equation of motion of the membrane $r = r_0(\eta)$ is known, then from these conditions the connection $t(\eta)$ between global and proper times and factor $\Gamma(t)$ follow easily:

\[ \Gamma(t) = \frac{f(r_0)\sqrt{1 + c^{-2}(r_0,\eta)^2}}{\sqrt{f(r_0) + c^{-2}(r_0,\eta)^2}} \]  \hspace{1cm} (8)

\[ \frac{dt}{d\eta} = \frac{\sqrt{f(r_0) + c^{-2}(r_0,\eta)^2}}{f(r_0)} \]  \hspace{1cm} (9)

The differential equation for the function $r_0(\eta)$ follows from Einstein-Maxwell equations with energy-momentum tensor and charge current concentrated on the surface of the membrane. It is:

\[ Mc^2 = m(r_0)c^2 \sqrt{1 + \left(\frac{d r_0}{c d\eta}\right)^2 + \frac{Q^2}{2r_0} - \frac{k m^2(r_0)}{2r_0}} \]  \hspace{1cm} (10)

Here $m(r_0) > 0$ is the effective rest mass of the membrane in the radially comoving frame. This quantity includes the membrane’s rest mass as well as all kinds of interaction mass-energies between membrane’s constituents, that is those intrinsic energies which are responsible for the tension. The constants $Q$ and $M$ are the total charge of the membrane and total relativistic mass of the system. These are the
same constants which appeared earlier in Eq.(2). The membrane’s energy density $\epsilon$ and tension $\tau$ are (see Appendix for a further clarification):

$$\epsilon = \epsilon_0(r_0)\delta[r - r_0(\eta)] \quad \tau = \tau_0(r_0)\delta[r - r_0(\eta)]$$ (11)

where

$$\epsilon_0 = \frac{m(r_0)c^2}{8\pi r_0^2} \left[ \frac{1}{\sqrt{1 + c^{-2}(r_0,\eta)^2}} + \frac{f(r_0)}{\sqrt{f(r_0) + c^{-2}(r_0,\eta)^2}} \right]$$ (12)

$$\tau_0(r_0) = \frac{dm(r_0)}{dr_0} \frac{r_0\epsilon_0(r_0)}{2m(r_0)}$$ (13)

The electromagnetic potentials have the form $A_r = A_\theta = A_\phi = 0$, $A_t = A_t(t,r)$ and for the electric field strength $\partial A_t/\partial r$ the solution is

$$\frac{\partial A_t}{\partial r} = \begin{cases} \frac{Q}{r} & \text{for } r > r_0(\eta) \\ 0 & \text{for } r < r_0(\eta) \end{cases}$$ (14)

The formulas (4)-(14) give the complete solution of the problem for the case of empty space inside the membrane.

Finally we would like to stress the following important point. As follows from discussion in Appendix, the signs of the square roots $\sqrt{1 + c^{-2}(r_0,\eta)^2}$ and $\sqrt{f(r_0) + c^{-2}(r_0,\eta)^2}$ coincide with the signs of the time component $u^0$ of the 4-velocity of the membrane evaluated from inside and outside of the membrane respectively. The component $u^0$ is a continuous quantity by definition and can not change the sign when passing through the membrane’s surface. Besides, for macroscopical objects we are interested in in this paper $u^0$ should be positive. Consequently the both aforementioned square roots should be positive. From another side it is easy to show that equation (10) can be written also in the following equivalent form

$$Mc^2 = mc^2 \sqrt{f(r_0) + \left( \frac{dr_0}{d\eta} \right)^2 + \frac{Q^2}{2r_0} + \frac{km^2}{2r_0}}$$ (15)

Then from this expression and from (10) follows that both square roots will be positive if and only if

$$Mc^2 - \frac{Q^2}{2r_0} - \frac{km^2}{2r_0} > 0$$ (16)

This is unavoidable constraint which must be adopted as additional condition for any physically realizable solution of the equation of motion (10) in classical macroscopical realm.

3. Nambu-Goto membrane with “antigravity” effect

To proceed further we must specify the function $m(r_0)$, which is equivalent to specifying an equation of state, as can be seen from (13).
Let us analyze the membrane with equation of state $\epsilon = \tau$. This model can be interpreted as “bare” Nambu-Goto charged membrane\textsuperscript{20,21}, or as Zeldovich-Kobzarev-Okun charged domain wall\textsuperscript{22}. It follows from (13) that for such type of membrane we have:

$$m = \sigma r_0^2$$

(17)

where $\sigma$ is an arbitrary constant. In this and next section we consider only the case of positive constants $\sigma$ and $M$:

$$\sigma > 0, \quad M > 0.$$ 

(18)

The sign of $Q$ is of no matter since the charge appear everywhere in square. Now we write the equation of motion (10) in the following form:

$$4 \left( \frac{d r_0}{c d \eta} \right)^2 - \left( \frac{k \sigma r_0}{c^2} + \frac{2M}{\sigma r_0^2} - \frac{Q^2}{c^2 \sigma r_0^3} \right)^2 = -4.$$ 

(19)

Formally this can be considered as the equation of motion of a non-relativistic particle with the “mass” equal to 8 moving in the potential $U(r_0)$,

$$U(r_0) = - \left( \frac{k \sigma r_0}{c^2} + \frac{2M}{\sigma r_0^2} - \frac{Q^2}{c^2 \sigma r_0^3} \right)^2$$

(20)

and under that condition that particle is forced to live on the “total energy” level equal to minus four.

For the existence of the stable stationary state we are interested in, the following conditions should hold:

(1) The gravitational field in the exterior region should correspond to the super-extreme RN metric:

$$Q^2 > kM^2.$$ 

(21)

(2) The potential $U(r_0)$ should have a local minimum at some value $r_0 = R_{\text{min}}$. The form (20) of $U(r_0)$ permit this if and only if

$$k \sigma^2 Q^6 < (M c^2)^4.$$ 

(22)

Under this restriction the potential $U(r_0)$ has three extrema, two maxima at points $r_0 = R^{(1)}_{\text{max}}$ and $r_0 = R^{(2)}_{\text{max}}$ and a minimum which is located between them: $R^{(1)}_{\text{max}} < R_{\text{min}} < R^{(2)}_{\text{max}}$. We show the shape of the potential $U(r_0)$ for this case in Fig.1.

The equation $U(r_0) = 0$ has only one real root and this is also the first local maximum $R^{(1)}_{\text{max}}$. The minimum and the second maximum are coming as two other roots of the equation $\frac{dU}{dr_0} = 0$.

The equation for $R_{\text{min}}$ is:

$$k \sigma^2 R^4_{\text{min}} - 4M c^2 R_{\text{min}} + 3Q^2 = 0.$$ 

(23)

This fourth order equation has only two real solutions and $R_{\text{min}}$ is the smaller one.
Fig. 1. The membrane’s motion can be described as the motion of a non-relativistic point particle in the potential $U(r_0)$.

(3) For the stationary position of the membrane at the minimum of the potential we must ensure the relation $U(R_{\min}) = -4$ which is:

$$\frac{k\sigma}{c^2} R_{\min} + \frac{2M}{\sigma} R_{\min}^{-2} - \frac{Q^2}{c^2 \sigma} R_{\min}^{-3} = 2$$

(24)

(the minus two in the r.h.s. of (24) would be incompatible with Eq.(23) under condition (18)).

(4) To have repulsive region it is necessary for the membrane’s radius $R_{\min}$ to be less than the minimum of the gravitational potential $f(r)$, that is less than the quantity $Q^2/Mc^2$. In this case outside of the membrane surface in the region $R_{\min} < r < Q^2/Mc^2$ we have the repulsive effect. Then we demand:

$$R_{\min} < \frac{Q^2}{Mc^2}.$$  (25)
(5) Also the additional constraint (16) should be satisfied. This means that for our stationary solution we have to satisfy the inequality:

\[ Mc^2 - \frac{Q^2}{2R_{\text{min}}} - \frac{k\sigma^2}{2}R_{\text{min}}^3 > 0 \]  

(26)

(6) We have also another condition: that the electric field nearby the membrane should be not too large, otherwise the stability of the model would be destroyed by the strong macroscopical consequences of quantum effects, e.g. by the intensive electron-positron pair creation. This condition (which was suggested by J.A. Wheeler long time ago, see the reference with this Wheeler’s proposal in the paper of Bekenstein\textsuperscript{23}) is:

\[ \frac{Q}{R_{\text{min}}^2} << \mathcal{E}_{\text{cr}} \]  

\[ \mathcal{E}_{\text{cr}} = \frac{m_e^2c^3}{\epsilon_e\hbar} \]  

(27)

where \( m_e \) and \( \epsilon_e \) are the electron’s mass and charge). \( \mathcal{E}_{\text{cr}} \) is the well known critical electric field above which the intensive process of pair creation starts.

To satisfy these six conditions we have to find a physically acceptable domain in the space of the four parameters \( M, Q, \sigma \) and \( R_{\text{min}} \). The point is that such domain indeed exists and it is wide enough. If we introduce the dimensionless radius of the stationary membrane \( x \) as

\[ \frac{k\sigma}{c^2}R_{\text{min}} = x \]  

(28)

then one can check directly that the first five of the above formulated conditions will be satisfied under the following three constraints:

\[ x < 1 \]  

(29)

\[ M = \frac{c^4}{k^2\sigma}(3x^2 - 2x^3) \]  

(30)

\[ Q^2 = \frac{c^8}{k^3\sigma^2}(4x^3 - 3x^4) \]  

(31)

The last two of these relations are just the equations (23) and (24) but written in the form resolved with respect to \( M \) and \( Q^2 \).

The formulas (29)-(31) shows that for the first five conditions it is convenient to take \( x < 1 \) and \( \sigma \) as independent parameters, and then to calculate the mass and charge necessary to obtain the model we need.

As for the last constraint (27) it gives some restriction also for parameter \( \sigma \):

\[ k\sigma^2 << \frac{x}{4 - 3x}\mathcal{E}_{\text{cr}}^2 \]  

(32)

The energy density \( \epsilon \) for the stationary state at \( r_0 = R_{\text{min}} \), expressed in terms of parameters \( x \) and \( \sigma \), is:

\[ \epsilon = \frac{\sigma c^2}{8\pi} \left(1 + \sqrt{x^2 - 2x + 1}\right)\delta(r - R_{\text{min}}) \]  

(33)
4. Summary

1. We showed that exists a possibility to have a spherically charged membrane in stable stationary state producing RN repulsive gravitational force outside its surface and having flat space inside. To construct such model one should take a pair of constants $0 < x < 1$ and $\sigma > 0$ satisfying the inequality (32) and calculate from (28) and (30)-(31) the membrane’s radius $R_{\text{min}}$, total mass $M$ and charge $Q$.

