MODERN APPROACHES TO
GENERALIZED THEORIES OF GRAVITY

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Introduction

General Relativity is the most coherent and consistent theoretical framework able to describe the gravitational field in a background-independent form and characterized by geometrodynamical features. Its innumerable merits notwithstanding, its match with other branches of physics may result still incomplete. In particular, the attempt to describe also other phenomena from a geometrical point of view and the comparison with quantum mechanics are two research directions, towards which the way is not straightforward.

Although General Relativity has been demonstrated to be more appropriate than any other theory of gravitation implying different predictions, an alternative theory of gravity sufficiently viable to be treated as a possible- but not necessarily likely-worth-considering candidate should fulfill some basic requirements. Self-consistency requires that no logical contradictions arise from basic assumptions. Completeness requires that the result of any experiment should be derived from first principles. Agreement with all know experiments implies that any new prediction does not contradict experimental evidence. Metric theories of gravity are among the most trusted alternative theories of gravity. A metric theory is a theoretical framework, according to which the spacetime is endowed with a metric, and the principle of equivalence is satisfied. In fact, the metric is the only tool able to leave proper room for all non-gravitational interactions in a self-consistent scenario.

Nevertheless, an important exception has to be discussed. In fact, the presence of torsion, as introduced by the Einstein-Cartan theory, is not detectable in experimental evidence, and EC theory is not a priori distinguishable from GR from an experimental point of view.

Within all the possible extensions of GR, this work is devoted to the analysis of three basic research lines.
a reformulation of GR, independent of quantum mechanics, which allows one to face some shortcomings, such as the classical singularity, or to describe some phenomena, directly observed or theoretically hypothesized, such as dark energy and dark matter;

an investigation of extended geometries, where non-Riemannian objects are taken into account, both from a macroscopic and a microscopic point of view;

a possible approach to quantum mechanics, such that the cosmological singularity can be viewed from a different perspective, and some compactification scenarios are interpreted as further microscopic degrees of freedom.

Of course, far from being an exhaustive treatment of all the modern research guidelines, these points have been picked up from a host of possible lines, whose applications in modern physics are nowadays extensively examined.

In particular, this thesis work consists of three chapters.

In the first chapter, we examine those modified theories of gravity, which allow for a metric formulation, but rely on a modification of the Einstein-Hilbert action. In the first part, we motivate and analyze the weak-field limit of a non-analytic modification of the Einstein-Hilbert Lagrangian for the gravitational field. After investigating the parameter space of the model, we impose constraints on the parameters characterizing this class of theories imposed by Solar-System data, i.e. we establish the validity range where this solution applies and refine the constraints by the comparison with planetary orbital periods. As a result, we claim that this class of models is viable within different astrophysical scales. In the second part, we analyze the dynamical implications of an exponential Lagrangian density for the gravitational field, as referred to an isotropic FRW Universe. Then, we discuss the features of the generalized deSitter phase, predicted by the new Friedmann equation. The existence of a consistent deSitter solution arises only if the ratio between the vacuum-energy density and that associated with the fundamental length of the theory acquires a tantalizing negative character. This choice allows us to explain the present universe dark energy as a relic of the vacuum-energy cancellation due to the cosmological constant intrinsically contained in our scheme. The corresponding scalar-tensor description of

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the model is addressed too, and the behavior of the scalar field is analyzed for both negative and positive values of the cosmological term. In the first case, the Friedmann equation is studied both in vacuum and in presence of external matter, while, in the second case, the quantum regime is approached in the framework of "repulsive" properties of the gravitational interaction, as described in recent issues in Loop Quantum Cosmology. In particular, in the vacuum case, we find a pure non-Einsteinian effect, according to which a negative cosmological constant provides an accelerating deSitter dynamics, in the region where the series expansion of the exponential term does not hold.

In the second chapter, we investigate some features of non-Riemannian geometries. In the first part, a novel analysis of the role of the local Lorentz group is implemented both in flat and in curved space-time, and the resulting dynamics is analyzed in view of the geometrical interpretation of the gauge potential. The Yang-Mills picture of local Lorentz transformations is first approached in a second-order formalism. For the Lagrangian approach to reproduce the second Cartan structure equation as soon as the Lorentz gauge connections are identified with the contortion tensor, an interaction term between the Lorentz gauge fields and the spin connections has to be postulated. The full picture involving gravity, torsion and spinors is described by a coupled set of field equations, which allows one to interpret both gravitational spin connections and matter spin density as the source term for the Yang-Mills equations. The contortion tensor acquires a propagating character, because of its non-Abelian feature, and the pure contact interaction is restored in the limit of vanishing Lorentz connections. Furthermore, the fine structure of Hydrogen-like atoms is investigated within this framework.

In the second part, a generalized connection, including Christoffel coefficients, torsion, non-metricity tensor and metric-asymmetricity object, is analyzed according to the Schouten classification. The inverse structure matrix is found in the linearized regime, autoparallel trajectories are defined and the contribution of the components of the connection are clarified at first-order approximation.

The third chapter is devoted to some applications of the polymer representation of quantum mechanics.
In th first part, within the framework of non-standard (Weyl) representations of
the canonical commutation relations, we investigate the polymer quantization of the Taub cosmological model. The Taub model is analyzed within the Arnowitt-Deser-Misner reduction of its dynamics, by which a time variable arises. While the energy variable and its conjugate momentum are treated as ordinary Heisenberg operators, the anisotropy variable and its conjugate momentum are represented by the polymer technique. The model is analyzed at both classical and quantum level. As a result, classical trajectories flatten with respect to the potential wall, and the cosmological singularity is not probabilistically removed. In fact, the dynamics of the wave packets is characterized by an interference phenomenon, which, however, is not able to stop the evolution towards the classical singularity.

In the second part, on the basis of Fourier duality and Stone-von Neumann theorem, we examine polymer-quantization techniques and modified uncertainty relations as possible 1-extraD compactification schemes for a phenomenological truncation of the extraD tower.
Publication List


PL3 N. Carlevaro, OML, G. Montani, Fermion dynamics by internal and space-time symmetries, Mod. Phys. Lett. A, in press.


PL18 OML, G. Montani, Electro-Weak Model within a 5-dimensional Lorentz group theory, in Proceeding of XI Marcel Grossmann Meeting on General Relativity, Berlin, July 2006 [gr-qc/0702025].

1 Modified gravity

In this chapter, we will analyze some features of modified theories of gravity, where the
Ricci scalar in the Einstein-Hilbert action is replaced by a generic function of it. After
reviewing the main features of this kind of models, we will focus our attention on the
weak-field limit of a particular class of models, where the gravitational action consists
of the Ricci scalar plus a non-analytic term, and on the cosmological implications of
an exponential gravitational action. The original works appeared on (PL1), (PL5),
(PL9), (PL10), (PL15).

1.1 Introduction

Although General Relativity (GR) is a well settled-down theory for the description of
dynamics, nevertheless, over the last decades, a wide number of approaches
have been developed to generalize it. These extended points of view are aimed not
only at addressing the quantization of the gravitational field, but also at establishing
proper deformations of the Einsteinian dynamics towards the space-time singularities.
From the very beginning, the possibility to reformulate GR by using a generic func-
tion of the Ricci scalar (see, for example, [1] for a recent review and the references
therein) has appeared as a natural issue offered by the fundamental principles estab-
lished by Einstein. Indeed, this kind of extended Einstein-Hilbert (EH) Lagrangian
preserves the fundamental features at the basis of General Relativity and Equivalence
principles, and induces a direct modification of the dynamics only (at least, as far
as the metric approach is concerned). Recently, modified $f(\mathcal{R})$ gravity has acquired
interest in view of the possibility to describe within this redefined dynamics some
unexpected features observed on the large-scale structure of the Universe, i.e. the Pi-
oneer anomaly, the galaxy rotation curve behaviour, the Universe acceleration, and,
finally, the removal of singular behaviours of the gravitational field [2, 3, 4, 5].
One of the most puzzling questions, which has come out from the modern un-
standing of the Universe evolution, is certainly the present value of the cosmological constant [6, 7]. In fact, the observations of the recession of SNIA, Super-Novae IA, (treated as standard candles) provide convincing indications for an accelerating Universe [8, 9]. This surprising behavior is guaranteed by a negative-pressure contribution, and the determination of the precise equation of state for the matter that is accelerating the Universe is the present challenge for cosmologists. However, data from the cosmic-microwave background suggest that the so-called Dark Energy has reliably the features of a cosmological constant, which corresponds to about 70 percent of the critical density of the Universe. Such an amount of the cosmological term is relevant for the actual dynamics, but extremely smaller than the vacuum value. Estimations of the vacuum energy yield indeed the Planckian value, corresponding to $10^{120}$ times the observed numbers. This striking contradiction between the theoretical predictions and the actual value suggest that, if the Universe acceleration is really due to a cosmological constant, then a precise mechanism of cancellation must be fixed for the vacuum energy density. We stress that no fundamental theory provides a convincing explanation for such a cancellation and therefore it is naturally expected to find it from specific features of the field dynamics. The main interesting proposals to interpret the presence of Dark Energy can be divided into two classes [10, 11]: those theories that make explicitly presence of matter and the other ones, which relay on modifications of the Friedmann dynamics.

However, it is important to remark that any modification of the EH Lagrangian is reflected onto a deformed gravitational-field dynamics at any length scale investigated or observed. Thus, the success of such $f(\mathcal{R})$ gravity in the solution of a specific problem has to match consistency with observation in other length scales [12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. In this respect, we emphasize that the small value of the present curvature of the Universe [22] leads us to believe that, independently of its specific functional form, the $f(\mathcal{R})$ term must be regarded as a lower-order expansion in the Ricci scalar. On the other hand, it is easily understood that the peculiarities of such an expansion will be extremely sensitive of the morphology of the deformed Lagrangian.
1.1 Introduction

1.1.1 Modified Gravitational Action

The dynamics of a gravitational field in vacuum associated with the metric tensor $g_{\mu\nu}$, $\mu = 0, 1, 2, 3$ is described by the Einstein-Hilbert (EH) action, $S_{EH}$,

$$S_{EH} = -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R,$$

where $\sqrt{-g} \equiv \text{det}g_{\mu\nu}$. The form of the gravitational action is fixed by the request that field equations contain second-order derivatives of the metric tensor only, and, by its variation with respect to the metric tensor $g_{\mu\nu}$, it is natural to recognize the well-known Einstein field equations

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 0, \quad (1.1a)$$

$$\Box \mathcal{R} = 0 \quad (1.1b)$$

where

$$\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu} \quad (1.2a)$$

$$\mathcal{R}_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\nu\sigma} - \Gamma^\sigma_{\mu\sigma} \Gamma^\rho_{\nu\rho} \quad (1.2b)$$

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) \quad (1.2c)$$

here $\Box \equiv g^{\rho\sigma} \nabla_\rho \nabla_\sigma$, $\nabla_\rho$ denotes covariant differentiation and $,$ denotes ordinary differentiation.

In the application of the variational principle, the natural assumption that $\delta g_{\mu\nu}$ vanishes on the boundaries of the considered space-time region has to be adopted unless boundary terms are included into the original action. We also note that we could get second-order equations from $S_{EH}$ because of the appearance of surface contribution to the Ricci scalar. This feature does not hold when, instead of $\mathcal{R}$, we take a generic function of it; such a more general choice is yet compatible with the 4-diffeomorphism invariance of the theory, but could lead to very different dynamical implications for the gravitational field.

**Jordan Frame**

We now fix our attention to the generalization of the previous scheme when the following gravitational action is taken into account

$$S_G = -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} f(\mathcal{R}) \quad (1.3)$$
whose variation with respect to $g^{\mu\nu}$ yields generalized Einstein equations
\begin{align}
 f'\mathcal{R}_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \nabla_\mu \nabla_\nu f' + \Box f' &= 0 \quad (1.4a) \\
 3\Box f' + f'\mathcal{R} - 2f &= 0 \quad (1.4b)
\end{align}
where $f'(\mathcal{R}) \equiv df(\mathcal{R})/d\mathcal{R}$: as expected, the new field equations contain higher-order derivatives, and, in particular, forth-order derivatives of the metric tensor appear.

Eq. (1.3) can be cast in a different form, by means of two auxiliary fields $A$ and $B$, which play the role of Lagrange multipliers [23]. Introducing the two Lagrange multipliers $A$ and $B$ allows one to rewrite (1.3) as
\begin{align}
 S &= -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} [B(\mathcal{R} - A) + f(A)] . \quad (1.5)
\end{align}
Variation with respect to $B$ leads to $\mathcal{R} = A$, while variation with respect to $A$ leads to $A = f'^{-1}(B)$ or equivalently $B = f'(A)$, $f'^{-1}$ being the inverse function of $f'$. It is possible to eliminate either $A$ or $B$ from (1.5), thus obtaining
\begin{align}
 S &= -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} [B(\mathcal{R} - f'^{-1}(B)) + f(f'^{-1}(B))] \quad (1.6)
\end{align}
or
\begin{align}
 S &= -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} [f'(A)(\mathcal{R} - A) + f(A)] , \quad (1.7)
\end{align}
respectively. From (1.7), it is clear that, if $f' < 0$, the Universe enters a repulsive-gravity regime. If we assume that gravity should not be repulsive, then we must require $f' > 0$. Equations (1.6) or (1.7) are equivalent, at least from a classical point of view, and are usually referred to as the Jordan-frame action in presence of the two auxiliary fields.

**Lagrangian approach for the FRW model** As an application of the generalized gravitational theory discussed in the previous section, let’s now consider the following FRW line element
\begin{align}
 ds^2 &= N(t)^2 dt^2 - a(t)^2 dl^2 , \quad (1.8)
\end{align}
$N(t)$ being the lapse function, $a(t)$ the cosmic scale factor of the Universe and $dl^2$ reading
\begin{align}
 dl^2 &= \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (1.9)
\end{align}
with \(0 < r < 1\), \(0 \leq \theta < \pi\), \(0 \leq \phi < 2\pi\) and \(k = 0, \pm 1\) denoting the sign of the spatial curvature [24].

Taking into account the homogeneity request (which implies the energy density \(\epsilon = \epsilon(t)\) and the pressure \(p = p(t)\)) and applying to a fixed volume of the expanding Universe the first thermodynamical principle, \(dU = \delta Q - pdV\), (with the isentropic character of the Universe, \(\delta Q = 0\)), we provide the following relation

\[
d(\epsilon a^3) = -3pa^2 da. \tag{1.10}
\]

In view of the homogeneity of the space-time, the action for the FRW model reduces to a 2-dimensional problem: in fact, the action for the cosmic scale factor \(a(t)\) and the lapse function \(N(t)\) reads

\[
S = -\frac{Vc^4}{16G\pi} \int dt N a^3 f(R) - V \int dt N a^3 \epsilon(a), \tag{1.11}
\]

where \(V\) is the volume of the space portion on which the action is taken.

So far, varying this action with respect to \(N\), we obtain

\[
a^3 f + Na^3 f' \frac{\partial R}{\partial N} + \frac{16G\pi}{c^4} \epsilon - 6f'' \frac{dR}{dt} \frac{a^2 \dot{a}}{N^2} - 6f' \left[ \frac{a^2 \ddot{a}}{N^2} + 2 \frac{a \dot{a}^2}{N^2} - 2 \frac{a^2 \dot{\dot{a}} \dot{N}}{N^3} \right] = 0, \tag{1.12}
\]

and, in the synchronous reference, \(N = 1\),

\[
\frac{1}{2} f + 3f' \frac{\ddot{a}}{a} - 3f'' \frac{dR}{dt} \frac{\dot{a}}{a} = -\frac{8G\pi}{c^4} \epsilon, \tag{1.13}
\]

which is the same as the 00-component of the generalized Einstein equations for the FRW metric, and reduces to the standard Friedmann equation when \(f(R) \equiv R\).

On the other hand, variation with respect to \(a\) leads to the generalized Euler-Lagrange equation

\[
\frac{\partial L}{\partial a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{a}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{a}} = 0. \tag{1.14}
\]

The validity of this equation requires that \(\delta \dot{a}\) vanish on the boundaries; nevertheless, a large class of variation functions is still available for the calculation. Expressing (1.14) for \(N = 1\), we find

\[
-\frac{1}{2} f + f' \left[ -\frac{\ddot{a}}{a} - 2 \frac{\dot{a}^2}{a^2} - 2 \frac{k}{a^2} \right] + 2f'' \frac{dR}{dt} \frac{\dot{a}}{a} + f''' \left( \frac{dR}{dt} \right)^2 + f'' \frac{d^2 R}{dt^2} = -\frac{8G\pi}{c^4} p \tag{1.15}
\]

which coincides with the \(ii\)-components of the generalized Einstein equations in the FRW metric. We stress that the equation above has been obtained making use of the
continuity equation (1.10).
Equations (1.13) and (1.15) describe the whole dynamics of the FRW Universe in a synchronous reference frame, when the gravitational Lagrangian is generalized, as in (1.10).
Finally, combining together (1.13) and (1.15), we can restate, for our general case, the well known equation for the universe acceleration
\[
\frac{1}{6} \left( f''(dR/dt)^2 + 3f'' \left[ \frac{d^2R}{dt^2} + \frac{\dot{a}}{a} \frac{dR}{dt} \right] + f'(-\frac{\dot{a}^2}{a^2} - 2k/a^2) - f \right) = -\frac{4\pi G}{3c^4}(\epsilon+3p). \tag{1.16}
\]
The results obtained in this section are at the ground of our cosmological investigation based on the generalized gravitational action [25].
To conclude, we note that (1.13), (1.15) and (1.10) are among them correlated; in fact, as it can be easily checked after straightforward calculation, differentiating (1.13) and using the continuity equation (1.10), we generate (1.15). Thus, as in the standard case, here we deal with the three unknowns $\epsilon$, $p$ and $a$ and two independent equations only arise for them: as a consequence, to develop a solution of our generalized FRW dynamics, the equation of state $p = (\gamma - 1)\epsilon$ is required, i.e.,
\[
\epsilon(a) = \mathcal{C}a^{-3\gamma}, \mathcal{C} = const., \tag{1.17}
\]
\[
p = (\gamma - 1)\mathcal{C}a^{-3\gamma}, \tag{1.18}
\]
$\gamma$ being the polytropic index.

**Einstein Frame**

It is possible to demonstrate that the non-linear gravitational Lagrangian (1.3) in the Jordan frame can be cast in a dynamically-equivalent form, i.e. the action for a scalar field in GR (with a rescaled metric) in the Einstein frame, by means of a suitable conformal transformation [26, 27] of the metric tensor.

We consider the following conformal scaling of the metric tensor
\[
g_{\mu\nu} \rightarrow e^{\varphi}g_{\mu\nu}. \tag{1.19}
\]
For the particular choice $\varphi = -\ln f'(A)$, action (1.7) reads
\[
s = -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \frac{3}{2} g^{\rho\sigma} \partial_{\rho} \varphi \partial_{\sigma} \varphi - V(\varphi) \right], \tag{1.20}
\]
where
\[ V(\varphi) = \frac{A}{f'(A)} - \frac{f(A)}{f'(A)^2}, \tag{1.21} \]
i.e. this action describes a scalar field minimally-coupled to the rescaled metric. We remark that the definition \( \varphi = -\ln f'(A) \) requires \( f' > 0 \). The analogy is fully completed if the further transformation \( \varphi \rightarrow k\phi \), where \( k \equiv \sqrt{\frac{16\pi G}{3c^3}} \), is performed. In fact, this transformation accounts for the right dimensions of a scalar field.

The analysis of the potential \( V(\phi) \) is relevant in gaining insight onto the parameters that appear in the expression of \( f(R) \). More precisely, the appearance of a minimum is expected to become crucial in the dynamics of the scalar field. In fact, in the FRW cosmological model, the total energy density of the scalar field follows the relation
\[ \frac{d}{dt} \left( \frac{\dot{\varphi}^2}{2c^2} + V(\varphi) \right) = -3H\frac{\dot{\varphi}^2}{c^2} < 0. \]
If we assume an expanding universe, \( H > 0 \), then, as a consequence, starting with a given value of the energy density, sooner or later, the friction due to the universe expansion settles down the scalar field near its potential minimum [28].

If a matter fluid is taken into account, the pertinent stress-energy tensor \( T_{\mu\nu} \) associated to the energy density \( \epsilon \), the pressure \( p \) and the four-velocity \( u_{\mu} \), \( T_{\mu\nu} = (\epsilon + p)u_{\mu}u_{\nu} - pg_{\mu\nu} \), has to be rescaled as
\[ T_{\mu\nu} \rightarrow e^{-\phi}T_{\mu\nu}, \quad T^{\mu}_{\quad \nu} \rightarrow e^{-2\phi}T^{\mu}_{\quad \nu}, \tag{1.22} \]
according to the conformal transformations induced by (1.19), i.e.,
\[ u_{\mu} \rightarrow e^{\phi/2}u_{\mu}, \quad \epsilon \rightarrow e^{-2\phi}\epsilon, \quad p \rightarrow e^{-2\phi}p. \tag{1.23} \]

Nevertheless, the interpretation of the equivalence between the two models gives rise to some remarks about the physical meaning of the transformation [29]. In the Jordan frame, gravity is described by the metric tensor only, while, in the Einstein frame, the rescaled metric tensor experiences the scalar field as a source matter field. These considerations entail the discussion of the role of matter fields coupled to gravity. In fact, non-linear theories of gravity in vacuum [30] leave room for ambiguity about which frame should be considered as the physical one, while the presence of matter fields sheds light on this indistinctness by the request of a minimal coupling with
gravity. The mathematical equivalence between the two theories is achieved dynamically, since the spaces of the classical (on shell) solutions are locally isomorphic. Without aiming at solving this interpretative ambiguity [31], throughout this paper we will try to investigate the role and the properties of matter field [32] in the determination of cosmological solutions [33, 34].

**Back transformations**

It is anyhow possible to go the other way round in order to investigate the expression of some potentials in the Jordan frame, by solving (1.21) for the unknown $f(R)$ (PL9). It is worth noting that the condition $f'' \neq 0$ has to be imposed to keep results consistent from a mathematical point of view; it automatically rules Einsteinian gravity (plus a cosmological constant) out of our investigation. (The condition $f \neq 0$ excludes trivial solutions). Because of the definition of (1.21), it is not always possible to obtain an explicit function $f(R)$, and, in most cases, only a parametric solution is obtained. The preliminary analysis of the simple case of a quadratic potential [35], $V(\phi) = m\phi^2$, restricts the range of potentials that can be studied in this perspective. In fact, in this case, (1.21) reads

$$f'R - f = \frac{2}{3}mf'^2(\log f')^2 :$$

(1.24)

differentiating with respect to $R$ and introducing the auxiliary function $f' = p$, one obtains the system

$$R = \frac{4}{3}mp(\log p) (1 + \log p),$$

$$f = \frac{2}{3}mp^2(\log p) (2 + \log p),$$

(1.25)

where $C$ is an arbitrary integration constant. From this example, we can see that there is a large class of inflationary potentials [36], which can not be solved with respect to the first derivative explicitly, and are therefore unsuitable for our investigation. However, the exponential case [37] offers a great variety of applications. For

$$V(\phi) = \alpha e^{-\frac{\sqrt{2}}{\lambda} \phi},$$

(1.26)

(1.21) rewrites

$$f'R - f = \alpha f'^2 e^{\lambda \log f'} \equiv \alpha f'^q,$$

(1.27)
1.1 Introduction

with \( q \equiv \lambda + 2 \): differentiating with respect to \( R \), we find the general solution

\[
f = \frac{1}{(\alpha q)^{\frac{1}{q-1}}} q - \frac{1}{q} R^{1+\frac{1}{q-1}} + C = \frac{1}{(\alpha q)^{\frac{1}{q-1}}} q - \frac{1}{q} R^{\frac{q}{q-1}} + C, \tag{1.28}
\]

where \( C \) is an arbitrary integration constant: we recover the original potential only for \( C \equiv 0 \). This solution is consistent for \( q \neq 0 \) and \( q \neq 1 \), i.e., \( \lambda \neq -2 \) and \( \lambda \neq -1 \), respectively. This solution is also consistent with the condition \( f'' \neq 0 \), as \( \frac{q}{q-1} \neq 1 \ \forall q \), and a linear gravitational Lagrangian, excluded by construction, is not a general solution for (1.27).

Contrastingly, it’s worth remarking that the special case \( \lambda = 0 \), i.e., a constant potential \( V(\phi) = \alpha \), is provided by both a linear gravitational Lagrangian plus a cosmological constant [38], and \( f = R^2/(4\alpha) \).

Finally, the trivial case \( V(\phi) = 0 \), obtained for \( \alpha = 0 \ \forall \lambda \), is provided only by a linear gravitational Lagrangian without a cosmological constant.

We can recover information about the cases \( \lambda = -2 \), \( \lambda = -1 \), \( C \neq 0 \) by studying the slightly different class of potentials [39]

\[
V(\phi) = \alpha_1 e^{-\sqrt{\lambda_1} \phi} + \alpha_2 e^{-\sqrt{\lambda_2} \phi}. \tag{1.29}
\]

Potential (1.29) is not in general solvable with respect to the first derivative explicitly, but admits the formal solution

\[
R = \alpha_1 (2 + \lambda_1) p^{1+\lambda_1} + \alpha_2 (2 + \lambda_2) p^{1+\lambda_2},
\]

\[
f = \alpha_1 (1 + \lambda_1) p^{2+\lambda_1} + \alpha_2 (1 + \lambda_2) p^{2+\lambda_2} + C, \tag{1.30}
\]

where, as above, \( C \) is an arbitrary integration constant, \( p \equiv f' \), and \( f'' \neq 0 \).

Some particular cases admit nevertheless an explicit solution. In fact, for \( \lambda_2 = -2 \), we find

\[
f = \frac{1}{(\alpha_1 r)^{\frac{1}{r-1}}} \frac{r - 1}{r} R^{1+\frac{1}{r-1}} + \alpha_2, \tag{1.31}
\]

while, for \( \lambda_2 = -1 \), we obtain

\[
f = \frac{1}{(\alpha_1 r)^{\frac{1}{r-1}}} \frac{r - 1}{r} (R - \alpha_2)^{1+\frac{1}{r-1}}, \tag{1.32}
\]

where, as above, \( f'' \neq 0 \) and \( r \equiv 2 + \lambda_1 \): \( r \neq 0 \) and \( r \neq 1 \), i.e., \( \lambda_1 \neq -2 \) and \( \lambda_1 \neq -1 \), respectively.

For \( \lambda_1 = 0 \) of (1.29), (1.21) is not solvable explicitly, and, differently from the previous
result, a linear gravitational Lagrangian is not a solution.

Particular cases of (1.29) can be analyzed. For $\lambda_1 = -\lambda_2 = 2 \equiv \lambda$, we find the solution $f = \frac{1}{(4\alpha_1)^{1/3}} \frac{2}{3} R^{4/3} + \alpha_2$, while, for $\lambda_1 = -\lambda_2 = 1 \equiv \lambda$, $f = \frac{1}{3\alpha_1^{3/2}} (R - \alpha_2)^{3/2}$. For $\alpha_1 = \alpha_2 \equiv \alpha$, the hyperbolic-cosine potential [39] [40] is recovered.

The Newtonian limit of such scenarios can be examined. In fact, for (1.26), the limit $q \to \infty$, i.e., $\lambda \to \infty$, induces a vanishing potential, and solution (1.28) gives the proper limit, $f \propto R$, i.e., a gravitational Lagrangian linear in the Ricci scalar, without a cosmological constant, which is also achieved by imposing $\alpha \equiv 0$ in (1.28). From a physical point of view, for $\lambda \gg 1$, (1.28) reduces to $f(r) \propto R^{1+\epsilon(\lambda)}$, with $\epsilon(\lambda) = 1/(1 + \lambda) << 1$, and (1.26) becomes negligible for small values of $\phi$. Furthermore, as discussed above, there are two classes of functions that can reproduce a constant potential, i.e., both a linear gravitational Lagrangian plus a cosmological constant and $f = R^2/(4\alpha)$. Anyhow, a quadratic gravitational Lagrangian can be shown to be inappropriate to fit solar-system data [41, 12].

### 1.1.2 Weak-field limit

The metric tensor $g_{\mu\nu}$ can be decomposed as a flat metric $\eta_{\mu\nu} \equiv (1, -1, -1, -1)$ plus a small perturbation $h_{\mu\nu}$, $h_{\mu\nu} \ll 1$, i.e. $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$.

Assuming a spherical symmetry, as fully discussed in [42], the most general expression for the metric tensor $g_{\mu\nu}$ is

\[
\begin{align*}
g_{tt} &= 1 + \Phi(r) \\
g_{rr} &= -1/(1 + \Psi(r)) \sim -1 + \Psi(r) \\
g_{\theta\theta} &= -r^2 \\
g_{\phi\phi} &= -r^2 \sin^2 \theta.
\end{align*}
\]

In the weak-field limit, the expansion (1.33) is assumed to hold.

The weak-field limit of EH gravity is obtained from the set of Einstein equations (1.1) considering only the first-order terms in $h$, thus neglecting the $O(h^2)$ contributions, because of the form of (1.2).

The weak-field limit of $f(R)$ gravity can be investigated, without loss of generality, according to the following scheme.
If \( f(R) = R + F \), where \( F \) is a generic function of the Ricci scalar and/or some other curvature scalars, the weak-field limit is recovered following the same procedure and the same approximation paradigm, since terms of order \( O(h) \) are always present by construction. This way, terms of order \( O(h^2) \) have to be neglected, and all other contributions lead to a weak-field expression whose comparison with the EH case is consistent.

The case of \( f(R) = a_1 R + a_2 R^2 + F \), where \( a_1 \) is dimensionless, \( a_2 \) has the dimension of length\(^2\) and \( F \) is a generic function of \( R \) and/or some other curvature scalars, has already inspired a large amount of work [43, 44, 45]. In the paper by [46], an \( R_{\mu\nu} R^{\mu\nu} \) term was also considered, \( a_1 \) was set equal to one, and a well-defined Newtonian limit was found. The particular choice of \( F = 0 \), i.e. a modified action \( f(R) = a_1 R + a_2 R^2 \) has also been widely investigated. For this modified action in the Jordan frame to reproduce an attractor in the Einstein frame, it is easily verified that \( a_1 > 0 \) and \( a_2 < 0 \). For another approach to the definition of the sign of these parameters, see, for example, [40] and all the literature referring to it. Applying the same scheme outlined in [46] for this simplified case, the following solution is found

\[
R = c_1 \frac{e^{ar}}{r} + c_2 \frac{e^{-ar}}{r} \tag{1.34a}
\]

\[
g_{00} = 1 + \tilde{\Phi}(r) \equiv 1 + \frac{c_1}{3} \frac{e^{ar}}{a^2 r} + \frac{c_2}{3} \frac{e^{-ar}}{a^2 r} + c_3 + \frac{1}{r} \tag{1.34b}
\]

\[
g_{rr} = -1 + \tilde{\Psi}(r) \equiv -1 + \frac{c_1}{3} \frac{e^{ar}}{a^2} \left( a - \frac{1}{r} \right) - \frac{c_2}{3} \frac{e^{-ar}}{a^2} \left( a + \frac{1}{r} \right) + c_4 \frac{1}{r} \tag{1.34c}
\]

where \( a \equiv +\sqrt{-a_1/6a_2} \) has the dimension of length\(^{-1}\), \( c_1 \) and \( c_2 \) are two integration constants with the dimension of length\(^{-1}\), \( c_4 \) is an integration constant with the dimension of length, and \( c_3 \) is a dimensionless integration constant, which can be set equal to zero.

These results have also been generalized for the case of a generic gravitational action \( f(R) \) analytical in the point \( R = 0 \) [47]. Because this function is analytical in \( R = 0 \), its series expansion in the vicinity of this point holds, and reads \( f(R) = \sum_{j=0}^{\infty} f_j R^j \), where the \( f_j \)'s are the \( j \)-th order Taylor coefficients of the series and have the dimension of length\(^{2j-2}\). Even though an infinite number of parameters has, in principle, to be fixed [48], the Newtonian limit of this model was shown to depend only on the parameters \( f_1 \) and \( f_2 \), for vanishing \( f_0 \).
If, on the contrary, the function \( f(R) \) is not described by such an expression, other schemes have to be followed.

### 1.2 Non-analytic modifications of the Einstein-Hilbert Lagrangian

On the one hand, the most immediate generalization is of course to deal with a function of the Ricci scalar analytic in the point \( \mathcal{R} = 0 \) (so that its Taylor expansion holds \([47]\)). This approach is equivalent to deal with a polynomial form \([46, 43, 44, 45]\), whose free parameters are available to fit the observed phenomena on different sectors of investigation. On the other hand, despite the appealing profile of such a choice, it is extremely important to observe that it is not the most general case (PL1), since real (non-integer) exponents of the Ricci scalar are in principle on the same footing as the simplest case \([49, 50, 23]\).

In the following, we will concentrate our attention on such an open issue, and we will develop a modified theory of the form \( f(\mathcal{R}) = \mathcal{R} + \gamma \mathcal{R}^\beta \), where \( \gamma \) and \( \beta \) are two free parameters to be constrained at physical level \([13, 14, 15, 16, 17, 18, 19, 20, 21, 12]\). In particular, the details of our model will lead us to deal with rational non-integer numbers for \( \beta \), and to restrict it in the most appropriate interval for physical interpretation at low curvature, \( 2 < \beta < 3 \).

We analyze the weak-field limit of our modified Lagrangian and derive the corresponding spherically-symmetric field equations. Here we retain only those non-integer powers in the Ricci scalar, which represent the dominant effect after the linear approximation. The choice \( 2 < \beta < 3 \) allows us to distinguish the dominant non-Einsteinian terms from the non-linear ones of GR. The explicit solution of the derived system is found, and its main features are analyzed in view of possible constraints from observational data \([21, 13, 14, 15, 16, 17, 18, 19, 20, 21, 51, 52, 53, 54, 55, 56, 57]\).

The main issue of our treatment is to demonstrate that our \( f(\mathcal{R}) \) model is appropriate both to fulfill Solar-System constraints and to provide a significant break-down of the Newton law at galactic scales.

The analysis is organized as follows. We provide the details of our model both in the
Jordan and in the Einstein frame. Then, we analyze the weak-field limit of the $f(\mathcal{R})$ theory in the Jordan frame, both in vacuum and in presence of matter. We discuss the uniqueness of the solutions and match the interior and the exterior solutions on the surface of the central mass (in a special case). Phenomenological estimations fix the constraints on the model and outline its validity at Solar-system scales. Our results are compared with those of other models, such as [41] and related literature. Discussion and conclusions follow. We consider the following modified gravitational action

$$S = -\frac{\rho^3}{16\pi G} \int d^3 x \sqrt{-g} f(\mathcal{R}) = -\frac{\rho^3}{16\pi G} \int d^3 x \sqrt{-g} (\mathcal{R} + \gamma \mathcal{R}^\beta), \quad (1.35)$$

where $\beta$ is dimensionless, and the parameter $\gamma$ has the dimensions of length $^{2\beta-2}$. We can define the characteristic length scale of our model as $L_\gamma \equiv |\gamma|^{1/(2\beta-2)}$.

A function $f(\mathcal{R})$ is analytic when it is an infinitely-differentiable function such that the Taylor series at any point $\mathcal{R}_0$ in its domain converges to $f(\mathcal{R})$ for $\mathcal{R}$ in a neighborhood of $\mathcal{R}_0$. It is then straightforward to verify that (1.35) is non-analytic in $\mathcal{R} = 0$ for non-integer, rational $\beta$, but still it is well-behaved in the limit $\mathcal{R} \to 0$. Similar models have been investigated, for example, in [50, 49], mostly from a cosmological point of view; in [23], the possibility of an irrational exponent is also envisaged. We stress that a similar behaviour is exhibited also by models such as $f(\mathcal{R}) = \mathcal{R} + \mathcal{R} \ln \mathcal{R}$, as in [58], where the modification of the EH term is non-analytical too, but it is not expressed as a power of the Ricci scalar.

It is interesting to remark that, for models like $f(\mathcal{R}) = 1/\mathcal{R}$ or $f(\mathcal{R}) = \mathcal{R} - \mu^4/\mathcal{R}$, as in [59], $f(0)$ is a diverging quantity, while (1.35) is not.

From a physical point of view, we address non-analytic modifications of the EH action, which vanish in the limit of small values of the space-time curvature.

We can gain further information on the value of $\beta$ by analyzing the conditions that allow for a consistent weak-field stationary limit in the Jordan frame. Furthermore, the study of the potential of the scalar field in the Einstein frame will define the whole parameter space $\{\beta \gamma\}$. 

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1.2.1 Jordan frame

Having in mind to investigate the weak-field limit of our theory to pursue its prediction at Solar-System scales, we can decompose the corresponding metric as \( g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \), where \( h_{\mu\nu} \) is a small (for our case, static) perturbation of the Minkowskian metric \( \eta_{\mu\nu} \). In this limit, the Einstein equations in the Jordan frame read

\[
R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = -\gamma R^{3-1};_{\mu\nu} + \gamma \beta \eta_{\mu\nu} \Box R^{3-1} = 0, \tag{1.36a}
\]
\[
R = 3\gamma \beta \Box R^{3-1}, \tag{1.36b}
\]

where \( (;) \) denotes covariant differentiation, and, for the sake of compactness, in this treatment we retain the usual notation of the curvature scalar \( R \) and the Ricci tensor \( R_{\mu\nu} \) calculated up to the weak-field limit, as we are evaluating the \( \mathcal{O}(h) \) orders.