2. The equation of motion (10) can be used also for the description of the oscillation of the membrane in the potential well ABC (see fig.1) above the equilibrium point C. If we slightly increase the total membrane’s energy $Mc^2$ then the potential $U(r_0)$ around its minimum (i.e. the point C and its vicinity) will be shifted slightly down but the level “minus four” in Eq.(20) on which the system lives will remain at the same position. Then the membrane will oscillate between the new shifted walls AC and CB.

3. It is easy to see that in the general dynamical state the membrane can live only inside the potential well ABC. All regions outside ABC are forbidden. In the region to the right from the point $R^{(2)}_{\text{max}}$ and above the potential $U(r_0)$ any location of the membrane is impossible due to the fact that inequality (16) is violated there.

This means that a membrane of considered type in principle can not have the radius (no matter in which state) greater than $R^{(2)}_{\text{max}}$. In turn for $R^{(2)}_{\text{max}}$, it is easy to obtain from the potential (20) the upper limit $R^{(2)}_{\text{max}} < \frac{c^2}{k\sigma} \left( \frac{4k^2 \sigma M}{c^4} \right)^{1/3}$.

The same violation of the inequality (16) take place in the domain between $R^{(1)}_{\text{max}}$ and $R^{(2)}_{\text{max}}$ and above the segment AB. The motion in the region to the left from the point $R^{(1)}_{\text{max}}$ and above the curve $U(r_0)$ is forbidden again due to the same violation of the condition (16). This means that a membrane of considered type in principle can not have the radius less than $R^{(1)}_{\text{max}}$. In particular there is no way for a membrane with positive effective rest mass $m$ to collapse to the point $r_0 = 0$ leaving outside the field corresponding to the RN naked singularity solution. This conclusion is in agreement with the main result of the paper [24].

4. Although we claimed that the stationary state of a membrane constructed is stable this stability should be understood in a very restrict sense, that is as stability in the framework of the dynamics described by the equation (10). We do not know what will happen to our membrane after the whole set of arbitrary perturbations will be given.

5. In general the arbitrary perturbations will change also the equation of state. We investigated a membrane with equation of state $\epsilon = \tau$. However this case can be considered only as “bare” Nambu-Goto membrane, by other words as a toy model. In the papers [20, 21, 25, 26, 27, 28] it was shown that arbitrary perturbations essentially renormalize the form of the equation of state of the strings and membranes. Moreover for the membranes [21] (differently from the strings) the fixed points of the renormalization group for the transverse and longitudinal perturbations does not coincide, which means that for the general “wiggly” membrane there is no equation of state of the type $\epsilon = \epsilon(\tau)$ at all.
6. We also would like to stress that for appearance of repulsive force the presence of electric field is of no principal necessity. For example the repulsive gravitational forces arise also in neutral viscous fluid and in the course of interaction between electrically neutral topological gravitational solitons.

7. From the conditions (21)-(26) also follows that in addition to the inequality (25) the radius $R_{\text{min}}$ of the shell in the stable stationary state cannot be less than $Q^{2}/MC^{2}$. A simple analysis shows that there is no way for $R_{\text{min}}$ to be arbitrarily small keeping some finite non-zero value for $M$ and $Q$.

Appendix

For the spherically symmetric case the metric is:

$$-(ds)^{2}_{\text{in}} = g_{00}c^{2}dt^{2} + g_{11}dr^{2} + r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2})$$ (34)

where $g_{00}$ and $g_{11}$ depend only on $t$, $r$ and the standard notation for the coordinates is:

$$(x^{0}, x^{1}, x^{2}, x^{3}) = (ct, r, \theta, \phi) .$$ (35)

The Electromagnetic tensor $F_{ik}$ has the form:

$$F_{ik} = A_{k,i} - A_{i,k}$$ (36)

and the Einstein-Maxwell equations are:

$$R^{i}_{k} - \frac{1}{2} R\delta^{i}_{k} = \frac{8\pi k}{c^{4}} T^{i}_{k}$$ (37)

$$(F^{ik})_{;i} = \frac{4\pi}{c} \rho u^{i}$$ (38)

The energy-momentum tensor for a spherical charged membrane with energy density $\epsilon$ and tangential tension $\tau$ is:

$$T^{k}_{l} = \epsilon u_{i}u^{k} - (\delta_{i}^{2}\delta_{2}^{k} + \delta_{i}^{3}\delta_{3}^{k})\tau + \frac{1}{4\pi}(F_{lj}F^{kl} - \frac{1}{4}\delta^{k}_{l}F_{lm}F^{lm})$$ (39)

and for the membrane’s 4-velocity $u^{i}$ we have:

$$u^{0} = u^{0}(t, r), \quad u^{1} = u^{1}(t, r), \quad u^{2} = u^{3} = 0 , \quad u^{i}u_{i} = -1 .$$ (40)

The main step is to define the 4-invariant charge and energy densities $\rho$ and $\epsilon$. After that, the tension $\tau$ follows automatically from the Einstein-Maxwell equations and from the equation of state. To construct $\rho$ and $\epsilon$ we apply the Landau-Lifschitz procedure.

---

*aWe use the notations in which the interval is written as $-ds^{2} = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$ and metric signature is $(-, +, +, +)$, i.e. the time-time component $g_{00}$ is negative. The norm of a time-like vector is negative. The Roman indices take values 0, 1, 2, 3. The Newtonian constant is denoted by $k$. The simple partial derivatives we designated by a comma, while covariant derivatives by semicolon.
The charge \(dq\) in the 3-volume element \(dV = \sqrt{g_{11}g_{22}g_{33}}\ dx^1 dx^2 dx^3\) is a 4-invariant quantity by definition (although \(dV\) is not a 4-scalar). The three-dimensional charge density \(\rho^{(3)}\) can be introduced by the relation \(dq = \rho^{(3)}dV\). Consequently, for the spherically symmetric membrane case it is:

\[
\rho^{(3)} = \frac{Q\delta(r - r_0)}{4\pi r^2 \sqrt{g_{11}}},
\]

(41)

where \(Q\) is the electric charge of the membrane and \(r_0\) is the membrane’s radius. Indeed it is easy to check that \(Q = \int \rho^{(3)}dV\) as it should be.

Since \(\rho^{(3)}dV\) is a 4-scalar the quantities \(\rho^{(3)}dV dx^i\) represent a 4-vector. With the use of the previous formula we obtain:

\[
cp^{(3)}dV dx^i = \frac{cQ\delta(r - r_0)}{4\pi r^2 u^0 \sqrt{-g_{00}g_{11}}} u^i \sqrt{-g} d^4x,
\]

(42)

where \(g\) is the 4-metric’s determinant. The last formula shows that the factor in front of \(u^i \sqrt{-g} d^4x\) is a 4-scalar. This scalar is nothing else but the 4-invariant charge density \(\rho\) which appeared in the Maxwell equation (38):

\[
\rho = \frac{cQ\delta[r - r_0(t)]}{4\pi r^2 u^0 \sqrt{-g_{00}g_{11}}},
\]

(43)

For the electric current \(j^k\) we have \(j^k = \rho u^k\).

The 4-scalar energy density \(\epsilon\) which figure in the energy-momentum 4-tensor (39) can be constructed exactly in the same way if we observe that the rest energy of the matter in a 3-volume element \(dV\) (i.e. the sum of the all kinds of the internal energies of this element in the reference system in which this element is at rest) is a 4-invariant quantity by definition. Then we can introduce the 3-dimensional rest energy density (the direct analogue of the previous charge density \(\rho^{(3)}\)) which under integration over 3-volume gives the total rest energy \(mc^2\) of the membrane. Then \(mc^2\) is the sum of the all kinds of internal energies of the membrane in the radially comoving system in which membrane is at rest. In this way we obtain:

\[
\epsilon = \frac{mc^2\delta[r - r_0(t)]}{4\pi r^2 u^0 \sqrt{-g_{00}g_{11}}}.
\]

(44)

Clearly the effective rest mass \(m\) of the membrane in the presence of a tension depends on the membrane radius \(r_0(t)\).

In the case of spherical symmetry the electromagnetic potentials \(A_i\) can be taken in the form:

\[
A_0 = A_0(t, r), \quad A_1 = A_2 = A_3 = 0,
\]

(45)

which gives only one nonvanishing component for the electromagnetic tensor \(F_{ik}\), namely \(F_{10}\) (and its antisymmetric partner \(F_{01}\)):

\[
F_{10} = A_{0,1}.
\]

(46)

\footnote{The \(\delta\)-function in curved metric (34) is defined by the usual relation \(\int \delta(r - r_0)dr = 1\). Such \(\delta\)-function has dimension \(cm^{-1}\).}
Now, we enter with definitions (34)-(36) and (39)-(46) into the Einstein-Maxwell equations (37)-(38) to calculate the solution. These calculations need special care since we are dealing with distributions in application to the non-linear theory. In general this is not a trivial task (see e.g. [31, 32, 33]), however, for particular case of spherical symmetry everything is tractable and can be done easily thanks to the specially simple structure of the field equations. The resulting solution contains four arbitrary constants of integration $M_{in}$, $Q_{in}$ and $M_{out}$, $Q_{out}$ which have an obvious interpretation as mass and charge of a central RN source and the total mass and charge of the whole system (the central body together with the membrane) respectively. The membrane’s charge $Q$ is simply the difference of $Q_{out}$ and $Q_{in}$:

$$Q = Q_{out} - Q_{in}.$$  \(\text{(47)}\)

To represent the solution in compact form we use the proper time $\eta$ of the membrane, denoting the membrane’s equation of motion as $r = r_0(\eta)$, and introducing the following notations:

$$\phi_{in}(r) = 1 - \frac{2kM_{in}}{c^2r} + \frac{kQ^2_{in}}{c^4r^2} \quad \{ \text{48} \}$$

$$\phi_{out}(r) = 1 - \frac{2kM_{out}}{c^2r} + \frac{kQ^2_{out}}{c^4r^2}$$

$$S_{in}(\eta) = \sqrt{\phi_{in}(r_0) + c^{-2}(r_0,\eta)^2} \quad \{ \text{49} \}$$

$$S_{out}(\eta) = \sqrt{\phi_{out}(r_0) + c^{-2}(r_0,\eta)^2}$$

We consider the global time $t$ in (34) as continuous quantity when passing through the membrane. Then the intervals inside, outside and on the membrane are:

$$-(ds^2)_{in} = -\Gamma^2(t)\phi_{in}(r)c^2dt^2 + \frac{dr^2}{\phi_{in}(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$  \(\text{(50)}\)

$$-(ds^2)_{out} = -\phi_{out}(r)c^2dt^2 + \frac{dr^2}{\phi_{out}(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$  \(\text{(51)}\)

$$-(ds^2)_{on} = -c^2d\eta^2 + r_0^2(\eta)(d\theta^2 + \sin^2\theta d\phi^2)$$  \(\text{(52)}\)