The structure of such field equations lead us to focus our attention on the restricted region of the parameter space \( 2 < \beta < 3 \). This choice is enforced by the fulfillment of the conditions by which all other terms are negligible with respect to the linear and the lowest-order non-Einsteinian ones.

1.2.2 Einstein frame

In the Einstein frame, according to (1.21), the potential \( V(\varphi) \) describing the dynamics of the scalar field reads

\[
V(\varphi) = e^{2k\varphi} (-1 + \beta) \left( \frac{-1 + e^{-k\varphi}}{-1 + \beta} \right)^{-1 + \beta} \gamma \tag{1.37}
\]

The appearance of a minimum in the potential is crucial, since the cosmological implementation of this picture into an isotropic and homogeneous Universe suggests one that such a minimum becomes, sooner or later, an attractive stable configuration for the system. In fact, the energy density of the scalar field is monotonically damped by the Universe expansion and the later stages of the Universe must be compatible with a weak perturbation of the GR scheme. As a matter of fact, a constant value of \( \varphi \) corresponds to an Einsteinian dynamics and its extremal-point character ensures the nearby stability of this configuration.

From the analysis of the scalar potential, we see that, for \( 2 < \beta < 3 \), there are two extremal points

\[
\varphi_0 = 0, \quad \varphi_{\beta} = \frac{1}{k} \log \left( \frac{-2 + \beta}{2(-1 + \beta)} \right),
\]

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and $V(\varphi)$ behaves like an attractor only for the cases

1. $\gamma > 0$, $\beta > 2$, $\beta = \frac{2n}{2m+1}$, $m, n \in \mathbb{Z}$: $\varphi_0$ is a minimum, $\varphi_\beta$ is a maximum;

2. $\gamma < 0$, $\beta > 2$, $\beta = \frac{2n}{2m+1}$, $m, n \in \mathbb{Z}$: $\varphi_0$ is a maximum, $\varphi_\beta$ is a minimum.

### 1.2.3 Weak-field limit

We can now study the weak-field stationary limit of the model in the Jordan frame.

The metric tensor $g_{\mu\nu}$ can be decomposed as a flat metric $\eta_{\mu\nu} \equiv (1,-1,-1,-1)$ plus a small perturbation $h_{\mu\nu}$, $h_{\mu\nu} \ll 1$, i.e. $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$.

The weak-field limit of EH gravity is obtained from the set of Einstein equations considering only the first-order terms in $h$, thus neglecting the $\mathcal{O}(h^2)$ contributions. In the case of $f(\mathcal{R}) \sim \mathcal{R} + \mathcal{R}^2$, and therefore also for an analytic $f(\mathcal{R})$, the Newtonian limit is recovered following the same procedure and the same approximation paradigm, since the non-Einsteinian contributions resum exactly to the $\mathcal{O}(h)$ terms [46, 43, 44, 45, 47].

From the analysis of the form of (1.36a)-(1.36b), we learn that it is possible to find a weak-field solution in the Jordan frame by solving the set of (1.36a-1.36b) up to next-to-leading order in $h$, i.e. up to $\mathcal{O}(h^{\beta-1})$, and neglecting the $\mathcal{O}(h^2)$ contributions only for the cases $2 < \beta < 3$.

It is important to remark that for the model $f(\mathcal{R}) \sim \mathcal{R}^n$, as in [41], field equations in the weak-field limit are solved according to none of these paradigms. In fact, terms of order $\mathcal{O}(h^n)$ and of order $\mathcal{O}(h^{n-1})$ appear, and one cannot in principle consider small perturbations to the Einsteinian leading terms [60].

These considerations motivate our claim concerning the choice of the function and the restriction of the parameter $\beta \in (2,3)$.
Vacuum Solution

The most general spherically-symmetric line element in the weak-field approximation reads

\[ ds^2 = (1 + \Phi)dt^2 - (1 - \Psi)dr^2 - d\Omega^2, \quad (1.38) \]

where \( \Phi \) and \( \Psi \) are the two generalized gravitational potentials, and \( d\Omega^2 \) is the solid-angle element.

Within this framework, the modified Einstein equations (1.36a)-(1.36b) rewrite

\[
\begin{align*}
R_{tt} - \frac{1}{2} R - \gamma \beta \nabla^2 R^{\beta-1} &= 0 , \\
R_{rr} + \frac{1}{2} R - \gamma \beta R^{\beta-1} ,_{rr} + \gamma \beta \nabla^2 R^{\beta-1} &= 0 , \\
R_{\theta\theta} + \frac{1}{2} r^2 R - \gamma \beta r R^{\beta-1} ,_{r} + \gamma \beta r^2 \nabla^2 R^{\beta-1} &= 0 , \\
R &= -3 \gamma \beta \nabla^2 R^{\beta-1},
\end{align*}
\]

where

\[
\begin{align*}
R &= \nabla^2 \Phi + \frac{2}{r^2} (r \Psi) ,_r , \\
R_{tt} &= \frac{1}{2} \nabla^2 \Phi , \\
R_{rr} &= -\frac{1}{2} \Phi ,_{rr} - \frac{1}{r} \Psi ,_r , \\
R_{\theta\theta} &= -\Psi - \frac{r}{2} \Phi ,_r - \frac{r}{2} \Psi ,_r , \\
R_{\phi\phi} &= \sin^2 \theta R_{\theta\theta} .
\end{align*}
\]

Here (,) denotes ordinary differentiation and \( \nabla^2 \equiv \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \). We stress again that the system of field equations above contains only \( \Phi \) and \( \Psi \) as fundamental variables, while the weak-field limit of the curvature scalar \( R \) is simply retained to reduce the order of the system and to allow for a straightforward discussion of the uniqueness of the solution.

System (1.39a)-(1.39d) is solved by

\[
\begin{align*}
R &= Ar^{\frac{2}{\beta-2}}, \\
\Phi &= \sigma + \frac{\delta}{r} + \Phi_\beta \left( \frac{r}{L_\gamma} \right)^{2\beta-1}_{\beta-2}, \\
\Psi &= \frac{\delta}{r} + \Psi_\beta \left( \frac{r}{L_\gamma} \right)^{2\beta-1}_{\beta-2},
\end{align*}
\]

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where

\[ A = \left[ -\frac{6\gamma\beta(3\beta-4)(\beta-1)}{(\beta-2)^2} \right]^{1/2-\beta}, \quad \text{(1.42a)} \]

\[ \Phi_\beta \equiv \left[ -\frac{6\beta(3\beta-4)(\beta-1)}{(\beta-2)^2} \right]^{1/2-\beta} \frac{(\beta-2)^2}{6(3\beta-4)(\beta-1)}, \quad \text{(1.42b)} \]

\[ \Psi_\beta \equiv \left[ -\frac{6\beta(3\beta-4)(\beta-1)}{(\beta-2)^2} \right]^{1/2-\beta} \frac{(\beta-2)}{3(3\beta-4)}, \quad \text{(1.42c)} \]

the integration constant \( \delta \) has the dimensions of length, and the dimensionless integration constant \( \sigma \) can be set equal to zero. The integration constant \( A \) has the dimensions of \( \text{length}^{2-\beta} \), and \( \Phi_\beta \) and \( \Psi_\beta \) are dimensionless, accordingly.

It is worth remarking that he trace equation (1.39d), solved by (1.41a), admits also the trivial solution \( R = 0 \). By the substitution \( R = (u(r)/r)^{1/(\beta-1)} \), (1.39d) rewrites as an Emden-Fowler equation, i.e.

\[ \frac{d^2u}{dr^2} = -\frac{1}{3\gamma\beta} u^{1/(\beta-1)} r^{\frac{\beta-2}{\beta-1}}. \quad \text{(1.43)} \]

Emden-Fowler equations admit a general power-law solution, and, for some appropriate exponents of the variables, also a particular (parametric) solution can be found. However, for the Emden-Fowler equation (1.43), no particular non-trivial (parametric) solution for the functional dependence of (1.43) on \( \beta \) is listed in the literature, so that solution (1.41a) is unique [51, 52, 53, 54, 55, 56].

The remaining field equations contain first- and second-order derivatives of the generalized gravitational potentials. Dealing with a solution that retains a Newton-like term (i.e. determined by proper Cauchy conditions, as in the next subsection) uniquely fixes the functional form of \( \Phi \) and \( \Psi \) once the trace solution is taken into account [51, 52, 53, 54, 55, 56].

Since the expression of \( \Phi \) and \( \Psi \) diverge for large \( r \), a validity range for the weak-field solution has to be defined (as discussed in the next section). Within this range, in the Newtonian approximation, the quantity \( c^2\Phi/2 \) has to be identified with the Newtonian potential, as geodesic motion has to depend only on the Christoffel coefficient \( \Gamma^r_{00} = \Phi_r/2 \). As a result, the integration constant \( \delta \) has to be set as \( \delta \equiv -r_S \), where \( r_S = 2GM/c^2 \) is the Schwarzschild radius of a central mass \( M \), being \( G \) the gravitational constant.
Moreover, we find that $A$ is well-defined only in the case $\gamma < 0$, $\beta = 2n/(2m + 1)$, i.e. only for configuration (2) of the scalar-tensor description. In this case, $A > 0$. It is remarkable that the limit $\gamma \to 0$, apparently restoring GR [61], is mapped, in the solution, into the opposite case $\gamma \to \infty$ [46]. Indeed, this situation in which the typical length scale associated to the parameter $\gamma$ is arbitrarily large would correspond to the physical intuition of an arbitrarily large scale for the validity of this solution.

**Dust Solution**

Even though Solar-System experiments take place in vacuum, it is interesting to consider an interior solution.

We remark that, in $f(\mathcal{R})$ theories, the presence of matter implies a differential relation between the Ricci scalar and the trace of the stress-energy tensor rather than an algebraic one, as in EH gravity, in the trace equation. Furthermore, the remaining field equations can be arranged as effective Einstein ones, with an effective stress-energy tensor, which contains also geometrical terms. Nevertheless, such an effective stress-energy tensor does not obey any energy condition, and the effective energy density is not, in general, positive definite [62, 63].

In presence of matter, field equations (1.36a) and (1.36b) rewrite

\begin{align}
R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R - \gamma \beta R^{\beta-1} ;_{\mu\nu} + \gamma \beta \eta_{\mu\nu} \Box R^{\beta-1} &= \frac{8\pi G}{c^4} T_{\mu\nu} \quad (1.44a) \\
3\gamma \beta \Box R^{\beta-1} - R &= \frac{8\pi G}{c^4} T, \quad (1.44b)
\end{align}

where $T_{\mu\nu}$ is the stress-energy tensor associated with external matter, and $T = g^{\mu\nu} T_{\mu\nu}$.

For our purposes, we will examine the particular case of a dust central mass of radius $\rho$, for which $T_{\mu\nu} = \epsilon u_{\mu} u_{\nu}$, with an energy density $\epsilon$ such that $\epsilon = \epsilon_0 r^{2/(\beta-2)}$ for $r \leq \rho$, $\epsilon = 0$ for $r > \rho$, $\epsilon_0$ constant. This way, it will be possible to find an analytic expression for the solutions of field equations.
In fact, in this case, field equations in the weak-field limit rewrite
\[
R_{tt} - \frac{1}{2} R - \gamma \beta \nabla^2 R^{3-1} = \frac{8\pi G}{c^4} \epsilon_0 r^{\frac{2}{3-2}}, \tag{1.45a}
\]
\[
R_{rr} + \frac{1}{2} R - \gamma \beta R^{3-1,rr} + \gamma \beta \nabla^2 R^{3-1} = 0 , \tag{1.45b}
\]
\[
R_{\theta\theta} + \frac{1}{2} r^2 R - \gamma \beta r R^{3-1,rr} + \gamma \beta r^2 \nabla^2 R^{3-1} = 0 , \tag{1.45c}
\]
\[
3\gamma \beta \nabla^2 R^{3-1} = -R - \frac{8\pi G}{c^4} \epsilon_0 r^{\frac{2}{3-2}}, \tag{1.45d}
\]
and are solved by
\[
R_{\text{mat}} = B r^{\frac{2}{3-2}}, \tag{1.46a}
\]
\[
\Phi_{\text{mat}} = \sigma' + \frac{\delta'}{r} + \Phi_{\beta mat} \left( \frac{r}{L_\gamma} \right)^{\frac{2}{3-2}}, \tag{1.46b}
\]
\[
\Psi_{\text{mat}} = \frac{\delta'}{r} + \Psi_{\beta mat} \left( \frac{r}{L_\gamma} \right)^{\frac{2}{3-2}}, \tag{1.46c}
\]
where
\[
\Phi_{\beta mat} \equiv \frac{(\beta - 2)^2}{6(\beta - 1)(3\beta - 4)} \left( B + \frac{8\pi G}{c^4} \epsilon_0 \right) L_\gamma^{\frac{2}{3-2}}, \tag{1.47a}
\]
\[
\Psi_{\beta mat} \equiv \frac{\beta - 2}{3(\beta - 4)} \left( B - \frac{8\pi G}{c^4} \epsilon_0 \right) L_\gamma^{\frac{2}{3-2}}, \tag{1.47b}
\]
and $B$ is defined as
\[
-3\gamma \beta \frac{2(\beta - 1)(3\beta - 4)}{(\beta - 2)^2} B^{3-1} = B + \frac{8\pi G}{c^4} \epsilon_0. \tag{1.48}
\]

The dimensionless integration constant $\sigma'$ can be set equal to 0, as in the vacuum case, and the integration constant $\delta'$ has to be set equal to 0 for a solution vanishing at $r = 0$.

Differently form the vacuum case, (1.48) does not impose any particular constraint on $\beta$, but only those values of $\beta$ have to be considered, for which the exterior solution is well-defined, in order to match the solutions on the surface of the central mass. Furthermore, the definition of $B$ from (1.48) is not unique. Since $2 < \beta < 3$, it is possible to associate (1.48) to a polynomial equation, whose number of solutions equals, in general, the degree of the polynomial itself. Whether an analytical expression for these solutions can be found depends on the degree of such an equation [57]. The case $\beta = 5/2$, the only value of $\beta$ for which (1.48) can be associated with an
Modified gravity

exactly-solvable (cubic) polynomial, has to be ruled out, since the exterior solution is not definite in this case. Anyhow, \( B = B(\beta, \gamma, \epsilon_0) \) tends to \( A = A(\beta, \gamma) \) as \( \epsilon_0 \) tends to 0. Once a solution of \((1.48)\) is chosen, then the solution of the field equations is unique \([51, 52, 53, 54, 55, 56]\).

Even though the case we analyze is straightforward, it properly allows us to treat the matching between the interior and the exterior solution as a probe of the applicability of our model. Furthermore, we remark that, if any other dependence of the matter distribution on the coordinate \( r \) had been chosen, no analytic expression for the solutions of the field equations could have been found \([51, 52, 53, 54, 55, 56]\).

Matching the exterior solution (and its derivatives) with the interior solution (and its derivatives) at the point \( r = \rho \) allows one to establish the relation between the matter density of the central mass \( \epsilon \), its mass \( M \) (through the Schwarzschild radius), its radius \( \rho \) and the parameter \( \beta \). The typical length scale \( L_\gamma \) is not fixed by this set of equations, and will be determined in the discussion on the viability of the model at Solar-System scales.

As a result, for the generalized gravitational potentials, the following set of conditions if found, which holds once a solution of \((1.48)\) is chosen:

\[
- \frac{r_s}{\rho} + \Phi_\beta \left( \frac{\rho}{L_\gamma} \right)^{\frac{\beta-1}{\beta-2}} = \Phi_{\beta \mathrm{mat}} \left( \frac{\rho}{L_\gamma} \right)^{\frac{\beta-1}{\beta-2}}, \tag{1.49a}
\]

\[
- \frac{r_s}{\rho} + \Psi_\beta \left( \frac{\rho}{L_\gamma} \right)^{\frac{\beta-1}{\beta-2}} = \Psi_{\beta \mathrm{mat}} \left( \frac{\rho}{L_\gamma} \right)^{\frac{\beta-1}{\beta-2}}, \tag{1.49b}
\]

\[
\frac{r_s}{\rho^2} + \Phi_\beta \left( \frac{\rho}{L_\gamma} \right)^{\frac{\beta}{\beta-2}} \left( \frac{\rho}{L_\gamma} \right)^{\frac{1}{\beta-1}} = \Phi_{\beta \mathrm{mat}} 2^{\frac{\beta-1}{\beta-2}} \left( \frac{\rho}{L_\gamma} \right)^{\frac{\beta}{\beta-1}}, \tag{1.49c}
\]

\[
\frac{r_s}{\rho^2} + \Psi_\beta \left( \frac{\rho}{L_\gamma} \right)^{\frac{\beta}{\beta-2}} \left( \frac{\rho}{L_\gamma} \right)^{\frac{1}{\beta-1}} = \Psi_{\beta \mathrm{mat}} 2^{\frac{\beta-1}{\beta-2}} \left( \frac{\rho}{L_\gamma} \right)^{\frac{\beta}{\beta-1}}. \tag{1.49d}
\]

1.2.4 Solar-System constraints

The most suitable arena where to evaluate the reliability and the validity range of the weak-field solution \((1.41a)-(1.41c)\) is, of course, the Solar System \([12, 13, 14, 15, 16, 17, 18, 19, 20, 21]\).

For this reason, we can specify \((1.41b)\) and \((1.41c)\) for the typical length scales involved in the problem. To this end, we split \( \Phi \) and \( \Psi \) of \((1.41b)\) and \((1.41c)\) into two terms,
the Newtonian part and a modification, respectively, i.e.
\[
\Phi \equiv \Phi_N + \Phi_M \equiv -r_s/r + \Phi_\beta(r/L_\gamma)^{2(\beta-1)(\beta-2)},
\]
\[
\Psi \equiv \Psi_N + \Psi_M \equiv -r_s/r + \Psi_\beta(r/L_\gamma)^{2(\beta-1)(\beta-2)},
\]
where we have specified \(\Phi_N\) and \(\Psi_M\) for the Schwarzschild radius of the Sun, \(r_s \equiv 2GM_s/c^2\), \(M_s\) being the Solar mass.

In fact, while the weak-field approximation of the Schwarzschild metric is valid within the range \(r_s << r < \infty\) because it is asymptotically flat, the modification term has the peculiar feature to diverge for \(r \to \infty\). It is therefore necessary to establish a validity range \(r_{min} << r << r_{max}\), where this solution is physically predictive [64]. The definition of this validity range is strictly related to the parameters \(\beta\) and \(L_\gamma\) involved in the model. Stringent constraints for \(L_\gamma\) can be found in the analysis of planetary orbital periods, which represent a severe test within the Solar System, because of the high precision of experimental data.

Validity range

Since we aim to provide a physical picture at least of the planetary region of the Solar System, we are led to require that \(\Phi_M\) and \(\Psi_M\) remain small perturbations with respect to \(\Phi_N\) and \(\Psi_N\), so that it is easy to recognize the absence of a minimal radius except for the condition \(r >> r_s\). A maximum radius \(r_{max}\) appears instead.

In fact, the definition of \(r_{max}\) requires further discussion. The typical distance \(r^*\) corresponds to the request
\[
| \Phi_N(r^*) | \sim \Phi_M(r^*), \quad | \Psi_N(r^*) | \sim \Psi_M(r^*).
\]
For \(r_s << r << r^*\), the system obeys thus Newtonian physics, and experiences the modification term as a correction.

Another maximum distance \(r^{**}\) can be defined, according to the request that the weak-field expansion in (1.38) should hold, regardless to the ratios \(\Phi_M/\Phi_N\) and \(\Psi_M/\Psi_N\). According to our definition of \(r_{min}\), \(r^{**}\) is defined as
\[
| \Phi_N(r^{**}) | << \Phi_M(r^{**}) \sim 1, \quad | \Psi_N(r^{**}) | << \Psi_M(r^{**}) \sim 1.
\]
We remark that that, within this scheme, \( r^* \) and \( r^{**} \) are defined as functions of \( \beta \) and \( L_\gamma \), i.e. \( r^* \equiv r^*(\beta, L_\gamma) \) and \( r^{**} \equiv r^{**}(\beta, L_\gamma) \) respectively

\[
    r^* \sim \left( r_s/\Phi_\beta \right)^{\frac{\beta-2}{3\beta-4}} L_\gamma^{\frac{2\beta-2}{3\beta-4}}, \quad r^{**} \sim L_\gamma/\Phi_\beta^{\frac{2\beta-2}{\beta-2}}.
\]  

(1.53)

It is important to stress that, for the validity of our scheme, the condition \( r^* >> r_s \) must hold, i.e. \( L_\gamma >> \Phi_\beta^{(\beta-2)/(2\beta-2)} r_s \).

**Planetary orbital periods**

At this level, neglecting the lower-order effects concerning the eccentricity of the planetary orbit, we can deal with the simple model of a planet moving on circular orbit around the Sun, and its orbital period \( T \) is given by \( T = 2\pi (r/a)^{1/2}, a \equiv \frac{\beta}{2} d\Phi/dr \) being the centripetal acceleration. For our model, from (1.41b), we get

\[
    T_\beta = \frac{2\pi}{(GM_s)^{1/2}} r^{3/2} \left( 1 + 2 \frac{\beta-1}{\beta-2} \Phi_\beta \frac{r^3}{\beta-2} \right)^{-1/2} L_\gamma^{-2/2}. \tag{1.54}
\]

Hence we evaluate the correction to the Keplerian period \( T_K = 2\pi r^{3/2}(GM_s)^{-1/2} \), compare it with the experimental data of the period \( T_{exp} \) and its uncertainty \( \delta T_{exp} \) and then impose that the correction be smaller than the experimental uncertainty, i.e.

\[
    \frac{\delta T_{exp}}{T_{exp}} \gtrsim \frac{|T_K - T_\beta|}{T_K} \sim \frac{\beta-1}{\beta-2} \Phi_\beta \frac{r_p^{\beta-2}}{r_s L_\gamma^{\beta-2}} \tag{1.55}
\]

where \( r_p \) is the mean orbital distance of a given planet from the Sun.

**Numerical estimations**

We now specify the previous considerations for and intermediate value of \( \beta \), allowed by the analysis of the weak-field limit consistency and the scalar-tensor description. To provide proper numbers, we choose a typical (non-peculiar) value of the parameter \( \beta \), say \( \beta = 8/3 \).

High-precision measurements are nowadays available for the distances between Solar-System planets and the Sun, so that the relative error in the orbital period is extremely small. According to this fact, we specify our analysis for example for the Earth [12], for which \( T_{exp} = 365.256363051 \text{days}, \delta T_{exp} = 5.0 \cdot 10^{-10} \text{days} \) and \( r_p = 149.6 \cdot 10^6 \text{km} =\)
This lower bound for $L_\gamma$ can be plugged into (1.51) and (1.52) to get an estimation of the distances $r^*$ and $r^{**}$. Since the constraints on the $tt$ and the $rr$ components of the metric tensor imply similar conditions, we restrict our analysis to the physically-relevant case of the $tt$ component, and, after direct calculation, obtain $r^* \sim 1.6 \cdot 10^{10} km \sim 1.1 \cdot 10^2 AU$ and $r^{**} \sim 1.5 \cdot 10^{12} km \sim 1.0 \cdot 10^4 AU$. Such values of $r^*$, $r^{**}$ and $L_\gamma$ are essentially stable with respect to the range of $\frac{8}{3} < \beta < 3$. In fact, in the vicinity of the greatest value $\beta = 3$, we would obtain $L_\gamma > 1.3 \cdot 10^{12} km \sim 8.7 \cdot 10^3 AU$, $r^* \sim 4.0 \cdot 10^{10} km \sim 2.7 \cdot 10^2 AU$, $r^{**} \sim 1.3 \cdot 10^{13} km \sim 8.7 \cdot 10^4 AU$.

Our analysis clarifies how the predictions of the corresponding equations for the weak-field limit appear viable in view of the constraints arising from the Solar-System physics. Indeed, the lower bound for $L_\gamma$ does not represent a serious shortcoming of the model, as we are going to discuss.

1.2.5 Comparison with other approaches

To better appreciate the features of this model, it might be useful to recall some aspects of other models, whose implications have been investigated at Solar-System scales.

The model $f(\mathcal{R}) = f_0 \mathcal{R}^n$, where $f_0$ is a constant with the dimensions of $\text{length}^{-2+2n}$, has been analyzed in [41], and in [65] it was shown that the solution of the Einstein field equations is not unique. In particular, in [41], the parameter $n$ has been constrained as $1 < n < 1 + 7.2 \cdot 10^{-19}$ from the combination of observational constraints arising both at Solar-System scales (such as the perihelion precession of Mercury) and at cosmological ones.

For $f(\mathcal{R}) = f_0 \mathcal{R}^n$, the constant $f_0$ factors out of the Einstein equations in vacuum, and, even if dimensionfull, does not contribute to the definition of a characteristic length scale. Contrastingly, the characteristic length $L_\gamma$ is in our scheme defined directly from the modified action, and can account directly for how significantly distant from EH gravity the model is.

Furthermore, in [41], it was shown that the two gravitational potentials in spherical
Modified gravity

symmetry are expressed as the sum of two power laws, which depend crucially on $n$ and tend to the Schwarzschild solution when $n$ tends to 1.

Cosmological implications of this model where further explored in [66], and, in [67], exact Friedmann solutions were found both in vacuum and in presence of a perfect fluid. It is interesting to remark that these results apply also to our model. In fact, as far as primordial cosmological investigation is concerned, for large $\mathcal{R}$, the $\mathcal{R}^\beta$ term dominates the EH term. In vacuum, for $2 < \beta < 3$, and in presence of matter, under suitable hypotheses on the equations of state, which allow one to keep $2 < \beta < 3$, it is therefore straightforward to assume that the early-time description of the Universe is that proposed, for example, in such investigation, without the very stringent constraints on $n$ for $f(\mathcal{R}) = f_0 \mathcal{R}^n$ arising from Solar-System observations.

Furthermore, in [68], it was shown that, taking a solution of the 00 component of the metric tensor in the weak-field limit in the form

$$g_{00} = 1 - \frac{r_s}{r} - \frac{r_s r^{k-1}}{L_c^k}, \quad \beta = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^2 - 4n + 2}, \quad (1.56)$$

a set of the free parameters $n$ and $L_c$, which has the dimensions of length, exists, which allows it to fullfill field equations up to the precision of $10^{-6}$. For $k \sim 0.817$ ($n \sim 3.5$) and $1AU < L_c < 4AU$, this model was illustrated to fit low surface brightness galaxies rotation curves.

These values of $k$ and $L_c$ where proved to be inconsistent with Solar-System tests in [12], but no numerical constraints for $k$ and $L_c$ were obtained.

If we specify our investigation scheme for this case, we evaluate the expression of the planetary orbital period $T_c$, $T_c = 2\pi(GM)^{-1/2}r^{3/2}(1 - (k - 1)r^k_p/L_c^k)^{-1/2}$, and we find that $r^c = L_c$. If we take $k \sim 0.817$, we get $\delta T_{exp}/T_{exp} \geq (k - 1)r^k_p/(2L_c)$, and, when the experimental data for the orbital period of the Earth are taken into account, a lower bound for $L_c$ is found, i.e. $L_c \geq 7.45 \cdot 10^{23} km \sim 5.0 \cdot 10^{15} AU$. On the contrary, if we chose $k$ according to the results of [41], i.e. $k \sim 2.2 \cdot 10^{-18}$, we would find for $L_c$ a value that is many orders of magnitude greater then the previous case. The cross-match of this values allows one to establish the validity range of the model.
1.2 Non-analytic modifications of the Einstein-Hilbert Lagrangian

1.2.6 Discussion

When adopting an $f(R)$ model to extend GR, we are addressing an appealing procedure to consistently derive unexpected features of standard geometrodynamics. However, the wide spectrum of possible effects allowed by such a reformulation could represent one of the shortcomings of the approach if no precise account of observational data is taken. Indeed, we have enough precision in the measurement of astrophysical systems to represent a severe test for the reliability of an extended approach for the gravitational-field dynamics. The present investigation has demonstrated how, in contrast with other analogous treatments [68, 12], the non-analytic case $2 < \beta < 3$ is not excluded by Solar-System tests.

Even though we have restricted our analysis here to the uncertainty of the Earth period, nevertheless in sheds light on the viability of the theory unambiguously. The lower bound of $L_\gamma$ is almost compatible already with the Solar-System scale, since it would predict non-Newtonian effects for outer regions only. However, it is clear that the corresponding value of $r^{**}$, where the theory would require a non-linear treatment, is manifestly incompatible with observations on the galactic scale.

Thus, we can obtain a more reliable estimation for the fundamental scale $L_\gamma$ by requiring that the value of $r^*$ overlap the typical galactic scale on which the rotation curves manifest the flatness of their behaviour, as outlined by $21cm$ Hydrogen line [69]. Indeed, taking $r^*$ in correspondence of the length of about $10Kpc \sim 3 \cdot 10^{17}km$ and an internal mass (assumed as spherically distributed) of $10^{11} M_\odot$, we get $L_\gamma \sim 4 \cdot 10^{17}km \sim 13Kpc$ and $r^{**} \sim 5 \cdot 10^{18}km \sim 1.7 \cdot 10^2 Kpc$ for $\beta = 8/3$. It is not among the purposes of this manuscript to discuss the detailed predictions of our theories on the galactic scales (we will address the analysis of this question in further investigations).

What we wish to emphasize here is that our proposal for a suitable value of the characteristic length $L_\gamma$ can predict small changes on Solar-System observables, while it could drive relevant modifications in higher-scale astrophysics.
1.3 Exponential Lagrangian Density and the Vacuum-Energy Problem

In this section, we will analyze some cosmological implications of an exponential Lagrangian density (PL5), (PL10, (PL15)). Such a choice for the modified gravitational Lagrangian will allow us to investigate the link between the cosmological-constant term (which may arise in $f(R)$ models as a geometrical contribution) and the problem of vacuum energy. The fact that $f(R)$ theories can mimic the effect of dark energy is a very well known result. On the other hand, dark energy can be interpreted as deriving from some kind of matter contribution.

Here, we address a mixture of these two points of view, with the aim of clarifying how the "non-gravitational" vacuum energy affects so weakly the present Universe dynamics [70]. In what follows, we determine the Friedmann equation corresponding to an exponential form for the gravitational-field Lagrangian density. The peculiar feature of our model is that the geometrical components contain a cosmological term too, whose existence can be recognized as soon as we expand the exponential form in Taylor series of its argument. An important feature of our model arises when taking a Planckian value for the fundamental parameter of the theory (as requested by the cancellation of the vacuum-energy density). In fact, as far as the Universe leaves the Planckian era and its curvature has a characteristic length much greater than the Planckian one, then the corresponding exponential Lagrangian is expandible in series, reproducing General Relativity (GR) to a high degree of approximation. As a consequence of this natural Einsteinian limit (which is reached in the early history of the Universe), most of the thermal history of the Universe is unaffected by the generalized theory. The only late-time effect of the generalized framework consists of the relic cosmological term actually accelerating the Universe. Indeed, our model is not aimed at showing that the present Universe acceleration is a consequence of non-Einsteinian dynamics of the gravitational field, but at outlining how it can be recognized from a vacuum-energy cancellation. Such a cancellation must take place in order to deal with an expandable Lagrangian term and must concern the vacuum-energy density as far as we build up the geometrical action only by means of fundamental units.

The really surprising issue fixed by our analysis is that the deSitter solution exists in presence of matter only for a negative ratio between the vacuum-energy density and
the intrinsic cosmological term, $\epsilon_{\text{vac}}/\epsilon_\Lambda$. We can take the choice of a negative value of the intrinsic cosmological constant, which predicts an accelerating deSitter dynamics. Nevertheless, in this case, we would get a vacuum-energy density greater than the modulus of the intrinsic term. This fact looks like a fine-tuning, especially if we take, as we will do below, a Planckian cosmological constant. The vacuum-energy density is expected to be smaller than the Planckian one by a factor $\mathcal{O}(1) \times \alpha^4$, where $\alpha < 1$ is a parameter appearing in non-commutative formulations of the relativistic particle. The analysis of the corresponding scalar-tensor model helps us shed light on the physical meaning of the sign of the cosmological term. In fact, for negative values of the cosmological term, the potential of the scalar field exhibits a minimum, around which scalar-field equations can be linearized. The study of the deSitter regime shows that a comparison with the modified-gravity description is possible in an off-shell region, i.e., in a region where the classical equivalence between the two formulations is not fulfilled. However, despite this apparently unphysical character, this choice is allowed by recent developments in Loop Quantum Cosmology (LQC). In fact, Ashtekar et al [71, 72] have shown that, over a critical value, fixed by the Immirzi parameter [73] $\gamma$, the effective cosmological energy density becomes negative, and gravity exhibits a repulsive character, so that a Big Bounce of the Universe is inferred. The reason why our scheme can solve the problem of the cosmological constant is that our generalized Friedmann equation not only acquires the negative ratio discussed above, but also states that its value is $(-1 + x)$, where $x$ is less/equal to the squared ratio between the Planckian length an the present Hubble radius.

We can summarize our point of view by the assumption that we want to build up a generalized gravitational action, which depends on the Planck length as the only parameter. This statement leads naturally to the exponential form of the Lagrangian density, hence it provides the framework of our proposal. By other words, the cancellation that takes place between the intrinsic term and the effective vacuum energy leaves a relic term, of order $10^{-120}$ times the present Universe Dark Energy, much smaller than the original.

It is worth stressing that this semi-cancellation could be treated in the usual Einstein-Hilbert (EH) scheme by introducing a positive cosmological term which compensates for the vacuum-energy density. The Einsteinian regime of the exponential Lagrangian density can be recovered after a series expansion. Nevertheless, two different possibilities are found: the series expansion either does not hold or brings puzzling predictions
about the cosmological term. Correspondingly, in the first case, an unlikely implication would appear when dealing with a non-Einsteinian physics on all astrophysical scales, and, in the second case, the expansion is only possible in the region $\Lambda >> R$, i.e., in the region where the cosmological constant dominates the dynamics, but for the fact that $R$ should be the same order of $\Lambda$. This contradiction can only be solved if a suitable cancellation mechanisms is hypothesized: here we find the constraint on the ratio $\epsilon_{\text{vac}}/\epsilon_\Lambda \ll 1$ in the deSitter regime in presence of matter.

The analysis is organized as follows. The interpretative problems of the vacuum energy will be introduced, and the necessity to establish a cut off will be approached, as an example, within the formalism of the modified canonical commutation relations predicted by the generalized uncertainty principle (GUP).

The features of an exponential gravitational action will then be investigated, as far as the deSitter regime is concerned, and the appearance of a negative energy density will be regarded to as a means to explain the present small value of the Universe vacuum energy.

The corresponding scalar-tensor model for an exponential Lagrangian density will be also considered, and, for the deSitter phase, the scalar field will be shown to admit a damped oscillating solution that tends to the fixed (minimum) value.

A proposal for the solution of the puzzle is eventually exposed, and the Universe acceleration will be related to the vacuum energy through the introduction of the dimensionless parameter $\delta$, which acts like a compensating factor between the energy density associated to the cosmological constant and that estimated for the vacuum energy in presence of a cut-off.

### 1.3.1 The vacuum-energy problem

As well known [74, 75, 76], the vacuum-energy density associated to a massless quantum field is a diverging quantity unless an appropriate normal ordering (which, on curved space-time, would depend on the metric properties of the manifold) can be found; however, if we fix a cut-off on the momentum variable, $P_{\text{max}} = \alpha \frac{h}{\hbar_{\text{pl}}}$ ($\alpha$ being a dimensionless parameter of order unity) then the vacuum energy density can be
estimated as follows
\[ \epsilon_{\text{vac}} = \int_0^{P_{\text{max}}} \frac{d^3p}{\hbar^3} c p = \int_0^{P_{\text{max}}} \frac{4\pi p^2 dp}{\hbar^3} c p = \pi \alpha^4 \epsilon_{\text{pl}}, \] (1.57)

where \( \epsilon_{\text{pl}} \equiv \hbar c/\ell_{\text{pl}}^4 \).

By other words, we would have to deal with a vacuum-energy density of a Planck-mass particle per Planck volume. However, no evidence appears today for such a huge cosmological term; in fact, recent observations on Supernova data [8, 9] indicate that the Universe is now accelerating with a non-definite equation of state [77]. The indication from cosmic microwave background anisotropies suggests one that the most appropriate characterization of such an equation of state be \( p \sim -\epsilon \) [78, 79, 80][6, 7]. However, when estimating this observed cosmological term, it is immediate to recognize that it is extremely smaller than the cut-off value. In fact, for the observed value of the constant energy density, we get the estimation
\[ \epsilon_{\text{today}} \sim 0.7 \epsilon_0 \simeq \frac{2c^2 H_0^2}{8\pi G} = \frac{1}{4\pi \alpha^4} \left( \frac{l_{\text{pl}}}{\ell} \right)^2, \] (1.58)

where \( \epsilon_0 \) denotes the present Universe critical density \( \epsilon_0 \sim \mathcal{O}(10^{-29})gcm^{-3} \), \( H_0 \sim 70Kms^{-1}Mpc^{-1} \) the Hubble constant, and \( \epsilon_{\text{today}} \) the present value of the vacuum-energy density; since \( L_H \equiv cH_0 \sim \mathcal{O}(10^{27}cm^{-1}) \), we see that a large factor \( 10^{-120} \) appears in (1.58), i.e. \( \epsilon_{\text{today}} \sim \mathcal{O}(10^{-120})\epsilon_{\text{vac}} \). It is well known that this striking discrepancy between the expected and the observed value of the vacuum energy constitutes one of the greatest puzzle of modern cosmology [70].