The matching conditions for these intervals through the membrane are:

$$[(ds^2)_{in}]_{r=r_0(\eta)} = [(ds^2)_{out}]_{r=r_0(\eta)} = (ds^2)_{on}$$  \(\text{(53)}\)

Using the relations (53), the factor $\Gamma(t)$ in (50) and the connection $t(\eta)$ between global and proper times can be expressed through the membrane’s radius $r_0(\eta)$:

$$\frac{dt}{d\eta} = \frac{S_{out}}{\phi_{out}(r_0)}, \quad \text{(54)}$$

$$\Gamma(t) = \frac{\phi_{out}(r_0)S_{in}}{\phi_{in}(r_0)S_{out}} \quad \text{(55)}$$
Namely the continuity conditions (53) and continuous character of the time variable \( t \) are responsible for the appearance of the term \( \Gamma^2(t) \) in \( g_{00} \) in Eq.(54). Since this term depends only on time, it can be easily removed by passing to the internal time variable \( t_{in} \) by the transformation

\[
\Gamma dt = dt_{in} ,
\]

which can be found with the help of (54) after the function \( r_0(\eta) \) became known. In terms of the variables \( (t_{in}, r) \) also the internal metric (50) takes the standard RN form.

As it was already mentioned, the membrane’s effective rest mass \( m \) which appeared in the energy density (44) depends on the membrane radius. The concrete form of the function \( m(r_0) \) is not known in advance and its specification is equivalent to the specification of the equation of state. For an arbitrary \( m(r_0) \) the Einstein-Maxwell equations (37)-(38) give the following equation of motion for the membrane:

\[
M_{out}c^2 - M_{in}c^2 = \frac{1}{2}(S_{in} + S_{out})mc^2 + \frac{QQ_{in}}{r_0} + \frac{Q^2}{2r_0} ,
\]

(57)

together with the condition that both square roots \( S_{in} \) and \( S_{out} \) defined by (49), should have the same sign. The provenance of this condition is due to the fact that the signs of \( S_{in} \) and \( S_{out} \) are nothing else but the signs of the time-component of \( u^0 \) of the membrane’s 4-velocity when it is seen from the inside \( (r \to r_0 - 0) \) and outside \( (r \to r_0 + 0) \) of the membrane surface respectively. In our approach (with continuous coordinates \( t, r \)) we can consider the 4-velocity \( u^i \) as a field continuous through the surface of the membrane. We can define \( u^i \) everywhere in space-time simply by smooth parallel transport from the membrane’s surface, no matter that the membrane is concentrated only at the points \( r = r_0 \). This concentration is ensured not by \( u^i \) but due to the \( \delta \)-functions in the densities \( \rho \) and \( \epsilon \). Since \( u^0 \) can not change sign passing through the membrane, \( S_{in} \) and \( S_{out} \) should have the same sign.

Of course, we need to know the fields \( u^0 \) and \( u^1 \) only on the membrane, and there they are:

\[
u^0 = t_{,\eta} ; \quad u^1 = c^{-1}r_{0,\eta}
\]

(58)

It is easy to check that the matching conditions (54) and (55) are nothing else but the demand that the normalization constraint \( u^iu_i = -1 \) should hold independently from which side we approach the surface of the membrane.

It is worth to be remarked that the Einstein-Maxwell equations also demand for the trajectory \( r_0(\eta) \) the second order (in time) differential equation of motion. However, this last one represents simply the result of the differentiation in time of the first order equation (57). Then this second-order equation we can forget safely.
The resulting expressions for the energy density and tension are:

\[
\epsilon = \frac{mc^2}{8\pi r_0^2} \left[ \frac{\phi_{\text{in}}(r_0)}{S_{\text{in}}} + \frac{\phi_{\text{out}}(r_0)}{S_{\text{out}}} \right] \delta[r - r_0(\eta)] \tag{59}
\]

\[
\tau = \frac{r_0}{2m} \frac{dm}{dr_0} \epsilon . \tag{60}
\]

The electric field \( F_{10} \) outside the membrane is:

\[
F_{10} = \frac{Q_{\text{out}}}{r^2}, \quad r > r_0 . \tag{61}
\]

Inside the membrane we have:

\[
F_{10} = \frac{Q_{\text{in}}}{r^2} \frac{dt_{\text{in}}}{dt}, \quad r < r_0 , \tag{62}
\]

where the factor \( \frac{dt_{\text{in}}}{dt} \) depends only on time and can be calculated from the relations (55) and (56). The origin of this factor is due to the fact that we use the time \( t \) as continuous global time including the region inside the membrane. If we describe the internal metric in terms of internal time \( t_{\text{in}} \) the field strength \( F_{10} \) would be simply \( Q_{\text{in}}/r^2 \).

The formulas (48)-(52), (54), (55) and (57)-(62) provide the complete solution of the problem. It is worth explaining briefly the main steps of our integration procedure that we applied to the Einstein-Maxwell equations.

As in any spherically symmetric problem it is convenient to use, instead of the full original Einstein equations (37), only its \( (0) \), \( (1) \) and \( (\bar{1}) \) components, and the hydrodynamical equations \( T_{k\bar{k}} = 0 \). All the remaining components of equations (37) after that will be satisfied identically either due to the Bianchi identities or due to the symmetry of the problem. Then the solution for \( g_{11} \) together with the basic eq.(57) follows from \( (0) \) and \( (\bar{1}) \) components of Einstein equations (37), and after that the solution for \( g_{00} \) follows from the difference of the \( (0) \) and \( (\bar{1}) \) components of (37). The solution for the electric field \( F_{10} \) is the result of the Maxwell equations (38). The hydrodynamical equations \( T_{k\bar{k}} = 0 \) give only two relations. The first one simply express the tension \( \tau \) in terms of other quantities and this is the formula (60). The second one results in the already mentioned second order differential equation for \( r_0(\eta) \) which represents the differentiation in time of the first order equation (57). Then this second order equation is of no importance.

We remark also that the procedure described above need a caution because the symbolic function are involved. Nevertheless everything going well under the following three standard operation rules with such functions:

1. \( \frac{d}{dx} \theta(x) = \delta(x) \),
2. \( F(x)\delta(x) = \frac{1}{2}[F(-0) + F(+0)]\delta(x) \),
3. \( \frac{d}{dx} \theta^2(x) = 2\theta(x)\delta(x) = \delta(x) \).
(To call the third rule as the standard one is a little exaggeration; however it works well and final results indeed coincide with those obtained in literature by different approaches). Originally we obtained the solution in global form using the step function \( \theta(x) \) and only after that we represented the results separately in the regions \( r > r_0 \) and \( r < r_0 \). However, since \( \theta(x) \) is defined also at the point \( x = 0 \) \([\theta(0) = 1/2]\), we found by the way the values for the metric and electric field also at the points of the membrane’s surface. Such global form is:

\[
\begin{align*}
\frac{1}{g_{11}} &= 1 - \frac{2kM_{\text{in}}}{c^2r} - \frac{2k(M_{\text{out}} - M_{\text{in}})}{c^2r}\theta[r - r_0(\eta)] + \frac{k}{c^4r^2}\{Q_{\text{in}} + Q\theta[r - r_0(\eta)]\}^2, \\
F_{10} &= \sqrt{-g_{00}g_{11}} \{Q_{\text{in}} + Q\theta[r - r_0(\eta)]\},
\end{align*}
\]

(63)

(64)

(65)

to which should be added the equation (57). This equation arise as self-consistency condition for the \((0_0)\) and \((1_0)\) components of Einstein equations, which can be verified by the direct substitution into these components of the above global expressions together with eqs. (58)-(60).

Finally it should be mentioned that the membrane’s equation of motion (57) can be written in the following two equivalent forms:

\[
\begin{align*}
mc^2S_{\text{in}} &= M_{\text{out}}c^2 - M_{\text{in}}c^2 - \frac{Q_{\text{in}}Q}{r_0} - \frac{Q^2}{2r_0} + \frac{k m^2}{2r_0}, \\
mc^2S_{\text{out}} &= M_{\text{out}}c^2 - M_{\text{in}}c^2 - \frac{Q_{\text{in}}Q}{r_0} - \frac{Q^2}{2r_0} - \frac{k m^2}{2r_0}.
\end{align*}
\]

(66)

(67)

Each of these two equations is equivalent to (57) which can be checked easily by simple algebraic manipulations. For practical calculations we can use only one of these equations, however, in addition it is necessary to ensure the same sign for both quantities \( S_{\text{in}} \) and \( S_{\text{out}} \). (For a membrane with empty space inside they both should be positive). More convenient is relation (66) which we write as

\[
M_{\text{out}}c^2 = M_{\text{in}}c^2 + mc^2\sqrt{\phi_{\text{in}}(r_0) + c^{-2}(r_0,\eta)^2} + \frac{Q_{\text{in}}Q}{r_0} + \frac{Q^2}{2r_0} - \frac{k m^2}{2r_0}.
\]

(68)

This is the equation obtained by Chase\(^{15}\) with the aid of a different derivation procedure which makes use of Gauss-Codazzi equations (see Israel\(^{16}\)).

Eqn.(68) is interesting because in spite of the fact that \( m \) depends on time (or on \( r_0 \)) this equation looks like an usual integral of motion, that is as if \( m \) was a constant. Relation (68) expresses the conservation of the total energy \( M_{\text{out}}c^2 \) of the system which is the sum of the five familiar constituents: 1) the rest energy of the central body, 2) the kinetic energy of the membrane together with its gravitational potential energy in the gravitational field of the central body, 3) the electric interaction energy
between membrane and central source, 4) the positive electric self-interaction energy of the membrane, and 5) the negative gravitational self-interaction energy of the membrane.

References

We describe the equation of motion of two charged spherical shells with tangential pressure in the field of a central Reissner-Nordstrom (RN) source. We solve the problem of determining the motion of the two shells after the intersection by solving the related Einstein-Maxwell equations and by requiring a physical continuity condition on the shells velocities.

We consider also four applications: post-Newtonian and ultra-relativistic approximations, a test-shell case, and the ejection mechanism of one shell.

This work is a direct generalization of Barkov-Belinski-Bisnovati-Kogan paper.

Keywords: Classical gravity; Exact solutions.

1. Introduction

The mathematical model that we analyze in this paper describes the dynamic evolution of two spherical shells of charged matter which freely move outside the field of a central Reissner-Nordstrom (RN) source. Microscopically these shells are assumed to be composed by charged particles which move on elliptical orbits with a collective variable radius. The angular motion, distributed uniformly and isotropically on the shell surfaces, is mathematically described by a tangential-pressure term in the energy momentum tensor of the Einstein equations. The definition of the shell implies that all the particles have the same following three ratios: energy/mass, angular momentum/mass, and charge/mass. Indeed, since the equations of motion for any
singled-out particle “a” are
\[ \frac{dt_a}{ds} = \frac{1}{-m_a c^2 g_{tt}(r_a)}(E_a + e_a A_0(r_a)) \] (1)
\[ \left( \frac{dr_a}{ds} \right)^2 = \frac{1}{m_a^2 c^4} (E_a + e_a A_0(r_a))^2 \left( \frac{1}{-g_{tt}(r_a) g_{rr}(r_a)} \right) - \left( \frac{l_a^2}{m_a^2 c^2 r^2} + 1 \right) \frac{1}{g_{rr}(r_a)} \] (2)
\[ \left( \frac{d\theta_a}{ds} \right)^2 = \frac{l_a^2}{m_a^2 c^2 r^4} - \frac{k_a^2}{m_a^2 c^2 r^4 \sin^2 \theta_a} \] (3)
\[ \frac{d\varphi_a}{ds} = \frac{k_a}{m_a c r^2 \sin^2 \theta_a} \] (4)

\((g_{tt} \text{ and } g_{rr} \text{ are the components of a spherical symmetric metric and } A_0 \text{ its electric potential; } k_a \text{ and } l_a \text{ are arbitrary constants}), \) it is easy to see that the radial motion for all particles is the same if
\[ \frac{E_a}{m_a} = \text{const}, \quad \frac{e_a}{m_a} = \text{const}, \quad \frac{|l_a|}{m_a} = \text{const}, \quad \forall a, \] (5)

where each const. does not depend on the index \(a\). Therefore, if at the beginning the particles are on the same radius \(r_a = R_0\), then the shell will evolve “coherently”, i.e. all particles will evolve with the same radius.

Now the problem we are interested in is to find the exchange of energy between the two shells after the intersection. Indeed the motion of the shells before and after the intersection can be easily deduced from the equation of motion for just one shell, which equation has been found many years ago by Chase\(^1\) with a geometrical method first used by Israel\(^2\).

What we achieve in the present paper is the determination of the constant parameters after the intersection knowing just the parameters before the intersection. Actually the unknown parameter is only one, \(m_{21}\), which is the Schwarzschild mass parameter measured by an observer between the shells after the intersection. This parameter is strictly related to the energy transfer which takes place in the crossing, and it is found imposing a proper continuity condition on the shells velocities.

In the model we assume that the emission of electromagnetic waves is negligible, and that there are no other interactions between the two shells apart the gravitational and electrostatic ones. In particular the shells, during the intersection, are assumed to be “transparent” each other (i.e. no scattering processes).

The paper is divided as follows: in Sec.2 we preliminarily discuss the one-shell case; in Sec.3, which is the central part of this article, we find the unknown parameter \(m_{21}\); then, Secs.4-7 are devoted to some applications: post-Newtonian approximation, zero effective masses case (i.e. ultra-relativistic case), test-shell case, and finally the ejection mechanism.

In this paper we deal only with the mathematical aspects of the problem; some astrophysical applications of charged shells in the field of a RN black hole have been considered in Ref.[3].
2. A gravitating charged shell with tangential pressure

The motion of a thin charged dust-shell with a central RN singularity was firstly studied by De La Cruz and Israel\(^4\), while the case with tangential pressure was achieved by Chase\(^1\) in 1970. All these authors used the extrinsic curvature tensor and the Gauss-Codazzi equations. However we followed a different way, indeed the same solution can be found also by using $\delta$ and $\theta$ distributions and then by direct integration of the Einstein-Maxwell equations (see Ref.[5] and the appendix in Ref.[6]). This method has the advantage of a clearer physical interpretation, and it is also straightforward in the calculations; however in the following we will give only the main passages.

Let there be a central body of mass $m_{in}$ and charge $e_{in}$ and let a spherical massive charged shell with charge $e$ move outside this body. It is clear in advance that the field internal to the shell will be RN, while externally we will have again a RN metric but with different mass and charge parameters $m_{out} = \frac{m_{in}}{c^2}$ and $e_{out} = e_{in} + e$.

Using the coordinates $x^0 = ct$ and $r$, which are continuous when passing through the shell, we can write the intervals inside, outside, and on the shell as

\[ -(ds)^2_{in} = -e^{iT(t)}f_{in}(r)c^2dt^2 + f_{in}^{-1}(r)dr^2 + r^2d\Omega^2 \]

\[ -(ds)^2_{out} = -f_{out}(r)c^2dt^2 + f_{out}^{-1}(r)dr^2 + r^2d\Omega^2 \]

\[ -(ds)^2_{on} = -c^2d\tau^2 + r_0(\tau)^2d\Omega^2 \]

where we denoted

\[ d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \]

and

\[ f_{in} = 1 - \frac{2Gm_{in}}{c^2r} + \frac{Ge_{in}^2}{c^4r^2} , \]

\[ f_{out} = 1 - \frac{2Gm_{out}}{c^2r} + \frac{G(e_{in} + e)^2}{c^4r^2} . \]

In the interval (8), $\tau$ is the proper time of the shell. The “dilaton” factor $e^{iT(t)}$ in (6) is required to ensure the continuity of the time coordinate $t$ through the shell. If the equation of motion for the shell is

\[ r = R_0(t) , \]

then joining the angular part of the three intervals (6)-(8), one has

\[ r_0(\tau) = R_0[t(\tau)] , \]

where the function $t(\tau)$ describes the relationship between the global time and the proper time of the shell. Joining the radial-time parts of the intervals (6)-(7) on the shell requires that the following relations hold:

\[ f_{in}(r_0) \left( \frac{dt}{d\tau} \right)^2 e^{iT(t)} - f_{in}^{-1}(r_0) \left( \frac{dr_0}{cd\tau} \right)^2 = 1 , \]
If the equation of motion for the shell — i.e. the function \( r_0(\tau) \) — is known, then the function \( t(\tau) \) follows from (13) and consequently \( T(t) \) can be deduced by (12). Thus the problem consist only in determining \( r_0(\tau) \), which can be done by direct integration of the Einstein-Maxwell equations

\[
\begin{align*}
R^k_i - \frac{1}{2}Rg^k_i &= \frac{8\pi G}{c^4} T^k_i \\
(\sqrt{-g}F^{ik})_{,i} &= \sqrt{-g}\frac{8\pi \rho u^i}{c^4}
\end{align*}
\]

(14)

with the energy-momentum tensor given by:

\[
T^k_i = \epsilon u^k u^i + (\delta_i^2 \delta_j^k + \delta_i^j \delta_k^j)p + T^{(el)}_i k
\]

(15)

\[
T^{(el)}_i k = \frac{1}{4\pi}(F^i_l F^{kl} - \frac{1}{4}F_{lm} F^{lm})
\]

(16)

Here on we employ the following notations:

\[-ds^2 = g_{ik}dx^idx^k, \quad g_{ik} \text{ has signature } (-,+,+,+)
\]

\[x^k = (ct, r, \theta, \phi) \quad i,j,k... = 0,1,2,3
\]

\[p \equiv p(R_0) = p_\theta = p_\phi = \text{tangential pressure} \quad (p_e = 0)
\]

\[F^i_k = A_{k,i} - A_{i,k}
\]

The above equations are to be solved for the metric

\[-ds^2 = g_{00}(t,r)c^2dt^2 + g_{11}(t,r)dr^2 + r^2d\Omega^2,
\]

(17)

and for the potential

\[A_0 = A_0(t,r), \quad A_1 = A_2 = A_3 = 0,
\]

(18)

As follows from the Landau-Lifshitz approach [7] (see Ref.[5]) the energy distribution of the shell is

\[\epsilon = \frac{M(t)c^2\delta[r - R_0(t)]}{4\pi r^2u^0\sqrt{-g_{00}g_{11}}},
\]

(19)

while its charge density is

\[\rho = \frac{c e\delta[r - R_0(t)]}{4\pi r^2u^0\sqrt{-g_{00}g_{11}}},
\]

(20)

where \( \delta \) is the standard \( \delta \)-function. In the absence of tangential pressure \( p \), the quantity \( M \) in Eqn.(19) would be a constant, but in presence of pressure, \( Mc^2 \) includes the rest energy along with the energy (in the radially comoving frame) of the tangential motions of the particles that produce this pressure.
It can be checked that the Einstein part of (14) actually lead to the solution (6)-(8) with, in addition, the “joint condition”
\[
\sqrt{f_{\text{in}}(r_0)} + \frac{dr_0}{cd\tau} \right)^2 + \sqrt{f_{\text{out}}(r_0)} + \frac{dr_0}{cd\tau} \right)^2 = 2 \left( \frac{m_{\text{out}} - m_{\text{in}}}{\mu(\tau)} \right) - e^2 + 2e_1 \mu(\tau) c^2 r_0,
\]
where we denoted
\[
\mu(\tau) = M[t(\tau)],
\]
while
\[
m_{\text{out}} - m_{\text{in}} = E/c^2
\]
is a constant which can be interpreted as the total amount of energy of the shell. Then, from the Maxwell side of (14) the only non-vanishing component of the electric field is
\[
F_{01} = -\frac{\sqrt{-g_{00}g_{11}}}{r^2} \{ e_{\text{in}} + e\theta[r - R_0(t)] \}
\]
(\(\theta(x)\) is the standard step function). Finally, the equations \(T_{k;k} = 0\) can be reduced to the only one relation:
\[
p = -\frac{dM}{dt} \frac{c^2 \delta[r - R_0(t)]}{8\pi r_a \sqrt{-g_{00}g_{11}}}
\]
We will not treat here the steady case (i.e. \(r_0 = \text{const}\)) which should be treated separately; thus in the following we will assume always \(r_0 \neq \text{const}\).