**Models for a minimal length** A more rigorous understanding for the parameter \( \alpha \) comes out from an approach based on GUP. Such theories implement modified canonical operators obeying the generalized relation
\[ [x, p] = i\hbar (1 + \frac{1}{\alpha^2} \frac{G}{c^3 \hbar} p^2). \] (1.59)

This commutation relation can be recognized on the ground of fundamental properties of the Minkowski space in presence of a cut-off, but it also comes out from quantum-gravity and string-theory approaches [81, 82]. As a consequence of non-commutative models, we deal with a notion of minimal length associated to a particle state. For
instance, in the case of a non-relativistic particle [83], we get the following limit for its wave-length

\[ \lim_{E \to \infty} \lambda(E) = \frac{4}{\alpha} l_{pl}, \quad (1.60) \]

\( E \) being the energy of the particle.

For a discussion of a maximum value for a relativistic-particle momentum at Planck scales, in the context of the k-Poincaré algebra, see [84, 85]. But it is worth noting that, in our case, the discussion above must be referred to a flat FRW background, and, therefore, all the observables correspond with physical quantities corrected by the presence of the scale factor.

Below, we will propose a solution to such a puzzle based on the peculiarity that a deSitter dynamics acquires in the context of an exponential form of the gravitational Lagrangian.

### 1.3.2 Jordan frame

Recent observations based on Supernova [8, 9] data indicate that the universe is now accelerating with a non-definite equation of state [77]. The indication from cosmic-microwave-background anisotropies suggests one that the most appropriate characterization of such an equation of state be \( p \sim -\epsilon \) [6, 7], i.e. a cosmological term. The appearance of a non-zero cosmological constant indicates that \( f(R = 0) \neq 0 \), while the EH action is a linear term in \( R \), with the same sign of the previous one. To deal with \( f(R) \) as a series expansion, we would have, in principle, to fix an infinite number of coefficients. However, in what follows, we address the point of view that only one characteristic length fixes the dynamics: the cosmological constant \( \Lambda \), apart from the Planck length, which can be constructed with the fundamental units \( G, c, h \) as \( l_{pl} \equiv \sqrt{Gh/c^3} \). As a consequence of this point of view, we fix the following explicit form for \( f(R) \)

\[ f(R) = \lambda e^{\mu R}, \quad (1.61) \]

where \( \lambda \) and \( \mu \) are two constants available for the problem. For a discussion of the local dynamical stability of a Universe described by this kind of Lagrangian, see [86]. Comparing the first two terms that come from the expansion of (1.61) (valid in the region \( \mu R \ll 1 \)), with the EH action plus a cosmological term, i.e.

\[ L = -\frac{\hbar}{16\pi l_{pl}^2} (R + 2\Lambda), \quad (1.62) \]
1.3 Exponential Lagrangian Density and the Vacuum-Energy Problem

we arrive at the following identifications

\[ \lambda = 2\Lambda, \quad \mu = \frac{1}{2\Lambda}. \]  

(1.63)

As required, our gravitational Lagrangian is fixed by one parameter only, which has to be provided by observational data.

When specified for the present choice of the Lagrangian density, the two field equations for the FRW model, (1.13) and (1.15), take the explicit expressions

\[ \Lambda e^{\mu R} + 3e^{\mu R} \frac{\ddot{a}}{a} - \frac{3}{2\Lambda} e^{\mu R} \frac{dR}{dt} \frac{\dot{a}}{a} = -\frac{8G\pi}{c^4} \epsilon \]  

\[ -\Lambda e^{\mu R} + e^{\mu R} \left[ \frac{\ddot{a}}{a} - 2\frac{a^2}{a^2} - 2\frac{k}{a^2} \right] + \frac{1}{\Lambda} e^{\mu R} \frac{dR}{dt} \frac{\dot{a}}{a} + \frac{1}{(2\Lambda)^2} e^{\mu R} \left( \frac{dR}{dt} \right)^2 + \frac{1}{2\Lambda} e^{\mu R} \frac{d^2 R}{dt^2} = -\frac{8G\pi}{c^4} p, \]  

(1.64)

(1.65)

respectively.

These two equations have to be coupled to the continuity equation (1.10) and to the equation of state \( p = (\gamma - 1)\epsilon \).

**deSitter regime**

As a first step of this generalized FRW dynamics, we face the study of the deSitter model, where a constant vacuum energy density is taken into account. To this end, we take a cosmic scale factor of the form

\[ a = a_0 e^{\sigma t}, \quad a_0 = \text{const}, \quad \sigma = \text{const}. \]  

(1.66)

It is easy to recognize that for such a choice, the Ricci scalar rewrites \( R = -12\sigma^2/c^2 \); hence, according to the equation of state of a cosmological constant \( p = -\epsilon \), equation (1.64) reduces to the simple form

\[ \epsilon = -\epsilon_\Lambda e^{-x} \left( 1 + \frac{x}{2} \right) \]  

\[ \epsilon_\Lambda \equiv \frac{c^4 \Lambda}{8\pi G}, \quad x \equiv \frac{6\sigma^2}{c^2 \Lambda}. \]  

(1.67)

(1.68)

The expression above has the surprising feature that the energy density would acquire a negative sign. This unphysical property of the model is formally removed as soon as we expand the exponential term in correspondence to small values of the dimensionless constant \( x \), and, restating the usual Friedmann relation, we get

\[ \epsilon = \epsilon_\Lambda \left( \frac{x}{2} - 1 \right) \Rightarrow \sigma^2 = \frac{8\pi G}{3c^2} (\epsilon + \epsilon_\Lambda). \]  

(1.69)
Thus, when the expansion rate of the Universe $\sigma$ is much smaller than the cosmological constant $\Lambda$, we get the usual Friedmann relation between matter and geometry. But, though such a standard relation is apparently reproduced as a low-curvature approximation for $x \ll 1$, nevertheless its inconsistency shows up when (1.69) is restated as

$$x = 2 \left( \frac{\epsilon}{\epsilon_A} + 1 \right).$$

(1.70)

We see that, by the expression above, for positive values of $\epsilon$ and $\epsilon_A$, the quantity $x$ has always to be greater than two, in clear contradiction with the hypothesis $x \ll 1$, at the ground of the derivation of (1.69). Though we are dealing with a surprising behavior, due to the negative ratio $\epsilon/\epsilon_A$, i.e., $\Lambda < 0$, however this feature offers an intriguing scenario. In fact, in the next section, we will apply relation (1.70) to treat the non-observability of the universe vacuum energy in connection with the present Universe acceleration.

We conclude this section by emphasizing that (1.67) admits a special vacuum solution ($\epsilon \equiv 0$), which corresponds to the relation $x \equiv \frac{6\sigma^2}{\epsilon \Lambda} = -2$. For the choice of a negative cosmological constant, $\Lambda = -|\Lambda|$, the equation above provides

$$\sigma^2 = \frac{c^2 |\Lambda|}{3}.$$

(1.71)

Thus we see that, in vacuum, our model has the peculiar feature of predicting a deSitter evolution in correspondence to a negative $\Lambda$ value. However it should be noted that, for this value of $x = -R/(2\Lambda)$, the exponential Lagrangian cannot be expanded, and we deal with the full non-perturbative regime with respect to the Einsteinian gravity. It is just the request to deal with an expandable Lagrangian that leads us to deal with the case $x \ll 1$ and to introduce an external matter field.

### 1.3.3 Einstein frame

The scalar-tensor formalism reviewed in Section 2 is here applied to the particular choice of the exponential Lagrangian density, in order to clarify the meaning of the relations found in the previous section.

The conformal scaling factor here reads

$$f(A) = \lambda e^{\mu A}, \quad f'(A) = e^{-\phi},$$

(1.72)
where \( A = -\phi/\mu \), and the potential rewrites
\[
V(\phi) = -2\Lambda e^\phi(\phi + 1). \tag{1.73}
\]

According to the results of section 4, the on-shell relation between the Einstein frame and the Jordan one is recognized in the identification 
\( A \equiv R \Rightarrow \phi \equiv -R/(2\Lambda) \).

Collecting all the terms together, we get the scalar tensor action
\[
S = -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[ R + \frac{3}{2} g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - 2\Lambda e^\phi(\phi + 1) \right]. \tag{1.74}
\]

Two remarks are now in order:

1. for the comparison of (1.74) with the usual form of the scalar field, a transformation \( \phi \to \sqrt{\frac{16\pi G}{3c^3}} \phi \) could be considered;\(^1\)

2. potential (1.73) is here referred to as one with \( \Lambda \) retaining its own sign. For a discussion of the case \( \Lambda > 0 \) in the quantum sector, where the potential admits no minimum, see Section 6. If one is interested in a description where a stable equilibrium is forecast for the field \( \phi \), the sign of the constant \( \Lambda \) should be reversed, i.e., \( \Lambda \to -|\Lambda| \), as illustrated in figure 1.

This way, smological term

\[\text{Figure 1.1: } V(\phi) \text{ vs } \phi \text{ with } \Lambda = -|\Lambda| \text{ (arbitrary units).}\]

\(^1\)We stress that the factor \( \sqrt{\frac{16\pi G}{3c^3}} \) that arises in front of the kinetic part of the Lagrangian density is independent of the choice of the function \( f \). We nevertheless perform this transformation in this section in order to keep the notation compact.
any more, but is a parameter of the theory, which will be tuned in order to reproduce the observational data.

For the considerations above, action (1.74) rewrites

\[
S = -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R^+ + \frac{1}{c} \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) + \epsilon_{\Lambda|\epsilon} c \sqrt{\frac{16\pi G}{3c^4}} \phi \left( \sqrt{\frac{16\pi G}{3c^4}} \phi + 1 \right) \right].
\] (1.75)

**Scalar Field**

Variation of 1.75 with respect to \( \phi \) leads to

\[
g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + \frac{dV}{d\phi} = 0.
\] (1.76)

If we specialize the rescaled metric tensor for the case of an isotropic Universe, we then deal with the FRW line element, and all the dynamical variables depend on time only, i.e.

\[
\ddot{\phi} + 3H \dot{\phi} + c^2 \frac{dV(\phi)}{d\phi} = 0,
\] (1.77)

where \( H \equiv \dot{a}/a \).

It’s worth remarking that, for the present choice of the negative constant, the potential \( V(\phi) \) admits now a minimum for \( \phi = \phi_0 \equiv -2 \sqrt{\frac{3c^4}{16\pi G}} \), and the corresponding linearized equation reads

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{2}{3} c^2 |\Lambda| \epsilon^{-2} \left( \phi + 2 \sqrt{\frac{3c^4}{16\pi G}} \right) = 0.
\] (1.78)

The appearance of this minimum is expected to become relevant in the dynamics of the scalar field because it is a well-known result that its total energy density follows the relation

\[
\frac{d}{dt} \left( \frac{\dot{\phi}^2}{2c^2} + V(\phi) \right) = -3H \frac{\dot{\phi}^2}{c^2} < 0,
\] (1.79)

where we are assuming an expanding universe, i.e., \( H > 0 \). In fact, starting with a given value of the energy density, sooner or later, the friction due to the universe expansion settles down the scalar field near its potential minimum [28].
**deSitter regime** All these results can apply to the deSitter phase, and a comparison with the issues of Section 4 can be addressed. Therefore, in what follows, we search for a solution of the linearized scalar field equation (1.78), in correspondence with the choice $a(t) = a_0 e^{\epsilon t}$ and $\epsilon(t) \equiv \epsilon = \text{const.}$

The linearized equation (1.78) then rewrites

$$\ddot{\phi} + 3\sigma \dot{\phi} + \frac{2}{3} c^2 |\Lambda| e^{-2} \left( \phi + 2\sqrt{\frac{3c^4}{16\pi G}} \right) = 0,$$

(1.80)

whose solution around the minimum is

$$\phi = -2\sqrt{\frac{3c^4}{16\pi G}} + e^{-\frac{2}{3} \sigma t} [C_+ \cos \beta_+ t + C_- \sin \beta_- t],$$

(1.81)

where $C_\pm$ are two arbitrary constants, and

$$\beta_\pm \equiv \mp i \sqrt{|\Lambda|} c^2 \sqrt{\frac{3x}{2} - \frac{8}{3} e^{-2}}.$$ (1.82)

the discriminant (1.82) is negative for $x < 0.24$, and, because of the prescription $x \ll 1$, it is always negative, so that the field $\phi$ tends, as expected, to $\sqrt{\frac{16\pi G}{3c^4}} \phi_0 = -2$.

It is worth remarking that the on-shell relation provides the identification $\sqrt{\frac{16\pi G}{3c^4}} \phi_0 = -\frac{6\sigma^2}{\Lambda c^2} = -2$.

**External Matter** In presence of a matter source described by the energy-momentum tensor $T_{\mu\nu}$, variation of 1.75 with respect to the metric tensor leads to the Einstein equations in the Einstein frame, i.e.

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = \frac{8\pi G}{c^4} \left( T_{\mu\nu} e^{-\sqrt{\frac{16\pi G}{3c^4}} \phi} + T_{\mu\nu}(\phi) \right).$$ (1.83)

If we specialize the rescaled metric tensor to the case of an isotropic Universe, then we deal with the FRW line element, and all the dynamical variables depend on time only.

For this case, taking the matter source in the form of a perfect fluid with rescaled quantities, the Einstein-scalar system above rewrites

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \left( e^{-2\sqrt{\frac{16\pi G}{3c^4}} \phi} \dot{\phi}^2 + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} V(\phi) \right)$$ (1.84)

$$2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 = -\frac{8\pi G}{c^2} \left( e^{-2\sqrt{\frac{16\pi G}{3c^4}} \phi} \dot{\phi}^2 + \frac{1}{2} V(\phi) \right)$$ (1.85)
Equations (1.84) and (1.85), i.e., the 00 and ii components of the Einstein equations, are not independent, but linked by the rescaled continuity equation

$$\frac{d}{dt} \left( e^{-2\sqrt{\frac{|\Lambda|}{16\pi G}} \phi} \dot{\epsilon}(t) \right) = (\dot{\epsilon}(t) + \dot{\phi}(t)) e^{-2\sqrt{\frac{|\Lambda|}{16\pi G}} \phi} \frac{1}{a^3} \frac{da^3}{dt}. \quad (1.86)$$

Solution (1.81) can now be inserted in (1.84): since the time derivative of the scalar field can be neglected in the vicinity of the minimum, the new Friedmann equation reads

$$\sigma^2 = \frac{8\pi G}{3e^2} (\dot{e}e^4 - \dot{e}|_{\Lambda}|e^{-2}), \quad (1.87)$$

and can be compared with (1.69): because of the conformal transformations (1.23) and (1.19), the two equations completely match. In fact, the different modifications to the two energy densities $\dot{e}e^4$ and $\dot{e}|_{\Lambda}|e^{-2}$ are due to the coupling of the scalar filed with matter and to the conformal transformation of the metric, respectively. Obviously, in both Jordan and Einstein frame, the metric structure remains that of a deSitter phase simply because the conformal factor $e^{-2}$ is nearly constant around the minimum. We recall that the value $\phi_0 = -2\sqrt{\frac{3e^4}{16\pi G}}$ would correspond to the choice $x = -R/(2\Lambda) = -2$ in the Jordan frame defined in section 4. However, we easily check that, in the scalar-tensor theory, such a choice can no longer be a vacuum solution of the theory. In fact, in absence of matter ($\epsilon \equiv 0$) we would deal with a negative cosmological constant as a source of the expansion rate of the universe. However, the correspondence between the Einstein and the Jordan frame takes place as far as we compare equation (1.87) when a constant energy density is included with relation (1.69) obtained for $x \ll 1$. Thus we are led to postulate an off-shell correspondence between the analysis developed for a deSitter space, in which the expansion rate of the universe is much smaller than the $|\Lambda|$ value, and the scalar-tensor approach near the stable configuration, as far as matter a source is included too. The off-shell correspondence provides us with a valuable tool to regard the potential defined in (1.75) as an attractive configuration in the exponential-Lagrangian dynamics. Collecting the two points of view together, we can claim that, when dealing with an exponential Lagrangian, a deSitter phase exists, such that $\epsilon \sim \epsilon|_{\Lambda}$ and it corresponds with general features in the space of the solution.
1.3.4 Proposal for an explanation of the cosmological term

Here we collect the issues of the previous sections together, in order to provide an explanation for the reason why the large value of the vacuum-energy density is today unobservable, or reduced to the actual cosmological constant $\mathcal{O}(10^{-120})$ orders of magnitude smaller than it. We specify that our scheme does not fix the present cosmological term, but simply outlines the mechanism for a cancellation of the original cut-off term.

Implementation of a coherent cosmological model In the light of the discussion above, which calls attention for a solution of the vacuum-energy problem, we are now ready to formulate our proposal for a coherent construction of our model. However, before piecing the jigsaw together, we must focus our attention on some relevant features, which arise from the previous analysis. The exponential Lagrangian is characterized, as established here, by a single parameter, according to which the expansion of the exponential term into power series holds, and which fixes the zero order term of such an expansion, i.e., a cosmological constant. The peculiar feature of this formulation consists of the following consideration. The exponential term is expandable only if $R/(2 | \Lambda |) \ll 1$, but this would imply that the dynamics must contain a cosmological term much greater than the Universe curvature, i.e., an inconsistency which apparently prevents us from recovering the Einstein limit. On the other hand, significant contributions from powers $R/(2 | \Lambda |) \leq 1$ would be predicted in a regime where the expansion of the exponential term does not hold. We analyzed the deSitter regime in some detail because it turns out analytically treatable and very useful for the investigation of the behavior of the exponential term, since we deal with $R = \text{const}$. The main result we get in the pure geometrical (Jordan) frame is that the vacuum dynamics admits a deSitter phase in correspondence with a certain negative value of $\Lambda$ and the additional presence of matter is observable instead only if its constant energy density and $\epsilon_A$ have opposite signs. Furthermore, the vacuum solution lives in the non-Einsteinian region ($x \ll 1$). On the contrary, in the scalar-tensor scheme, when the non-Einsteinian features are recast as matter source, we find an attractive picture in correspondence with the same (on-shell) value $x = -2$, but it turns out admissible only in presence of matter. These two different
representations of the new cosmological dynamics can match only if we assume that
the system evolution is always concerned with a constant matter contribution and if
such a source nearly cancels the negative cosmological term, so that we fix $x \ll 1$.

The universal features of such a matter contribution and its constant value suggest
one to identify it with the vacuum energy discussed in the previous sections. Moreover,
the cancellation required to get $x \ll 1$ is the natural scenario in which a relic
dark energy can be recognized.

The reason why the cancellation proposed between the $\Lambda$ term and the vacuum energy
density provides the right order of magnitude of the dark-energy contribution can be
recognized in the following fact. By the structure of our model, the relic constant en-
dergy density must be a factor $O(R/(2 | \Lambda |))$ smaller than the dominant contribution
$O(\epsilon_{\Lambda})$. Thus if we take the vacuum energy density close to the Planckian value (as
suggested in Section 4) then the actual ratio $R/(2 | \Lambda |)$ is of order $O(10^{-120})$. Such
a quantity behaves like $O(\epsilon_{\Lambda} \frac{R}{L_H})$, where $L_H \sim O(10^{27} cm)$ is the present Hubble radius
of the universe. However, it must be remarked that such a consideration holds in the
case $\epsilon_{\Lambda}$ and the vacuum energy density are the only contributions.

If, as below, an additional physical matter field is added, then the relic dark energy
contribution is simply constrained to be less than the factor $R/(2 | \Lambda |)$ of the vacuum
energy.

**Friedmann dynamics in the scalar-tensor scheme**

Dividing the source energy density into the form

$$\epsilon_{\text{mat}} = \epsilon_{\text{vac}} + \rho(t),$$

where $\rho(t)$ is a generic field contribution, then the Friedmann equation for the scalar-
tensor scheme in proximity of the minimum $\phi_0$ reads

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \left( (\epsilon_{\text{vac}} + \dot{\rho}(t))e^4 - \delta_{|\Lambda|}e^{-2} \right).$$

Since the compatibility of the Jordan- and the Einstein-frame approaches requires that
the expansion rate of the Universe be much smaller than the corresponding parameter
$\Lambda$, then we are led to account for the non-exact cancellation of the vacuum-energy
density by the small parameter $\delta \ll 1$ as follows.

If we take $\dot{\epsilon}_{\text{vac}}e^4 = e^{-2\delta_{|\Lambda|}}(1 + \frac{\delta}{2}) = \pi e^{4\alpha^4}\epsilon_{pl}$, i.e., $| \Lambda | \sim 8\pi^2\alpha^4e^6/l_{pl}^2$, eq. (1.89)
restates
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \left( \dot{\rho}(t)e^4 - \dot{\epsilon}_{\Lambda}e^{-\delta} \right). 
\]
(1.90)

Thus, when the constant energy density dominates, we recognize \( \delta = O\left( \frac{L_{Pl}^2}{\epsilon_{\Lambda}} \right) \), since now \( \epsilon_{\Lambda} \) has a Planckian value. We note that the factor \( e^6 \) appearing in the expression of \( \Lambda \) is of course present only in the scalar-tensor theory, because of the rescaling of the involved energy densities.

**Friedmann dynamics in the f(R)-frame**

On the other hand, this picture can be recovered even in the original Jordan frame, as far as we observe that, for a Planckian value of \( \Lambda \), the exponential Lagrangian is expandable in power series immediately after the Planckian era of the Universe. In fact, as far as we fix \( \epsilon_{\Lambda} \) at Planckian scales, then, as emphasized above, we automatically get for \( \delta \equiv x \) of order \( O\left(10^{-120}\right) \). By other words, even in the Jordan frame, our model is able to explain the vacuum-energy cancellation and to determine the amplitude of the compensating factor \( \delta \) simply by the assumption that the gravitational action contain the Planck length as the only fundamental parameter.

If we now introduce a pure matter contribution, \( \epsilon_{\text{mat}} \ll |\epsilon_{\text{vac}}| \), it is easy to recognize that the standard Friedmann equation with the present cosmological constant is recovered:
\[
H_0^2 = \frac{\Lambda c^2}{6} \left( \frac{\epsilon_{\text{vac}} + \epsilon_{\text{mat}}}{\epsilon_{\Lambda}} + 1 \right) = \frac{8\pi G}{3c^2} \epsilon_{\text{mat}} + \frac{\delta \Lambda c^2}{6}. 
\]
(1.91)

All our considerations refer here to the deSitter solution, and, therefore, \( \epsilon_{\text{mat}} \) is to be regarded as constant. However, it is naturally expected that the Friedmann equation with a small cosmological term arises as low-energy curvature of this theory for any dependence on \( \epsilon_{\text{mat}} \); in fact, for our choice of \( \epsilon_{\Lambda} \), the Lagrangian density of the gravitational field explicitly reads
\[
L = \frac{\hbar}{l_{Pl}^4} \pi \alpha^4 e^{-\frac{r_{pl}^2}{16\pi\alpha^2}}. 
\]
(1.92)

From this expression for the gravitational-field Lagrangian density, we recognize that, as far as the typical length scale \( D \gg l_{Pl} \) of the curvature (\( R \sim 1/D^2 \)), we can address the expansion in terms of small quantity \( l_{Pl}^2/D^2 \)
\[
L \sim \frac{\hbar}{l_{Pl}^4} \pi \alpha^4 - \frac{\hbar}{16\pi l_{Pl}^2} R + O\left( \frac{1}{D^2} \right) 
\]
(1.93)
This approximated Lagrangian density would provide for the FRW metric the following Friedmann equation

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \left[ \epsilon_{\text{mat}}(t) + \epsilon_\Lambda \left( \frac{\epsilon_{\text{vac}}}{\epsilon_\Lambda} + 1 \right) \right].
\] (1.94)

Such an approximated equation, isomorphic to (1.90), allows us to reproduce all the considerations developed about the exact deSitter case. However, the analysis performed above is relevant in the Jordan frame in outlining the necessity of the constraint \( \epsilon_{\text{vac}}/\epsilon_\Lambda \sim -1 \).

In fact, the exact deSitter case clarified that, for positive \( \Lambda \) values, this relation is the only one able to provide the consistency of the Friedmann equation according to an exponential Lagrangian density. This feature could not be recognized by an approximated analysis, as in (1.94).

### 1.3.5 Hints for the quantum regime of the model

As discussed in Section 5, the potential of the scalar field admits a minimum if the sign of the parameter \( \Lambda \) is reversed. The present paragraph, on the contrary, is aimed at investigating a model where the potential term admits no minimum, i.e., a model with an absolute maximum and a slow-rolling regime, as described in figure 2.

![Figure 1.2: V(ϕ) vs ϕ with Λ > 0 (arbitrary units).](image)
We stress that in this case the negative ratio $\epsilon_{\text{vac}}/\epsilon_\Lambda$ could take place only in correspondence with the apparently unphysical request $\epsilon_{\text{vac}} < 0$.

A late-time solution $\phi(t)$ can be looked for, according to the potential profile, such that $\dot{\phi}(t) \to 0$ in the limit $\phi(t) \to -\infty$ and $V(\phi(t) \to -\infty) \to 0$. In what follows, we discuss the behavior of our scalar-tensor model near the cosmological singularity and provide some hints about its quantum dynamics. The case $\Lambda > 0$, as remarked above, admits no stable configuration; the analysis below, however, would apply also in the case $|\Lambda| < 0$, because the behavior of the scalar field in the vicinity of the singularity would hold as well, but for the fact that, for $\phi \to \infty$ no stable configuration would be reached.

In absence of external matter, i.e., $\epsilon(t) = 0$, the Friedmann equation (1.84) simplifies as

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \phi^2}{3c^2} = \frac{8\pi G \phi^2}{2} :$$

(1.95)

if only the positive root is taken into account, the scale factor $a$ acquires the form

$$a = a_0 e^{\sqrt{\frac{4\pi G \phi}{3c^2}}},$$

(1.96)

where $a_0$ is an integration constant. This way, the scale factor $a$ tends to 0 as the field $\phi$ tends to $-\infty$.

As a next step, in this approximation, the linearized equation for the field $\phi$ (1.78) reads

$$\ddot{\phi} + 3H \dot{\phi} = 0,$$

(1.97)

where $H = \dot{a}/a$, so that

$$\dot{\phi} = \frac{\dot{\phi}_0}{a_0^3} e^{-\sqrt{\frac{12\pi G}{c^2}}} \phi,$$

(1.98)

$\dot{\phi}_0$ being an integration constant. Consequently, the time dependence of $\phi(t)$ and $a(t)$ can be found, i.e.,

$$\phi = \sqrt{\frac{c^2}{12\pi G}} \ln \left[ \sqrt{\frac{12\pi G}{c^2}} \frac{\dot{\phi}_0}{a_0^3} (t - t_0) \right],$$

(1.99)

$$a = \frac{\dot{\phi}_0}{a_0^3} \sqrt{\frac{12\pi G}{c^2}} t^{1/3}.$$  

(1.100)

As requested, at the time $t_0 = 0$, the field (1.99) tends to $-\infty$ [87].

So far, it is possible to verify that the potential $V(\phi)$ and its first derivative could be neglected in (1.84) and (1.78): in fact, its contribution at early times is of order
\( \mathcal{O}(t^{3/2} \ln(t + 1)) \), which can be ignored in the presence of the leading-order terms \( \mathcal{O}(t^{-2}) \) due to both \((\dot{a}/a)^2\) and \(\phi^2\).

Recent studies in Loop Quantum Gravity (LQG) outlined that the expectation value of the Hamiltonian operator in a given state is, in general, different from the classical Hamiltonian contribution and this difference is responsible for systematic quantum corrections to the classical energy density involved in the problem. In particular, application of this quantum scheme to the isotropic FRW Universe (in the presence of a massless scalar field, which plays the role of time) provided modified relations between the Hubble parameter and the energy density of the Universe; this effective cosmological dynamics is mapped into the original Friedmann equation as soon as we allow the energy density of the Universe to become negative over critical values, i.e., the following correspondence takes place

\[
\epsilon \rightarrow \epsilon_{\text{eff}} \equiv \epsilon \left(1 - \frac{\epsilon}{\epsilon_{\text{crit}}} \right), \quad \epsilon_{\text{crit}} = \frac{\sqrt{3}}{16\pi^2\gamma^3} \epsilon_{\text{pl}}, \quad (1.101)
\]

where \(\epsilon_{\text{crit}}\) (with \(\gamma\) the Immirzi parameter) is a critical value of the energy density two orders below the Planck scale, over which the matter contribution becomes negative, thus illustrating a repulsive nature of the gravitational field near the (removed) cosmological singularity. In fact, in a standard Friedmann dynamics, this peculiar matter source induces a bounce in the dynamics of the scale factor, solving the singularity problem and opening interesting perspectives on the cyclic evolution of the closed Universe.

These developments can apply to the scalar-tensor model equivalent to the choice of an exponential gravitational action. In particular, as hinted by (1.99), a region can be found, where the potential \(V(\phi)\) can be neglected. If external matter is absent, the results of LQG can apply to our scheme in such a region, by modifying the Friedmann equation (1.84), i.e.,

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{c^2} \epsilon_{\text{eff}}(\phi), \quad (1.102)
\]

where \(\epsilon_{\text{eff}}(\phi) = \epsilon(\phi) \left(1 - \frac{\epsilon(\phi)}{\epsilon_{\text{crit}}} \right)\), according to (1.101). In presence of external matter, on the other hand, we obtain

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{c^2} \left(\epsilon_{\text{eff}}(\phi) + \epsilon(t)e^{-2\sqrt{\frac{16\pi G}{3c^4}} \phi} \right). \quad (1.103)
\]

We note that we can so far analyze the implication of the dynamical equivalence between modified gravity and scalar tensor approaches. In fact, the on-shell request
1.3 Exponential Lagrangian Density and the Vacuum-Energy Problem

gives
\[ \phi = -\mu R = -\frac{R}{2\Lambda} \quad (1.104) \]
and, by the power-law (1.100), \( R = 1/(3\ell^2) \), which does not match the solution (1.99) found for \( \phi \), where the functional dependence on time is logarithmic. Nevertheless, as a general trend, the curvature scalar diverges as \( \phi \) tends to \(-\infty\), i.e., at very early times the on-shell relation is qualitatively satisfied.

This analysis shows that near the cosmological singularity our scalar-tensor theory takes the form of general relativity in the presence of a massless scalar field. This fact allows us to infer some possible issues of its quantization [88].

As a result, we can claim that our proposed non-Einsteinian scheme is characterized by a non-singular behavior when the corresponding scalar-tensor picture is canonically quantized. In fact, the possibility to neglect the potential field as the big-Bang is classically approached is mapped by the results discussed in [89, 90] into a Big-Bounce. However, two relevant questions have to be faced here

1. A LQG formulation for the generalized \( f(R) \) gravity is not yet viable and the correspondence between the Jordan and the Einstein frame on quantum level cannot be addressed;

2. The non-singular feature we established here in view of the possibility to neglect the potential term near the cosmological singularity can be extended to a wide class of scalar-tensor theories corresponding to the \( f(R) \) formulation.

In particular, by the calculations above, the potential term is negligible in the asymptotic behavior towards the singularity as far as \( V(\phi) \) evaluated for (1.99) behaves as \( O(t^{-2+\beta}) \), with \( \beta > 0 \). The condition on the potential term, which satisfies such a request, can be easily stated as

\[ \lim_{\phi \to -\infty} \frac{V(\phi)}{\phi^\theta e^{-2\sqrt{\frac{12\Lambda\phi}{c^2}}} e^{\phi}} = 0, \quad (1.105) \]

\( \forall \theta > 0 \). Let’s remark that the behaviour of a potential term \( \sim e^{-2\phi} \) would correspond to a generalized gravitaitonal Lagrangian linear in the \( R \) variable.

1.3.6 Discussion

After deriving the Einstein equations for a generalized gravitational action and specifying the results for an FRW metric, the particular choice of an exponential La-
grangian density has been analyzed. The free parameters of such a Lagrangian density have been fixed as functions of the cosmological constant, and, in the deSitter regime, the ratio between the vacuum-energy density and the geometrical contribution has been illustrated to acquire a negative sign, which has been the springboard for the investigation of the relation between the vacuum-energy and cut-off approaches to the geometrical description of the Universe. In particular, the cut-off introduced in the vacuum-energy density has been linked with the modified commutation relation following from a generalized uncertainty principle, and has been fixed at Planck scales. The negative sign of the ratio $\epsilon_{vac}/\epsilon_{\Lambda}$ not only explains the non-observability of the cut-off vacuum-energy density and is in line with the LQC prediction of the Big Bounce in an FRW metric, but also allows one to recover the standard Friedmann equation in the deSitter phase, when the matter contribution is taken into account, and for any choice of the matter terms.

Studying some aspects of the pertinent scalar-tensor description has allowed us to investigate further connotations of the implementation of such a scheme. In particular, the physical meaning of the sign of the cosmological constant has been explained to provide interesting hints about cosmological implications. For $\Lambda < 0$, the accelerating Universe is predicted to stabilize around the minimum of the scalar potential, while, for $\Lambda > 0$, a possible connection with LQC has been envisaged.

The main issue of our analysis has consisted in fixing the link between the vacuum-energy cancellation and the present Universe Dark Energy. By other words, we have guessed that the actual acceleration, observed via SNIA, is due to the relic of the original huge vacuum energy, after its mean value has been compensated for by the intrinsic cosmological constant $\Lambda$ contained in the exponential Lagrangian. We will address the theoretical explanation of the phenomenologically-suggested fine tuning, $\delta \sim O(10^{-120})$, as a prospective investigation.
2 Non-Riemannian Geometries

The investigation of possible extensions of General Relativity has already inspired a large amount of work. When non-Riemannian geometries are taken into account, in the most general setting, an asymmetric metric tensor, torsion and non-metricity tensor arise. These objects lead to a modification of the gravitational Lagrangian, and their coupling with spinors can also be analyzed. Within this framework, a larger variety of geometrical objects other than the Ricci scalar is now available, and they provide one with a generalized scenario, where several features absent in General Relativity can be investigated.

In this chapter, after motivating the physical bases of our investigation, we will first review the general features of non-Riemannian geometries, and then we will present the original results appeared in (PL3), (PL4), (PL7), (PL8), (PL9), (PL13), (PL17). Throughout this chapter, we will adopt the following notation:

- lower-case Greek letters from the beginning of the alphabet (i.e. $\alpha, \beta$) denote 4-dimensional inertial indices, $\alpha = 0, 1, 2, 3$;
- lower-case Greek letters from the middle of the alphabet (i.e. $\mu, \nu$) denote general 4-dimensional world indices, $\mu = 0, 1, 2, 3$;
- lower-case Latin letters from the beginning of the alphabet (i.e. $a, b$) denote 4-dimensional bein indices, $a = 0, 1, 2, 3$;
- lower-case Latin letters from the middle of the alphabet (i.e. $m, n$) denote the rank of tensors or the dimension of spaces;
- lower-case Latin letters from the second half of the alphabet (i.e. $r, s$) denote spinor indices, $r = 1, 2$;
• repeated indices are summed over for tensor objects, and the summation symbol is usually omitted (i.e. \( v^\mu v_\mu \equiv \sum_{\mu=0}^{3} v^\mu v_\mu, \ u^a u_a \equiv \sum_{a=0}^{3} u^a u_a \));

• lower-case Latin letters from the first part of the alphabet (i.e. \( i, j \)) denote indices within summation, to which peculiar attention has to be devoted;

• \( \tilde{\cdot} \) denotes quantities involving torsion;

• \( * \) denotes quantities involving non-metricity;

• \( ^\wedge \) denotes quantities involving torsion, non-metricity and/or other objects;

• \( \wedge \) denotes exterior product;

• \( \star \) denotes the Hodge operator for forms

  \[ \star \eta = \frac{1}{p!} \eta_{m_1...m_p} \varepsilon^{m_1...m_p} m_{p+1}...m_n dx^{m_{p+1}} \wedge ... \wedge dx^{m_n}, \]

where \( \eta \) is a p-form on a n-dimensional manifold);

• \( \tilde{\cdot} \) denotes adjoint spinors (i.e. \( \tilde{\psi} = \psi^\dagger \gamma^0 \), where \( \psi \) is a spinor, \( \psi^\dagger \) the conjugate spinor, \( \tilde{\psi} \) the adjoint spinor and \( \gamma^0 \) the zero-th Dirac matrix).

### 2.1 Introduction

In non-flat spaces the notion of the parallel transport of vector fields needs the introduction of connections, which also define the covariant derivative. By means of the connections, we can define the equation of parallel transport as follows: on a 4-dimensional manifold \( M^4 \), given a curve \( \gamma(t) \) passing for a point \( P \in M^4 \), the parallel-transported vector field of the vector field \( V^\rho(P) \) along \( \gamma(t) \) is the solution of the equation

\[
\frac{dV^\rho}{dt} = -\tilde{\Gamma}^\rho_{\mu\nu} V^\nu \dot{\gamma}^\mu,
\]

where \( \tilde{\Gamma}^\rho_{\mu\nu} \) denote general affine connections. Moreover, the general covariant derivative \( \tilde{\nabla}_\mu \) of a tensor field \( V^\rho_\nu(x) \) is defined as

\[
\tilde{\nabla}_\mu V^\rho_\nu = \partial_\mu V^\rho_\nu + \tilde{\Gamma}^\rho_{\mu\sigma} V^\sigma_\nu - \tilde{\Gamma}^\sigma_{\mu\nu} V^\rho_\sigma,
\]

where \( \partial_\mu \) indicates the ordinary partial derivative. In fact, the derivative of a vector must be evaluated at two different spacetime points and it is therefore necessary
to transport the displaced vector back to its original position for comparison. In particular, if the vector $V^\rho$ is parallel-transported along the infinitesimal displacement $dx^\mu$, the change due to this transport is given by $-\bar{\Gamma}_\mu^\nu V^\nu dx^\mu$, which leads to the correct definition of the covariant derivative. In this respect, one may define the curvature tensor as the result of parallel transporting a vector $V^\rho$ around a closed path $\xi^\mu$,

$$\Delta V^\rho = \frac{1}{2} V^\nu \bar{R}^\rho_{\sigma\rho\nu} \int \xi^\mu dx^\sigma .$$  \hspace{1cm} (2.3)

Consider now transporting the infinitesimal vector $l^\rho$ along $m^\rho$ and compare it with transporting $m^\rho$ along $l^\rho$. Let us define the vector $A^\rho = l^\rho + m^\rho - \bar{\Gamma}_\mu^\rho l^\mu m^\mu$ and the vector $B^\rho = m^\rho + l^\rho - \bar{\Gamma}_\mu^\rho m^\mu l^\mu$. Their difference is $C^\rho = 2\bar{\Gamma}_\rho_{[\mu\nu]}l^\mu m^\nu$. One can easily realize that the vectors $A_\rho$ and $B_\rho$ do not form a close parallelogram [91, 92]. The non-closure of parallelograms in space-time is due to the antisymmetric part of affine connections which define the torsion tensor

$$T^\rho_{\mu\nu} = \bar{\Gamma}_\rho_{[\mu\nu]} .$$  \hspace{1cm} (2.4)

In general, connections $\bar{\Gamma}_\rho_{\mu\nu}$ are non-tensor quantities; on the other hand, their antisymmetric part transforms like a tensor, as fas as the most general metric-compatible form of connections are concerned.