The joint condition (21) can be written in several different forms: two of them, which will be useful in the following, are
\[
\sqrt{f_{\text{in}}(r_0)} + \frac{dr_0}{cd\tau} \right)^2 = \left( \frac{m_{\text{out}} - m_{\text{in}}}{\mu(\tau)} \right) + \frac{G \mu^2(\tau) - e^2 - 2e_1}{2\mu(\tau)c^2 r_0},
\]
and
\[
\sqrt{f_{\text{out}}(r_0)} + \frac{dr_0}{cd\tau} \right)^2 = \left( \frac{m_{\text{out}} - m_{\text{in}}}{\mu(\tau)} \right) - \frac{G \mu^2(\tau) + e^2 + 2e_1}{2\mu(\tau)c^2 r_0}.
\]
As in Ref. 5, all the radicals encountered here are taken positive, since for astrophysical considerations only these cases are meaningful. To proceed further, we must specify the equation of state, i.e. the function \(\mu(\tau)\). Here we consider a particle-made shell, therefore the sum of kinetic and rest energy of all the particles is
\[
Me^2 = \sum_a \left( m_a c^2 \sqrt{1 + \frac{p_a^2}{m_a^2 c^2}} \right),
\]
where \(p_a\) is the tangential momentum of each particle (the electric interaction between the particles is already taken into account by the self-energy term of, e.g.,
From the definition of the shell (see Introduction) it follows:
\[ \frac{p_a^2}{m_a^2} = \frac{l_a^2}{m_a^2 R_0^2} = \text{const}, \]
the square root in (28) does not depend on the index \( a \); then defining
\[ \sum_a m_a c^2 = mc^2, \quad \sum_a |l_a| = L, \]
formula (28) can be re-written (remembering definition (22) too) as
\[ \mu(\tau) = \sqrt{m^2 + \frac{L^2}{c^2 r_0^2(\tau)}}. \]
Thus, now, one can determine the function \( r_0(\tau) \) from equation (21) [or from one of the equivalent forms (26)-(27)] if the initial radius of the shell and the six arbitrary constants \( m_{in}, m_{out}, m, e_{in}, e \) and \( L \) are specified. Accordingly with (19), (25), (22) and (30), the equation of state that relates the shell energy density \( \epsilon \) to the tangential pressure \( p \) is
\[ p = \frac{\epsilon}{2 m^2 c^2 R_0^2} \left( 1 + \frac{L^2}{m^2 c^2 R_0^2} \right)^{-1} \]
as in the uncharged case, i.e. the presence of the charges do not modify the relation between energy density and pressure (indeed the presence of the charge is hidden in the equation of motion). Note that when the shell expands to infinity \( (R_0 \to \infty) \) the angular momentum becomes irrelevant and the equation of state tends to the dust case \( p << \epsilon \).

3. The shells intersection
Let us now consider the case of two shells which move in the field of a central charged mass. The generalization from the previous (single-shell) case is straightforward if the shells do not intersect: indeed the outer shell do not affect the motion of the inner one, while the inner one appears from outside just as a RN metric. Therefore the principal aim of this section is to consider the intersection eventuality and to predict the motion of the two shells after the crossing, having specified the initial conditions before the crossing. After the intersection one has a new unknown constant that has to be found by imposing opportune joining conditions as now we are going to explain (the analysis follows step by step the Ref.5’s one).
Let us previously analyze the space-time in the \((t, r)\) coordinates (which are continuous through the shells). We define the point \( O \equiv (t_*, r_*) \) as the intersection point; then the space-time is divided in four regions (see Fig.1):
\[ COB \ (r > R_1, r > R_2), \]
\[ COA \ (R_1 < r < R_2), \]
\[ AOD \ (r < R_1, r < R_2), \]
\[ BOD \ (R_2 < r < R_1). \]
Intersections of self-gravitating charged shells

Correspondingly to these regions we have the metric in form (13) but with different coefficients $g_{00}$ and $g_{11}$:

\[
\begin{align*}
    g^{(COB)}_{00} &= -f_{\text{out}}(r), & g^{(COB)}_{11} &= f_{\text{out}}^{-1}(r) \\
    g^{(COA)}_{00} &= -e^{T_{1}(t)} f_{12}(r), & g^{(COA)}_{11} &= f_{12}^{-1}(r) \\
    g^{(AOD)}_{00} &= -e^{T_{0}(t)} f_{in}(r), & g^{(AOD)}_{11} &= f_{in}^{-1}(r) \\
    g^{(BOD)}_{00} &= -e^{T_{2}(t)} f_{21}(r), & g^{(BOD)}_{11} &= f_{21}^{-1}(r)
\end{align*}
\]

The dilaton factor $T(t)$ allows to cover all the space-time with only one $t$-coordinate; here, $f_{\text{in}}$ and $f_{\text{out}}$ are the same as those in (9) while $f_{12}$ and $f_{21}$ are given by similar expressions:

\[
\begin{align*}
    f_{12} &= 1 - \frac{2Gm_{12}}{c^{2}r} + \frac{G(e_{\text{in}} + e_{1})^{2}}{c^{4}r^{2}} \\
    f_{21} &= 1 - \frac{2Gm_{21}}{c^{2}r} + \frac{G(e_{\text{in}} + e_{2})^{2}}{c^{4}r^{2}}
\end{align*}
\]

As we said, the parameters $m_{\text{in}}$, $m_{12}$, $m_{\text{out}}$, $e_{\text{in}}$, $e_{1}$, $e_{2}$ are assumed to be specified at the beginning, while $m_{21}$ is the actual unknown constant which has yet to be determined from the joining conditions on $(t_{*}, r_{*})$.

**Before the intersection**

Let us write the equation of motion for the two shells before the intersection (shell-1 inner and shell-2 outer). This can be made easily adapting the (27) and (26) to the
present case:
\[
\sqrt{f_{12}(r_1) + \left(\frac{dr_1}{cd\tau_1}\right)^2} = \frac{(m_{12} - m_{in})}{M_1} - \frac{GM_1^2 + e_1^2 + 2e_{in}e_1}{2M_1 c^2 r_1}
\] (39)
for shell 1, while for shell 2
\[
\sqrt{f_{12}(r_2) + \left(\frac{dr_2}{cd\tau_2}\right)^2} = \frac{(m_{12} - m_{in})}{M_2} + \frac{GM_2^2 - e_2^2 - 2(e_{in} + e_1)e_2}{2M_2 c^2 r_2}
\] (40)
with
\[
M_1 = \sqrt{m_1^2 + \frac{L_1^2}{c^2 r_1^2}}, \quad M_2 = \sqrt{m_2^2 + \frac{L_2^2}{c^2 r_2^2}}.
\] (41)
Here, \(\tau_1\) and \(\tau_2\) are the proper times of the first and second shells respectively, while \(r_1(\tau_1) = R_1[t(\tau_1)]\) and \(r_2(\tau_2) = R_2[t(\tau_2)]\). Now we have to impose the joining conditions for the intervals on both the shells. For the first shell (on curve AO) one has:
\[
e^{T_1(t)}f_{12}(r_1) \left(\frac{dt}{dr_1}\right)^2 - f_{12}^{-1}(r_1) \left(\frac{dr_1}{cd\tau_1}\right)^2 = 1
\] (42)
\[
e^{T_0(t)}f_{in}(r_1) \left(\frac{dt}{dr_1}\right)^2 - f_{in}^{-1}(r_1) \left(\frac{dr_1}{cd\tau_1}\right)^2 = 1;
\] (43)
while for the second shell:
\[
f_{out}(r_2) \left(\frac{dt}{dr_2}\right)^2 - f_{out}^{-1}(r_2) \left(\frac{dr_2}{cd\tau_2}\right)^2 = 1
\] (44)
\[
e^{T_1(t)}f_{12}(r_2) \left(\frac{dt}{dr_2}\right)^2 - f_{12}^{-1}(r_2) \left(\frac{dr_2}{cd\tau_2}\right)^2 = 1.
\] (45)

If all free parameters and initial data to Eqs.(39)-(41) were specified and if the functions \(r_1(\tau_1)\) and \(r_2(\tau_2)\) were derived, then their substitution in (42)-(45) gives the functions \(\tau_1(t)\), \(\tau_2(t)\) and \(T_1(t)\), \(T_0(t)\), which is enough for determining the motion of the shells before the intersection. Therefore the intersection point \((t_*, r_*)\) can be found by solving the system
\[
\begin{align*}
& r_1(\tau_1(t_*)) = r_* \\
& r_2(\tau_2(t_*)) = r_*
\end{align*}
\] (46)
which we assume that has a solution.
After the intersection

The equation of motion for the shells after the intersection time \( t_+ \) can be constructed in the same way again by turning to Eqns. (26) and (27), and introducing the new parameter \( m_{21} \) which characterize the “Schwarzschild mass” seen by an observer in the region \( BOD \). We use Eq. (26) for (now outer) shell 1 and Eq. (27) for (now inner) shell 2:

\[
\sqrt{f_{21}(r_1) + \left( \frac{dr_1}{cd\tau_1} \right)^2} = \frac{(m_{out} - m_{21})}{M_1} + \frac{GM_1^2 - e_1^2 - 2e_1(e_in + e_2)}{2M_1c^2r_1},
\]

\[(47)\]

\[
\sqrt{f_{21}(r_2) + \left( \frac{dr_2}{cd\tau_2} \right)^2} = \frac{(m_{21} - m_{in})}{M_2} - \frac{GM_2^2 + e_2^2 + 2e_2e_in}{2M_2c^2r_2}.
\]

\[(48)\]

Naturally, \( M_1(r_1) \) and \( M_2(r_2) \) are given by the same expression of (41) but now they have to be calculated on \( r_1(\tau_1) \) and \( r_2(\tau_2) \) after the intersection.

Joining the intervals on the first shell (on curve \( OB \)) yields

\[
f_{out}(r_1) \left( \frac{dt}{d\tau_1} \right)^2 - f_{out}^{-1}(r_1) \left( \frac{dr_1}{cd\tau_1} \right)^2 = 1
\]

\[(49)\]

\[
edT_2(t) f_{21}(r_1) \left( \frac{dt}{d\tau_1} \right)^2 - f_{21}^{-1}(r_1) \left( \frac{dr_1}{cd\tau_1} \right)^2 = 1.
\]

\[(50)\]

Then, joining the second shell (on curve \( OB \)) we obtain:

\[
edT_2(t) f_{21}(r_2) \left( \frac{dt}{d\tau_2} \right)^2 - f_{21}^{-1}(r_2) \left( \frac{dr_2}{cd\tau_2} \right)^2 = 1
\]

\[(51)\]

\[
edT_0(t) f_{in}(r_2) \left( \frac{dt}{d\tau_2} \right)^2 - f_{in}^{-1}(r_2) \left( \frac{dr_2}{cd\tau_2} \right)^2 = 1.
\]

\[(52)\]

Since the initial data to Eqns. (47) and (48) have already been specified (from the previous evolution), then the evolution of the shells after the intersection would be determined from Eqns. (47)-(52) if parameter \( m_{21} \) were known. Thus we need an additional physical condition from which we could determine \( m_{21} \).