Because of this property, the presence of a torsion field modifies the notion of the Principle of Equivalence; indeed, we are not referring to the equivalence between inertial and gravitational mass, which is preserved since the theory remains geometric, but to the formulation of the Equivalence Principle [93] according to which, once defined an inertial frame in a point, the physical laws are the same as those of special relativity. In presence of torsion, the latter, behaving like a tensor, can not be set to zero by a convenient coordinate choice. Therefore, since we expect torsion to be a source of some “force”, it is not possible to define an inertial frame in any point, which is a necessary condition for the applicability of the principle.

The metric tensor, connections and the Einstein tensor

Let us now introduce a metric defined by the square modulus of a vector $V^\rho$ as

$$\|V\|^2 = g_{\mu\nu} V^\mu V^\nu ,$$  \hspace{1cm} (2.5)
here \( g_{\mu\nu} \) denotes the symmetric metric tensor defining the square of the infinitesimal interval \( ds \) as

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu.
\]  

(2.6)

It is possible to establish a relation between connections, torsion and metric tensor of the form [94]

\[
\tilde{\Gamma}^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} [\partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma}] + \frac{1}{2} [\mathcal{T}^\rho_{\mu\nu} - \mathcal{T}^\rho_{\nu\mu}] = \Gamma^\rho_{\mu\nu} + \mathcal{K}^\rho_{\mu\nu}.
\]  

(2.7)

Here \( \Gamma^\rho_{\mu\nu} \) denote the Christoffel Symbols (which are symmetric in the first two indices) and \( \mathcal{K}^\rho_{\mu\nu} \) identifies the contortion tensor defined by

\[
\mathcal{K}^\rho_{\mu\nu} = \frac{1}{2} [\mathcal{T}^\rho_{\mu\nu} - \mathcal{T}^\rho_{\nu\mu}].
\]  

(2.8)

and it is antisymmetric in the last two indices. A space endowed with affine connections (2.7) is called Einstein-Cartan (EC) Space \( U^4 \). In such a space, using the definition of connections (2.7), one can write the curvature tensor [92, 95] in presence of torsion. It reads

\[
\tilde{R}^\sigma_{\mu\rho\nu} = \partial_\nu \tilde{\Gamma}^\sigma_{\mu\rho} - \partial_\rho \tilde{\Gamma}^\sigma_{\mu\nu} + \tilde{\Gamma}^\gamma_{\nu\rho} \tilde{\Gamma}^\sigma_{\mu\gamma} - \tilde{\Gamma}^\gamma_{\nu\mu} \tilde{\Gamma}^\sigma_{\rho\gamma}.
\]  

(2.9)

Such a curvature can be easily expressed through the Riemann Tensor \( R^\sigma_{\mu\rho\nu} \) (curvature tensor depending only on metric), the covariant derivative \( \nabla_\mu \) (torsionless case of eq. (2.2)) and contortion as

\[
\tilde{R}^\sigma_{\mu\rho\nu} = R^\sigma_{\mu\rho\nu} + \nabla_\nu \mathcal{K}^\sigma_{\rho\mu} + \nabla_\rho \mathcal{K}^\sigma_{\mu\nu} + \mathcal{K}^\gamma_{\sigma\nu} \mathcal{K}^\gamma_{\mu\rho} - \mathcal{K}^\gamma_{\sigma\rho} \mathcal{K}^\gamma_{\mu\nu}.
\]  

(2.10)

Similar formulas can be written for the Ricci tensor and for the scalar curvature with torsion:

\[
\tilde{R}_{\mu\rho} = \tilde{R}^\nu_{\mu\rho\nu} = R_{\mu\rho} + \nabla_\sigma \mathcal{K}^\sigma_{\rho\mu} - \nabla_\rho \mathcal{K}^\sigma_{\mu\sigma} + \mathcal{K}^\gamma_{\sigma\nu} \mathcal{K}^\gamma_{\mu\rho} - \mathcal{K}^\gamma_{\rho
u} \mathcal{K}^\gamma_{\mu\sigma},
\]  

(2.11)

(it is worth underline that it is not symmetric) and

\[
\tilde{R} = g^{\mu\rho} \tilde{R}_{\mu\rho} = R + 2 \nabla^\sigma \mathcal{K}^\sigma_{\rho\mu} - \mathcal{K}^\sigma_{\mu\sigma} \mathcal{K}^\mu_{\rho\gamma} + \mathcal{K}^\mu_{\gamma\sigma} \mathcal{K}^\mu_{\sigma\rho},
\]  

(2.12)

where \( R_{\mu\rho} \) and \( R \) are constructed with the Christoffel Symbols as usual in GR. In this scheme, the Einstein tensor in presence of torsion is defined according the standard picture as

\[
\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R},
\]  

(2.13)

and, by such a definition, one can show [91] that the antisymmetric part of the Einstein tensor is related to torsion field by the following relation

\[
\tilde{G}_{[\mu\nu]} = (\nabla_\rho + 2 T\sigma_{\rho\sigma}) T^\rho_{\mu\nu}.
\]  

(2.14)
2.1 Introduction

The non-metricity tensor

It is worth noting that more general affine connections can be implemented. We underline that, in order to maintain the correct behaviour of the covariant derivative (2.2), any tensor $A^o_{\nu\mu}$ can be added to connections [95]. A particular choice corresponds to define the affine-connection coefficients as

$$\hat{\Gamma}^o_{\mu\nu} = \tilde{\Gamma}^o_{\mu\nu} + \frac{1}{2} \left[ Q^o_{\mu\nu} - Q^o_{\nu\mu} + Q^o_{\nu\mu} \right],$$

where we have introduced the tensor of non-metricity defined as

$$Q^\nu_{\mu\rho} = \nabla^\nu g^{\mu\rho},$$

here $\nabla^\nu$ denotes the covariant derivative $\nabla^\nu V^\rho = \partial^\nu V^\rho + \tilde{\Gamma}^\rho_{\mu\nu} V^\rho$. We remark that non-metricity does not preserve lengths and angles under parallel displacement. To conclude, we summarize the space characterization is presence of the tensor quantities introduced above

\begin{center}
\begin{tabular}{c|c|c|c}
General Linear space $L^4$ & $Q_{\mu\nu\rho}\equiv0$ & Einstein-Cartan space $U^4$ & $T_{\mu\nu\rho}\equiv0$ & Riemann space $V^4$
\end{tabular}
\end{center}

2.1.1 Einstein-Cartan Theory and non-dynamical torsion

Completely neglected in the first formulation of the theory of GR by Einstein, the introduction of torsion was later implemented by Einstein himself [96], Eddington [97], Schrödinger [98] and principally Cartan [99, 100, 101], where torsion was connected with the spin angular momentum.

In the original Einstein-Cartan Theory (ECT), the geometric Lagrangian density is assumed to be composed by the curvature scalar $\tilde{R}$ (generating the generalization in presence of torsion of the Einstein-Hilbert (EH) Action) and the matter Lagrangian is taken into account simply through the minimal coupling rule: $\eta_{\mu\nu} \rightarrow g_{\mu\nu}, \partial_{\mu} \rightarrow \nabla_{\mu}$. The minimal substitution will be applied to matter field only, but not to gauge fields of internal symmetry groups [91]. This way, the gravitational action corresponds to the EH Action written in the EC space, i.e.,

$$\tilde{S}_{EH} = -\frac{1}{2} \int d^4x \sqrt{-g} \tilde{R} = -\frac{1}{2} \int d^4x \sqrt{-g} \ g^{\nu\rho} \delta^\mu_\sigma \left( \partial_\mu \tilde{\Gamma}^\sigma_{\nu\rho} - \partial_\nu \tilde{\Gamma}^\sigma_{\mu\rho} - \tilde{\Gamma}^\sigma_{\mu\rho,\nu} + \tilde{\Gamma}^\sigma_{\nu\rho,\mu} \right).$$

(2.17)
Non-Riemannian Geometries

Being $\varphi$ the matter field, after the minimal coupling procedure generating the total Lagrangian density $\mathcal{L} = \mathcal{L}(\varphi, \partial \varphi, g, \partial g, T)$, one can define the usual EMT as

$$T^{\mu\nu} = \delta \mathcal{L}/\delta g_{\mu\nu}. \quad (2.18)$$

Accordingly, one can suppose [91] to define an analogous quantity related to the contortion (or torsion) field, \textit{i.e.},

$$s_{\rho}^{\mu\nu} = \delta \mathcal{L}/\delta T^\rho_{\mu\nu}. \quad (2.19)$$

Such a tensor is constructed from the matter fields $\varphi$, but may also depend on metric and torsion [95]. If Dirac fermions minimally coupled to torsion are considered, $s_{\rho}^{\mu\nu}$ corresponds to the spin energy potential.

Considering now the variational principle

$$\delta \mathcal{S}_{EH} + \delta \mathcal{S}_M + \delta \mathcal{S}_T = 0, \quad (2.20)$$

where $\delta \mathcal{S}_M$ and $\delta \mathcal{S}_T$ are defined, according to the previous expressions, as

$$\delta \mathcal{S}_M = \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}, \quad (2.21)$$

$$\delta \mathcal{S}_T = \frac{1}{2} \int d^4x \sqrt{-g} s_{\rho}^{\mu\nu} \delta T^\rho_{\mu\nu}, \quad (2.22)$$

one can derive the field equation of the system. Using the definition (2.18), one obtains

$$G^{\mu\nu} - (\nabla_{\rho} + 2T^\rho_{\rho\sigma}) \left[ \tilde{T}^{\mu\nu\rho} + \tilde{T}^{\rho\mu\nu} + \tilde{T}^{\rho\nu\mu} \right] = T^{\mu\nu}, \quad (2.23a)$$

$$T_{\mu\nu\rho} = 2 \left( s_{\rho[\mu\nu]} + s_{[\mu} g_{\nu]\rho} \right), \quad (2.23b)$$

where $s_{\mu} = s_{\rho}^{\mu\rho}$ and $\tilde{T}^{\mu\nu\rho}$ is the modified torsion tensor defined as

$$\tilde{T}^{\mu\nu\rho} = T^{\mu\nu\rho} + T^{\sigma}_{\mu\rho} \sigma g^{\mu\rho} - T^{\mu}_{\sigma\rho} \sigma g^{\nu\rho}. \quad (2.24)$$

In vacuum, eq. (2.23b) implies $T_{\mu\nu\rho} = 0$. One can easily see that torsion is proportional to the spin energy potential and it vanishes in vacuum. In this picture, torsion obeys to an algebraic equation, instead of a differential one, and it acquires a non-propagating dynamics. Torsion is inextricably bound to matter and cannot propagate through the vacuum as a wave or via any interaction of non-vanishing range. At the same time, we can underline that, because of such a character, one is able
to substitute everywhere spin for torsion and cast out effectively torsion from the
formalism. In particular, using eq. (2.22) and (2.23b), torsion leads to the contact
spin-spin interaction which can be expressed by the classical potential \( V(s) \sim s^2 \).

It is worth noting that we can define a quantity analogous to \( s^\mu_{\rho\nu} \), but related
to the contortion field. In this case, one can recognize it as the proper spin angular
momentum tensor. Denoting such a tensor with \( \tau^\rho_{\mu\nu} \), after some manipulation, one
can recast eq. (2.23b) in the form

\[
\mathcal{T}^\rho_{\mu\nu} + \delta^\rho_\mu \mathcal{T}^\sigma_{\nu\sigma} - \delta^\rho_\nu \mathcal{T}^\sigma_{\mu\sigma} = \tau^\rho_{\mu\nu}.
\]

\[ (2.25) \]

### 2.1.2 Fundamental statements

Gauge theories describe all physical interactions, but the gravitational one. Many
ttempts to construct a gauge model of gravitation exist, in particular the papers
by Utiyama [102] and by Kibble [103] were the starting points for various gauge
approaches to gravitation. As a result, Poincaré gauge theory (PGT) [104, 91, 105, 106,
107, 108, 109, 110, 111, 112] is a generalization of the Einstein scheme of gravity,
in which not only the energy-momentum tensor, but also the spin of matter plays
a dynamical role when coupled to Lorentz connections, in a non-Riemannian space-
time. Anyway, up to now, neither Poincaré gauge theory (PGT) nor other gauge
approaches to the gravitational interaction have led to a consistent quantum scheme
of the gravitational field.

To include spinor fields consistently, it is necessary to extend the framework of Gen-
eral Relativity (GR), as already realized by Hehl \textit{et al.} [91]; this necessity is strictly
connected with the non existence in GR of an independent concept of spin angular
momentum for physical fields, as the Lorentz group has not an independent status of
gauge group in GR.

The Einstein-Cartan Theory (ECT) theory accounts for both mass and spin of mat-
ter as sources of the gravitational field, and represent a description of gravity which
is more suitable than GR from a microscopical point of view. In fact, fundamental
interactions other than gravity are usually described within a theoretical framework,
where matter is described by matter fields, and, spin, symmetries and conservation
laws are properly encoded. In GR, contrastingly, matter can be described by point
particles, fluids and light rays. This fundamental difference notwithstanding, spin
effects are negligible for macroscopic matter, so that the observational predictions
of EC theory are regarded as the same as GR, from a phenomenological point of view [109]. Furthermore, ECT is a special case of Poincaré gauge theory. However, PGT is much more general than ECT and encompasses also propagating Lorentz connections.

The gauge freedom of the gravitational interaction is the invariance under diffeomorphisms [113], strictly connected with the general-covariance principle. Moreover, the equivalence of any local reference system introduces another gauge freedom, mathematically expressed by the invariance of the model under local Lorentz transformations. But the former gauge freedom reabsorbs the latter, depriving the local Lorentz gauge group of independent gauge connections. Thus the dynamics of the gravitational field reduces to that of tetrads, while spin connections (introduced as gauge fields of local Lorentz transformations) have no longer an independent role in this framework, being only a particular combination of the tetrad fields and their derivatives.

The reasons of this have to be searched in the physical content of the two gauge transformations. From a mathematical point of view, using the tetrad formalism allows one to distinguish between diffeomorphisms and local Lorentz transformations, the first acting as a pull-back on the tetrad fields, while the second as a Lorentz rotation of the local basis; but, as we shall see, in the case of isometric coordinate transformations, the former can be restated in terms of the latter, and spin connections no longer have the proper change to ensure local Lorentz invariance. In fact, under isometric diffeomorphisms, spin connections transform as tensors, and cannot be gauge potentials for such diffeomorphism-induced local rotations.

Indeed, in the coordinate formalism, an infinitesimal diffeomorphism reads

\[ x^\mu \rightarrow x'^\mu = x^\mu + \alpha^\mu (x), \]  

(2.26)

where \( \alpha^\mu (x) \) are four \( C^\infty \) functions, while an infinitesimal Lorentz (isometric) rotation has the form

\[ x^\mu \rightarrow x'^\mu = x^\mu + \alpha^\mu_\nu x^\nu, \]  

(2.27)

where \( \alpha^\mu_\nu \) are the six infinitesimal rotational parameters. For local Lorentz transformations, (2.27) can be reabsorbed in (2.26), i.e.,

\[ x^\mu \rightarrow x'^\mu = x^\mu + \alpha^\mu_\nu (x) x^\nu = x^\mu + \tilde{\alpha}^\mu (x). \]  

(2.28)
This point of view is easily generalized to the case of scalar or macroscopic matter, which can be consistently approximated to spin-less matter, while gives rise to striking differences for spin-$\frac{1}{2}$ matter fields: fermions are described by a spinor representation of the Lorentz group, while the diffeomorphism group has no finite spinor representation. For the study of classical spinning particles interacting with the gravitational field, see also [114]. The transformation law of fermions under local Lorentz rotations can no way be reabsorbed in the transformation law of tensor fields under the group of diffeomorphisms. Accordingly, the framework has to be further extended to consistently include fermions, giving back to the Lorentz group its status of independent gauge group [91], with its own connection fields, not directly related with the tetrad fields, as they are in General Relativity.

Nevertheless, it is worth stressing that infinite representations of such a group do exist, as shown in [115, 116] and the references therein. Finite spinors can introduced either by making use of the non-linear representations of the double covering of the general-coordinate-transformation group, which are linear when restricted to the Poincaré subgroup, or by introducing a bundle of cotangent frames and defining in this space the action of a physically-distinct Lorentz group. In the second case, after generalizing the Lorentz group, infinite-dimensional linear spinor representations or finite-dimensional non-linear spinor representations can be found. In [116], infinite-component spinor and tensor fields (so-called manifields) are introduced: these manifields are are then lifted to the proper corresponding representation via the introduction of infinite-component frame-fields.

Poincaré Gauge Theory is the theoretical framework that enables one to take into account both translations and Lorentz transformations, and spinor dynamics in curved space-time is recovered by the introduction of compensating fields that restore local invariance [104]. The present proposal [117, 118, 119, 120, 121] differs from PGT also because we obtain differential dynamical equations for the gauge connections, rather than algebraic ones. Nevertheless, a contact interaction is recovered in the limit of vanishing Lorentz connections, where the relation between the spin connections and the spin density becomes algebraic. For another propagative approach to torsion, see, for example, [122, 123, 124].

Furthermore, it is worth recalling that the teleparallel theory of gravity can be
treated physically as a gauge theory of translations. In fact, teleparallel gravity can be understood within the framework of metric-affine gravitational theories [125], and it is picked up from such other models by reducing the affine symmetry group to the translation subgroup, i.e. by imposing vanishing curvature and non-metricity. Within the framework of a metric-affine approach to teleparallel gravity, the introduction of spinless matter, characterized only by the energy-momentum, can be illustrated to be completely consistent with teleparallelism, while the case of spinning matter sources exhibits a consistency problem, and teleparallelism appears to be not applicable in the second case [126]. On the other hand, formulations have been proposed, in which gravity can be understood as deriving from translation symmetry (see, for example [126, 127, 128] and all the references therein).

The features of GR that make it resemble to a gauge theory are reviewed, and the tools which will support our analysis, i.e., the tetrad formalism and Cartan structure equations, are revised, and the properties of local Lorentz transformations are outlined.

PGT is then described from both the gauge and the geometrical points of view. The geometrical structure of PGT and the geometrical meaning of gauge fields and conserved quantities will follow from the comparison. This is mostly devoted at stressing those features that allow us to examine differences and consider similarities with the Lorentz-gauge proposal.

We briefly revise the main features of Teleparallel geometry, understood as a limit of PGT, and particular attention will be devoted to the case of parallelizable manifolds. In fact, in this case, the interpretation of the scheme as a gauge theory of translations can be conceived as complementary of a gauge theory of the Lorentz group.

After these remarks, we will show the possibility to restate isometric diffeomorphisms in terms of local Lorentz rotations, both in their finite and infinitesimal character. Since, in our approach, the introduction of a Lorentz gauge field is based on the fact that spinors behave as a representation of the Lorentz group [117], translations (i.e., non-isometric infinitesimal diffeomorphisms) are not included in this gauge picture, because, from this point of view, translations are not distinguishable from generic diffeomorphisms.

A gauge theory for the local Lorentz group on flat space-time is implemented. After
defining the space where local Lorentz transformations take place, we will implement Lorentz gauge transformations for spinor fields, from which the non-Abelian character of the gauge field will be inferred, and a suitable bosonic Lagrangian density will be established, accordingly. Finally, Dirac and Yang-Mills equations will be derived, where the interaction between spinors and the gauge fields shows up [117, 122, 123, 124, 118, 119, 120, 121].

The previous results will be generalized to curved space-time in the sixth Section. Local Lorentz transformations in curved space-time will be analyzed, and two different approaches (second- and first-order formalism, respectively) to implement a Lorentz gauge theory in such a scenario will be followed. Dirac, Yang-Mills, and Einstein equations will be derived in both cases. In particular, the relation between suitable bein projections of the contortion field and Lorentz gauge fields will be hinted in the second-order formalism, and the geometrical hypotheses for such an identification will be investigated in the first-order formalism. An interaction term between the Lorentz gauge field and the spin connections has to be postulated in order to restore the proper mathematical identification from the second Cartan structure equation. First- and second-order approaches will be eventually compared in the linearized regime.

2.1.3 General Relativity as a gauge theory

This section is aimed at describing GR as a gauge model, i.e., at pointing out the features that render the comparison possible, as well as those that make this theory isolated form the gauge picture, such as the metric field, which has no analogues in other known physical interactions.

Metric tensor and tetrad field

Let $M^4$ be a 4-dimensional pseudo-Riemannian manifold, and $e$ a one-to-one map on it, $e : M^4 \rightarrow TM^4_x$, which sends tensor fields on $M^4$ in tensor fields in the Minkowskian tangent space $TM^4_x$: the fields $e^a_\mu$ are called tetrads or vierbein, and are an orthonormal basis for the local Minkowskian space-time; Greek indices ($\mu = 0, 1, 2, 3$) change as tensor ones under general coordinate transformations, while Latin indices

55
(i = 0, 1, 2, 3) refer to local Lorentz transformations. The following relations between the tetrads and the metric field \( g_{\mu\nu} \) of the manifold \( M^4 \) hold

\[
g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \quad e^a_\mu e^a_\mu = \delta^a_\mu, \quad e^a_\mu e^\nu_a = \delta^{\nu}_\mu, \tag{2.29}
\]

where \( \eta_{ab} \) is the metric tensor in the local Minkowski frame. Eq. (2.29) states that, given the tetrad field \( e^a_\mu \), the metric tensor \( g_{\mu\nu} \) is uniquely determined, and all metric properties of the space-time are expressed by the tetrad field, accordingly. It is worth noting that the converse is not true: there are infinitely many choices of the local basis that reproduce the same metric tensor, because of the local Lorentz gauge invariance. This is also the reason why there are more components in \( e^a_\mu \) than in the metric \( g_{\mu\nu} \), the difference being six, i.e., the number of independent rotations in four dimensions.

Rewriting usual tensor equations in a frame formalism allows us to project tensor fields from the 4-dimensional manifold to the Minkowskian space-time, thus emphasizing the local Lorentz invariance of the scheme. Moreover, to assure that the projected derivative of a tensor field be invariant under local Lorentz transformations, the connection 1-forms \( \omega^a_b \) must be introduced, which take values in the adjoint representation of the Lorentz group. They define the covariant exterior derivative operator \( d^{(\omega)} \), whose action on Lorentz-valued differential forms is

\[
d^{(\omega)} v^{a_1,...,a_n}_{c_1,...,c_m} = dv^{a_1,...,a_n}_{c_1,...,c_m} + \sum_{i=1}^{\eta} \omega^a_{b_i} \wedge v^{a_1,...,b_i,...,a_n}_{c_1,...,c_m} + \sum_{j=1}^{m} \omega^a_{d_j} \wedge v^{a_1,...,a_n}_{c_1,...,d_j,...,c_m}. \tag{2.30}
\]

For later purposes, it’s worth remarking that the introduction of the tetrad formalism enables one to include spinor fields in the dynamics, as spinor fields are a particular representation of the Lorentz group.

**Structure equations**

Connection 1-forms lead to the usual definition of the curvature 2-form \( R^a_b \):

\[
d^{(\omega)} \circ d^{(\omega)} v^a = R^a_b \wedge v^b, \tag{2.31}
\]

or

\[
R^a_b = d\omega^a_{b} + \omega^a_{c} \wedge \omega^c_{b}, \tag{2.32}
\]
which is the first Cartan structure equation, and can be rewritten as the Bianchi identity
\[ d(\omega) R^a_b = 0. \]  
(2.33)

In this formalism, the action for GR consists of the lowest-order non-trivial scalar combination of the Riemann curvature 2-form (2.31) and the tetrad fields, that is the Hilbert-Einstein action
\[ S(e, \omega) = \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}. \]  
(2.34)

Variation with respect to connections leads to the second Cartan structure equation in the torsion-less case,
\[ de^a + \omega^a_b \wedge e^b = 0, \]  
(2.35)

which links the tetrad fields to the spin connections and leads to the identity
\[ e^b \wedge R^a_b = 0. \]  
(2.36)

Variation with respect to tetrads leads to the following equations:
\[ \epsilon_{abcd} e^b \wedge R^{cd} = 0, \]  
(2.37)

which give the dynamical Einstein field equations, once the solution of the second Cartan structure equation (2.35) is considered.

We remark that, under local Lorentz transformations, spin-connections transform like a Lorentz gauge vector, and the Riemann 2-form is preserved by such a change. Therefore, in flat space-time, we deal with non-zero spin connections, but a vanishing curvature 2-form.

In both flat and curved space-time, spin connections exhibit the right behavior to play the role of Lorentz gauge fields, and GR has the features of a gauge theory, despite some relevant shortcomings outlined below.

**Lie Derivatives and Killing Vectors**

A coordinate transformation \( x \rightarrow x' \) that does not modify the form of the metric defines the isometry group of a given space. For the infinitesimal transformation \( x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu \), the invariance of the metric is expressed by the Killing equation \( \xi_{\mu, \nu} + \xi_{\nu, \mu} = 0 \). Expanding \( \xi^\mu \) in power series of \( x \), we find that global Poincaré
transformations have the form $\xi^\mu = \epsilon^\mu + \omega^\mu_\nu x^\nu$, and are defined in terms of ten constant parameters $\omega^\mu_\nu = -\omega^\nu_\mu$ and $\epsilon^\mu$, i.e. these transformations are translations and Lorentz rotations.

It is possible to associate a vector field to a one-parameter family of diffeomorphisms [113]. In fact, given a manifold $M$, a one-parameter group of diffeomorphisms $\phi_t$ is a $C^\infty$ map from $\mathbb{R} \times M \to M$ such that for any fixed $t \in \mathbb{R}$, $\phi_t : M \to M$ is a diffeomorphism, $\phi_t \circ \phi_s = \phi_{t+s}$, and $\phi_{t=0}$ is the identity. Then for any point $p \in M, \phi_t(P) : \mathbb{R} \to M$, there is a curve that passes through $p$ at $t = 0$. This way, $v_p$ is the tangent to this curve at $t = 0$. This way, the vector field $v$ is associated to the one-parameter group of transformations, and can be regarded as the generator of these transformations.

Given a manifold $N$ and a function $f$ on it, $f : N \to \mathbb{R}$, such that $f \circ \phi : M \to \mathbb{R}$, $\phi^*$ is defined as $(\phi^* v)(f) = v(f \circ \phi)$.

The Lie derivative $\mathcal{L}_v$ of a smooth tensor field $T^{a_1...a_m}_{b_1...b_n}$ at the point $p$ with respect to $v$ is defined as

$$
\mathcal{L}_v T^{a_1...a_m}_{b_1...b_n} = \lim_{t \to 0} \frac{\phi_t^* T^{a_1...a_m}_{b_1...b_n} - T^{a_1...a_m}_{b_1...b_n}}{t},
$$

and is a linear map from tensors of rank $(m, n)$ to tensors of rank $(m, n)$. Furthermore, $\mathcal{L}_v T^{a_1...a_m}_{b_1...b_n} = 0$ everywhere if $\phi_t$ is a symmetry transformation for $T^{a_1...a_m}_{b_1...b_n}$ for all $t$.

When specified for the features of the gravitational field, the discussion above leads to the definition of isometries for the metric tensor. In fact, if $\phi_t : M \to M$ is a one-parameter group of isometries, $\phi_t^* g_{\mu\nu} = g_{\mu\nu}$, the vector field $\xi$ that generates $\phi_t$ is a Killing vector field, and $\xi$ satisfies the Killing equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.

Despite the use of the Lie derivative is the most appropriate and synthetic approach to the characterization of the properties of the local translation group, in what follows we will retain the usual space-time covariant formulation in order to keep the mathematical formulation of the gauge theory of the Lorentz group as close as possible to its physical implementation at the level of field dynamics, in agreement with common approaches [113].
2.1 Introduction

Physical interpretation

So far, it is possible, on the one hand, to appreciate the similarity with gauge theories, the role of connection fields being played by spin connections, $\omega^a_{\mu b}$, and, on the other hand, to remark that the presence of the tetrad field, introduced by the Principle of General Covariance, is an ambiguous element for a gauge paradigm. This scenario would be appropriate if the theory were based on two independent degrees of freedom. Since spin connections can be uniquely determined as functions of tetrad fields, (2.35), then this correlation opens a puzzle in the interpretation of these connections as the only fundamental fields of the gauge scheme.

Nonetheless, it is just the introduction of fermions that requires to treat local Lorentz transformations as the real independent gauge of GR. In fact, when spinor fields are taken into account, their transformations under the local Lorentz symmetry imply that the Dirac equation is endowed with non-zero spin connections even on Minkowskian space-time, as developed in section 5.

Because of the behavior of spinor fields, it becomes crucial to investigate whether diffeomorphisms can be reinterpreted to some extent as local Lorentz transformations.

2.1.4 Poincaré gauge theory

In this section, we analyze a well-established proposal to connect the presence of torsion with the local nature of the Poincaré symmetry. This approach represents a comparison scheme for the present developments of the model outlined here. PGT will be described from both a gauge and a geometrical point of view, and particular attention will be payed to the physical meaning of field equations, which predict a contact interaction, i.e., a non-propagating gauge field.

Gauge approach

Let us consider an infinitesimal global Poincaré transformation in Minkowski space,

$$x^\mu \rightarrow x'^\mu = x^\mu + \tilde{\epsilon}^\mu_\nu x^\nu + \tilde{\epsilon}^\mu,$$  

(2.39)
and the consequent transformation law for spinor fields

\[ \psi(x) \rightarrow \psi'(x) = \left( 1 + \frac{1}{2} \varepsilon^{\mu \nu} M_{\mu \nu} + \varepsilon^{\mu} P_{\mu} \right) \psi(x), \]  

(2.40)

where the generators \( M_{\mu \nu} = L_{\mu \nu} + \Sigma_{\mu \nu} \) and \( P_{\mu} \) obey Lie-algebra commutation relations. If the matter Lagrangian density is assumed to depend on the spinor field and on its derivatives only, \( L = L(\psi, \partial_{\mu} \psi) \), and if the equations of motion are assumed to hold, the conservation law \( \partial_\mu J^\mu = 0 \) is found, where

\[ J^\mu = \frac{1}{2} \varepsilon^{\nu \lambda} M_{\nu \lambda}^\mu - \varepsilon^{\nu} T^\mu_{\nu}, \]  

(2.41)

where the canonical energy-momentum and angular-momentum tensors are defined, respectively, as

\[ T^\mu_{\nu} = \frac{\partial L}{\partial \psi_{\nu, \mu}} \partial_{\nu} \psi - \delta_{\nu}^\mu L, \]  

(2.42)

\[ M^\mu_{\nu \lambda} = (x_{\nu} T^\mu_{\lambda} - x^\lambda T^\mu_{\nu}) - S^\mu_{\nu \lambda} \equiv \]  

\[ \equiv (x_{\nu} T^\mu_{\lambda} - x^\lambda T^\mu_{\nu}) + \frac{\partial L}{\partial \psi_{\nu, \mu}} \Sigma_{\nu \lambda} \psi. \]  

(2.43)

Because the parameters in (2.41) are constant, according to the Noether theorem, conservation laws for the energy-momentum current and for the angular-momentum current, together with the related charges, are established:\footnote{If the integration on the boundaries of the 3-space brings vanishing contributions.}

\[ \partial_{\nu} T^\nu_{\nu} = 0 \Rightarrow P^\nu = \int d^3 x T^{0 \nu} \]  

(2.44)

\[ \partial_{\nu} M^\mu_{\nu \lambda} = 0 \Rightarrow M_{\nu \lambda} = \int d^3 x M^0_{\nu \lambda}. \]  

(2.45)

When transformations are locally implemented, eq.s (2.41)-(2.45) do not hold any more, and compensating gauge fields have to be introduced in order to restore local invariance. As a first step, a covariant derivative \( D_a \psi \) is defined as

\[ D_a \psi = e_a^\mu D_\mu \psi = e_a^\mu (\partial_\mu + A_\mu) \psi = e_a^\mu \left( \partial_\mu + \frac{1}{2} A_\mu^{ab} \Sigma_{ab} \right) \psi, \]  

(2.46)

where the compensating fields \( e^k_\mu \) and \( A^{ij}_\mu \), and the generator \( \Sigma_{ij} \) have been taken into account. This way, the Lagrangian density depends on the covariant derivative.
of the fields, instead of the ordinary one, $L = L(\psi, D_\mu \psi)$; covariant derivatives (2.46) do not commute, but satisfy the commutation relation

$$[D_\mu, D_\nu] \psi = \frac{1}{2} F^\nu_{\mu\rho} \Sigma_{\rho \psi},$$

$$[D_\sigma, D_\tau] \psi = \frac{1}{2} F^\sigma_{\tau\rho} \Sigma_{\tau \psi} - F^\rho_{\tau\mu} D_\mu \psi,$$  \hspace{1cm} (2.47)

where $F^\mu_{\rho \sigma}$ and $F^\rho_{\tau \mu}$ are the Lorentz field strength and the translation field strength, respectively.

Covariant energy-momentum and spin currents, $T^\nu_{\mu}$ and $S^\nu_{\mu}$, can be found, in analogy with the global case, after the substitution $\partial_\nu \rightarrow D_\nu$, and are found to be equivalent, if the equation of motion for matter fields are assumed to hold, to the dynamical currents $\tau^\nu_{\mu}$ and $\sigma^\nu_{\mu}$:

$$T^\nu_{\mu} = \frac{\partial L}{\partial D_\mu \psi} D_\nu \psi - \delta^\nu_{\mu} L \equiv \tau^\nu_{\mu} = e^\nu_{\mu} \frac{\partial L}{\partial e^\nu_{\mu}},$$  \hspace{1cm} (2.48)

$$S^\nu_{\mu} = - \frac{\partial L}{\partial D_\mu \psi} \Sigma_{\mu \psi} \equiv \sigma^\nu_{\mu} = - \frac{\partial L}{\partial A^\mu_{\mu}}.$$  \hspace{1cm} (2.49)

As outlined in [91], it is possible to infer the inadequacy of special relativity to describe the behavior of matter fields under local Poincaré transformations. Global Poincaré transformations preserve distances between events and the metric properties of neighboring matter fields: comparing field amplitudes before performing the transformation, and then transforming the result, or comparing the transformed amplitudes of the fields is equivalent. This property is known as rigidity condition, as matter fields behave as rigid bodies under this kind of transformations. On the contrary, it can be shown that the action of local Poincaré transformations can be interpreted as an irregular deformation of matter fields, thus predicting different phenomenological evidences for the field and for the transformed field.

The compensating gauge fields $e^\nu_{\mu}$ and $A^\mu_{\mu}$, introduced to restore local invariance, describe geometrical properties of the space-time, as it can be easily argued from a geometrical point of view.

**Geometrical approach**

---

2.1 Introduction
The geometrical approach to PGT can be carried out by considering the most general
metric-compatible linear connections,

\[ \Gamma^\mu_{\nu\rho} = \left\{ \frac{\mu}{\nu,\rho} \right\} - K^\mu_{\nu\rho}, \quad (2.50) \]

where \( \left\{ \frac{\mu}{\nu,\rho} \right\} \equiv \left\{ \frac{\mu}{\rho,\nu} \right\} \) are the Christoffel symbols, and \( K^\mu_{\nu\rho} \equiv K^\mu_{\rho\nu} \) the contortion tensor, with 24 independent components, which can be written as a function of the
torsion field \( T^\mu_{\nu\rho} \),

\[ K^\mu_{\nu\rho} = -\frac{1}{2} \left( T^\mu_{\nu\rho} - T^\mu_{\rho\nu} + T^\mu_{\nu\rho} \right). \quad (2.51) \]

Geometric covariant derivatives are defined as

\[ D_\mu \psi = (\partial_\mu + \omega_\mu) \psi = \left( \partial_\mu + \frac{1}{2} \omega^{ab}_\mu \Sigma_{ab} \right) \psi, \quad (2.52) \]

where the spin connections \( \omega^{ab}_\mu \) consist of the bein projections of the Ricci rotation
coefficients and the contortion field, respectively: \( \omega_{ab\mu} = R_{a\mu} + K_{a\mu} \).