This condition follows from the fact that the Christoffel symbols (i.e. the accelerations) of the shells have only finite discontinuities (finite jumps), therefore the relative velocity of the shells must remain continuous through the crossing point.

In the presence of two shells, we can construct one more invariant than in the single shell case (where only \( u_iu^i = -1 \) was possible): the scalar product between the two 4-velocities of the shells. We can also avoid to apply the parallel transport if we evaluate the 4-velocities on the intersection point \( (t_+, r_+) \). The continuity condition can be found imposing that the scalar product has to have the same value when evaluated in both the two limits \( t \to t_-^+ \) and \( t \to t_+^- \).
Determination of $Q$.

Let us start determining the quantity
\[ Q \equiv \{ g_{00}^{(COA)} u_{AO}^0 u_{CO}^0 + g_{11}^{(COA)} u_{AO}^1 u_{CO}^1 \} \big|_{t=t^*, r=r_1=r_2=r_*}, \] (53)
which is the scalar product of the two 4-velocities evaluated in the intersection point from the region $AOC$ (along the curves $AO$ and $CO$). Written explicitly, the unit tangent vector to trajectory $AO$ is
\[ u^i_{AO} = (u^0_{AO}, u^1_{AO}, u^2_{AO}, u^3_{AO}) = \left( \frac{dt}{dr_1}, \frac{dr_1}{cdr_1}, 0, 0 \right)_{t \leq t^*}, \] (54)
while for the trajectory $CO$ we have
\[ u^i_{CO} = (u^0_{CO}, u^1_{CO}, u^2_{CO}, u^3_{CO}) = \left( \frac{dt}{dr_2}, \frac{dr_2}{cdr_2}, 0, 0 \right)_{t \leq t^*}. \] (55)

The fact that these are actually unit vectors follows from the joining equations (42) and (45).

The components of the vector (54) can be easily expressed from Eqs.(39) and (42) as
\[ \left( \frac{dt}{dr_1} \right)_{t \leq t^*} = e^{-T_1(t)/2} M_1(r_1) f_{12}(r_1) \left( m_{12} - m_{\text{in}} - \frac{GM_1^2(r_1) + e_1^2}{2c^2 r_1} \right) \] (56)
\[ \left( \frac{dr_1}{cdr_1} \right)_{t \leq t^*} = \delta_1 M_1(r_1) f_{12}(r_1) \sqrt{\left( m_{12} - m_{\text{in}} - \frac{GM_1^2(r_1) + e_1^2}{2c^2 r_1} \right)^2 - M_1^2(r_1) f_{12}(r_1)} \] (57)
where
\[ \delta_1 = \text{sgn} \left( \frac{dr_1}{cdr_1} \right)_{t \leq t^*}. \] (58)

Analogously, for the components of vector (55), we obtain the following expressions from Eqs.(40) and (45):
\[ \left( \frac{dt}{dr_2} \right)_{t \leq t^*} = e^{-T_1(t)/2} M_2(r_2) f_{12}(r_2) \left( m_{\text{out}} - m_{12} + \frac{GM_2^2(r_2) - e_2^2 - 2e_2(e_1 + e_1)}{2c^2 r_2} \right) \] (59)
Intersections of self-gravitating charged shells

\[
\left( \frac{d r_2}{cd\tau_2} \right)_{t \leq t_*} = \frac{\delta_2}{M_2(r_2)f_{12}(r_2)} \cdot \sqrt{m_{\text{out}} - m_{12} + \frac{GM_2^2(r_2) - e_2^2 - 2e_2(e_{1n} + e_1)}{2c^2r_2}} - M_2^2(r_2)f_{12}(r_2)
\]

(60)

\[
\delta_2 = \text{sgn} \left( \frac{d r_2}{cd\tau_2} \right)_{t \leq t_*}.
\]

(61)

Thus, from the preceding results, we obtain:

\[
Q = -\frac{1}{m_{12}}w_{2f/J}.
\]

\[
\left\{ \left( m_{12} - m_{1n} = \frac{GM_2^2 + e_2^2 + 2e_1e_{1n}}{2c^2r_*} \right) \left( m_{\text{out}} - m_{12} + \frac{GM_2^2 - e_2^2 - 2e_2(e_{1n} + e_1)}{2c^2r_*} \right) + \right.
\]

\[
-\delta_1\delta_2 \sqrt{\left( m_{12} - m_{1n} = \frac{GM_2^2 + e_2^2 + 2e_1e_{1n}}{2c^2r_*} \right)^2 - M_2^2 f_{12}}
\]

\[
\left. \left( m_{\text{out}} - m_{12} + \frac{GM_2^2 - e_2^2 - 2e_2(e_{1n} + e_1)}{2c^2r_*} \right)^2 - M_2^2 f_{12} \right\};
\]

(62)

Determination of \(Q'\).

It is possible to apply the same procedure to the region \(BOD\) (i.e. after the intersection time), finding the quantity

\[
Q' \equiv \{ g_{00}^{(BOD)}u_0^{0DB}u_0^{0DD} + g_{11}^{(BOD)}u_1^{1DB}u_1^{1DD} \}_{t = t_*, r_1 = r_2 = r_*}.
\]

(63)

Now the unit tangent vectors to trajectories \(OB\) and \(OD\) are:

\[
u_0^{1DB} = \left( u_0^{0DB}, u_1^{0DB}, u_2^{0DB}, u_3^{0DB} \right) = \left( \frac{dt}{dt_1}, \frac{dR_1}{cd\tau_1}, 0, 0 \right)_{t \geq t_*},
\]

(64)

\[^a\text{Obviously, when we say } t \geq t_*, \text{ we tacitly assume before a (possible) second intersection.}\]
\[ u_{OD}^i = (u_{OD}^0, u_{OD}^1, u_{OD}^2, u_{OD}^3) = \left( \frac{dt}{d\tau}, \frac{dr_2}{c d\tau}, 0, 0 \right)_{t \geq t_*} \]  

from the joining conditions (50) and (51) it is possible to see that these are actually unit vectors. The components of \( u_{OB}^i \) can be deduced from Eqs.(47) and (50), while the components of \( u_{OD}^i \) from Eqs.(48) and (51). Then, using the metric in the region \( BOD \), it is possible to calculate the scalar product

\[
Q' = \frac{-1}{M_1 M_2 f_1} \cdot \left\{ \left( m_{\text{out}} - m_{21} + \frac{GM_1^2 c_1^2 + 2 c_1 (e_{in} + e_{21})}{2c^2 r_*} \right) \left( m_{21} - m_{\text{in}} - \frac{GM_2^2 c_2^2 + 2 c_2 e_{in}}{2c^2 r_*} \right) + \right. \\
\left. - \delta_1' \delta_2' \sqrt{\left( m_{\text{out}} - m_{21} + \frac{GM_1^2 c_1^2 + 2 c_1 (e_{in} + e_{21})}{2c^2 r_*} \right)^2 - M_1^2 f_{21}} \\
\sqrt{\left( m_{21} - m_{\text{in}} - \frac{GM_2^2 c_2^2 + 2 c_2 e_{in}}{2c^2 r_*} \right)^2 - M_2^2 f_{21}} \right\},
\]

where \( \delta_1' \) and \( \delta_2' \) have been defined as in (58) and (61), but for \( t \geq t_* \). We introduced these symbols only for generality, but actually we are interested only in the case with

\[
\delta_1' = \delta_1, \quad \delta_2' = \delta_2.
\]

The necessary continuity requirement is thus

\[
Q = Q',
\]

then, since \( r_* \) is assumed to be known, this equation allows to find \( m_{21} \).

**Physical meaning of \( Q \) and \( Q' \).**

Using standard definition for the shell velocities before the intersection one has

\[
\left( \frac{v_1}{c} \right)^2 = \frac{g_{11}^{(COA)}(r_1)}{-g_{00}^{(COA)}(r_1)} \left( \frac{dr_1}{c dt} \right)^2,
\]

\[
\left( \frac{v_2}{c} \right)^2 = \frac{g_{11}^{(COA)}(r_2)}{-g_{00}^{(COA)}(r_2)} \left( \frac{dr_2}{c dt} \right)^2,
\]

This is the only possible case if one excludes \( v_1(t^*) = v_2(t^*) = 0 \), because there are non discontinuities in the velocities.
and similarly for the velocities after the intersection,

\[
\left(\frac{v_1'}{c}\right)^2 = \frac{g^{(BOD)}_{11}(r_1)}{-g^{(BOD)}_{00}(r_1)} \left(\frac{dr_1}{cdt}\right)^2
\]

\[
\left(\frac{v_2'}{c}\right)^2 = \frac{g^{(BOD)}_{11}(r_2)}{-g^{(BOD)}_{00}(r_2)} \left(\frac{dr_2}{cdt}\right)^2
\]

Then it is easy to obtain from the definitions (53) and (63), that \(cQ\) and \(cQ'\) can be re-written as

\[
Q = \left\{ \begin{array}{c}
v_1 v_2 / c^2 - 1 \\
\sqrt{1 - v_1^2 / c^2} \sqrt{1 - v_2^2 / c^2}
\end{array} \right\}_{t=t, r_1=r_2=r_*}
\]

and

\[
Q' = \left\{ \begin{array}{c}
v_1' v_2' / c^2 - 1 \\
\sqrt{1 - (v_1')^2 / c^2} \sqrt{1 - (v_2')^2 / c^2}
\end{array} \right\}_{t=t, r_1=r_2=r_*}
\]

**Determination of \(P\) and \(P'\).**

First of all it is convenient to introduce new symbols to simplify the expressions of \(Q\) and \(Q'\). With

\[
q_1 \equiv -\frac{GM_1^2 + e_1^2 + 2e_1 e_{in}}{2e^2 r_*},
\]

\[
q_2 \equiv \frac{GM_2^2 - e_2^2 - 2e_2 (e_{in} + e_1)}{2e^2 r_*},
\]

and

\[
q_1' \equiv \frac{GM_1^2 - e_1^2 - 2e_1 (e_{in} + e_2)}{2e^2 r_*},
\]

\[
q_2' \equiv -\frac{GM_2^2 + e_2^2 + 2e_2 e_{in}}{2e^2 r_*},
\]

then \(Q\) and \(Q'\) can be re-written as

\[
Q = \frac{-1}{M_1 M_2 f_{12}} \cdot \left\{ (m_{12} - m_{in} + q_1) (m_{out} - m_{12} + q_2) + \\
\delta_1 \delta_2 \sqrt{(m_{12} - m_{in} + q_1)^2 - M_1^2 f_{12}} \\
\sqrt{(m_{out} - m_{12} + q_2)^2 - M_2^2 f_{12}} \right\}
\]

\[\text{It is also worth noting that } \sqrt{Q^2 - 1/Q} = -|v_1/c - v_2/c|/(1 - v_1 v_2/c^2), \text{ which is the relative velocity definition of two "particles" in relativistic mechanics.}\]
and

\[ Q' = \frac{-1}{M_1 M_2 f_{21}} \times \left\{ (m_{\text{out}} - m_{\text{21}} + q_1') (m_{\text{21}} - m_{\text{in}} + q_2') + \right. \\
\left. -\delta_1'\delta_2' \sqrt{(m_{\text{out}} - m_{\text{21}} + q_1')^2 - M_1^2 f_{12}} \times \right. \\
\left. \sqrt{(m_{\text{21}} - m_{\text{in}} + q_2')^2 - M_2^2 f_{12}} \right\}, \tag{76} \]

Now, in principle is possible to find \( m_{\text{21}} \) by squaring and solving \( Q = Q' \) (which is a quartic equation). However the procedure is cumbersome and moreover it is not possible with Eq.\( (68) \) alone to determine the sign of the roots. Fortunately, as in the non-charged case, it is possible to follow another easier way. Indeed, it is possible to introduce two other invariants, say \( P \) and \( P' \), similar to \( Q \) and \( Q' \), which are constructed using the scalar products of the 4-velocities of the two shell, but now taking the limit to \((t_*, r_*)\) from the AOD and COB regions respectively.