The gauge potentials \( e_a^\mu \) are generally interpreted as the relation between the orthonormal frame (Greek indices) and the coordinate one (Latin indices), while the
introduction of the gauge potentials \( \omega^{ab}_\mu \) is connected with the corresponding rotations of the orthonormal basis at neighboring points: this induces a change in the derivative
operator, i.e.

\[ \partial_\mu \rightarrow D_\mu \equiv \partial_\mu + \frac{1}{2} \omega^{ab}_\mu \Sigma_{ab}. \quad (2.53) \]

Torsion contributes to the gravitational dynamics, according to its gravitational ac-
tion: it has been illustrated [129] that the most general form for a Lagrangian \( L_T \)
(which allows for equation of motion that are at most of second order in the field
derivatives) is

\[ L_T = AT_{abc} T^{abc} + BT_{abc} T^{bac} + CT_a T^a \equiv b_{abc} T^{abc}. \quad (2.54) \]

where \( T_a = T^b_{ba} \) and \( b_{abc} = a( A T_{abc} + B T_{[abc]} + C \eta_{a[b} T_{c]} \). The values of the parameters
\( A, B, C \) are to be determined according to the Physics that has to be described, and
some relevant examples are discussed in [130, 111, 112, 109].
2.1 Introduction

Discussion

The comparison of gauge and geometrical approaches leads to the identification of the Lorentz gauge fields $A_{\mu}^{ab}$, which accounts for local Lorentz transformations, with the spin connections $\omega_{\mu}^{ab}$, and the fields $e_{a}^{\mu}$, which describe translations, with the components of the tetrad field. This way, the identifications of the Lorentz field strength with curvature, and that of the translation field strength with torsion, are straightforward.

After the introduction of covariant derivatives, the coordinate representation of the generators of translations, $P_{\mu} \rightarrow \partial_{\mu}$, changes as $P_{\mu} \rightarrow D_{\mu}$ with the implementation of local transformations; the variation of matter fields, in the two cases, differs by a local Lorentz rotation, thus mixing up the concept of translations and Lorentz rotations [131].

The analysis of [132, 133], based on the conception of Poincaré symmetry as a purely inner symmetry on Minkowskian background, sheds light on the relation between gravity and the structure of spacetime. According to this scheme, the global action of the Poincaré group is analogous to the description of the action of inner-symmetry groups as groups of generalized rotations in the field space. The interpretation of the resulting theory as a gauge theory of gravitation is achieved by requiring that the corresponding local invariance show up as invariance under general coordinate transformations and local $SO(3, 1)$ frame rotations.

Furthermore, the usual Einstein-Hilbert action for gravity in four dimensions is not invariant under the translational part of the Poincaré group, while is invariant under the Lorentz group.

Because of these unpleasant features, the straightforward interpretation of PGT as a gauge theory of gravity implies a non-trivial modification of the geometrodynamics. Field equations read:

$$\frac{1}{e} D_{\mu} \left( ee_{a}^{\nu} e_{b}^{\mu} \right) = S_{ab}^{\nu}, \quad (2.55a)$$

$$R_{ab}^{\mu} - \frac{1}{2} e_{a}^{\mu} P_{cd} = T_{\mu}^{a}. \quad (2.55b)$$

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The first equation provides the expression of the connections of the rotation group as a function of the connections of the translation group and the matter fields, while the second equation is the Einstein dynamical equation: tensor fields involved in the equations above must satisfy identities (2.33) and (2.36). By the geometrical identification of covariant gauge derivatives, \((2.49)\) becomes an algebraic relation between spin and torsion: since the relation is not differential, torsion is not predicted to propagate, but its existence is bound to the presence of spin-\(\frac{1}{2}\) matter fields, i.e., torsion is not a real dynamical field.

For later convenience, it will be useful to restate the description of PGT in a slightly different formalism, which allows for a better explanation of the role of spin, so that the analogies and differences with the Lorentz gauge theory will be more noticeable. Eq. \((2.40)\) can be written as

\[
\psi (x) \rightarrow \psi' (x) = \left(1 + \frac{1}{2} \epsilon^{\mu \nu} \Sigma_{\mu \nu} + \epsilon^\mu P_\mu \right) \psi (x),
\]

where \(\epsilon^\mu \equiv \tilde{\epsilon}^\mu + \tilde{\epsilon}_\nu \delta^\mu_\rho x^\rho\), \(\epsilon^\alpha \beta = \tilde{\epsilon}^\alpha \beta\), and the generators of translations and spin rotations satisfy the relations:

\[
\begin{align*}
[\Sigma_{ab}, \Sigma_{cd}] &= \eta_{c[a} \Sigma_{b]d} - \eta_{d[a} \Sigma_{b]c}, \\
[\Sigma_{ab}, P_c] &= -\eta_{c[a} \partial_{b]}, \quad [P_a, P_b] = 0.
\end{align*}
\]

The advantage of \((2.56)\) consists in keeping pure rotations separated from translations. The orbital angular momentum is this way kept independent of the spin angular momentum: the former is strictly related with the energy-momentum, thus with the rotation-dependent part of \(\epsilon^\mu\), while the latter is connected with the pure-rotation parameter \(\epsilon^{\mu \nu}\). In fact, if the analogy is drawn between a generic diffeomorphism and a global Poincaré transformation, it is impossible to perform translations and rotations independently, but, when a localized symmetry is considered, this becomes possible, because the parameters defining the transformation are allowed to vary freely.

### 2.1.5 Teleparallelism

An interesting limit of PGT is Weitzenböck or teleparallel geometry, defined by the requirement

\[
R^a_{\mu \nu} (A) = 0.
\]
Teleparallel geometry (see, for example, [109] for a hand-on review and all the references therein) can be interpreted, to some extents, as complementary to Riemannian curvature vanishes, and torsion remains to characterize the parallel transport. The physical interpretation of such a geometry relies on the fact that there is a one-parameter family of teleparallel Lagrangians which is empirically equivalent to GR [130, 111].

### 2.1 Introduction

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**Lagrangian and Field equations**

Within this framework, the gravitational field is described by tetrads \( e^a \_\mu \) and Lorentz connections \( A^{ab} \_\mu \), where (2.58) has to be taken into account. For our purposes, it is useful to consider the class of Lagrangians quadratic in torsion

\[
L_{TP} = b L_T + \lambda^{\mu\nu} R_{\mu\nu} - L_M, \tag{2.59}
\]

where \( \lambda^{\mu\nu} \) are Lagrange multipliers introduced to ensure condition (2.58) in the variational formalism, and \( L_T \) has been defined in (2.60a). Variation of (2.59) with respect to \( e^a \_\mu \), \( A^{ab} \_\mu \) and \( \lambda^{\mu\nu} \) lead to the following field equations:

\[
\begin{align}
4 \nabla_\mu (e^a \_\beta_{\mu\rho}) - 4 b^b \beta_{\epsilon a} T_{bca} + e_a \_\mu e L_T &= \tau_i^\mu, \tag{2.60a} \\
4 \nabla_\mu \lambda_{ab}^{\mu\rho} - 8 e^a \_{\beta_{ab}} \rho^\nu &= \sigma^\mu_{ab}, \tag{2.60b} \\
R_{\mu\nu}^{ab} &= 0. \tag{2.60c}
\end{align}
\]

Equation (2.60c) ensures (2.58) from variational principles.

Equation (2.60a) is a dynamical equation for \( e^a \_\mu \).

The only role of (2.60b) is to determine the Lagrange multipliers \( \lambda_{ab}^{\mu\nu} \), and the non-uniqueness of \( \lambda_{ab}^{\mu\nu} \) is related to an extra gauge freedom in the theory. In fact, the gravitational Lagrangian (2.59) is, by construction, invariant under the local Poincaré transformations, and, up to a four-divergence, under the transformations [134]

\[
\delta \lambda_{ab}^{\mu\nu} = \nabla_\rho \varepsilon_{ab}^{\mu\nu\rho} \rightarrow \delta \varepsilon_{ab}^{jk} \nabla_0 \varepsilon_{ab}^{jk} + \varepsilon_{ab}^{jk} \delta \lambda_{ab}^{0j} = \nabla_i \varepsilon_{ab}^{ij} \tag{2.61}
\]

where the gauge parameter \( \varepsilon_{ab}^{\mu\nu\rho} = - \varepsilon_{ab}^{\mu\nu\rho} \) is completely antisymmetric in its upper indices, and \( \varepsilon_{ab}^{\mu\nu} \equiv \varepsilon_{ab}^{\mu\nu\rho}(\text{the invariance of (2.60b) follows directly from } R_{\mu\nu}^{ab} = 0). \)

One can show that the only independent parameters of the \( \lambda \) symmetry are \( \varepsilon_{ab}^{ij} \), so that the six parameters \( \varepsilon_{ab}^{ijk} \) are not independent of \( \varepsilon_{ab}^{ij} \) and can be completely discarded, leaving 18 independent gauge parameters, which can be used to fix 18
multipliers $\lambda_{ab}^{\mu\nu}$, whereupon the remaining 18 multipliers are determined by the independent field equations (2.60b) (at least locally). The gauge structure of such a one-parameter teleparallel theory is believed to be still problematic [135, 136]

**Orthonormal frames**

If a manifold is parallelizable (which is a quite strong topological restriction), the vanishing of curvature implies that the parallel transport is path independent, so that the resulting tetrads are globally well defined. In such an orthonormal teleparallel frame, the connection coefficients vanish:

$$A_{\mu}^{ab} = 0. \quad (2.62)$$

This construction is not unique, but it defines a class of orthonormal frames, related to each other by global Lorentz transformations. In such a frame, the covariant derivative reduces to the partial one, and the torsion takes the simple form: $T_{\mu}^{a\nu} = \partial_{\nu}e_{\mu}^a - \partial_{\mu}e_{\nu}^a$.

Equation (2.62) defines a particular solution of the condition $R_{\mu\nu}^{ab}(A) = 0$. Since a local Lorentz transformation of the tetrad field induces a non-homogeneous change in the connection,

$$e_{\mu}^a = A_{\mu}^{ab}e_{\mu}^b \quad \Rightarrow \quad A_{\mu}^{ab} = \Lambda_{c}^{a}A_{\mu}^{bc} + \Lambda_{c}^{a}\partial_{\mu}\Lambda_{bc},$$

it follows that the general solution of $R_{\mu\nu}^{ab}(A) = 0$ has the form $A_{\mu}^{ab} = \Lambda_{c}^{a}\partial_{\mu}\Lambda_{bc}$. Thus, the choice (2.62) breaks local Lorentz invariance, and represents a gauge fixing condition.

**Discussion**

In (2.59), the teleparallel condition is ensured by the presence of the Lagrange multiplier. Equation (2.60b) merely serves to determine the multiplier, while the non-trivial dynamics is completely contained in (2.60a). So far, teleparallel theory (on parallelizable manifolds) may also be described by imposing the gauge condition (2.62) directly in the action. The resulting theory is defined in terms of the tetrad field only, and may be thought of as the gauge theory of translations.

The consistency of teleparallel gravity when spinning matter is taken into account has also been discussed within the framework of the teleparallel limit of PGT...
[137, 138]. In [137], an inconsistency, due to frame dependence, was illustrated to arise for every gauge theory of the Poincaré group that admits a teleparallel limit in the absence of spinning matter. Furthermore, in [138], a restricted class of transformations was found, according to which the frame invariance of the gravitational Lagrangian does not lead to inconsistencies, even as far as Standard-Model particles are concerned, and experimental aspects were analyzed.

### 2.1 Introduction

In [137], an inconsistency, due to frame dependence, was illustrated to arise for every gauge theory of the Poincaré group that admits a teleparallel limit in the absence of spinning matter. Furthermore, in [138], a restricted class of transformations was found, according to which the frame invariance of the gravitational Lagrangian does not lead to inconsistencies, even as far as Standard-Model particles are concerned, and experimental aspects were analyzed.

#### 2.1.6 Generalized Lagrangian in MAG

For the sake of completeness, and for comparison with some results of this chapter, we briefly review the most general parity-preserving Lagrangian in metric-affine gravity, when torsion and non-metricity are taken into account.

Different groups have already added, within a metric-affine framework, different quadratic pieces to the Hilbert–Einstein–type Lagrangian, see [139, 140, 141, 142, 143, 144, 145, 146, 147, 148], e.g., and references given there. The end result of all these deliberations is the most general parity conserving quadratic Lagrangian which is expressed in terms of the $4 + 3 + 11$ irreducible pieces (see [149]) of $Q_{\alpha\beta}$, $T^\alpha$, $R^\alpha_{\beta}$, respectively:

$$V_{\text{MAG}} = \frac{1}{2\kappa} \left[ -a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda \eta + T^\alpha \wedge \left( \sum_{l=1}^{3} a_l^{(l)} T_\alpha \right) 
+ 2 \left( \sum_{l=2}^{4} c_l^{(l)} Q_{\alpha\beta} \right) \wedge \vartheta^\alpha \wedge T^\beta + Q_{\alpha\beta} \wedge \left( \sum_{l=1}^{4} b_l^{(l)} Q^\alpha_{\beta} \right) 
+ b_5 \left( \sum_{l=3}^{4} Q_{\alpha\gamma} \wedge \vartheta^\alpha \right) \wedge \left( \sum_{l=2}^{5} T^\alpha \wedge \vartheta_\beta \right) \right]$$

$$- \frac{1}{2\rho} R^{\alpha\beta} \wedge \left( \sum_{l=1}^{6} w_l^{(l)} W_{\alpha\beta} + w_7 \vartheta_\alpha \wedge (e_\gamma)^{(5)} W^\gamma_{\beta} \right)$$

$$+ \sum_{l=1}^{5} z_l^{(l)} Z_{\alpha\beta} + z_6 \vartheta_\gamma \wedge (e_\alpha)^{(2)} Z^\gamma_{\beta} + \sum_{l=7}^{9} z_l \vartheta_\alpha \wedge (e_\gamma)^{(l-4)} Z^\gamma_{\beta} \right)$$

(2.63)

The constant $\lambda$ is the cosmological constant, $\rho$ the strong gravity coupling constant, the constants $a_0, \ldots, a_3, b_1, \ldots, b_5, c_2, c_3, c_4, w_1, \ldots, w_7, z_1, \ldots, z_9$ are dimensionless. We have introduced in the curvature square term the irreducible pieces of the antisymmetric part $W_{\alpha\beta} := R_{[\alpha\beta]}$ and the symmetric part $Z_{\alpha\beta} := R_{(\alpha\beta)}$ of the curvature.
2–form. In $Z_{\alpha\beta}$, we have the purely post–Riemannian part of the curvature. Note the peculiar cross terms with $c_I$ and $b_5$.

Esser [150], in the component formalism, has carefully enumerated all different pieces of a quadratic MAG Lagrangian, for the corresponding nonmetricity and torsion pieces, see also Duan et al. [141]. Accordingly, Eq.(2.63) represents the most general quadratic parity–conserving MAG–Lagrangian. All previously published quadratic parity–conserving Lagrangians are subcases of (2.63). Hence (2.63) is a safe starting point for our future considerations.

We analyze Yang–Mills type Lagrangians. Since $V_{\text{MAG}}$ is required to be an odd 4–form, if parity conservation is assumed, we have to build it up according to the scheme $F \wedge *F$, i.e. with one Hodge star, since the star itself is an odd operator. Also the Hilbert–Einstein type term is of this type, namely $\sim R^{\alpha\beta} \wedge *(\partial_\alpha \wedge \partial_\beta)$, as well as the cosmological term $\sim \eta = *1$. Thus $V_{\text{MAG}}$ is homogeneous of order one in the star operator. It is conceivable that in future one may want also consider parity violating terms with no star appearing (or an even number of them) of the (Pontrjagin) type $F \wedge F$. Typical terms of this kind in four dimensions would be

$$R^{\alpha\beta} \wedge (\partial_\alpha \wedge \partial_\beta), \quad 1, \quad T^\alpha \wedge T_\alpha, \quad Q_{\alpha\beta} \wedge \partial^\alpha \wedge T^\beta, \quad R^{\alpha\beta} \wedge R_{\alpha\beta}. \quad (2.64)$$

The first term of (2.64), e.g., represents the totally antisymmetric piece of the curvature $R^{[\gamma\delta\alpha\beta]} \partial_\gamma \wedge \partial_\delta \wedge \partial_\alpha \wedge \partial_\beta$. Such parity–violating Lagrangians have been studied in the past, see, e.g., [151, 152] and [153, 154].

### 2.2 The role of the Lorentz group

In what follows, we will discuss separately the two cases of flat and curved spacetime as far as the implementation of the geometrical gauge proposal for the Lorentz symmetry is concerned (PL3), (PL4), (PL8), (PL13), (PL17). Here, we wish to fix some key points, which are at the ground of the physical motivation for such a Lorentz gauge theory as a non-Riemannian effect. The local Lorentz invariance of GR has the role of a real gauge symmetry in the sense that the corresponding changes play no physical role in the space-time dynamics. Therefore, spin connections are not physical fields, but just gauge potentials, subject to such transformations that do not alter the curvature tensor, describing the Einsteinian meaning of the gravitational interaction. The diffeomorphism invariance of GR is related to the formal transcription of a
physical property, i.e., the covariance of physical laws under changes of the reference system. Such an invariance is well formulated in terms of tensor quantities, vanishing or not in every reference frame. Spin connections transform like vectors, as far as diffeomorphisms are concerned, and can never vanish on a curved spacetime. Thus, if we are able to show, as we will argue in the following, that diffeomorphism invariance induces local Lorentz rotations, then we can conclude that spin connections can no longer be regarded to as good gauge potentials for these rotations, because of their transformation properties. In particular, on flat spacetime, these variables can be taken as vanishing, by choosing the tetrad vectors as $e^a_\mu = \delta^a_\mu$, such that they must remain identically zero under diffeomorphisms. For these coordinate transformations, which can be interpreted as local rotations, this behavior makes them inappropriate to restore local Lorentz invariance. In such a scheme, the request for external fields of Lorentz gauge connections appears well grounded. This picture remains still valid in curved spacetime, as long as we accept that the ambiguity of spin-connection transformations must be solved in favor of their tensor nature, when diffeomorphism-induced rotations are implemented. By other words, under real gauge transformations, spin connections feel a gauge transformation, but the external fields transform only according to their Lorentz indices. Nevertheless, when local rotations are induced by coordinate changes, the only fields able to restore the request of Lorentz invariance are the latter. In fact, the nature of gauge potentials is naturally lost by gravitational connections, behaving like tensors only. We are now going to demonstrate that, both in the infinitesimal and in the finite case, such correspondence between coordinate changes and local rotations takes place only if we deal with isometric diffeomorphisms. This request is naturally expected if we want to reproduce a Lorentz symmetry, and we stress that it always holds for this restricted class of diffeomorphisms on a Minkowski space (at the ground of the present paradigm).

**Finite gauge transformations**  
Let us consider the coordinate transformation corresponding to the special case of an isometry of the spacetime, i.e., $x' = x'(x)$. The conditions ensuring invariance of the metric tensor in term of the bein vectors is

$$e^a_\mu dx'^\nu = \Lambda^a_b(x)c^b_\nu(x)dx^\nu,$$  

(2.65)
where Λ denotes a local rotation. This condition is easily restated in terms of the following gradients

\[
\frac{dx^\mu'}{dx^\nu} = \Lambda^a_b(x)e^b_\nu(x)e^\mu'(x). \tag{2.66}
\]

As far as isometric diffeomorphisms are concerned, we get the key relation

\[
e^a_\nu(x) = \frac{dx^a'}{dx^\nu} = \Lambda^a_b(x)e^b_\nu(x). \tag{2.67}
\]

This way, we see that the isometric component of a diffeomorphism is formally indistinguishable from a local rotation of the bein basis, i.e., the physical implementation of a local Lorentz gauge.

**Infinitesimal case**  The explanation of the puzzling scenario depicted at the end of Section (2.1.3) is hinted by the relation between diffeomorphisms and local Lorentz transformations. A generic diffeomorphism \( \phi : M^4 \rightarrow M^4 \) maps the orthonormal basis fields \( e_\mu^a \) into \( \phi^* \left( e_\mu^a \right) \), which are not orthonormal in every point of the manifold, and do not represent any family of physically-realizable observers; on the contrary, an isometric diffeomorphism induces orthonormal transformed basis: an isometry generates a local Lorentz transformation of the basis. An infinitesimal isometric diffeomorphism, described by

\[
x'^\mu = x^\mu + \xi^\mu(x), \quad \text{and} \quad \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{2.68}
\]

induces a transformation of the basis vectors,

\[
e'^a_\mu(x') = e^\nu_a(x) + e^\nu_a(x)\partial_\nu \xi^\mu(x). \tag{2.69}
\]

Since

\[
e'^a_\mu(x) = \Lambda^b_a(x)e^\mu_b(x) = \left( \delta^b_a + \epsilon^b_a \right)e^\mu_b(x), \tag{2.70}
\]

the rotation coefficients \( \epsilon^b_a \) of an infinitesimal Lorentz transformation can be written as a function of both the tetrad fields and the generic displacement \( \xi^\mu(x) \).

On curved space-time, there exist two different local Lorentz transformations, which coincide in general flat space-time footnote It is worth remarking that active and passive translations are discussed within the framework of translation gauge too, as outlined, for example, in [127].. Active Lorentz transformations are due to the action of the Lorentz group on vectors
2.2 The role of the Lorentz group

$V^\mu$ and spinors $\psi$ on the tangent bundle, i.e., $V^\mu \to \Lambda(x)^\mu_a V^\nu$ and $\psi \to S(\Lambda)(x)\psi$, respectively, and are mathematically represented by a Lorentz matrix depending on position and defined everywhere,

$$W^{a_1a_2\ldots}_{b_1b_2\ldots} = \Lambda(x)^{a_1}_{c_1} \ldots \Lambda^{-1}(x)^{d_1}_{a_1} \ldots W^{c_1c_2\ldots}_{d_1d_2\ldots},$$

$$\psi' = S(\Lambda(x))\psi, \quad \bar{\psi}' = \bar{\psi}S^{-1}(\Lambda(x)). \quad (2.71)$$

Passive Lorentz transformations are due to isometric diffeomorphisms of the manifold, which pull-back the local basis in the generic point $P$,

$$\phi_\ast (e^a) = \Lambda(x)^a_b e^b \Rightarrow \Lambda(P)^a_b = e^a_\mu \frac{\partial x^\mu}{\partial x'^b} \bigg|_{x'=P} e^\nu_b. \quad (2.72)$$

While active transformations do not involve coordinates and are defined everywhere once the matrix function $\Lambda(x)^a_b$ is assigned, passive transformations can be reduced to a local Lorentz transformation only in the generic point $P$, acting as a pure diffeomorphisms in the other points on the manifold.

Active and passive Lorentz transformations can be demonstrated to coincide in curved space-time too, as it can be inferred by the comparison of the transformation laws for the tetrad field under Lorentz and world transformations. In fact, for local Lorentz transformations, one gets

$$e^a_\mu(x') = \Lambda^a_b(x') e^a_\mu(x'), \quad (2.73)$$

while, for world transformations,

$$e^a_\mu(x) \to e'^a_\mu(x') = e^a_\mu(x) \frac{\partial x^\mu}{\partial x'^\nu} \approx e^a_\mu(x) + e^a_\mu(x) \frac{\partial \xi^\mu}{\partial x'^\nu}. \quad (2.74)$$

The comparison leads to

$$e'^a_\mu(x') = e^a_\mu(x') + e^b_\mu(x') e^a_b, \quad (2.75)$$

where

$$e^a_b \equiv -D_b \xi^a - R^a_{bc} \xi^c \quad (2.76)$$

and $\lambda_{abc} = R_{abc} - R_{bac}$ are the anholonomy coefficients. To pick up local Lorentz transformations from the set of generic diffeomorphisms, the isometry condition $\nabla_{(\mu} \xi_{\nu)} = 0$ must be taken into account, so that in (2.76) only the anti-symmetric part of $D_b \xi^a$ does not vanish. Finally, we get

$$\epsilon_{ab} = D_{[a} \xi_{b]} - R_{abc} \xi^c, \quad (2.77)$$

which is anti-symmetric, i.e. $\epsilon_{ab} = -\epsilon_{ba}$. 

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Discussion These considerations helped us focus general attention on the use of local passive Lorentz rotations, and search a consistent formulation for the physical nature of the corresponding gauge fields. The difference between flat and curved spacetime is emphasized because, in the former case, we are allowed to extract physical information on the proposed gauge fields in the simple case of vanishing spin connections, while, in the second case, details on the interaction between gravity and gauge fields can be outlined by fixing the extended dynamical equations.

2.2.1 Flat Space-Time

This section is dedicated to the construction of a gauge description of the pure Lorentz group on a flat Minkowski space-time. The choice of flat space is due to the fact that, in this case, the Riemann curvature tensor vanishes and, consequently, the usual spin connections $\omega^{ab}$ can be set to zero (but, in general, they are allowed to be non-vanishing in view of local Lorentz invariance). This allows one to introduce Lorentz-valued connections as the gauge field of passive local Lorentz transformations on flat space-time as far as the correspondence between an infinitesimal diffeomorphism and a local Local rotation is recovered, as shown in the previous section.

Tensors

Let $M^4$ be a 4-dimensional flat manifold: the metric tensor $g_{\mu\nu}$ reads

$$ g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} = \eta_{\alpha\beta} \epsilon_{\mu}^{\alpha} \epsilon_{\nu}^{\beta}, \quad \text{(2.78)} $$

where $\epsilon_{\mu}^{\alpha}$ are bein vectors, $x^\alpha$ are Minkowskian coordinates, and $y^\mu$ are generalized coordinates. For an infinitesimal generic diffeomorphism

$$ y^\mu \equiv x^\alpha \rightarrow y'^\mu(y) = y^\mu(x^\alpha) = x^\alpha + \xi^\alpha(x^\gamma) \quad \text{(2.79)} $$

and for an infinitesimal local Lorentz transformation

$$ x^\alpha \rightarrow x'^\alpha = x^\alpha + \epsilon^\alpha_{\beta}(x^\gamma)x^\beta, \quad \text{(2.80)} $$

the behavior of a vector field $V_{\alpha} \rightarrow V'_{\alpha}$ must be the same: from the comparison of the two transformation laws

$$ V'_{\mu}'(y') = V_{\mu}(y') + \frac{\partial \xi^\gamma(y)}{\partial y^\mu} V_{\gamma}(y'), \quad \text{(2.81)} $$
V'_α(x'^γ) = V_α(x^γ) + \epsilon_α^β V_β(x^γ), \quad (2.82)

respectively, the identification

\[ \epsilon_α^β \equiv \frac{\partial \xi_α(x^γ)}{\partial x^β} \quad (2.83) \]

is possible, where the isometry condition

\[ \partial_β ξ_α + \partial_α ξ_β = 0 \quad (2.84) \]

has been taken into account in order to restore the proper number of degrees of freedom of Lorentz transformations, 10, out of that of generic diffeomorphisms, 16.

The coordinate transformation that induces vanishing Christoffel symbols in the point \( P \) is

\[ x_α^P = x_α^{tb} + \frac{1}{2} \left[ \Gamma_α^β_δ \right]_P x_β^{tb} x_δ^{tb}, \quad (2.85) \]

where \( tb \) refers to the tangent bundle: the comparison with a generic diffeomorphism (2.79) leads to the identification in the point \( P \)

\[ y^μ(x^α)_P = x_α^{tb} + \frac{1}{2} \left[ \Gamma_α^β_δ \right]_P x_β^{tb} x_δ^{tb} - ξ^α, \quad (2.86) \]

i.e. the coordinates of the tangent bundle are linked point by point to those of the Minkowskian space through the relation (2.85), and they differ for the presence of the infinitesimal displacement \( ξ \).

### Spinors

The action describing the dynamics of spin-\( \frac{1}{2} \) fields on a 4-dimensional flat Minkowski manifold \( M^4 \) is

\[ S = \frac{i}{2} \int d^4x \left( \overline{\psi} \gamma^α ∂_α \psi - (∂_α \overline{\psi}) \gamma^α \psi \right), \quad (2.87) \]

where \( \gamma^α \) are Dirac matrices, and is invariant under global Lorentz transformations, which act on the spinor fields \( \psi \) and \( \overline{\psi} \) as

\[ \psi \rightarrow S(Λ) \psi, \quad \overline{\psi} \rightarrow \overline{\psi} S^{-1}(Λ), \quad (2.88) \]

where \( S(Λ) \) is a nonsingular function of the Lorentz matrix \( Λ \), if the \( γ \) matrices transform like vectors,

\[ S(Λ) \gamma^α S^{-1}(Λ) = Λ^α_β γ^b, \quad (2.89) \]
$S(\Lambda)$ for infinitesimal proper transformations $((\Lambda)^a_b = \delta^a_b + \epsilon^a_b$, with $\epsilon^{ab} = -\epsilon^{ba}$ and $\epsilon^a_b \ll 1$) reads

$$S(\Lambda) = 1 - \frac{i}{4} \epsilon^{ab} \Sigma_{ab},$$

(2.90)

where $\Sigma_{ab}$ are the generators of the Lorentz group2.

If this scenario is generalized, i.e., if accelerated coordinates, which are related to Minkowskian coordinates through (2.78), are taken into account, spinor fields, differently from vector fields, have to recognize the isometric components of the diffeomorphism (2.83) as a local Lorentz transformation. In fact, bein vectors $e_\mu^a$ defined in (2.78) form a basis on $M^4$, and, as a vector field, are the projection operators that map each point of the manifold to the tangent bundle, as defined in (2.86), thus characterizing the space where local Lorentz transformations live.3 In an accelerated frame, Lorentz-valued connections have to be introduced for matter fields:

$$L = \frac{1}{2} e^\mu_a \left[ \bar{\psi} \gamma^a \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^a \psi \right],$$

(2.91)

where the projectors $e_\mu^a$ from the target space to the tangent physical space are present; local Lorentz transformations for spinor fields read

$$\psi(x) \rightarrow S(\Lambda)(x)\psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)S^{-1}(\Lambda)(x),$$

(2.92)

where $S(\Lambda)(x)$ is a non-singular matrix $\forall x$. Let us assume that the $\gamma$ matrices transform locally as vectors, i.e.,

$$S(\Lambda)(x)\gamma^a S^{-1}(\Lambda)(x) = (\Lambda^{-1})^a_b (x)\gamma^b :$$

(2.93)

---

2In analogy with the formalisms of particle Physics and renormalization techniques [155, 75], a suitable coupling constant could be attributed to the symmetry induced by the Lorentz group. Anyhow, because of the technical character of this analysis, here we prefer follow the notation of the great majority of the works [113, 95], also in metric-affine gravity. Nonetheless, it is worth remembering that such a coupling constant should be very small, as this kind of interaction has not been detected experimentally yet [156]. For some issues related to the use of such a coupling constant, see also [157].

3It is worth remarking that, from an operationally-motivated point of view, bein vectors characterize a family of observers; in particular, the vector $e^i_\mu$, at each event, is tangent to the world line of the observer at that point, while the 3-dimensional basis vectors $e^i_\mu$ ($i = 1, 2, 3$) are aligned along the principal axis of the experimental device. In the accelerated frame, the diffeomorphism-induced Lorentz group is local, i.e., at each point of the manifold, there is a different action of the Lorentz group: the equivalence of every local basis (≡observer) introduces on the flat manifold $M^4$ a gauge freedom connected with pure local Lorentz rotations.
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for an infinitesimal local Lorentz transformation

\[ S(\Lambda) = 1 - \frac{i}{4}e^{ab}(x)\Sigma_{ab}, \]  

(2.94)

covariant gauge derivatives

\[ D_a\psi = e^\mu_a D_\mu\psi = e^\mu_a \left( \partial_\mu \psi - \frac{i}{4}A^bc_{\mu}\Sigma_{bc}\psi \right), \]  

(2.95a)

\[ \overline{D}_a\psi = e^\mu_a \overline{D}_\mu\psi = e^\mu_a \left( \partial_\mu \overline{\psi} + \frac{i}{4}\overline{\psi}\Sigma_{bc}A^{bc}_{\mu} \right), \]  

(2.95b)

assure invariance under local Lorentz transformations,

\[ \gamma^\mu D_\mu \Psi \rightarrow S(\Lambda)\gamma^\mu D_\mu \Psi \]  

(2.96)

provided that the gauge fields transform as

\[ A^b_a \rightarrow \Lambda(x)^a_c A^c_d \Lambda^{-1}(x)^d_b + \Lambda(x)^a_c d \Lambda^{-1}(x)^c_b, \]  

(2.97)

or, in the infinitesimal case,

\[ \omega_\mu \rightarrow S^{-1}(\Lambda)\omega_\mu S(\Lambda) - S^{-1}(\Lambda)\partial_\mu S(\Lambda), \]  

(2.98)

i.e., as Yang-Mills gauge fields.

The implementation of local Lorentz symmetry, \( \partial_\mu \rightarrow D_\mu \), leads to the interaction-Lagrangian density

\[ \mathcal{L}_{int} = \frac{1}{8} \left( \overline{\psi} \gamma^a \Sigma_{bc} A^b_{\mu} e^\mu_a \psi - \overline{\psi} \Sigma_{bc} \gamma^a A^b_{\mu} e^\mu_a \psi \right), \]  

(2.99)

which can be equivalently rewritten as

\[ \mathcal{L}_{int} = \frac{1}{8} \epsilon^\mu_a \overline{\psi} \left\{ \gamma^a, \Sigma_{bc} \right\} \psi A^b_{\mu} = -j^b_{bc} A^b_{\mu}, \]  

(2.100)

where curl brackets \( \{ \} \) indicate the anti-commutator, and, because

\[ \left\{ \gamma^a, \Sigma^{bc} \right\} = 2\epsilon^{abc} \gamma^d, \]  

(2.101)

we get

\[ j^a_{\mu} = -\frac{1}{4} \epsilon^{ab} \Sigma_{cd} \psi^{c,d}_{\mu} \]  

(2.102)

where \( j^d_A = \overline{\psi} \gamma^d \psi \) is the spinor axial current: the spinor axial current interacts with the gauge field \( A_\mu \), and is the source of the gauge field itself.
An action for the gauge field has to be added: since the curvature 2-form of the Lorentz gauge connections

\[ F^{ab} = dA^{ab} + A^a_e \wedge A^{cb}, \quad (2.103) \]

is not invariant under gauge transformations, as usual in Yang-Mills gauge theories, the gauge invariant action for the model will be

\[ S(A) = \frac{1}{32} \int tr \star F \wedge F = -\frac{1}{4} \int d^4x \det{\{ e \}} F^{ab}_{\mu\nu} F^{\mu\nu}_{ab}, \quad (2.104) \]

where \( \star \) denotes the Hodge operator. From a physical point of view, the most natural action is (2.104) [75]. Anyhow, from a mathematical and more abstract point of view, it would also be possible to introduce the irreducible pieces of \( F \) with different weights. We will further discuss this possibility within the framework of curved spacetime.

**Field equations**  Collecting all the terms together, we obtain the complete action

\[ S(A) + S_{FM}(\psi, \bar{\psi}, A) = \int d^4x \det{\{ e \}} \left\{ -\frac{1}{4} \left( F^{ab}_{\mu\nu} F^{\mu\nu}_{ab} \right) + \frac{i}{2} e^e_a \left[ \bar{\psi} \gamma^a D_\mu \psi - D_\mu \bar{\psi} \gamma^a \psi \right] \right\}, \quad (2.105) \]

where the covariant derivatives (2.95) have been introduced. It is straightforward to verify that this expression naturally fits all the features of a Yang-Mills gauge description. In fact, the covariant derivatives (2.95) assure invariance under local Lorentz transformations, in terms of a gauge transformation, for the spinor part of the action, and the term (2.104) also is invariant under such transformations. According to this picture, it will be natural to obtain the typical field equations of a Yang-Mills theory. Furthermore, it is worth remarking that the introduction of different irreducible pieces of \( F \) with different weights would spoil such gauge description.

Since we are dealing, for the moment, with flat space-time, tetrad vectors are not dynamical fields, but only projectors from the target space to the general physical space, then they will appear only in the expression of the invariant volume of the space-time and in scalar products: no variation with respect to them will be needed for field equations. Actually the only real dynamical field is the Lorentz-valued connection 1-forms \( A^a_b \). In fact, if, in analogy with GR, the curvature 2-form saturated on bein vectors is considered as an action for the model, a trivial theory is obtained. Variation of (2.106) with respect to the tetrad field would provide the total energy-momentum.

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tensor accounting for the dynamics and interactions of the vector field $A$ and the spinor field $\Psi$, respectively. Such a variation will indeed be crucial in casting Einstein equations on curved space-time (where the gravitational action has to be added), but here the vier-bein variables are regarded simply as 'kinematic' variables only.

Variation with respect the field $A_{\mu}^a$ leads to the dynamical equations

$$D_\mu F_{\mu a}^b = J_{\nu a}^b,$$

which are the Yang-Mills equations for the non-Abelian gauge field of the Lorentz group on flat space-time. The source of this gauge field is the conserved density of spin of the fermion matter.

Variation with respect to the spinor fields $\psi$ and $\bar{\psi}$ and relation (2.101) lead to the usual Dirac interaction equations for the spinor field and for the adjoint field:

$$\epsilon^a_{\mu} \left[i\gamma^a \partial_\mu + \frac{1}{8} \{ \gamma^a, \Sigma_{cd}\} A_{\mu}^{cd}\right] \psi = \epsilon^a_{\mu} \left[i\gamma^a \partial_\mu + \frac{1}{4} \epsilon^{ab}_{cd} \gamma^5 \gamma^b A_{\mu}^{cd}\right] \psi = 0, \quad (2.107)$$

and

$$\epsilon^a_{\mu} \bar{\psi} \left[i\gamma^a \bar{\partial}_\mu - \frac{1}{8} \{ \gamma^a, \Sigma_{cd}\} A_{\mu}^{cd}\right] = \epsilon^a_{\mu} \bar{\psi} \left[i\gamma^a \bar{\partial}_\mu - \frac{1}{4} \epsilon^{ab}_{cd} \gamma^5 \gamma^b A_{\mu}^{cd}\right] = 0. \quad (2.108)$$

Field equations illustrate that the dynamics for a spinor field in an accelerated frame differs from the standard Dirac dynamics for the spinor-gauge field interaction term, i.e., spinor fields are not free fields any more. For the analysis of the Dirac equation in non-inertial systems in flat spacetime, see also [158].