More explicitly, we define

\[ P \equiv \left\{ g^{(AOD)}_{00} u_{AO}^0 u_{OD}^0 + g^{(AOD)}_{11} u_{AO}^1 u_{OD}^1 \right\}_{t=t^*, r=r_1=r_2=r_*}, \tag{77} \]

and

\[ P' \equiv \left\{ g^{(COB)}_{00} u_{CO}^0 u_{OB}^0 + g^{(COB)}_{11} u_{CO}^1 u_{OB}^1 \right\}_{t=t^*, r=r_1=r_2=r_*}. \tag{78} \]

Then, the same continuity requirement of Eq.\( (68) \) implies that it must hold also that

\[ Q = P, \quad P = P'. \tag{79} \]

Following the same method used to find \( Q \) and \( Q' \), after some calculations, one arrives to

\[ P = \frac{-1}{M_1 M_2 f_{21}} \times \left\{ (m_{12} - m_{\text{in}} + p_1) (m_{21} - m_{\text{in}} + p_2) + \right. \\
\left. -\delta_1'\delta_2' \sqrt{(m_{12} - m_{\text{in}} + p_1)^2 - M_1^2 f_{\text{in}}} \times \right. \\
\left. \sqrt{(m_{21} - m_{\text{in}} + p_2)^2 - M_2^2 f_{\text{in}}} \right\}, \tag{80} \]
and

\[ P' = \frac{1}{M_1 M_2 f_{\text{out}}} \cdot \left\{ (m_{\text{out}} - m_{21} + p'_1) (m_{\text{out}} - m_{12} + p'_2) + \right. \\
- \delta'_1 \delta'_2 \sqrt{(m_{\text{out}} - m_{21} + p'_1)^2 - M^2_{\text{out}} f_{\text{out}}} \\
\left. \sqrt{(m_{\text{out}} - m_{12} + p'_2)^2 - M^2_{\text{out}} f_{\text{out}}} \right\}, \] (81)

where we have denoted

\[ p_1 \equiv \frac{G M_1^2 e_1^2 - 2 e_1 e_{\text{in}}}{2 c^2 r_*}, \]

\[ p_2 \equiv \frac{G M_2^2 e_2^2 - 2 e_2 e_{\text{in}}}{2 c^2 r_*}, \]

and

\[ p'_1 \equiv - \frac{G M_1^2 e_1^2 + 2 e_1 (e_{\text{in}} + e_2)}{2 c^2 r_*}, \]

\[ p'_2 \equiv - \frac{G M_2^2 e_2^2 + 2 e_2 (e_{\text{in}} + e_1)}{2 c^2 r_*}. \]

**Determination of \( m_{21} \): the energy transfer**

Thus the complete set of continuity conditions at the point of intersection can be written as

\[ Q = Q', \quad Q = P, \quad Q = P'. \] (82)

It turns out that this three quartic equations for the unknown parameter \( m_{21} \) have only one common root. It is possible to find the solution using hyperbolic functions (see Appendix). The final result is remarkably simple:

\[ m_{21} = m_{\text{in}} + m_{\text{out}} - m_{12} - \frac{e_1 e_2}{c^2 r_*} - \frac{G M_1 M_2}{c^2 r_*} Q, \] (83)

or equivalently, in terms of \( f_{21} \):

\[ f_{21} = f_{\text{in}} + f_{\text{out}} - f_{12} + 2 \frac{G^2 M_1 M_2}{e^4 r^4_*} Q. \] (84)

It can be easily seen from Eqn.(83) that the charge \( e_{\text{in}} \) of the central singularity does not affect the result (but it affects the equation of the motion of the shells and thus \( Q \)). Formula (83) solves the problem of determining the mass parameter \( m_{21} \) from the quantities specified at the evolutionary stage before intersection. It is then
possible to determine the energy transfer between the shells. Indeed the energy of shell 1 and 2 before the intersection are, respectively

\[ E_1 = (m_{12} - m_{in})c^2, \quad E_2 = (m_{out} - m_{12})c^2, \]  

(85)

while, after the intersection

\[ E'_1 = (m_{out} - m_{21})c^2, \quad E'_2 = (m_{21} - m_{in})c^2. \]  

(86)

The conservation of total energy is automatically ensured by the above formulas, indeed

\[ E_1 + E_2 = E'_1 + E'_2. \]  

(87)

Then it is natural to define the exchange energy as

\[ \Delta E = E'_2 - E_2 = -(E'_1 - E_1). \]  

(88)

Then, from Eqn.(83) and the above definitions, it follows that

\[ \Delta E = -\frac{e_1 e_2}{r^*} - \frac{G M_1 M_2}{r^*} Q. \]  

(89)

It is also useful (especially for the Newtonian approximation) to use Eqn.(73) and re-express \( \Delta E \) as:

\[ \Delta E = -\frac{e_1 e_2}{r^*} - \frac{G M_1 M_2}{r^*} \left\{ \frac{v_1 v_2 / c^2 - 1}{\sqrt{1 - v_1^2 / c^2} \sqrt{1 - v_2^2 / c^2}} \right\} \bigg|_{r=r^*}. \]  

(90)

4. Post-Newtonian approximation

For slow velocities of the shells it is interesting to consider the Post-Newtonian limit of Eqn.(90):

\[ \Delta E = \frac{G m_1 m_2 - e_1 e_2}{r^*} + \frac{1}{2c^2} \left\{ \frac{G m_1 m_2}{m_1 r^*} [v_1(r^*) - v_2(r^*])^2 + \frac{G m_2 L_1^2}{m_1 r^*} + \frac{G m_1 L_2^2}{m_2 r^*} \right\} + o \left( \frac{1}{c^4} \right). \]  

(91)

It is worth noting that only the zeroth order in \( 1/c^2 \) changes with respect to the uncharged case (because of the Coulomb term \(-e_1 e_2/r^*\)), while all the other orders remain unchanged, being of kinetic origin; \( m_1 \) and \( m_2 \) are the rest masses of the shells, indeed we have used for the masses \( M_1 \) and \( M_2 \) the definitions (41).

It can be also useful to re-express all the quantities in a Newtonian language and consider only the zeroth order in \( 1/c^2 \), e.g. we can expand the energy as

\[ E = mc^2 + \mathcal{E} + o \left( \frac{1}{c^2} \right), \]  

(92)
where \( m \) and \( \mathcal{E} \) do not depend on \( c \). Therefore, similarly, we can define at the first order in \( 1/c^2 \)

\[
m_{12} - m_{\text{in}} = m_1 + \frac{\mathcal{E}_1}{c^2}, \quad m_{\text{out}} - m_{12} = m_2 + \frac{\mathcal{E}_2}{c^2},
\]

\[
m_{\text{out}} - m_{21} = m_1 + \frac{\mathcal{E}_1'}{c^2}, \quad m_{21} - m_{\text{in}} = m_1 + \frac{\mathcal{E}_2'}{c^2}.
\]

Then it follows also that the energy conservation law takes the form

\[
\mathcal{E}_1 + \mathcal{E}_2 = \mathcal{E}_1' + \mathcal{E}_2',
\]

and Eqn.\((88)\) becomes

\[
\mathcal{E}_1' = \mathcal{E}_1 - \Delta \mathcal{E}, \quad \mathcal{E}_2' = \mathcal{E}_2 + \Delta \mathcal{E},
\]

where \( \Delta \mathcal{E} = (\Delta E)_{c \to \infty} \). Thus from the above formulas and definitions it is clear that

\[
\Delta \mathcal{E} = \frac{Gm_1m_2 - e_1e_2}{r_*}.
\]

5. Pressureless shells with zero effective masses \((L_1 = L_2 = 0 \text{ and } M_1 = M_2 = 0)\)

It is interesting also to consider the case in which the motion of the particles of the shells is only radial (i.e. \( L_1 = L_2 = 0 \)) and the rest masses are negligible with respect to the kinetic energies and to the charges —indeed this is the case for two shells composed by (ultra)relativistic electrons and positrons. In this case the effective masses can be replaced by

\[
M_1 = M_2 = \lambda,
\]

where \( \lambda \) is a parameter arbitrary small. From Eqn.\((89)\), with \( Q \) expressed by formula \((62)\), it is easy to find that the energy transfer in this case is

\[
\Delta E = -\frac{e_1e_2}{r_*} + \frac{e_1^2r_*}{2Gf_{12}} (f_{\text{in}} - f_{12})(f_{12} - f_{\text{out}}) + o(\lambda^2),
\]

having assumed that the shells have opposite-directed velocities, i.e.

\[
\delta_1\delta_2 = -1.
\]

Otherwise, if the shells goes in the same direction, i.e.

\[
\delta_1\delta_2 = 1,
\]

then Eqn.\((89)\) becomes simply

\[
\Delta E = -\frac{e_1e_2}{r_*} + o(\lambda^2);
\]

obviously the previous formulas make sense only if \( r_* \) exists. We want to underline the presence of the term \( o(\lambda^2) \), because, strictly speaking, a charge cannot have zero rest mass, therefore we are in the case of just small effective masses. As expected, in
the case of vanishing charges \((e_1 = e_2 = 0)\), Eqn.\((102)\) gives zero at \(\lambda = 0\) because this is the case of two photon-shells which go in the same direction and therefore cannot never intersect.