The present goal is to extend this formulation on a curved space-time manifold, on which non-vanishing Ricci rotation coefficients appear.

**Generalized Pauli Equation**

We now investigate the effects that the gauge fields $A_{\mu}^{ab}$ can generate in flat space-time. In particular, we treat the interaction between connections $A_{\mu}^{ab}$ and the 4-spinor $\psi$ of mass $m$, in order to generalize the well-known Pauli Equation, which corresponds to the motion equation of an electron in presence of an electro-magnetic field [155, 95].

The implementation of the gauge model in flat space, i.e., $\partial_\mu \rightarrow D^{(A)}_\mu$ leads to the fermion Lagrangian density

$$\mathcal{L}_F = \frac{i}{2} \bar{\psi} \gamma^a \epsilon^\mu_a \partial_\mu \psi - \frac{i}{2} \epsilon^\mu_a \bar{\partial}_\mu \bar{\psi} \gamma^a \psi - m \bar{\psi} \psi + \mathcal{L}_{\text{int}}, \quad (2.109)$$

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and, to study the interaction term, we start from the explicit expression
\[ \mathcal{L}_{\text{int}} = \frac{1}{4} \bar{\psi} \epsilon^{abcd}_{\alpha \beta} \gamma_5 \gamma_i^{d} A^{\alpha \beta}_{b} \psi . \]  
(2.110)

Here \( a = \{0, i\} \), and we consider the role of the gauge fields by analyzing its components \( A^{0i}_0, A^{ij}_0, A^{0i}_j, A^{ij}_j \). We now impose the time-gauge condition \( A^{0i}_0 = 0 \) associated to this picture and neglect the term \( A^{0i}_0 \) since it sums over the completely anti-symmetric symbol \( \epsilon^{0}_{\alpha \beta \mu} \equiv 0 \). The interaction Lagrangian density rewrites now
\[ \mathcal{L}_{\text{int}} = \psi^{\dagger} C_0 \gamma^0 \gamma_5 \gamma^0 \psi + \psi^{\dagger} C_i \gamma^0 \gamma_5 \gamma^i \psi , \]  
(2.111)
with the following identifications
\[ C_0 = \frac{1}{4} \epsilon^{k}_{ij0} A^{ij}_k , \quad C_i = \frac{1}{4} \epsilon^{k}_{ij0} A^{0j}_k . \]  
(2.112)

Varying now the total action built up from the fermion Lagrangian density wrt \( \psi^{\dagger} \), we get the Modified Dirac Equation
\[ (i \gamma^0 \gamma^0 \partial_0 + C_i \gamma^0 \gamma_5 \gamma^i + i \gamma^0 \gamma^i \partial_i + C_0 \gamma^0 \gamma_5 \gamma^0) \psi = m \gamma^0 \psi , \]  
(2.113)
which governs the dynamics of the 4-spinor \( \psi \) interacting with the gauge fields described here by the \( C_0 \) and \( C_i \).

**Stationary solutions**  Let us now look for stationary solutions of the Dirac Equation expanded as
\[ \psi(r, t) \rightarrow \psi(r) e^{-i \mathcal{E} t}, \quad \psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix} , \quad \psi^{\dagger} = (\chi^{\dagger}, \phi^{\dagger}) , \]
where \( \mathcal{E} \) denotes the spinor total energy and the 4-component spinor \( \psi(r) \) is expressed in terms of the two 2-spinors \( \chi(r) \) and \( \phi(r) \) (here \( r \) denotes the radial vector and \( r = |r| \)). Using now the standard representation of the Dirac matrices,
\[ \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
where \( \sigma_\alpha \) denote Pauli matrices, the Modified Dirac Equation (2.113) splits into two coupled equations (here we write explicitly the 3-momentum \( p^i \)):
\[ (\mathcal{E} - \sigma_i C^i) \chi - (\sigma_i p^i + C_0) \phi = m \chi , \]  
(2.114a)
\[ (\mathcal{E} - \sigma_i C^i) \phi - (\sigma_i p^i + C_0) \chi = -m \phi . \]  
(2.114b)
2.2 The role of the Lorentz group

**Low-energy limit**  Let us now investigate the non-relativistic limit by splitting the spinor energy in the form

\[ E = E + m . \]  

(2.115)

Substituting this expression in the system (2.114), we note that both the \(|E|\) and \(|\sigma_i C^i|\) terms are small in comparison wrt the mass term \(m\) in the low-energy limit. Then, eq. (2.114b) can be solved approximately as

\[ \phi = \frac{1}{2m} (\sigma_i p^i + C_0) \chi . \]  

(2.116)

It is immediate to see that \(\phi\) is smaller than \(\chi\) by a factor of order \(\frac{v}{m}\) (i.e. \(\frac{v}{c}\) where \(v\) is the magnitude of the velocity): in this scheme, the 2-component spinors \(\phi\) and \(\chi\) form the so-called small and large components, respectively [159].

Substituting the small components (2.116) in eq. (2.114a), after standard manipulation we finally get

\[ E \chi = \frac{1}{2m} \left[ p^2 + C_0^2 + 2C_0 (\sigma_i p^i) + \sigma_i C^i \right] \chi . \]  

(2.117)

This equation exhibits strong analogies with the electro-magnetic case. In particular, it is interesting to investigate the analogue of the so-called Pauli Equation used in the spectral analysis of the energy levels as in the Zeeman effect [159]:

\[ E \chi = \left[ \frac{1}{2m} (p^2 + e^2 A^2 + 2eA_i p^i) + \mu_B (\sigma_i B^i) - e \Phi^{(E)} \right] \chi , \]  

(2.118)

where \(\mu_B = e/2m\) is the Bohr magneton (here \(e\) denotes the electron charge) and \(A_i\) are the vector-potential components, \(B^i\) being the components of the external magnetic field and \(\Phi^{(E)}\) the electric potential.

**Corrections for one-electron atoms**  Let us now neglect the second order term \(C_0^2\) in eq. (2.117) and implement the symmetry

\[ \partial_\mu \rightarrow \partial_\mu + A_\mu^{U(1)} + A_\mu^{ab} \Sigma_{ab} , \]  

(2.119)

with a vanishing electromagnetic vector potential, i.e., \(A_i = 0\). This way, we introduce a Coulomb central potential

\[ V(r) = Ze^2/4\pi\varepsilon_0 r , \]  

(2.120)
where \( Z \) is the atomic number and \( \varepsilon_0 \) is the vacuum dielectric constant. Substituting now \( E \to E - V(r) \) in eq. (2.117), we can derive the total Hamiltonian of the system:

\[
H_{\text{tot}} = H_0 + H',
\]

(2.121)

where

\[
H_0 = \frac{p^2}{2m} - \frac{Ze^2}{(4\pi \varepsilon_0)r}, \quad H' = H_1 + H_2,
\]

(2.122)

\[
H_1 = C_0 (\sigma_i p^i) / m, \quad H_2 = \sigma_i C^i / 2m,
\]

(2.123)

which characterizes the electron dynamics in a one-electron atom. The solutions of the unperturbed Hamiltonian are the well-known modified two-components Schrödinger wave function

\[
H_0 \psi_n \ell m_\ell m_s = E_n \psi_n \ell m_\ell (r) \chi_{\frac{1}{2}, m_s, \ell} \quad E_n = -m (Z\alpha)^2 / 2n^2,
\]

(2.124)

using the unperturbed basis \(| n; \ell m_\ell s m_s \rangle \).

Since \( H_1 \) and \( H_2 \) have to be treated like perturbations, the gauge fields can be considered as independent, in the low-energy (linearized) regime. The analysis of \( H_1 \) can be performed substituting the operator \( \sigma_i p^i \) with \( J_i p^i \), where the \( J_i \)'s are the components of the total angular momentum operator (in fact, \( L_\alpha p^\alpha = 0 \)). \( H_1 \) is diagonal in the basis \(| n; \ell s j m_j \rangle \). According to tensor analysis, we decompose the term \( J_i p^i \) into spherical-harmonic components. In particular, the Cartesian tensor operator \( p_i \) can be factorized into three components \( V^{(k)}_q \) where \( q = 0, \pm 1 \) (\( k = 1 \) for any vector operator) and, by the harmonics formalism, we can use the relation between \( V^{(k)}_q \) and \( Y_{l=0}^{m=q} \) to decompose the correction matrix element \( \langle H_1 \rangle \) into

\[
\langle H_1 \rangle = \frac{c_{\ell j} C_0}{m} m_j \langle n; \ell s j m_j | V^{(1)}_0 | n; \ell s j m_j \rangle + \\
+ \frac{c_{\ell j} C_0}{m} \sqrt{(j \mp m_j)(j \pm m_j + 1)} \langle n; \ell s j m_j | V^{(1)}_{\pm 1} | n; \ell s j m_j \pm 1 \rangle,
\]

(2.125)

where \( c_{\ell j} \) are the Clebsch-Gordan coefficients to change the basis \(| n; \ell m_\ell s m_s \rangle \) into \(| n; \ell s j m_j \rangle \). According to Wigner-Eckart theorem, for each component we get

\[
\langle V^{(1)}_0 \rangle \sim \langle j 1; m_j 0 | j 1; j' m'_j \rangle,
\]

\[
\langle V^{(1)}_{+1} \rangle \sim \langle j 1; m_j (+1) | j 1; j' (m'_j + 1) \rangle,
\]

\[
\langle V^{(1)}_{-1} \rangle \sim \langle j 1; m_j (-1) | j 1; j' (m'_j - 1) \rangle.
\]
obtaining the following selection rules

\[ j' = j + 1 , \quad m'_j = m_j . \]  

These conditions correspond to dealing with in- and out- states with the same parity. Anyhow, since \( J_\alpha p^\alpha \) is a pseudo-scalar operator, it connects states of opposite parity, and no transition is eventually allowed.

The analysis of \( H_2 \) requires a different approach. We assume that the gauge fields are directed along the \( z \) direction. This way, only the component \( C_3 \) is considered and, for the sake of simplicity, we impose that only one between \( A^{02}_1 \) and \( A^{01}_2 \) contributes, in order to recast the correct degrees of freedom. The effect of \( C_3 \) corresponds to that of an external magnetic-like field generated by the fields \( A^{03}_i \), which can be considered as the vector bosons (spin-1 and massless particles) of such an interaction. \( H_2 \) is now diagonal in the unperturbed basis \( | n; \ell \, m, s \, m_s \rangle \) and produce an energy-level split of the order

\[ \Delta E = \frac{C_3}{m} m_s , \]  

where \( m_s = \pm \frac{1}{2} \). Nevertheless, since we are dealing with sin-1 and massless gauge bosons, the usual electric-dipole selection rules [159] can be used. This way, we have to impose \( \Delta m_s = 0 \) and no correction to the well-known transitions results to be detectable.

Collecting all the results together, we conclude that no new spectral line arises. Because of this properties of the Hamiltonian, it is not possible to evaluate an upper bound for the coupling constant of the interaction.

### 2.2.2 Curved Space-Time

The considerations developed in the previous sections can be generalized to curved space-time; the torsion-less assumption of GR, perfectly realized by the Hilbert-Palatini action, does not allow for an independent gauge field of the Lorentz group. The connections \( A^a_b \) have been introduced on a general flat manifold in order to restore local invariance for the Lagrangian density of spinor fields under passive local Lorentz transformations. Generalizing the framework to curved space-time will provide a geometrical interpretation for the new connection fields, which will be identified with the field \( K^a_b \). The need to introduce local Lorentz gauge fields in curved space-time is aimed at restoring local invariance of the spinor Lagrangian density under
passive local Lorentz transformations, while spin connections allow one to recover the proper Dirac algebra for Dirac matrices. The generalization consists in considering the space-time $M^4$ as a curved manifold, on which the tetrad basis is consists of dynamical fields, which describe pure gravity; local Lorentz transformations are still considered as a gauge freedom, so that Lorentz-valued connection fields have to be introduced.

In the next paragraphs, within the framework of curved space-time, the relation between the gauge field of the Lorentz group and the geometrical properties of metric-compatible space-times will be investigated. In particular, in the second-order approach, the possibility of identifying the contortion field with Lorentz connections will be investigated, while, in the first-order approach, the geometrical hypotheses for the introduction of torsion as a Lorentz gauge field will be addressed. The two approaches will be compared in the linearized regime.

**Second-order approach**

If we consider Riemann-Cartan spaces, endowed with the affine connections (2.50), we look for an operator $D_\mu$ which allows for

$$D_\mu \gamma_\nu = 0; \tag{2.128}$$

such an operator is found to be

$$D_\mu A = \nabla_\mu A - [\Gamma_\mu, A] \tag{2.129}$$

for a generic geometrical object, and

$$D_\mu \psi = \partial_\mu \psi - \Gamma_\mu \psi, \quad D_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \bar{\psi} \Gamma^\mu \tag{2.130}$$

for spinor fields, so that the matter Lagrangian density reads

$$L_M = -\frac{i}{2} \bar{\psi} \gamma^\mu e_\mu^a D_\mu \psi + H.C., \tag{2.131}$$

where $H.C.$ denotes Hermitian conjugation.

By substitution of (2.50) in (2.129), after standard manipulation one finds the expression for the connections

$$\Gamma_\mu = \Gamma^R_\mu + \Gamma^K_\mu = \frac{1}{2} e^a_{[\mu} \Sigma^{ab]} + \frac{1}{2} A^{a\delta}_{\mu} \Sigma_{ab}, \tag{2.132}$$
where the tetrad projection of the Ricci coefficients and of the contortion field are defined, respectively,

\[ R_{ab\mu} = R_{abc}\xi^{c}_{\mu}, \]  
\[ A_{ab\mu} \equiv -K_{\rho\sigma\mu}e_{a}^{\rho}e_{b}^{\sigma}. \]  

The connections \( \Gamma_{\mu} \) defined in (2.129) split up into two different terms, the spin connections \( \Gamma_{\mu}^{R} \), which restore the commutation relations of the Dirac matrices in the physical space-time, and the gauge connections \( \Gamma_{\mu}^{K} \), which reestablish invariance under local Lorentz transformations, respectively. If \( R = 0 \), the scenario depicted above reduces to the results of (2.2.1), so that the gauge connections \( \Gamma_{\mu}^{K} \) can be interpreted as the real gauge fields of the local Lorentz group, for they are non-vanishing quantities even in flat space-time, as requested for any gauge field. Furthermore, formula (2.77) illustrates once more that \( R_{abc} \) cannot be a gauge field, for it defines gauge transformations on the tangent bundle.

Since, in a gauge setting, gauge connections are primitive objects, the total action \( S \equiv S(e, A, \psi) \) must depend on the independent fields \( \psi, e \), and \( A \), such as

\[ S = S(e, A, \psi) = -\frac{1}{2} \int \text{det}(e) d^4x [R(e) - (L_{M} - \frac{1}{4} F_{\mu\nu}(A) F^{\mu\nu}(A))], \]  

where gauge-fixing terms have not been included, as it will be convenient, for our purposes, to work in the restricted space of conserved currents.

Variation of the action with respect to the independent fields leads to field equations.

Variation with respect to bein vectors, \( \delta e_{\mu}^{a} \), leads to the bein projection of the Einstein equations, with Yang-Mills tensor \( T^{\mu\nu} \) as source

\[ (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)e_{a}^{\nu} = T_{\mu a}, \]  

while variation with respect to the field \( \delta A_{\mu}^{ab} \) brings Yang-Mills equations, with the spinor current density as a source:

\[ D_{\mu} F^{\mu\nu a}_{\mu b} = J^{\nu a}_{\mu b}. \]  

Finally, the Dirac equation for the spinor field \( \psi \) is obtained after variation with respect to the adjoint field, \( \delta \psi \),

\[ \gamma^{\mu} D_{\mu} \psi = 0, \]  

and vice-versa.

The comparison between local Lorentz transformations and gauge transformations
allows one to obtain the expression for conserved quantities. This way, since the current density \((2.137)\) admits the conservation law

\[ D_\mu J^{\mu ab} = 0, \quad (2.139) \]

a conserved (gauge) charge\(^4\) can be defined

\[ Q^{ab} = \int d^3x J^{0ab} = \text{const}; \quad (2.140) \]

on the other hand, the b ein projection of the spin term of the angular momentum tensor \(M^{\mu \nu}\), the conserved quantity for Lorentz transformations in flat space time, reads

\[ M^{ab} = \int d^3x \pi_r \Sigma_{r,ab} \psi_s = \text{const.,} \quad (2.141) \]

which coincides with \((2.140)\), provided that \(\pi_r\) is the density of momentum conjugate to the field \(\psi_r\), i.e., \(\pi_r = \partial L/\partial \psi_r\). This identification is possible only on flat space-time, because of the definition of the parameter \(e^{ab}\) \((2.77)\), which points up the remarkable features of local Lorentz transformations on the tangent bundle.

**First-order approach**

If one relaxes the torsion-less assumption, the second Cartan structure rewrites

\[ de^a + \tilde{\omega}^a_b \wedge e^b = T^a \quad (2.142) \]

where \(T^a\) is the torsion 2-form; this equation is solved by the connections

\[ \tilde{\omega}^a_b = \omega^a_b + K^a_b, \quad (2.143) \]

where \(K^a_b\) is the contortion 1-form, such that \(T^a = K^a_b \wedge e^b\), while \(\tilde{\omega}^a_b\) are the usual connection 1-forms. As a result, new 1-forms appear in the dynamics, which reestablish the proper degrees of freedom for the connections of the Lorentz group.

\(^4\)This quantity is a conserved one if one assumes that the fluxes through the boundaries of the space integration vanish.
In GR, nevertheless, these connections do not describe any physical field: after substituting the solution (2.143) of the structure equation into the Hilbert-Palatini action\(^5\), one finds that the connections \(K^a_b\) appear only in a non-dynamical term, i.e.,

\[
S(\epsilon, \omega) = \frac{1}{2} \int \epsilon_{abcd} \epsilon^a \land \epsilon^b \land \left( R^{cd} + K^c_f \land K^{fd} \right). \tag{2.144}
\]

Since the other terms vanish because of the structure equation, the connections \(K^a_b\) themselves vanish after variation, unless spinors are taken into account: in this case, the connections \(K^a_b\) become proportional to the spin density of the matter, thus giving rise to the Einstein-Cartan model, where the usual four-fermion term arises.

As far as the formulation of a Lorentz gauge theory is concerned, we will denote the connection 1-forms for local Lorentz transformations, independent of any other fields, with the quantities \(A^a_b\), and the standard connections of GR, which depend on the gravitational field and on spinors (if matter is taken into account) with \(\omega^a_b\). Then the total connections eventually rewrite

\[
C^a_b = \omega^a_b + A^a_b. \tag{2.145}
\]

From the comparison of (2.143) and (2.145), it can be inferred that the presence of the fields \(A\) is connected with the appearance of torsion in space-time, so that, if the proper geometrical interpretation has to be attributed to the fields \(A\), the interaction term between the spin connections \(\omega\) and the fields \(A\)

\[
S_{\text{int}} = 2 \int \epsilon_{abcd} \epsilon^a \land \epsilon^b \land \omega^c_f \land A^{fd}, \tag{2.146}
\]

has to be postulated. The action describing the dynamics of the fields \(A^a_b\) is just the same introduced in section (2.2.1), (2.104), while the action that accounts for the connections \(\omega^a_b\) can be taken as (2.34), since its presence is due to the existence on curved space-time of a particular local Lorentz transformation connected to the

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\(^5\)Let \(S(q_i, Q_j)\) be an action depending on two sets of dynamical variables, \(q_i\) and \(Q_j\). The solutions of the dynamical equations are extrema of the action with respect to both the two sets of variables: if the dynamical equations \(\partial S / \partial q_i = 0\) have a unique solution, \(q_i^{(0)}(Q_j)\) for each choice of \(Q_j\), then the extrema of the pullback \(S(q_i(Q_j), Q_j)\) of the action to the set of solution are precisely the extrema of the total total action \(S(q_i, Q_j)\). For an application of this theorem, see, for example, [160].
invariance under diffeomorphisms. Collecting all the terms together, we get

\[
S(e, \tilde{\omega}, A, \psi, \bar{\psi}) = \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge \tilde{R}^{cd} + \\
- \frac{1}{32} \int tr \star F \wedge F - \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge \tilde{\omega}^c \wedge A^{fd} + \\
+ \frac{1}{2} \int \epsilon_{abcd} e^a \wedge e^b \wedge c^c \wedge \left[ i\bar{\psi} \gamma^d \left( d - i \frac{4}{4} (\tilde{\omega} + A) \right) \psi - i \left( d + i \frac{4}{4} (\tilde{\omega} + A) \right) \bar{\psi} \gamma^d \psi \right].
\]

(2.147)

Two cases can be distinguished, where the properties of the Lorentz connections are defined from a geometrical point of view, according to the absence or presence of spinors.

If fermion matter is absent, the action reduces to

\[
S(e, \tilde{\omega}, A) = \frac{1}{2} \int \epsilon_{abcd} e^a \wedge e^b \wedge \tilde{R}^{cd} - \frac{1}{32} \int tr \star F \wedge F - \int \epsilon_{abcd} e^a \wedge e^b \wedge \tilde{\omega}^c \wedge A^{fd}. \tag{2.148}
\]

Variation with respect to the connections \( \tilde{\omega} \) gives, after standard calculations,

\[
d^{(\tilde{\omega})} e^a = A^a_b \wedge e^b, \tag{2.149}
\]

which admits the solution

\[
\tilde{\omega}^a_b = \omega^a_b + A^a_b, \tag{2.150}
\]

were \( \omega^a_b \) are the usual connection 1-forms: because of the analogy with the solution of the second Cartan structure equation (2.143), the connection \( A \) can be identified with the 1-form \( K \).

Since solution (2.150) is unique, action (2.148) can be pulled back to the given solution to obtain the reduced action for the system, which now depends on the gravitational field and on the independent connections of the Lorentz group only. Namely, we have:

\[
S(e, A) = \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} - \frac{1}{32} \int tr \star F \wedge F + \\
- \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge \omega^c \wedge A^{fd} - \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge A^c \wedge A^{fd}, \tag{2.151}
\]

where \( \omega = \omega(e) \) and \( R = R(\omega) \) denote the Ricci spin connections and the Riemann curvature 2-form, respectively.

Variation with respect to the gravitational field and the connections of the Lorentz
2.2 The role of the Lorentz group

group leads to

\[
e^{a}_{bcd} e^{b} \wedge R^{cd} = M^{a} + \epsilon^{a}_{bcd} e^{b} \wedge (\omega^{c}_{f} + A^{c}_{f}) \wedge A^{f d}, \tag{2.152a}
\]

\[
d^{(A)} \star F^{f d} = \epsilon_{abc} [d e^{a} \wedge e^{b} \wedge (\omega^{c}_{f} + 2A^{c}_{f})], \tag{2.152b}
\]

where $M^{a}$ is the energy-momentum 3-form of the field $A$, which can be explicitly obtained after variation of the Yang-Mills-like action with respect the gravitational 1-form.

Eqs (2.152) describe the coupled system of gravitational- and Lorentz-connection fields, i.e., they do not couple only through the energy-momentum tensor of the connection field. In fact, the presence of the interaction term (2.146) yields non-standard couplings both on the rhs of the Einstein equations and in the rhs of the Yang-Mills dynamical equations, and, in particular, while in flat space, the only source for the Lorentz connection fields $A$ is the density of spinor matter, in curved space-time also the gravitational spin connections become a source for this field. As a result, the gravitational field is a source for the torsion of space-time.

When the matter contribution is taken into account, variation of (2.147) with respect to the connections $\tilde{\omega}$ leads to

\[
d^{(\tilde{\omega})} e^{a} = A^{a}_{b} \wedge e^{b} - \frac{1}{4} \epsilon^{a}_{bcd} e^{b} \wedge e^{c} j^{d}_{(A)}, \tag{2.153}
\]

where $j^{a}_{(A)} = \bar{\psi} \gamma_{5} \gamma^{a} \psi$, i.e., the spinor axial current, deeply modifies eq. (2.149). In fact, the presence of spinor matter prevents one from identifying the connections $A$ as the only generator of torsion, since all the terms in the rhs of the second Cartan structure equation have to be interpreted as torsion. This way, both the fields $A$ and the spinor axial current contribute to the torsion of space-time. It is worth noting that, if the field $A$ vanishes, we obtain the usual result of PGT, i.e. the Einstein-Cartan contact theory, in which torsion is directly connected with the density of spin and does not propagate.

Eq. (2.153) admits the unique solution

\[
\tilde{\omega}^{a}_{b} = \omega^{a}_{b} + A^{a}_{b} + \frac{1}{4} \epsilon^{a}_{bcd} e^{c} j^{d}_{(A)}, \tag{2.154}
\]

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which can be inserted in the total action (2.147), thus bringing the result

\[
S(e, A, \psi, \bar{\psi}) = \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} - \frac{1}{32} \int tr \star F \wedge F + \\
+ \frac{1}{2} \int \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge \left[ \bar{\psi} \gamma^d \left( d - \frac{i}{4} (\omega + A) \right) \psi - i \left( d + \frac{i}{4} (\omega + A) \right) \bar{\psi} \gamma^d \psi \right] \\
- \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge A_f^c \wedge A_f^d - \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge \omega^{[c} \wedge A^{d]} + \\
- \frac{3}{16} \int e_a \wedge e_b \wedge e_c \wedge A^{[ab} j^{c]}(A) - \frac{3}{16} \int d^4 x \eta_{ab} j^a(A) j^b(A), 
\]

(2.155)

where the last term is the usual four-fermion interaction of the Einstein-Cartan scheme.

Variation with respect to the remaining fields leads to a generalization of the dynamical equations (2.152), where now the contribution of fermions is present too, i.e.,

\[
\epsilon_{bcd} e^b \wedge R^{cd} = M^a + \epsilon_{bcd} e^b \wedge (\omega_f + A_f^c) \wedge A_f^d + \\
+ \frac{9}{8} e_b \wedge e_c \wedge A^{[ab} j^{c]}(A) - \frac{1}{16} \epsilon_{bcd} e^b \wedge e^c \wedge e^d \eta_{fg} j^{f}(A) j^{g}(A) + \\
- 3 \epsilon_{bcd} e^b \wedge e^c \wedge \left[ i \bar{\psi} \gamma^d \left( d - \frac{i}{4} (\omega + A) \right) \psi - i \left( d + \frac{i}{4} (\omega + A) \right) \bar{\psi} \gamma^d \psi \right], 
\]

(2.156a)

\[
d^{(A)} \star F^f = \epsilon_{abc} e^d \wedge e^f \wedge (\omega^{cf} + 2 A_c^f) + \frac{27}{4} e^d \wedge e^f \wedge e_c \wedge j^c(A). 
\]

(2.156b)

Consequently, the density of spin of the fermion matter is present in the source term of the Yang-Mills equations for the Lorentz connection fields, and the Einstein equations contain in the rhs not only the energy-momentum tensor of the matter, but also a four-fermion interaction term. The dynamical equations of spinors are formally the same as those of the Einstein-Cartan model with the adjoint of the interaction with the connections of the Lorentz group A.

As a result, the Einstein-Cartan contact interaction is recovered in the limit of vanishing Lorentz connections, thus shedding light on the existence of independent connections of the Lorentz group on curved space-time, which modifies profoundly the dynamics of the gravitational field both in absence and in presence of fermion matter.

In particular, in the first case, the connections A are in strict relation with the torsion tensor modifying the Riemannian structure of ordinary space-time, while, in the second case, the presence of fermions already modifies the structure of space-time, and the Lorentz connections A contribute to the torsion tensor with a boson term.
2.2 The role of the Lorentz group

Moreover, the bosonic and fermionic parts of torsion interact, the latter being a source for the boson part of torsion, and the former the mediator of the interaction between two-fermion torsion terms.

In the most general metric structure, curvature, torsion and non-metricity are present (see for example [161] for the relation between Riemannian curvature and generalized curvature). In [128], the most general parity-conserving quadratic Lagrangian has been established for this metric structure, in terms of the irreducible pieces of non-metricity, torsion and curvature, and a cosmological term is also included.

In the curvature square term, the irreducible pieces of the symmetric and antisymmetric parts of the curvature 2–form have been introduced. Moreover, Since $V_{\text{MAG}}$ is required to be an odd 4–form, if parity conservation is assumed, it has to be built up with one Hodge star, according to the scheme $F \wedge \ast F$, since the star itself is an odd operator, as well as the Hilbert-Einstein type term and the cosmological term. Thus $V_{\text{MAG}}$ is homogeneous of order one in the star operator. This model has also been analyzed in [127], where its has been compared with Einstein-Cartan theory, PGT, the implications of the coupling with a scalar field, and the possibility that these models, possibly combined with a suitable symmetry-breaking mechanism, might lead to a consistent quantization program is also envisaged.

The works [130, 129, 162, 163] (see also [92] and the references therein for a review about torsion gravity) deal with models where only torsion and curvature are considered. In particular, the irreducible terms under the action of the Lorentz group are classified as

\[ T^\mu = T_{\mu \sigma}^\sigma \]  
\[ t_{\mu \nu \rho} = T_{\mu \nu \rho} + T_{\nu \rho \mu} - \frac{1}{3} (T_{\nu} g_{\mu \rho} + T_{\mu} g_{\nu \rho}) + \frac{2}{3} g_{\mu \rho} T_{\nu} \]  
\[ a_\mu = \frac{1}{3} \epsilon_{\mu \nu \rho \sigma} T^{\nu \rho \sigma}, \]

i.e. the trace, a traceless part and an antisymmetric part, respectively. The total action accounting for the presence of torsion $L_{\text{tors}}$ was taken as

\[ L_{\text{tors}} = \alpha t_{\mu \nu \rho} t^{\mu \nu \rho} + 4 \beta T_{\mu} T^{\mu} + \gamma a^\mu a_\mu, \]

where three constants have been introduced. Furthermore, the procedure was repeated also for the curvature scalar, and parity-violating terms were excluded. When the scalar invariant is added, as a result, a ten-parameter Lagrangian is found.
Our simplified action (2.155) is forced by the coupling allowed by a geometrical interpretation of this Lorentz gauge theory.

Comparison

To better understand the physical implications of first- and second-order approaches, a comparison between field equations should be accomplished. In particular, in the first-order approach, the role of the gravitational field as a source of torsion should be analyzed, since it has no analogs in GR. Because the gravitational field acts like a source term, it should be compared with a "current" term, which can be worked out from the second-order formalism. For the sake of simplicity, we will restrict our analysis to the linearized regime in the transverse-traceless (TT) gauge.

If we consider the case of small perturbations $h_{\mu\nu}$ of a flat Minkowskian metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

(2.159)

and the corresponding expression of the tetrad field as a sum of the Minkowskian bein projection $\delta^a_\mu$ and the infinitesimal perturbation $\zeta^a_\mu$,

$$e^a_\mu = \delta^a_\mu + \zeta^a_\mu,$$

(2.160)

the following identifications hold

$$\eta_{\mu\nu} = \delta^a_\mu \delta_{a\nu}$$

(2.161a)

$$h_{\mu\nu} = \delta_{a\mu} \zeta^a_\nu + \delta_{a\nu} \zeta^a_\mu.$$  

(2.161b)

In the linearized regime, all quantities will be truncated at the first order of $\zeta$.

Because of the interaction term (2.146) postulated in the first-order approach, it is possible to solve the structure equation and to express the connections as a sum of the pure gravitational connections plus other contributions, both in absence and in presence of spinor matter.

The connections $\omega^{ab}_\mu = e^{b\nu} \nabla_\mu e^a_\nu$ rewrite, because of the linearization,

$$\omega^{ab}_\mu = \delta^{ab} \left( \partial_\nu \zeta^a_\mu - \tilde{\Gamma}^a_\mu_{\rho} \delta^b_\rho \right),$$

(2.162)

where $\tilde{\Gamma}^a_\mu_{\rho}$ are the linearized Christoffel symbols, i.e.,

$$\tilde{\Gamma}^a_\mu_{\rho} = \frac{1}{2} \delta^{ab} \left( \zeta_{\sigma\mu,\nu} + \zeta_{\sigma\nu,\mu} - \zeta_{\mu\nu,\sigma} \right).$$

(2.163)
The Einstein Lagrangian density for $g_{\mu\nu}$ in the TT gauge reads

$$L = (\partial_{\rho} h_{\mu\nu}) (\partial^{\rho} h^{\mu\nu}) , \quad (2.164)$$

from which $M^\tau_{\alpha\beta}$, the spin-current density associated with the spin angular momentum operator, can be evaluated for a Lorentz transformation of the metric. In fact, if we consider the transformation

$$g_{\mu\nu} \rightarrow \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x^{\sigma'}}{\partial x^\sigma} g_{\rho'\sigma'} , \quad (2.165)$$

where $x'^\rho = x^\rho + \epsilon^\rho \tau x^\tau$, then the current reads

$$M^{\tau\rho\nu} = \frac{\partial L}{\partial h_{\mu,\tau}} \Sigma_{\mu\nu}^{\rho\sigma\tau} h_{\rho\sigma} = (\eta^{\chi^\mu,\nu,\tau} + \eta^{\chi^\nu,\mu,\tau}) \Sigma_{\mu\nu}^{\rho\sigma\tau} (\eta_{\pi\rho}^{\zeta,\mu} + \eta_{\pi\sigma}^{\zeta,\rho}) , \quad (2.166)$$

where

$$\Sigma_{\mu\nu}^{\rho\sigma\tau} = \eta^{\chi^\mu,\nu} (\delta^{\chi^\rho,\mu}_\nu \delta^{\chi^\sigma,\nu}_\nu + \delta^{\chi^\rho,\nu}_\nu \delta^{\chi^\sigma,\mu}_\mu) . \quad (2.167)$$

The two quantities (2.162) and (2.166) do not coincide: in fact, (2.162) is linear in the $\zeta$ terms, while (2.166) is second order in $\zeta$ by construction.

Since the connections $\omega$, on the rhs of (2.152b), acquire the physical meaning of a source for torsion, they can be interpreted as a spin-current density. Nevertheless, (2.162) is linear in $\zeta$, since the interaction term (2.146) is linear itself; as suggested by the comparison with gauge theories, and with (2.166) in particular, the interaction term should be quadratic. In this case, however, it would be very difficult to split up the solution of the structure equation as the sum of the pure gravitational connections plus other contributions.

### 2.2.3 Discussion

The considerations developed in the above have been prompted by observing that GR admits two physically different symmetries, namely the diffeomorphism invariance, defined in the world space-time, and the local Lorentz invariance, associated to the tangent fiber. Such two symmetries reflect the different behaviors of tensors and spinors, respectively, when global Lorentz transformations become local, i.e., while tensors do not experience the difference between the two transformations, spinors do.

In the Lorentz-gauge proposal, the diffeomorphism invariance concerns the metric
structure of the space-time and it finds in the vier-bein fields the natural gauge counterpart, though the gauge picture holds on a qualitative framework. On the other hand, the real gauge symmetry corresponds to local rotations in the tangent fiber and admits a geometrical gauge field induced by the space-time torsion and its properties. In the present analysis, the keypoint has been fixing the equivalence between isometric diffeomorphisms and local Lorentz transformations. In fact, under, the action of the former, spin connections behave like a tensor and are not able to ensure invariance under the correspondingly-induced local rotations.

This picture can lead one to infer the existence of a (metric-independent) gauge field of the Lorentz group, identified with the connection 1-forms $A$, which interact with spinors. The usual connection 1-forms could not be identified with the suitable gauge fields, for they are not a primitive object (they depend on bein vectors) and define local Lorentz transformations on the tangent bundle.

The mathematical identification of the Lorentz gauge field with the contortion field follows from the Cartan structure equation if a (unique) interaction term between the gauge field and the spin connection is introduced. This interaction term induces a Riemannian source to Yang-Mills equations; thus, the real vacuum dynamics of the Lorentz gauge connections takes place on a Minkowski space only, when the Riemannian curvature and the spin currents provide negligible effects, and spin connections can be chosen as vanishing. In fact, it is the geometrical interpretation of torsion as a gauge field that generates the non-vanishing part of Lorentz connections on flat space-time.

Differently from PGT, translations are not treated as a gauge freedom for spinors, because they have no spinor representation, so that spinors are not expected to distinguish between generic diffeomorphisms and translations. Furthermore, the Yang-Mills nature of Lorentz gauge connections enables one to predict propagating fields.

Despite these fundamental differences, a pure contact interaction for spinor fields is recovered for vanishing Lorentz connections: in this case, the Cartan structure equation provides non-zero torsion even when gauge bosons are absent. From this point of view, PGT can be qualitatively interpreted as the first-order approximation of the Lorentz-gauge scheme, when the carrier of the interaction is not observable, because of the weakness of its interaction.