6. The intersection of a test shell with a gravitating one

**One-shell case**

Let us consider firstly the case of a test shell on the RN field. This limit has the only aim to show that the shell’s equation of motion (26) actually reduce to the simple test-particle case; the limit can be obtained by putting

\[
m \to \lambda m, \quad e \to \lambda m, \quad L \to \lambda L, \quad (m_{\text{out}} - m_{\text{in}})c^2 \to \lambda E
\]

with \(\lambda \to 0\). Then, considering also (30), we find that Eqn.\((26)\) becomes

\[
E = \mu c^2 \sqrt{m^2 + L^2 c^2 R_0^2(t)} + \frac{e e_{\text{in}}}{r_0} - \lambda \frac{G \sqrt{2} - e^2}{2r_0},
\]

now, putting \(\lambda = 0\) the self-energy term is killed; then re-writing Eqn.\((104)\) using the more familiar Schwarzschild time \(t\) [and notation (11)],

\[
E = c^2 \sqrt{m^2 + \frac{L^2}{c^2 R_0^2(t)}} \left( \frac{f_{\text{in}}(r_0)}{f_{\text{in}}(R_0)} - \frac{d r_0}{c dt} \right)^2 + \frac{e e_{\text{in}}}{R_0} + o(\lambda),
\]

it is easy to recognize that Eqn.\((105)\) coincides with the first integral of motion of a test-charge particle on the Reissner-Nordstrom background, where \(E\) is the conserved energy of the particle, \(m\) the rest mass, \(e\) the charge and \(L\) the angular momentum.

**Two-shell case, with one test-shell**

Now we can deal with the more interesting two-shell case, in which shell-2 is considered “test”. To gain this limit we have to put

\[
m_2 \to \lambda m_2, \quad e_2 \to \lambda m_2, \quad L_2 \to \lambda L_2, \quad (m_{\text{out}} - m_{12})c^2 \to \lambda E_2, \quad (m_{21} - m_{\text{in}})c^2 \to \lambda E_2'.
\]

Then, using Eqn.\((83)\) with \(Q\) given by formula (62), one obtains

\[
\Delta E = -\frac{\delta_1}{r_*} + \frac{1}{r_* f_{12}} \cdot \left\{ \left( E_1 - \frac{G M_1^2 + e_1^2 + 2 e_1 e_{\text{in}}}{2 c^2 r_*} \right) \left( E_2 - \frac{e_2 (e_{\text{in}} + e_1)}{c^2 r_*} + \lambda \frac{G M_2^2 - e_2^2}{2 c^2 r_*} \right) + \right.
\]

\[
-\delta_1 \delta_2 \sqrt{\left( E_1 - \frac{G M_1^2 + e_1^2 + 2 e_1 e_{\text{in}}}{2 c^2 r_*} \right)^2 - M_1^2 f_{12}}
\]

\[
- \left. \sqrt{\left( E_2 - \frac{e_2 (e_{\text{in}} + e_1)}{c^2 r_*} + \lambda \frac{G M_2^2 - e_2^2}{2 c^2 r_*} \right)^2 - M_2^2 f_{12}} \right\}.
\]
Thus, only the self-energy terms of shell-2 are killed by $\lambda = 0$.

Now, it is worth noting the following fact: shell-1 does not have any discontinuity when it intersect the shell-2 (this is natural because shell-2 is “test” and does not affect the metric), on the other hand shell-2 undergoes a discontinuity in the metric when it cross shell-1 and consequently it has an actual discontinuity in the velocity. It is easy to calculate this gap; indeed using the definition (70) of velocity $v_2$ [with the time $\frac{dt}{\gamma_2}$ given by the joint condition (45)], with metric coefficient (34), and with the help the first integral of motion (40), one finds

$$v_2^2(r_2) = 1 - f_{\text{out}}(r_2) \left( \frac{E_2}{M_2(r_2)} - \frac{e_2(e_1 + e_{\text{in}})^2}{M_2(r_2)r_2^2} \right)^{-2} + o(\lambda), \quad t \leq t_*,$$

where we have used $f_{12} = f_{\text{out}} + o(\lambda)$; in the same way, using (72), (51), (36), and (48), the velocity $v'_2$ (after the intersection) is

$$[v'_2(r_2)]^2 = 1 - f_{\text{in}}(r_2) \left( \frac{E'_2}{M_2(r_2)} - \frac{e_2(e_1 + e_{\text{in}})^2}{M_2(r_2)r_2^2} \right)^{-2} + o(\lambda), \quad t \geq t_*,$$

where $E'_2$ can be expressed in function of $E_2$ with the help of (107). From the previous formulas it is clear that in general

$$v'_2(r_*) - v_2(r_*) \neq 0.$$

### 7. Shell ejection

The exchange in energy of the shells during the intersection makes possible that one initially bounded shell can acquire enough energy to escape to infinity.

The shell ejection mechanism can take place also in the Newtonian regime. In this case, from Eqs.(96)-(97) it results that

$$E'_1 = E_1 - \frac{Gm_1m_2 - e_1e_2}{r'_*}, \quad E'_2 = E_2 + \frac{Gm_1m_2 - e_1e_2}{r'_*},$$

and then, after the first intersection

$$\begin{cases} E''_1 = E'_1 + \frac{Gm_1m_2 - e_1e_2}{r'_*} = E_1 - (Gm_1m_2 - e_1e_2) \left( \frac{1}{r'_*} - \frac{1}{r''_*} \right) \\ E''_2 = E'_2 - \frac{Gm_1m_2 - e_1e_2}{r''_*} = E_2 + (Gm_1m_2 - e_1e_2) \left( \frac{1}{r'_*} - \frac{1}{r''_*} \right), \end{cases}$$

where we have denoted the radius of the first and second intersection with $r'_*$ and $r''_*$ respectively. In the following we will consider only the case

$$Gm_1m_2 - e_1e_2 > 0,$$

this is e.g. the case in which the two shells have opposite charges. Thus, also in the case $E_1, E_2 < 0$, if

$$r''_* > r'_*,$$
and if the initial condition were in such a way that $r'_n$ is enough small and $r''_n$ not too much close to $r'_n$, then it is possible to have $E''_2 > 0$, i.e. the ejection of the second shell.

Let us now assume that $r''_n > r'_n$, and consider a “semi-relativistic” case in which at the first intersection we use the full relativistic formulas\(^4\),

\[
\begin{aligned}
E'_1 &= E_1 - \frac{M_1(r'_n)M_2(r'_n)}{r'_n}(-Q) + \frac{e_1e_2}{r'_n} \\
E'_2 &= E_2 + \frac{M_1(r'_n)M_2(r'_n)}{r'_n}(-Q) - \frac{e_1e_2}{r'_n},
\end{aligned}
\]

while at the second intersection we use the Newtonian approximation,

\[
\begin{aligned}
E''_1 &= E'_1 + \frac{Gm_1m_2-e_1e_2}{r'_n} \\
&= E_1 - \left[\frac{M_1(r'_n)M_2(r'_n)}{r'_n}(-Q) - e_1e_2 \right] - \frac{Gm_1m_2-e_1e_2}{r'_n}, \\
E''_2 &= E'_2 - \frac{Gm_1m_2-e_1e_2}{r'_n} \\
&= E_2 + \left[\frac{M_1(r'_n)M_2(r'_n)}{r'_n}(-Q) - e_1e_2 \right] - \frac{Gm_1m_2-e_1e_2}{r'_n}.
\end{aligned}
\]

This approximation is always justified if the radius of the second intersection $r''_n$ is enough large. Now, it is remarkable that whatever the value of $r'_n$ is, the first term in the square brackets in Eqn.\((116)\) satisfies the inequality

\[
\frac{M_1(r'_n)M_2(r'_n)(-Q) - e_1e_2}{r'_n} > \frac{Gm_1m_2-e_1e_2}{r'_n}. \tag{117}
\]

Comparing the expressions \((116), (117)\) and \((112)\) it is possible to see that in the relativistic regime the shell ejection possibility is even greater than in the Newtonian case. Furthermore, it is worth noting that the presence of the charge do not change qualitatively the pure gravitational analysis, but just magnifies the ejection effect.

### 7.1. $Gm_1m_2-e_1e_2 < 0$ case

Let us consider also briefly the case in which the shells are equal-signed charged and the repulsion overcome the gravity attraction, i.e. $Gm_1m_2-e_1e_2 < 0$. In this case the ejection can happen only after an odd number of intersections.

E.g. after three intersections, from the previous formulas we have, in the Newtonian approximation:

\[
\mathcal{E}_{1''} = \mathcal{E}_1 - (Gm_1m_2-e_1e_2) \left(\frac{1}{r''_n} - \frac{1}{r''_n} + \frac{1}{r''_n}\right). \tag{118}
\]

Obviously this formula has a meaning only if

\[
\frac{1}{r''_n} < \frac{1}{r''_n} - \frac{1}{r''_n} + \frac{1}{r''_n}, \tag{119}
\]

\(^4\)Remember that $-Q = 1 + o(1/c^2)$.
otherwise the ejection happens at the first intersection (and then there would not be other crossings, and no \( r''^*, r'''^* \), or never more; if Eqn. (119) is true, then it means that the barycenter of the two shells is falling into the center singularity.

8. Conclusions

We have found the energy exchange between two charged crossing shells (formula (90)). Then we have studied special cases of physical interest in which the formulas simplify: the non relativistic case, the massless shells, the test shell, and finally the ejection mechanism in a semi-Newtonian regime: we found that the ejection mechanism is more efficient in the charged case than in the neutral one if the charges have opposite sign (because the energy transfer is larger due to the Coulomb interaction).

References

Bibliography


Interdicis mihi inspectione rerum naturae, a toto abductum redigis in partem? Ego non quaeram quae sint initia universorum? [...] qua ratione tanta magnitudo in legem et ordinem venerit? quis sparsa collegerit, confusa distinxerit, in una deformitate iacentibus faciem diviserit? unde lux tanta fundatur? >>

Seneca, Ep.LXV.

<<Do you forbid me to inquire into the universe? Do you compel me to withdraw from the whole and restrict me to a part? May I not ask what are the beginnings of all things? [...] How a so mighty magnitude was brought under the control of law and order, who gathered together the scattered atoms, who separated the disordered elements and assigned an outward form to elements that lay in the one vast shapelessness? Or whence came all the expanse of light?>> Answering to the question: Why to study physics? Isn’t it a waste of time?)