As far as short-range interacting torsion is concerned, it is worth remarking that a wide class of models also exists, where the only modes of the torsion tensor which
interact with matter are massive scalars. In the valuable analysis presented in [164], for example, the parameter space of the model has been properly constrained, such that torsion decays quickly into matter fields, and no long-range fields are generated, which could be discovered by ground-based or astrophysical experiments. Within this work, in particular, the possibility to introduce propagating torsion and its observational consequences are discussed: possible actions for torsion, consisting in powers and derivatives of torsion, are hypothesized, and the interaction with external matter fields is investigated. Among these possibilities, a single matter-interacting scalar mode is picked up, according to the request that no arbitrary constraints on the dynamics should be present. The main difference between this proposal and our work relies on the choice of a suitable action for torsion and on the motivations for such a choice. In fact, while, in the relevant approach developed in [164] several actions for torsion are taken into account, whose implications are evaluated to be not detectable in astro-physical or ground based experiments, our proposal for the action accounting for the presence of torsion is based on attributing a proper geometrical interpretation to Lorentz connections.

2.3 The Schouten Classification

We will propose an approach to a classification of affine-connection geometries with an asymmetric metric tensor, according to the Schouten scheme, starting from the definition of all objects and finding the compatibility conditions between them. The overwhelming majority of the approaches to geometries with an asymmetric metric (see [94] for a review of classical results) are physically motivated by starting from a Lagrangian to derive both field equations and the definition of the connection in terms of an asymmetric metric, by means of a Palatini variational principle. However, such an approach is restricted from the very beginning by a fixed Lagrangian, able to provide only a fixed class of geometries. Alternatively, a classification of such geometries in analogy with that of Schouten [165, 166] for the class of affine-connection geometries with a symmetric metric would provide a proper understanding of the structure and variety of possible geometries with an asymmetric metric.

In a generalized Schouten classification, the connection consists of four components, i.e., Christoffel coefficients, the metric-asymmetricity object, generalized contortion
tensor, and non-metricity tensor. The role of these components can be analyzed according to the different aspects of physics that are to be investigated. For a comparison between the macroscopic and the microscopic approaches in the case of torsion, see [121].

In [167], two mechanical (macroscopic) examples are discussed, the case of a homogeneous disc rolling without sliding on a horizontal plane, and that of a homogeneous ball rolling without sliding on a sphere. Here, the non-holomic connection introduced by Schouten is analyzed in the context of Wagner’s proposal for the curvature tensor. On the other hand, two main research lines can be outlined from a quantum point of view: that of Unified-Field theories, and that of Geometry-plus-Matter theories. In general, these two approaches make different predictions.

In the first case, (see [97] and [168] for the earliest attempts, and [169] for a recent review), all fields are geometrical. In [170], torsion is assumed vanishing, because it is linked with spin [91], and matter is related with the non-symmetric part of the metric, while dilaton with the non-metricity object.

In the second case, the coupling of matter fields with geometry has to be evaluated, and the features of the connection components have to be interpreted. In [171], the connection is established to be antisymmetric in the first two indices only for a metric-compatible affine connection, and covariant differentiation for spinor fields is investigated. The correlation between torsion and Yang-Mills fields is widely explored in [172]: the mapping between these two kinds of fields is discussed in terms of differential geometry. As a result, Riemann-Cartan geometries are shown to reproduce Yang-Mills equations, while Yang-Mills connections are illustrated to induce richer structures.

After reviewing the Schouten classification, we will find the inverse of the ”structure matrix”, which links the generalized connection with all the metric objects, in the linear approximation. The expression of autoparallel trajectories will then be evaluated in the first-order approximation (PL7), (PL9).
2.3 The Schouten Classification

2.3.1 The original Schouten classification

According to the Schouten classification [165], a non-Riemannian geometry of general setting\(^6\), with a symmetric metric tensor \(g_{\mu\nu}\) and two different affine connections, \(\Pi^\mu_{\nu\sigma}\) for the parallel transportation of covectors \(a_\mu\) and \(\Theta^\mu_{\nu\sigma}\) for the parallel transportation of vectors \(v^\mu\), is characterized by the presence of three tensors of rank \((1,2)\) which are responsible for the non-Riemannian character:

- a difference tensor between the two connections \(S^\mu_{\nu\sigma} = \Pi^\mu_{\nu\sigma} - \Theta^\mu_{\nu\sigma}\);
- a torsion tensor \(T^\mu_{\nu\sigma} = 2\Pi^\mu_{[\nu\sigma]}\) (square brackets denote antisymmetrisation)
- a non-metricity object\(^7\) \(N^\mu_{\nu\sigma} = g^{\mu\epsilon}g_{\nu\sigma|\epsilon}\) due to the incompatibility of metric and connection in general.

From the definition of the non-metricity object, one gets the connection \(\Theta^\mu_{\nu\sigma}\)

\[
\Theta^\mu_{\nu\sigma} = \Gamma^\mu_{\nu\sigma} + A^\mu_{\nu\sigma},
\]

where

\[
\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\epsilon} (g_{\nu\epsilon,\sigma} + g_{\sigma\epsilon,\nu} - g_{\nu\sigma,\epsilon})
\]

is the metric connection, and \(A^\mu_{\nu\sigma}\) is the so-called affine-deformation tensor, defined as

\[
A^\mu_{\nu\sigma} = -S^\mu_{\nu\sigma} + K^\mu_{\nu\sigma} - D^\mu_{\nu\sigma},
\]

\[
K^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\epsilon} [T^\epsilon_{\sigma\rho}g_{\epsilon\nu} + T^\epsilon_{\nu\rho}g_{\epsilon\sigma} + T^\epsilon_{\nu\sigma}g^\mu_{\epsilon}] 
\]

\[
D^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\epsilon} (N_{\nu\epsilon,\sigma} + N_{\sigma\epsilon,\nu} - N_{\nu\sigma,\epsilon})
\]

being the contortion tensor and the non-metricity tensor, respectively.

The case of Riemannian geometry is the totally degenerate case, \(S^\mu_{\nu\sigma} = T^\mu_{\nu\sigma} = N^\mu_{\nu\sigma} = 0\).

\(^6\)In this section, lower case Greek letters for the whole alphabet will denote world indices, since no ambiguity may occur.

\(^7\)where \(|\) denotes covariant derivation with respect to the connection \(\Theta\)
Curvature tensor

The most important result of the Schouten classification is that the curvature tensor \( \hat{R}^\alpha_{\beta\rho\sigma} \) for the connection \( \Theta^\mu_{\nu\sigma} \) contains the Riemannian curvature \( R^\alpha_{\beta\rho\sigma} \) of the metric connection \( \Gamma^\mu_{\nu\sigma} \), decoupled from non-Riemannian contributions,

\[
\hat{R}^\alpha_{\beta\rho\sigma} = R^\alpha_{\beta\rho\sigma} + 2A^\alpha_{\beta[\sigma;\rho]} + 2A^\alpha_{\varepsilon[\rho}A^\varepsilon_{\beta\sigma]} \tag{2.172}
\]

where ; denotes the covariant derivative with respect to \( \Gamma^\mu_{\nu\sigma} \) (underlined indices are not affected by antisymmetrisation). The decoupling of the Riemannian part from the non-Riemannian one is important for the analysis of extra structures on non-Riemannian spaces, especially in cases when such structures are known not to exist on Riemannian manifolds, so that they may be expected to be compatible with objects of non-Riemannian character. From the physical point of view, it means that a field theory based on such a non-Riemannian geometry always contains Riemannian gravity (general relativity with an appropriate Lagrangian) and extra fields as non-Riemannian (non-gravitational) effects: a non-Riemannian geometry always contains the Riemannian part, as represented in general relativity, with its Lagrangian, and extra fields due to the non-Riemannian structures.

2.3.2 The Generalized Schouten classification of non-Riemannian geometries with an asymmetric metric tensor

Let us now consider now an affine-connection geometry, with connections \( \Pi^\mu_{\nu\sigma} \) and \( \Theta^\mu_{\nu\sigma} \), as described above, and with an asymmetric metric tensor \( g_{\mu\nu} \), \( g_{\mu\nu} \neq g_{\nu\mu} \). A generalized Schouten classification can be constructed for this case. An asymmetric metric tensor \( g_{\mu\nu} \) can always be split into its symmetric part \( s_{\mu\nu} \) and antisymmetric part \( a_{\mu\nu} \),

\[
g_{\mu\nu} = s_{\mu\nu} + a_{\mu\nu} \tag{2.173}
\]

The metric tensor \( s_{\mu\nu} \) is assumed to lower and raise tensor indices together with its inverse \( s^{\mu\nu} \). Similarly to the case of the symmetric metric, the analysis of the incompatibility between metric and connection\(^8\) \( g_{\mu\nu;\rho} = N_{\mu\nu\rho} \) brings about the following

---

\(^8\)from now on, \( | \) will denote covariant derivation with respect to the metric part of the connection.
expression for the connection $\Pi^\theta_{\kappa\lambda}$:

$$
\Pi^\theta_{\kappa\lambda} (\delta^\sigma_\theta \delta^\nu_\nu \delta^\rho_\rho + g^\sigma_\kappa \delta^\nu_\kappa a_\theta_\nu + g^\sigma_\kappa \delta^\nu_\kappa a_\rho_\theta) = \Gamma^\sigma_\nu_\rho + \Delta^\sigma_\nu_\rho + K^\sigma_\nu_\rho - D^\sigma_\nu_\rho,
$$

(2.174)

where $\Gamma^\sigma_\nu_\rho$ is the usual Christoffel connection, while

$$
\Delta^\sigma_\nu_\rho = \frac{1}{2} s^\sigma_\mu (a_\mu_\nu_\rho + a_\rho_\mu_\nu - a_\nu_\rho_\mu)
$$

(2.175a)

$$
C^\sigma_\nu_\rho = \frac{1}{2} [s^\sigma_\mu (T^\nu_\mu_\rho g_\rho_\nu + T^\nu_\rho_\mu g_\rho_\nu) + T^\nu_\rho g_\nu^\sigma]
$$

(2.175b)

$$
D^\sigma_\nu_\rho = \frac{1}{2} s^\sigma_\mu (N^\mu_\nu_\rho + N^\mu_\rho_\nu - N^\mu_\nu_\rho)
$$

(2.175c)

are the metric-asymmetricity object, the generalized contortion tensor, and the non-metricity tensor, respectively, specified for the metric (2.173). The determinant of the "hypercubic" structure matrix $J^{\delta^\kappa_\nu_\lambda}_{\nu_\rho_\mu}$,

$$
J^{\delta^\kappa_\nu_\lambda}_{\nu_\rho_\mu} = \delta^\kappa_\theta_\delta^\nu_\nu \delta^\mu_\rho + g^\kappa_\kappa \delta^\nu_\kappa a_\theta_\nu + g^\kappa_\kappa \delta^\nu_\kappa a_\rho_\theta,
$$

(2.176)

is related to the existence of solutions of the system of inhomogeneous linear algebraic equations (2.174) for the unknowns $\Pi^\alpha_\beta_\gamma$, similar to the case of usual quadratic matrices.

**Generalized curvature tensor**

When the determinant is different from zero, the system has a non-trivial solution, which can be expressed through the inverse structure matrix $(J^{-1})^{\alpha_\nu_\rho}_{\sigma_\mu_\lambda}$, $(J^{-1})^{\alpha_\nu_\rho}_{\sigma_\beta_\gamma} J^{\delta^\mu_\nu_\lambda}_{\mu_\nu_\rho}$ = $\delta^\mu_\nu_\nu \delta^\nu_\kappa$, as

$$
\Pi^\alpha_\beta_\gamma = (\Gamma^\sigma_\nu_\rho + \Delta^\sigma_\nu_\rho + K^\sigma_\nu_\rho - D^\sigma_\nu_\rho) (J^{-1})^{\alpha_\nu_\rho}_{\sigma_\beta_\gamma}.
$$

(2.177)

This way, the curvature tensor for $\Pi^\alpha_\beta_\gamma$ can be written in the form

$$
\hat{R}^\alpha_\beta_\rho_\sigma = \hat{M}^\alpha_\beta_\rho_\sigma + 2 \hat{A}^\alpha_\beta_\rho_\sigma \tilde{\alpha}^\beta_\rho_\sigma + 2 \hat{A}^\alpha_\beta_\rho_\sigma \tilde{\alpha}^\rho_\beta_\sigma,
$$

(2.178)

where $\hat{M}^\alpha_\beta_\rho_\sigma$ is the curvature tensor for an affine connection $\hat{\Gamma}^\alpha_\beta_\gamma = (\Gamma^\sigma_\nu_\rho + \Delta^\sigma_\nu_\rho) (J^{-1})^{\alpha_\nu_\rho}_{\sigma_\beta_\gamma}$, $\hat{A}^\alpha_\beta_\gamma = (K^\sigma_\nu_\rho - D^\sigma_\nu_\rho) (J^{-1})^{\alpha_\nu_\rho}_{\sigma_\beta_\gamma}$ is a generalized affine deformation tensor and || is the covariant derivative with respect to $\hat{\Gamma}^\alpha_\beta_\gamma$. Eq. (2.178) is a generalisation of (2.172) for the case of an asymmetric metric. The extraction of the Riemannian curvature tensor $R^\alpha_\beta_\rho_\sigma$ from $\hat{M}^\alpha_\beta_\rho_\sigma$ gives a direct analogue of (2.172)

$$
\hat{R}^\alpha_\beta_\rho_\sigma = R^\alpha_\nu_\mu_\lambda (J^{-1})^{\alpha_\nu_\mu_\lambda}_{\xi_\beta_\sigma} + \Sigma^\alpha_\beta_\rho_\sigma (\hat{A}^\alpha_\beta_\rho_\sigma, \Delta^\alpha_\beta_\rho_\sigma, (J^{-1})^{\alpha_\nu_\rho}_{\sigma_\beta_\gamma}),
$$

(2.179)

where $\Sigma^\alpha_\beta_\rho_\sigma$ is a tensor constructed from generalised affine deformation tensor, metric asymmetricity object and the inverse structure matrix and their derivatives.
Linear approximation

The determinant of $J_{\theta^\nu \rho}^{\alpha \sigma \lambda}$ has been calculated in a perturbation expansion in terms of small asymmetric metric, $|a_{\mu \nu}| \ll |s_{\mu \nu}|$. Then, in linear approximation, the matrix

$$J_{\theta^\nu \rho}^{\alpha \sigma \lambda} = \delta^\sigma_\theta \delta^\kappa_\nu \delta^\lambda_\rho + s^{\alpha \lambda}_\rho \delta^\kappa_\theta \delta^\nu_\rho + s^{\alpha \kappa}_\rho \delta^\lambda_\nu \delta^\theta_\rho$$

(2.180)

has its inverse as

$$(J^{-1})^{\alpha \nu \rho}_{\sigma \beta \gamma} = \delta^\alpha_\sigma \delta^\nu_\beta \delta^\rho_\gamma - s^{\alpha \nu}_\beta \delta^\rho_\gamma + s^{\alpha \rho}_\beta \delta^\gamma_\nu$$

(2.181)

Eqs (2.174)-(2.179) are the main relations describing the structure of affine-connection geometries with an asymmetric metric.

2.3.3 Autoparallel trajectories

So far, it is possible to analyze the expression of autoparallel trajectories. Be $v^\mu = dx^\mu/d\lambda$ the tangent vector to the curve $x^\mu = x^\mu(\lambda)$. Because of the solution (2.181), the autoparallel equation

$$\frac{dv^\alpha}{d\lambda} + \Pi^{\alpha}_{\beta \gamma} v^\beta v^\gamma = 0$$

(2.182)

simplifies as

$$\frac{dv^\alpha}{d\lambda} + (\Gamma^{\sigma}_{\nu \rho} + \Delta^{\sigma}_{\nu \rho} + C^{\sigma}_{\nu \rho} - D^{\sigma}_{\nu \rho}) (J^{-1})^{\alpha \nu \rho}_{\sigma \beta \gamma} v^\beta v^\gamma = 0.$$  

(2.183)

Because of the symmetries of (2.181), (2.182) rewrites

$$\frac{dv^\alpha}{d\lambda} + \Gamma^{\alpha}_{\nu \rho} v^\nu v^\rho + \Delta^{\sigma}_{\nu \rho} (\gamma^{\alpha \nu}_{\sigma \gamma} a^\nu_\rho v^\gamma + s^{\alpha \rho} a^{\beta \sigma} v^\nu v^\beta) +$$

$$+ \frac{1}{2} s^{\mu \nu} (T^{e}_{\nu \rho} g_{\epsilon \rho} + T^{e}_{\rho \nu} g_{\epsilon \nu}) \delta^{\alpha}_{\epsilon} w^\nu w^\sigma + \frac{1}{2} T^{e}_{\nu \rho} g_{\epsilon} \sigma (s^{\alpha \nu}_{\gamma} a_{\gamma \sigma} u^\rho u^\gamma + s^{\alpha \rho} a_{\beta \sigma} u^\nu u^\beta) +$$

$$- \frac{1}{2} s^{\sigma \mu} (N_{\mu \nu} + N_{\nu \mu} - N_{\nu \mu}) (\delta^{\alpha}_{\sigma} u^\nu u^\rho - s^{\alpha \nu}_{\gamma} a_{\gamma \sigma} u^\rho u^\gamma - s^{\alpha \rho} a_{\sigma \beta} u^\beta u^\nu) = 0.$$  

(2.184)

If we expand the previous expression and keep only the terms linear in $a$, we obtain

$$\frac{dv^\alpha}{d\lambda} + \Gamma^{\alpha}_{\nu \rho} v^\nu v^\rho + \frac{1}{2} s^{\mu \nu} (T^{e}_{\nu \rho} s_{\epsilon \rho} + T^{e}_{\rho \nu} s_{\epsilon \nu}) \delta^{\alpha}_{\epsilon} u^\rho u^\sigma +$$

$$+ \frac{1}{2} s^{\mu \nu} (T^{e}_{\nu \rho} a_{\epsilon \rho} + T^{e}_{\rho \nu} a_{\epsilon \nu}) \delta^{\alpha}_{\epsilon} u^\nu u^\sigma + \frac{1}{2} T^{e}_{\nu \rho} s_{\epsilon} \sigma (s^{\alpha \nu}_{\gamma} a_{\gamma \sigma} u^\rho u^\gamma + s^{\alpha \rho} a_{\beta \sigma} u^\nu u^\beta) +$$

$$- \frac{1}{2} s^{\sigma \mu} (N_{\mu \nu} + N_{\nu \mu} - N_{\nu \mu}) (\delta^{\alpha}_{\sigma} u^\nu u^\rho - s^{\alpha \nu}_{\gamma} a_{\gamma \sigma} u^\rho u^\gamma - s^{\alpha \rho} a_{\sigma \beta} u^\beta u^\nu) = 0.$$  

(2.185)

Assuming that $a \sim T \sim N$, we find, at first order,

$$\frac{dv^\alpha}{d\lambda} + \Gamma^{\alpha}_{\nu \rho} v^\nu v^\rho + \frac{1}{2} s^{\mu \nu} (T^{e}_{\nu \rho} s_{\epsilon \rho} + T^{e}_{\rho \nu} s_{\epsilon \nu}) - \frac{1}{2} s^{\sigma \mu} (N_{\mu \nu} + N_{\rho \mu} - N_{\nu \mu}) \delta^{\alpha}_{\sigma} u^\rho u^\sigma = 0.$$  

(2.186)
It’s interesting to notice that torsion and the non-metricity tensor contribute at this approximation order, while the metric-asymmetry object provides a negligible contribution.

### 2.3.4 Discussion

A classification of non-Riemannian geometries with an asymmetric metric tensor has been proposed according to the Schouten classification. By adopting this approach for affine-connection geometries with an asymmetric metric, the structure and variety of such geometries can be investigated in a fully-geometrical formalism without adopting any variational principle. The definition of autoparallel trajectories at first order has been established: these results can be compared with those of [122], where autoparallel trajectories are derived from a modified Lagrangian, and torsion is shown to be relevant even at zeroth order, playing the same role as the gravitational field.
3 Polymer representation of quantum mechanics

In this chapter, we will analyze some features of the polymer representations of quantum mechanics. This kind of representation of quantum mechanics is based on assuming a different representation of the canonical commutation relations, and it is an interesting tool in the description of scenarios, where an underlying discretized structure is somehow hypothesized. In particular, we will focus our attention on cosmology and on extra-dimensional scenarios. In the first case, the application of the polymer representation of quantum mechanics to the Taub cosmological model will help us shed light on the kind of mechanism, according to which the cosmological singularity is removed in Loop Quantum Cosmology. In the second case, we will gain insight on the possibility to consider a 5D Kaluza-Klein model, where the tower of extra-D modes is truncated. The original works appeared on (PL2), (PL14).

3.1 Introduction

After implementing the canonical commutation relations from a quantum point of view, it is possible to find out the representations of such operators. In the coordinate representation, the wave function $\psi_\alpha(x)$ for the state $|\alpha>$ is given by $\psi_\alpha(x) = <x|\alpha>$ while, in the momentum representation, we get $\psi_\alpha(p) = <p|\alpha>$. These two representations are linked by Fourier duality, i.e., it is possible to pass from a representation to the other one by Fourier transform. The probability of finding the system in a given state, say $|\alpha'>$, is given by $P_{\alpha'} = \int_D dx |\psi_{\alpha'}(x)|^2 < \infty$, and the equivalent version in the momentum representation, so that wave functions belong to the Hilbert space $\mathcal{H} = L^2(D, dx)$, where the integration domain $D$ (and therefore, the specification of $L^2$) depends on the physics to be described. Since the momentum $p$ is the generator of translations, its action in the coordinate and
in the momentum representations is given by $P = -i\hbar\partial/\partial x$ and $i\hbar\partial/\partial p$, respectively, while $x$ can be shown to behave as a multiplicative operator.

### 3.1.1 Difference operators versus differential operators

It is also possible to implement a quantum-mechanical system by introducing difference operators instead of differential operators [173], i.e.,

$$D_{x_1,x_2}f(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad x_2 \neq x_1, \quad x_1, x_2 \in \mathbb{R}, \quad (3.1)$$

and the differential case should be recovered in the limiting process $x_2 \to x_1$. In particular, the derivative can be replaced by two kinds of operators

- additive operators $D^a$, i.e.
  $$D^a f(x) = \frac{f(x + a) - f(x - a)}{(x + a) - (x - a)} = \frac{f(x + a) - f(x - a)}{2a}, \quad a \in \mathbb{R} \quad (3.2)$$

- multiplicative operators $D^q$, i.e.
  $$D^q f(x) = \frac{f(qx) - f(q^{-1}x)}{qx - q^{-1}x} = \frac{1}{x} \frac{f(qx) - f(q^{-1}x)}{q - q^{-1}}, \quad q \in \mathbb{R}. \quad (3.3)$$

Considering difference operators instead of differential operators can be consistent for those cases, where differential operators do not exist, or where a discretized underlying structure is hypothesized, i.e., if a lattice is considered. According to the previous definitions, two kinds of lattices can be recognized, respectively, i.e.

- uniform $a$-lattices, i.e. $\mathbb{L}_a = \{x_0 + ja | j \in \mathbb{Z}, x_0 \in \mathbb{R}\}$

- uniform $q$-lattices, i.e. $\mathbb{L}_q = \{x_0 q^j | j \in \mathbb{Z}, x_0 \in \mathbb{R}, x_0 \neq 0\}$.

The introduction of a scale is closely related to the definition of a scale, and the consequence continuum limit. The relevance in introducing a scale is the possibility to focus the attention from the points of the lattice to the intervals defined by the lattice, with the aim of approximating continuous functions on $\mathbb{R}$ with functions that are constants on such intervals. For any given scale, one can approximate functions on the lattice, and one can pass from one scale to the next one by a coarse-graining map.
3.1.2 Weyl Quantization

Weyl quantization consists in assuming canonical commutation relation for the two operators \( \hat{p}, \hat{q} \), and in establishing a different (Weyl) representation of the operators. One can thereafter implement a quantization programme, and then recover information about the standard quantization method via the so-called GNS construction.

**Weyl Systems**  Given a symplectic vector space \((E, \omega)\), i.e., a vector space \(E\) endowed with a symplectic (non-degenerate, skew-symmetric, bilinear) form \(\omega\), a Weyl system is the strongly-continuous map \(W\) from \(E\) to unitary transformations on some Hilbert space \(\mathcal{H}\)

\[ W : E \rightarrow U(\mathcal{H}) \]  

and the Weyl form of the commutation relations reads

\[ W(e_1)W(e_2) = e^{i\omega(e_1,e_2)}W(e_2)W(e_1), \]

where the cocycle of the representation is determined by the the symplectic structure \(w\).

Complex coordinates, and the construction of a Fock space, with creation and annihilation operators, can be defined by the introduction of a complex form \(J : E \rightarrow E\), \(J^2 = -1\). An inner product on \(E\) can be defined by using \(J\) and \(\omega\).

It is possible to decompose the vector space \(E\) as \(\mathcal{L} \oplus \mathcal{L}^*\), where \(\mathcal{L} \subset E\) is a Lagrangian (both isotropic and coisotropic) subspace of \(E\). According to the von neumann theorem, the Hilbert space \(\mathcal{H}\) is the space of square-integrable functions \(\phi\) on \(\mathcal{L}\) endowed with the translation-invariant Lebesgue measure \(d\mu\), i.e., \(\mathcal{H} = L^2(d\mu, \mathcal{L})\). In this decomposition, vectors on \(E\) can be defined as \(e = (\alpha, \beta), \beta \in \mathcal{L}, \alpha \in \mathcal{L}^*\), and the action of \(W\) on the functions \(\phi\) reads

\[ U(\alpha)\phi(q) \equiv W((\alpha,0))\phi(q) = e^{i\alpha q}\phi(q) \]
\[ V(\beta)\phi(q) \equiv W((0,\beta))\phi(q) = \phi(q - \beta). \]

The vacuum expectation values of the operators \(U\) and \(V\) depends on the metric \(g\) constructed out of \(J\), i.e., \(g(e_1,e_2) = w(e_1,Je_2)\).

**The Stone-von Neumann Uniqueness theorem**  The Stone-von Neumann Uniqueness theorem states that any unitary irreducible representation of the Weyl commutation relation on \(\mathbb{C}^n\) is isomorphic to the Schroedinger representation. Furthermore, as

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corollary, it is also possible to show that any representation of the Weyl commutation relation on \( \mathbb{C}^n \) is the direct sum of copies of the Schrödinger representation.

### 3.1.3 GNS Construction

The GNS construction allows one to gain insight onto different representations of a given algebra \([174]\).

Given a function \( \phi \) and an algebra \( \mathcal{A} \), \( \phi \) is called a linear form over \( \mathcal{A} \) if

\[
\phi(\alpha A + \beta B) = \alpha \phi(A) + \beta \phi(B), \quad \forall A, B \in \mathcal{A}, \forall \alpha, \beta \in \mathbb{C}
\] (3.8)

If

\[
\|xy\| \leq \|x\| \|y\|, \quad \forall x, y \in \mathcal{A},
\] (3.9)

then \( \mathcal{A} \) is a Banach algebra. If \( \mathcal{A} \) is a Banach algebra, a linear form \( \phi \) is bounded if

\[
|\phi(A)| \leq c \|A\|
\] (3.10)

and the lowest bound for \( c \) is the norm of \( \phi \).

If

\[
(Ax, y) = (y, Ax) = (x, A^* y), \quad \forall A \in \mathcal{A},
\] (3.11)

then \( \mathcal{A} \) is a * algebra. If \( \mathcal{A} \) is a * algebra, with unit, the linear form \( \phi \) is a state if it is

- real \( \phi(A^*) = \phi(A) \)
- positive \( \phi(A^* A) \geq 0 \)
- normalized \( \|\phi\| = 1 \).

From a mathematical point of view, a state is a normalized positive linear form. A positive linear form over a Banach * algebra with unit is bounded, and \( \|\phi\| = \phi(I) \).

Furthermore, it satisfies the Schwarz inequality

\[
|\phi(AB)|^2 \leq \phi(AA^*) \phi(BB^*).
\] (3.15)

In fact, if we assume that the self-adjoint elements of \( \mathcal{A} \) correspond to physical observables, and that the unit element \( I \) correspond to the trivial observable, whose value is 1 for any physical state, then such a linear form can be interpreted as an expectation functional over physical observables.
Each positive linear form $\omega$ over a $*$ algebra $A$ defines a Hilbert space $\mathcal{H}_\omega$ and a representation $\pi_\omega$ of $A$ by linear operators acting on $A$.

Since $A$ is a linear space over the field $\mathbb{C}$, $\omega$ defines an Hermitian semi-definite product on $A$, i.e.

$$< A|B> = \omega(A, B), \quad <A|A| \geq 0, \quad |<A|B>|^2 \leq <B|B><A|A>, \quad \forall A, B \in A. \tag{3.16}$$

The set $\mathcal{J} \subset A$, $\mathcal{J} = \{X \in A : \omega(X^*X) = 0\}$, is a left ideal, and is called the Gelfand ideal of the state. Eliminating this set from $A$, i.e., considering $A/\mathcal{J}$ allows us to obtain linear space equipped with Hermitian positive-definite scalar product, and a vector $\psi$ in this space corresponds to the equivalence class $[A]$ of the elements of $A$ modulo $\mathcal{J}$, $\psi = \{A + \mathcal{J}\}$, with

$$<\psi|\psi> \equiv \|\psi\|^2 > 0. \tag{3.17}$$

It is worth remarking that the scalar product defined in (3.16) does not depend on $[A]$. Because of that, the action of the representation $\pi_\omega(A)$, defined on $A/\mathcal{J} \subset \mathcal{H}_\omega$, is

$$\pi_\omega(A)\psi = [AB] \quad if \quad \psi = [B]. \tag{3.18}$$

A representation $\pi$ is called cyclic if a cyclic vector $\Omega \in \mathcal{H}$ exists. A vector $\Omega$ is called cyclic if $\pi(A)\Omega$ is dense in $\mathcal{H}$. If $A$ has a unit, $\Omega = [I]$. Furthermore

$$\omega(A) = <\Omega|\pi_\omega(A)|\Omega>, \quad \tag{3.19}$$

and it is sometimes referered to as the vacuum state. Similarly, any vector $\psi \in \mathcal{H}_\omega$ defines a state

$$\omega_\psi(A) = <\psi|\pi_\omega(A)|\psi>. \quad \tag{3.20}$$

The GNS theorem (named after Gel'fand, Najmark and Segal) states that, given a $C^*$ algebra $A$ endowed with unit, and a linear positive functional $\omega$ in a compact subset of $A$, the triple $(\mathcal{H}(\omega), \pi_\omega, \Omega)$ exists and is unique.

### 3.1.4 Polymer Representation of quantum mechanics

The abstract techniques described in the previous sections can be specified for the case of a physical system, where a cutoff has to be somehow introduced, and the removal of the cutoff can be analyzed within the framework of the proper GNS construction.
To this end, it is possible to fix a physical meaning to the tools previously established.

The polymer representation of quantum mechanics [175, 176, 177, 178] consists in defining abstract kets, labeled by a real number, and then considering a suitable finite subset of them, whose Hilbert space is defined by the corresponding inner product [177]. This procedure can be shown to be an inequivalent representation of the Weyl algebra wrt the ordinary Schroedinger one. This representation helps one gain insight onto some particular features of quantum mechanics, when an underlying discrete structure is somehow hypothesized. The request that the Hamiltonian associated to the system be of direct physical interpretation defines the polymer phase space, and the continuum limit can be recovered by the introduction of the concept of scale [178].

In the simplest toy model, we can consider a particle moving on the real line \( \mathbb{R} \), so that the symplectic vector space will be its phase space \( \Gamma = \mathbb{R}^2 \), which can be automatically decomposed in terms of the \( q \) space \( (\mathcal{L}) \) and the \( p \) space \( (\mathcal{L}^*) \). Accordingly, the Hilbert space \( \mathcal{H} \) of the functions \( \phi \) will be \( \mathcal{H} = L^2(\mathbb{R},d\mu) \). The exponentiated versions of the operators \( q \) and \( p \),

\[
U(\alpha) = e^{i\hbar \alpha q}, \quad V(\beta) = e^{i\hbar \beta p}
\] (3.21)

generates the Weyl form of the commutation relation

\[
U(\alpha)V(\beta) = e^{i\hbar \alpha \beta}V(\beta)U(\alpha) : \nonumber
\] (3.22)

The Weyl algebra associated to this system if the set of finite linear combinations of the generators.

The introduction of a cutoff can be related to the particular choice of the complex structure \( J_d \)

\[
J = \begin{pmatrix} 0 & -d^2 \\ 1/d^2 & 0 \end{pmatrix}, \tag{3.23}
\]

where the parameter \( d \) has the dimensions of a length. It is possible to show that the Hilbert space \( \mathcal{H} \) of the functions \( \phi \) for the system is one with gaussian measure, \( \mathcal{H}_d = L^2(\mathbb{R},d\mu_d) \), where

\[
d\mu_d = \frac{1}{d\sqrt{\pi}} e^{-\frac{q^2}{d^2}} dq. \tag{3.24}
\]
Furthermore, the action of (3.21) on the functions $\phi$ and their vacuum expectation values are

\[ U(\alpha)\phi(q) = e^{i\frac{q}{\sqrt{d^2}}\alpha}q, \quad < U(\alpha) > = e^{\frac{1}{4}\frac{d^2\alpha^2}{\hbar^2}} \tag{3.25} \]

\[ V(\beta)\phi(q) = e^{\frac{\beta}{d^2}(q-\beta/2)}\phi(q), \quad V(\beta) > = e^{\frac{1}{4}\frac{\beta^2}{d^2}} \tag{3.26} \]

and the value in the exponents of the vacuum expectation values depends on the metric $g_d(e_1, e_2) = \omega(e_1, J\omega_2)$, for $e_1 = e_2 = (0, \alpha)$ and $e_1 = e_2 = (\beta, 0)$, respectively. The states of $\mathcal{H}_d$ generated by the action of the operator $U(\alpha)$ on the vacuum $\phi_0(q) = 1$ are

\[ \phi_\alpha(q) = U(\alpha)\phi_0(q) = e^{i\frac{q}{\sqrt{d^2}}\alpha}. \tag{3.27} \]

and the inner product, defined by (3.24), is

\[ < \phi_\mu, \phi_\nu > = \int \frac{1}{d\sqrt{\pi}} e^{-\frac{q^2}{d^2}} dq e^{-i\frac{\mu}{\sqrt{d^2}}q} e^{i\frac{\nu}{\sqrt{d^2}}q}. \tag{3.28} \]

Information about the Schroedinger representation of the model, $\phi(q) \in \mathcal{H}_s = L^2(\mathbb{R}, dq)$, can be recovered by the mapping

\[ \psi(q) = K\phi(q) = \frac{e^{-\frac{q^2}{2d^2}}}{d^{1/2}\sqrt{\pi^{1/4}}} \phi(q), \quad \psi_0(q) = \frac{e^{-\frac{q^2}{2d^2}}}{d^{1/2}\sqrt{\pi^{1/4}}} \phi_0(q) = \frac{e^{-\frac{q^2}{2d^2}}}{d^{1/2}\sqrt{\pi^{1/4}}}. \tag{3.29} \]

For finite values of $d$, the mapping is well-defined, so that all the $d$ representations are unitarily equivalent to the Schroedinger one. Contrastingly, in both limits $d \to 0$ and $1/d \to 0$, the descriptions are not unitarily equivalent to the Schroedinger one any more.

**The limit $1/d \to 0$ - A representation** In this limit, the operators $\hat{U}(\alpha)$ become discontinuous with respect to $\alpha$. Since the continuity of these operators is one of the hypothesis of the Stone-Von Neummann theorem, the uniqueness result does not apply here. The representation is inequivalent to the standard one.

Let us now analyze the other operator, i.e. the action of the operator $\hat{V}(\beta)$ on the basis $\phi_\alpha(q)$:

\[ \hat{V}(\beta) \cdot \phi_\alpha(q) = e^{-\frac{\beta^2}{2d^2} - i\frac{q\beta}{\hbar}} e^{(\beta/d^2 + i\alpha/\hbar)q} \]

which in the limit $1/d \to 0$ goes to,

\[ \hat{V}(\beta) \cdot \phi_\alpha(q) \rightarrow e^{i\frac{\beta}{\hbar}} \phi_\alpha(q) \]
that is continuous on $\beta$. Thus, in the limit, the operator $\hat{p} = -i\hbar \partial_q$ is well defined. Also, note that in this limit the operator $\hat{p}$ has $\phi_\alpha(q)$ as its eigenstate with eigenvalue given by $\alpha$:

$$\hat{p} \cdot \phi_\alpha(q) \mapsto \alpha \phi_\alpha(q)$$

To summarize, the resulting theory obtained by taking the limit $1/d \to 0$ of the ordinary Schrödinger description has the following features: the operators $U(\alpha)$ are well defined but not continuous in $\alpha$, so there is no generator (no operator associated to $q$). The basis vectors $\phi_\alpha$ are orthonormal (for $\alpha$ taking values on a continuous set) and are eigenvectors of the operator $\hat{p}$ that is well defined.

**The limit $d \to 0$ - B representation** We may redefine both the basis and the measure. We could consider, instead of a half-delta with support $\beta$, a Kronecker delta or characteristic function with support on $\beta$:

$$\chi'_\beta(q) := \delta_{q,\beta}$$

These functions have a similar behavior with respect to the product as the half-deltas, namely: $\chi'_\beta(q) \cdot \chi'_\alpha(q) = \delta_{\beta,\alpha}$. The main difference is that neither $\chi'$ nor their squares are integrable with respect to the Lebesgue measure (having zero norm). In order to fix that problem we have to change the measure so that we recover the basic inner product with the new basis. The needed measure turns out to be the discrete counting measure on $\mathbb{R}$. Thus any state in the ‘half density basis’ can be written (using the same expression) in terms of the ‘Kronecker basis’.

**Duality** Note that in this B-polymer representation, both $\hat{U}$ and $\hat{V}$ have their roles interchanged with that of the A-polymer representation: while $U(\alpha)$ is discontinuous and thus $\hat{q}$ is not defined in the A-representation, we have that it is $V(\beta)$ in the B-representation that has this property. In this case, it is the operator $\hat{p}$ that can not be defined. We see then that given a physical system for which the configuration space has a well defined physical meaning, within the possible representation in which wave-functions are functions of the configuration variable $q$, the A and B polymer representations are radically different and inequivalent.

Furthermore, it is also true that the A and B representations are equivalent in a different sense, by means of the duality between $q$ and $p$ representations and the $d \leftrightarrow 1/d$ duality: The A-polymer representation in the “$q$-representation" is equivalent
to the B-polymer representation in the “$p$-representation”, and conversely. When
studying a problem, it is important to decide from the beginning which polymer
representation one should be using. This has as a consequence an implication on
which variable is naturally “quantized” (even if continuous): $p$ for A and $q$ for B. There
could be for instance a physical criteria for this choice. For example a fundamental
symmetry could suggest that one representation is more natural than another one.

In the other polarization, namely for wavefunctions of $p$, the picture gets reversed: $q$
is discrete for the A-representation, while $p$ is for the B-case. Let us end this section
by noting that the procedure of obtaining the polymer quantization by means of
an appropriate limit of Fock-Schrödinger representations might prove useful in more
general settings in field theory or quantum gravity.

**Kynematics**

One can start by considering abstract kets $|\mu >, \mu \in \mathbb{R}$, and a suitable subset defined
by $\mu_i \in \mathbb{R}, i = 1, 2, ..N$. These kets are assumed to be an orthonormal basis, i.e.,
$< \mu | \nu > = \delta_{\mu \nu}$, along which any state $\phi$ can be projected. This defines a Hilbert
space $\mathcal{H}_{pol}$, on which two basic operators act, the symmetric ”label” operator, $\hat{\epsilon}$, such
that $\hat{\epsilon}|\mu > = \mu |\mu >$, and a one-parameter family of unitary operators, $\hat{s}(\lambda)$, such that
$\hat{s}(\lambda)|\mu > = |\mu + \lambda >$. Because all kets are orthonormal, $\hat{s}(\lambda)$ is discontinuous, and
cannot be obtained from any Hermitian operator by exponentiation. It is worth not-
ing that this Hilbert space is not separable$^1$.

For the toy model of a 1-dimensional system, whose phase space is described by the
variables $p$ and $q$, the polymer representation techniques find interesting applications
when one of the two variables is supposed to be discrete. This discreteness will affect
both wave functions, obtained by projecting the physical state on the $p$ or $q$ basis
(polarization), and the operators associated to the canonical variables, acting on them.

For later purposes, we will discuss only the case of a discrete position variable $q$,
and the corresponding momentum polarization.

In this case, wave functions are given by
\[ \psi_\mu(p) = < p|\mu > = e^{ip\mu}. \]  
(3.30)

$^1$A Hilbert space is separable if and only if it admits a countable orthonormal basis.
Accordingly, the "label" operator $\hat{\epsilon}$ is easily identified with $\hat{q}$, i.e.,

$$\hat{q}\phi_\mu = -i \frac{\partial}{\partial p}\phi_\mu = \mu\psi_\mu,$$  \hspace{1cm} (3.31)

while the "shift" operator does not exists, as discussed previously.

It can be shown that corresponding Hilbert space is $\mathcal{H}_{\text{pol}} = L^2(\mathbb{R}_B, d\mu_H)$, i.e., the set of square-integrable functions defined on the Bohr compactification of the real line $\mathbb{R}_B$, with a Haar measure $d\mu_H$. Since the kets $|\mu\rangle$ are arbitrary but finite, the wave functions can be interpreted as quasi-periodic function, with the inner product

$$<\psi_\mu|\psi_\lambda> = \int_{\mathbb{R}_B} d\mu_H \bar{\psi}_\mu(p)\psi_\lambda(p) = \lim_{L\rightarrow\infty} \frac{1}{2L}\int_{-L}^{L} dp \bar{\psi}_\mu(p)\psi_\lambda(p) = \delta_{\mu,\lambda}.$$  \hspace{1cm} (3.32)

**Dynamics** The Hamiltonian operator $H$ describing a quantum-mechanical system is usually a function of both coordinate and momentum, i.e.

$$H = H(q, p) = \frac{p^2}{2m} + V(q)$$  \hspace{1cm} (3.33)

while, in the particular case of a discrete position variable in the momentum polarization, $p$ cannot be implemented as an operator, so that some restrictions on the model have to be required.

As a first step, a suitable approximation for the kinetic term has to be provided. For this purpose, it is useful to restrict the arbitrary kets $|\mu_i\rangle$, $i \in \mathbb{R}$ to $|\mu_i\rangle$, $i \in \mathbb{Z}$, i.e., to introduce the notion of regular graph $\gamma_{\mu_0}$, defined as a numerable set of equidistant points, whose separation is given by the parameter $\mu_0$, $\gamma_{\mu_0} = \{ q \in \mathbb{R} | q = n\mu_0, \forall n \in \mathbb{Z} \}$. The associated Hilbert space $\mathcal{H}_{\gamma_{\mu_0}}$ is separable. Because of the regular graph $\mu_0$, the eigenfunctions of $\hat{p}_{\mu_0}$ must be of the form $e^{im\mu_0 p}$, $m \in \mathbb{Z}$, which are Fourier modes, of period $2\pi/\mu_0$. The inner product (3.32) is equivalent to the inner product on a circle $S^1$ with uniform measure, i.e.,

$$<\phi(p)|\psi(p) >_{\mu_0} = \frac{\mu_0}{2\pi} \int_{-\pi/\mu_0}^{\pi/\mu_0} \hat{\phi}(p)\psi(p),$$  \hspace{1cm} (3.34)

with $p \in (-\pi\mu_0, \pi/\mu_0)$, so that $\mathcal{H}_{\gamma_{\mu_0}} = L^2(S^1, dp)$. Within this space, it is possible to construct an approximation for the "shift" operator, i.e. a regulated operator $\hat{p}_{\mu_0}$,

$$\hat{p}_{\mu_0}|\mu_n>= \frac{i}{2\mu_0} (|\mu_{n+1}> - |\mu_{n-1}>).$$  \hspace{1cm} (3.35)
More precisely, the polymer paradigm can be understood as the formal substitution

\[ p \rightarrow \frac{1}{\mu_0} \sin(\mu_0 p), \]  

(3.36)

where the incremental ratio (3.35) has been evaluated for exponentiated operators. The Hamiltonian operator \( H_{\mu_0} \), which lives in \( \mathcal{H}_{\gamma_0} \), reads \( H_{\mu_0} = \frac{p_0^2}{2m} + V(\dot{q}) \), where the action of the new multiplication operator \( \hat{p}_{\mu_0} \) on wave functions in the momentum polarization is

\[ \hat{p}_{\mu_0}^2 \psi(p) = \frac{2}{\mu_0} [1 - \cos(p\mu_0)], \]  

(3.37)

while the differential operator \( q \) is well defined.

**Continuum Limit**  
The physical Hilbert space of such theories can be constructed as the continuum limit of effective theories at different scales, and can be illustrated to be unitarily isomorphic to the ordinary one, \( \mathcal{H}_S = L^2(\mathbb{R}, dp) \).

To this end, it is useful to remark that it is impossible to obtain \( \mathcal{H}_S \) starting from a given graph \( \gamma_0 = \{ q_k \in \mathbb{R} | q_k = ka_0, \forall k \in \mathbb{Z} \} \) by dividing each interval \( a_0 \) into \( 2^n \) in new intervals of length \( a_n = a_0/2^n \), because \( \mathcal{H}_S \) cannot be embedded into \( \mathcal{H}_{pol} \).

It is however possible to go the other way round and to look for a continuous wave function that is approximated by a wave function over a graph, in the limit of the graph becoming finer. In fact, if one defines a scale \( C_n \), i.e., a decomposition of \( \mathbb{R} \) in terms of the union of closed-open intervals that have lattice points as end points and cover \( \mathbb{R} \) without intersecting, one is then able to approximate continuous functions with functions that are constant on these intervals. As a result, at any given scale \( C_n \), the kinetic term of the Hamiltonian operator can be approximated as in (3.37), and effective theories at given scales are related by coarse-graining maps. In particular, it is necessary to regularize the Hamiltonian, treated as a quadratic form, as a self-adjoint operator at each scale by introducing a normalization factor in the inner product. The convergence of microscopically-corrected Hamiltonians is based on the convergence of energy levels and on the existence of completely normalized eigencoverctors compatible with the coarse-graining operation.
3.1.5 Link with Loop Quantum Cosmology

LQG, as well as LQC, are both based on a Hamiltonian formulation of GR with basic variables an SU(2) valued connection and the conjugate momentum variable which is a densitised triad\(^2\), a derivative operator quantised in the full LQG theory in the form of fluxes. By using connection-triad variables, arising from a canonical transformation of Ashtekar  variables, it is possible to make an analogy with gauge theories.

The connection carries information about the spatial curvature, in the form of the spin-connection, and the extrinsic curvature, while the densitised triad carries information about the spatial geometry, encoded in the three-metric. More precisely, we have

\[
A^i_a = \Gamma^i_a + \gamma K^i_a ,
\]

\[
E^a_i = \sqrt{|q|} e^a_i ,
\]

i.e. the connection, \(A^i_a\), is related to the ADM variables through \(\Gamma^i_a = -(1/2)e^{ijk}e^b_j(2\partial_k e^b_i + e^c_k e^i_a \partial_c e^b_i)\), the spin-connection compatible with the co-triad, \(\gamma\), Barbero-Immirzi parameter and \(K^i_a\), the extrinsic curvature one-form \(K^i_a = e^{bi} K_{ab}\), and the densitised triad, \(E^a_i\), is related to the three-metric, \(q_{ab}\), \(e^a_i\) is a physical triad, dual \((e^a_i e^j_a = \delta^j_i)\) to the co-triad, \(e^j_i\), and satisfying \(q_{ab} = e^a_i e^j_b \delta_{ij}\) , and

The Poisson bracket of the densitised triad and the connection reads

\[
\{ A^i_a(x), E^b_j(y) \} = \kappa \gamma \delta^b_i \delta^a_j \delta^3(x, y) ,
\]

where \(\kappa \equiv 8\pi G\).

---

\(^2\)The variables \(E^a_i\) and \(A^i_a\) were introduced by Barbero [179] as an alternative to the complex Ashtekar [89] variables. Both real and complex connections have been used for canonical gravity. The complex connection has SL(2,C) as gauge group, while the real connection has SU(2) as gauge group. Mathematical techniques can only cope with a quantum theory based on SU(2).
3.1 Introduction

The triad along an $S$ surface, such as

$$h_e(A) = \mathcal{P} \exp \int ds \dot{\gamma}^\mu(s) A^i_\mu(\gamma(s)) \tau_i ,$$

(3.40a)

$$F(S, f) = \int_S \epsilon_{abc} E^a f_i dx^a dx^b ,$$

(3.40b)

where $\mathcal{P}$ indicates a path ordering of the exponential, $\gamma^\mu$ is a vector tangent to the edge and $\tau_i = -i\sigma_i / 2$, with $\sigma_i$ the Pauli spin matrices, and $f_i$ an SU(2) valued test function.

LQC These expressions can be simplified a lot by restricting the analysis to homogeneous and isotropic geometries, for which

$$q = [a(t)]^2 [(1 - kr^2)^{-1} dr^2 + r^2 d\Omega^2] .$$

(3.41)

The symmetry reduced variables $E, A$ are given by $A^i_a = V_0^{-1/3} c^0 \omega^i_a$ and $E^a_i = pV_0^{-2/3} \det E^0_i \epsilon^a_i$, where the vector fields $E^a_i$ are dual to the 1-forms: $E^a_i \epsilon^a_j = \delta^a_j$.

The symmetric connections and triads can be decomposed using basis one-forms and vector fields obtained by Bianchi models.

In the case of a spatially flat background, derived from the Bianchi I model, the isotropic connection can be expressed in terms of the dynamical component of the connection $\tilde{c}(t)$, and the densitised triad can be decomposed in terms of the remaining dynamical quantity $\tilde{p}(t)$ left after symmetry reduction using the Bianchi I basis vector fields $X^a_i = \delta^a_i$. More precisely, we have

$$A^i_a = \tilde{c}(t) \omega^i_a ,$$

(3.42a)

$$E^a_i = \sqrt{q} \tilde{p}(t) X^a_i ,$$

(3.42b)

where $\omega^i_a$ is a basis of left-invariant one-forms $\omega^i_a = dx^i$, $q$ stands for the determinant of the fiducial background metric, $q = \omega^a_i \omega_b_i$.

In terms of the metric variables with three-metric $q_{ab} = a^2 \omega^i_a \omega_b_i$, the dynamical quantity is just the scale factor $a(t)$. Given that the Bianchi I basis vectors are $X^a_i = \delta^a_i$, and since the basis vector fields are spatially constant in the spatially flat
model, we obtain
\begin{equation}
|\tilde{p}| = a^2 , \quad (3.43a)
\end{equation}
\begin{equation}
\tilde{c} = \text{sgn}(\tilde{p})\gamma \frac{\dot{a}}{N} , \quad (3.43b)
\end{equation}
where the absolute value is taken because the triad has an orientation. In what follows, the lapse function, which is a constant due to spatial homogeneity, will be set equal to 1. Thus, GR can be formulated as a gauge theory in Ashtekar variables.

The canonical variables \(\tilde{c}, \tilde{p}\) are related through \(\{\tilde{c}, \tilde{p}\} = \frac{\kappa \gamma}{3} V_0\), where \(V_0\) the volume of the elementary cell adapted to the fiducial triad.

Defining the triad component \(p\), determining the physical volume of the fiducial cell, and the connection component \(c\), determining the rate of change of the physical edge length of the fiducial cell, as
\begin{equation}
p = V_0^{2/3} \tilde{p} , \quad c = V_0^{1/3} \tilde{c} , \quad (3.44)
\end{equation}
respectively, we obtain
\begin{equation}
\{c, p\} = \frac{\kappa \gamma}{3} , \quad (3.45)
\end{equation}
independent of the volume \(V_0\) of the fiducial cell.

Thus, the basic configuration variables in LQC are holonomies of the connection along a line segment \(\mu_0 0^1 e_0^a\) and the flux of the triad
\begin{equation}
h_i^{(\mu_0)}(A) = \cos \left( \frac{\mu_0 c}{2} \right) 1 + 2 \sin \left( \frac{\mu_0 c}{2} \right) \tau_i , \quad (3.46a)
\end{equation}
\begin{equation}
F_S(E, f) \propto p , \quad (3.46b)
\end{equation}
the basic momentum variable is the triad component \(p\). Here, \(1\) is the identity \(2 \times 2\) matrix and \(\tau_i = -i \sigma_i/2\) is a basis in the Lie algebra \(\text{SU}(2)\) satisfying the relation \(\tau_i \tau_j = (1/2)\epsilon_{ijk} \tau^k - (1/4)\delta_{ij}\).

Quantization procedure In quantum theory, following Dirac, one first constructs a kinematical description [71]. The Hilbert space is the space \(L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})\) of square integrable functions on the Bohr compactification of the real line. To specify states concretely, it is convenient to work with the representation in which the operator \(\hat{p}\) is diagonal. Eigenstates of \(\hat{p}\) are labelled by a real number \(\mu\) and satisfy the orthonormality relation:
\begin{equation}
< \mu_1 | \mu_2 > = \delta_{\mu_1, \mu_2} . \quad (3.47)
\end{equation}
Since the right side is the Kronecker delta rather than the Dirac delta distribution, a typical state in this Hilbert space can be expressed as a countable sum; $| \Psi > = \sum_n c^{(n)} | \mu_n >$ where $c^{(n)}$ are complex coefficients and the inner product is given by

$$\langle \Psi_1 | \Psi_2 \rangle = \sum_n \overline{c_1^{(n)}} c_2^{(n)} .$$

(3.48)

The fundamental operators are $\hat{p}$ and $\exp \frac{i\lambda c}{2}$:

$$\hat{p} | \mu \rangle = \frac{\kappa \gamma \hbar | \mu |}{6} | \mu \rangle ,$$

(3.49)

$$\exp \frac{i\lambda c}{2} | \mu > = | \mu + \lambda >$$

(3.50)

where $\lambda$ is any real number. Since the holonomy $h_k^{(\lambda)}$ of the gravitational connection $A^k_a$ along a line segment $\lambda c e^k_a$ is given by:

$$h_k^{(\lambda)} = \cos \frac{\lambda c}{2} \mathbb{I} + 2 \sin \frac{\lambda c}{2} \tau_k$$

(3.51)

the corresponding holonomy operator has the action:

$$h_k^{(\lambda)} | \mu > = \frac{1}{2} ( | \mu + \lambda > + | \mu - \lambda > ) \mathbb{I} + \frac{1}{i} ( | \mu + \lambda > - | \mu - \lambda > ) \tau_k .$$

(3.52)

### 3.2 Cosmology

The necessity for a quantum theory of gravity arises from fundamental considerations, and, in particular, from the space-time singularity problem. In fact, the classical theory of gravity implies the well known singularity theorems, among which the cosmological one [180]. The canonical quantization of gravity, which exhibits a host of difficulties both at technical and interpretative levels, is based on the Heisenberg representation of the Weyl algebra [181, 182]. On the other hand, the background-independent formulation of canonical quantum gravity based on Yang-Mills formalism has recently appeared [89]. Anyhow, smearing such variables in the holonomy-flux representation is an important step towards canonical quantum gravity [183, 184]. The scenario induced by such an algebra is illustrated to be equivalent to the so-called polymer representation of quantum mechanics [175, 176, 177, 178], as soon as a mechanical system is taken into account.

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3Here $\mathbb{I}$ is the unit $2 \times 2$ matrix and $\tau_k$ is a basis in the Lie algebra $SU(2)$ satisfying $\tau_i \tau_j = \frac{1}{2} \varepsilon_{ijk} \tau^k - \frac{1}{4} \delta_{ij}$. Thus, $2i \tau_k = \sigma_k$, where $\sigma_i$ are the Pauli matrices.
3.2.1 The Taub model

This work is aimed at investigating the quantization of the Taub model in the polymer representation of quantum mechanics (PL2). The Taub Universe arises as a particular case of the Bianchi IX model, i.e. the most general scheme allowed by the homogeneity constraint [185]. In the Bianchi IX model, the Universe dynamics towards the classical singularity is summarized by the chaotic motion of a particle. More precisely, this particle bounces an infinite number of times against the potential walls of a triangular domain, on a two-dimensional plane. The two-dimensional plane describes the configuration space of the particle (Universe) dynamics. The Taub model consists in restricting the dynamics to that of a one-dimensional particle bouncing against a wall, when only one degree of freedom is taken into account.

The relevance of the Taub universe in quantum cosmology is due to the fact that it is a necessary step towards the more general Bianchi IX model. The advantage of this model is that it is a generalization of other isotropic models. In particular, it has been used to test the validity of the minisuperspace scheme [186] and to explore the application of the extrinsic cosmological time [187]. Furthermore, the Taub model has also been investigated within the framework of a generalized uncertainty principle in [188], where the cosmological singularity has been shown to be probabilistically removed.

The polymer representation of quantum mechanics is relevant in treating the quantum-mechanical properties of a background-independent canonical quantization of gravity. In fact, the holonomy-flux algebra used in Loop Quantum Gravity reduces to a polymer-like algebra, when a system with a finite number of degrees of freedom is taken into account [175, 176]. From a quantum-field theoretical point of view, this is substantially equivalent to introducing a lattice structure on the space [189, 190, 191]. Loop Quantum Cosmology [192, 193, 194] can be regarded as the implementation of this quantization technique in the minisuperspace dynamics [195, 196].

The Taub model is approached in the scheme of an Arnowitt-Deser-Misner (ADM) reduction of the dynamics in the Poincaré plane. As a result, a time variable naturally emerges, and the Universe is described by an anisotropy-like variable. The anisotropy variable and its conjugate momentum are quantized within the framework of the
polymer representation. More precisely, the former appears as discretized, while the latter cannot be implemented as an operator in an appropriate Hilbert space directly, but only its exponentiated version exists. The analysis is performed at both classical and quantum levels. The modifications induced by the cutoff scale on ordinary trajectories are analyzed from a classical point of view. On the other hand, the quantum regime is explored in detail by the investigation of the evolution of the wave packets of the universe.

Two main conclusions can be inferred.

- An interference between the wave packets and the potential wall appears. Nevertheless, the classical cosmological singularity is not probabilistically removed. In fact, the wave function of the universe is not strictly localized away from it, and the wave packets fall into it following a classical trajectory.

- The comparison between the polymer approach and the Generalized Uncertainty Principle (GUP) model illustrates that the corresponding interference phenomena are produced in a complementary way. This feature appears both at classical level, as it is immediately recognized analyzing the modifications of the equations of motion, and in the quantum regime, as the behavior of the wave packets is investigated.

**Homogeneous cosmological models** Homogeneity reduces the configuration space of General Relativity to three dimensions. The homogeneous cosmological models [185], the Bianchi Universes, are such that the symmetry group acts simply transitively\(^4\) on each spatial manifold. The Bianchi IX model, together with Bianchi VIII, is the most general one and its line element reads, in the Misner parametrization [197],

\[
ds^2 = N^2 dt^2 - e^{2\alpha} (e^2\gamma)_{ij} \omega^i \otimes \omega^j, \tag{3.53}
\]

where \(N = N(t)\) is the lapse function and the left invariant 1-forms \(\omega^i = \omega^i_\alpha dx^\alpha\) satisfy the Maurer-Cartan equation \(2d\omega^i = \epsilon^i_{jk} \omega^j \wedge \omega^k\). The variable \(\alpha = \alpha(t)\) describes the isotropic expansion of the Universe and \(\gamma_{ij} = \gamma_{ij}(t)\) is a traceless symmetric matrix, \(\gamma_{ij} = \text{diag}(\gamma_+ + \sqrt{3}\gamma_-), \gamma_+ - \sqrt{3}\gamma_- , -2\gamma_+)\), which determines the anisotropy changes via \(\gamma_\pm\). The classical singularity appears for \(\alpha \to -\infty\), since the determinant of the 3-metric is given by \(h = \det e^{\alpha + \gamma_{ij}} = e^{3\alpha}\).

---

\(^4\)Let \(G\) a Lie group, \(G\) is said to act simply transitively on the spatial manifold \(\Sigma\) if, for all \(p, q \in \Sigma\), there is a unique element \(g \in G\) such that \(g(p) = q\).
Figure 3.1: The dynamical-allowed domain $\Gamma_Q(u,v)$ in the Poincaré complex upper half-plane where the dynamics of the Universe is restricted, towards the classical singularity, by the potential.

**Canonical Analysis.** The Hamiltonian constraint for this model is obtained performing the usual Legendre transformation. As well-known [197, 198, 199, 200], the dynamics of the Universe towards the singularity is described by the motion of a two-dimensional particle (the two physical degrees of freedom of the gravitational field) in a dynamically-closed domain. Such a domain depends on the time variable $\alpha$ in the Misner picture, while it is stationary in the Misner-Chitré framework defined by the variables [201]

$$\alpha = -e^\tau \xi, \quad \gamma_+ = e^\tau \sqrt{\xi^2 - 1} \cos \theta, \quad \gamma_- = e^\tau \sqrt{\xi^2 - 1} \sin \theta,$$

(3.54)

with $\xi \in [1, \infty)$ and $\theta \in [0, 2\pi]$. In fact, the dynamically-allowed domain becomes independent of $\tau$, which behaves like a time variable. In terms of these new variables, the Hamiltonian constraint rewrites

$$H = -p_\tau^2 + p_\xi^2(\xi^2 - 1) + \frac{p_\theta^2}{\xi^2 - 1} \approx 0.$$

(3.55)

**ADM Reduction.** Let us perform the ADM [202] reduction of the dynamics. This scheme relies on the idea to solve the classical constraint with respect to a given momentum, before implementing any quantization algorithm. This paradigm allows us to dynamically separate the six-dimensional phase space of the model. In particular, a time variable arises and an effective Hamiltonian, which will depend only on the
physical degrees of freedom of the system (the anisotropy-like variables), comes out. We solve explicitly the constraint \( H = 0 \) with respect to \( p_r \), and thus we consider the variable \( \tau \) as the time coordinate for the dynamics (we adopt the time gauge \( \dot{\tau} = 1 \)), obtaining

\[-p_r = \sqrt{p_{\xi}^2(\xi^2 - 1) + \frac{p_{\theta}^2}{\xi^2 - 1}}. \tag{3.56}\]

The dynamics of such a system is equivalent to a billiard ball on a Lobatchevsky plane [203], as we can see by means of the Jacobi metric\(^5\). It is possible to choose the so-called Poincaré representation in the complex upper half-plane [204] by using new variables \((u, v)\), defined as

\[\xi = \frac{1 + u + u^2 + v^2}{\sqrt{3}v}, \quad \theta = -\tan^{-1}\left(\frac{\sqrt{3}(1 + 2u)}{-1 + 2u + 2u^2 + 2v^2}\right). \tag{3.57}\]

The dynamical-allowed domain \( \Gamma_Q = \Gamma_Q(u, v) \) is plotted in Fig. 1. It is worth noting that the three corners in the Misner picture are replaced by the points \((0, 0)\), \((-1, 0)\) and \(v \to \infty\) in the \((u - v)\) plane. In this scheme, the ADM “constraint” is simpler than the previous one (3.56), and becomes

\[-p_r \equiv H_{\text{ADM}} = v\sqrt{p_u^2 + p_v^2}. \tag{3.58}\]

The Taub Universe corresponds to the Bianchi IX one in the particular case of \( \gamma_- = 0 \) [185]. The phase space of this model is four-dimensional and its dynamics is equivalent to the motion of a particle in a one-dimensional domain. Considering such a domain corresponds to taking only one of the three equivalent potential walls of the Bianchi IX model. As we can see from (3.54) and (3.57), this particular case appears for \( \theta = 0 \Rightarrow u = -1/2 \) (\( \xi = (v^2 + 3/4)/\sqrt{3}v \)), and the ADM Hamiltonian (3.58) rewrites

\[H^T_{\text{ADM}} = vp_v, \tag{3.59}\]

being \( v \in [1/2, \infty) \), as shown in Fig. 1. The Hamiltonian above (3.59) can be further simplified defining a new variable \( x = \ln v \), and becomes

\[H^T_{\text{ADM}} = p_x \equiv p, \tag{3.60}\]

which will be the starting point of our analysis. Within this framework, the Taub model is therefore described by a two-dimensional system in which the variable \( \tau \) is

\(^5\)This approach reduces the equations of motion of a generic system to a geodesic problem on a given manifold.
considered as the time, while the variable \( x \) describes the single degree of freedom of the Universe, i.e. the shape change. It is worth stressing that the classical singularity now appears for \( \tau \to \infty \).

**Physical variables** Let us clarify the physical meaning of our variables. The configuration variable \( x \) is related to the Universe anisotropy \( \gamma_+ \) via the expression (3.54), for \( \theta = 0 \) and \( \xi = (v^2 + 3/4)/\sqrt{3}v \), as

\[
\gamma_+ = \frac{e^\tau}{\sqrt{3}v} \left( v^2 - \frac{3}{4} \right) = \frac{e^{\tau-x}}{\sqrt{3}} \left( e^{2x} - \frac{3}{4} \right).
\]  

(3.61)

By this equation, a monotonic relation between the anisotropy of the Universe \( \gamma_+ \) and our (classical) configuration variable \( x = \ln v \in [x_0 \equiv \ln(1/2), \infty) \) appears, and, therefore, the variable \( x \) can be regarded as a measure of the model anisotropy. In particular, the isotropic shape of the Taub Universe (\( \gamma_+ = 0 \)) comes out for a particular value of \( x \), i.e. \( x = \ln(\sqrt{3}/2) \), and, in this case, we get the closed Friedmann-Robertson-Walker Universe.

### 3.2.2 WDW Dynamics

To better understand the modifications induced on the ordinary dynamics by the polymer representation, we briefly summarize the WDW wave packet dynamics for the Taub model. In this case, the Hamiltonian is simply (3.60); the associated Schroedinger eigenvalue equation can be solved directly: the eigenfunctions in the position representation are just plane waves. This way, wave packets (3.68) can be analytically calculated, with no upper limit for the energy \( k \). The result is plotted in Fig. 3. As we can see from the picture, the wave packets follow the ordinary classical trajectories described in the previous Section. The probability amplitude to find the particle (Universe) is peaked around these trajectories. In this respect, no privileged regions arise, namely no dominant probability peaks appear in the \((\tau-x)\) plane. As a matter of fact, the “incoming” Universe \((\tau < 0)\) bounces at the potential wall \((x = x_0)\) and then falls towards the classical singularity \((\tau \to \infty)\). Therefore, as well-known, the WDW formalism is not able to shed light on the necessary quantum resolution of the classical cosmological singularity. As we will see below, this picture is slightly modified in the polymer representation.
3.2 Cosmology

Figure 3.2: The WDW wave packet $|\Psi(x,\tau)\rangle$ for the Taub model, i.e. $a = 0$ ($k_0 = 0.1, \sigma = 1$).

3.2.3 Polymer Taub Universe

We will now apply the polymer discretization technique to the description of the Taub model. In particular, we will specify the Hamiltonian (3.60) for the case of a discretized $x$ space. As a result, the conjugate variable will not be implemented to operator directly, in the corresponding Hilbert space. Furthermore, the momentum space will be compactified, the compactification scale depending on the lattice characteristic length.

The modifications to the Taub universe induced by the polymer representation will be investigated at both classical and quantum level.

Classical Analysis

Let us now discuss the polymer dynamics of a Taub universe at classical level. By means of the substitution (3.36), the Taub Hamiltonian reads

$$H = \frac{1}{a_n} \sin(a_n p).$$

(3.62)

From now on, we will take into account the discussion about the definition of a scale, and, for the sake of compact notation, we will drop the index $n$ from $a_n$. From a classical point of view, the equations of motion are

$$\dot{x} = \{x, H\} = \cos(ap),$$

(3.63a)

$$\dot{p} = \{p, H\} = 0,$$

(3.63b)
Figure 3.3: Semiclassical equations of motion for the Taub Universe: ordinary trajectory (blue line, $a = 0$) and polymer trajectories (green ($\cos aA = 1/2$) and red ($\cos aA = 1/3$) dashed lines).

where dot denotes differentiation with respect to the time variable $\tau$. The equations of motion are immediately solved as

$$x(\tau) = \cos(ap)\tau, \quad (3.64a)$$

$$p(\tau) = A, \quad (3.64b)$$

where $A$ is a constant.

As well understood, the system (3.64) describes a free particle (Universe) bouncing against a wall.

In the ordinary case, i.e. for $a = 0$, the model can be interpreted as a photon in the Lorentzian minisuperspace, and the classical trajectory in the $(\tau - x)$ plane is its light-cone. More precisely, the incoming particle ($\tau < 0$) bounces on the wall ($x = x_0$) and falls into the classical cosmological singularity ($\tau \to \infty$).

Contrastingly, in the discretized case, i.e. for $a \neq 0$, the one-parameter family of trajectories flattens, i.e. the angle between the incoming trajectory and the outgoing one is greater than $\pi/2$ since $p \in (-\pi/a, \pi/a)$ (see Fig. 2). As these trajectories diverge rather than converging, we expect the polymer quantum effects to be reduced with respect to the classical case, as we will verify below.

**Quantum Regime**

We now investigate the quantum behavior of the model. After analyzing the mathematical requirements of the polymer representation and their physical implications
for the model, we apply the methods introduced above to the Taub Universe. In particular, we choose a discretized $x$ space, and solve the corresponding eigenvalue problem in the $p$ polarization.

Even though the bulk of the discussion of the relation of the polymer representation at different scales is based on the properties of the Hamiltonian as a quadratic form, we can nevertheless apply this paradigm to the Taub model, which is described by a linear Hamiltonian (3.60), after the well-known procedure, established in [205]. In fact, squaring the Hamiltonian leads to squared eigenvalues without affecting the corresponding eigenfunctions.

We are now ready to analyze the Schroedinger equation $i\partial_\tau \Psi = p\Psi$ for the wave function $\Psi = \Psi(p,\tau)$ corresponding to (3.60), where the configuration variable $x$ is defined in the domain $x \in [x_0 \equiv \ln(1/2), \infty)$.

Considering the time evolution for the wave function $\Psi$ as given by $\Psi_k(p, \tau) = e^{-ik\tau}\psi_k(p)$ and the results of [205], we obtain the following eigenvalue problem

$$
(p^2 - k^2)\psi_k(p) = \left[ \frac{2}{a^2}(1 - \cos(ap)) - k^2 \right]\psi_k(p),
$$

(3.65)

where, in the last step, the substitution (3.37) has been taken into account. This eigenvalue problem is solved by

$$
k^2 = k^2(a) = \frac{2}{a^2}(1 - \cos(ap)) \leq k^2_{\text{max}} = \frac{4}{a^2}
$$

(3.66a)

$$
\psi_{k,a}(p) = A\delta(p - p_{k,a}) + B\delta(p + p_{k,a})
$$

(3.66b)

$$
\psi_{k,a}(x) = A[\exp(ip_{k,a}x) - \exp(ip_{k,a}(2x_0 - x))].
$$

(3.66c)

(3.66b) is the momentum wave function, with $A$ and $B$ two arbitrary integration constant, and (3.66c) is the coordinate wave function, where an integration constant has been eliminated by imposing suitable boundary conditions. Moreover, we have defined the modified dispersion relation

$$
p_{k,a} = \frac{1}{a}\arccos\left(1 - \frac{k^2a^2}{2}\right)
$$

(3.67)

from (3.66a). Furthermore, we stress that $k^2$ is bounded from above, as illustrated in (3.66a), but it is its square root, considered for its positive determination, which accounts for the time evolution of the wave function.
3.2.4 Taub Wavepackets

We will now gain insight onto the physical implications of the model by constructing suitable wave packets $\Psi(x, \tau)$. In fact, analyzing the dynamics of such wave packets allows us to give a precise description of the evolution of the Taub model. Such an evolution will be preformed in both the polymer and Wheeler-DeWitt (WDW) approaches. More precisely, the latter will be considered the proper continuum limit of the polymer representation, as illustrated above. Wavepackets are a superposition of eigenfunctions (3.66c), such as

$$ \Psi(x, \tau) = \int_0^{k_{max}} dk A(k) \psi_{k,a}(x) e^{-ik\tau}, \quad (3.68) $$

where $A(k)$ is a Gaussian weighting function, i.e. $A(k) = \exp[-(k - k_0)^2/2\sigma^2]$.

Polymer Dynamics

We are now ready to analyze the modifications brought by the polymer representation in the quantized Taub Universe. Two cases can be distinguished, i.e. the case $k_0a \sim O(1)$, for which it is not possible to recover the ordinary representation of the momentum operator, and the case $k_0a \ll 1$, for which such a treatment is feasible. For $k_0a \sim O(1)$, we get remarkable modifications of the wave packet evolution. From a probabilistic point of view, however, such modifications do not remove the cosmological singularity. The case $k_0a \ll 1$, contrastingly, can be considered as the semiclassical limit of the polymer approach.

Peaked Weighting function. Let us now investigate the first case, $k_0a \sim 1$, where the implementation of the polymer substitution (3.36) does not lead to the ordinary Schroedinger dynamics. Furthermore, we stress that the choice of the value for the standard deviation $\sigma$ in the Gaussian weighting function can be relevant for detecting the effects of the polymer paradigm.

In fact, if the weighting function is very sharply peaked around any value $k_0$, the resulting wave packet will be well-approximated by a purely monochromatic wave, for which a narrow neighborhood of $k_0$ is selected. As a consequence, the ordinary dispersion relation is effectively reproduced by the deformed one, (3.67). In fact,
narrowing the range of $k$ is equivalent to expand the deformed Hamiltonian (3.62) around a given value of the momentum. This kind of behavior is explicitly illustrated in Fig. 4, where it is possible to appreciate a small interference phenomenon between the incoming (outgoing) wave and the wall. This feature can be interpreted as a relic of the polymer modifications of the Taub Universe dynamics, as it will be clearer in the next analysis.

**Spread Weighting function.** On the basis of the previous analysis, the effects of the polymer substitution show up when broad wave packets are considered, i.e. when a large neighborhood of $k_0$ is taken into account by the Gaussian weighting function. In this case, it is possible to appreciate all the modifications induced by the deformed Hamiltonian (3.62). As a result, a strong interference phenomenon appears between the incoming (outgoing) wave and the wall. However, as a matter of fact, such an interference phenomenon is not able to localize the wave packet in a determined region of the configuration space. This way, the probability density to find the Universe far away the singularity is not peaked, i.e. the cosmological singularity of this model is not tamed by the polymer representation from a probabilistic point of view. Consequently, the incoming particle (Universe) is initially ($t < 0$) localized around the classical polymer trajectory (3.64). It then bounces against the wall ($x = x_0$), where the wave packet spreads in the "outer" region, regains the classical polymer trajectory ($t > 0$) and eventually falls into the cosmological singularity ($t \to \infty$). This way, we claim

Figure 3.4: The peaked polymer wave packet $| \Psi(x, \tau) |$ for the Taub model, with $k_0a = 1/2$ ($a = 50$, $k_0 = 0.01$, $\sigma = 0.0125$).
Figure 3.5: The spread polymer wave packet $|\Psi(x, \tau)|$ for the Taub model, with $k_0a = 1/2$ ($a = 50$, $k_0 = 0.01$, $\sigma = 0.125$).

that the classical singularity is not solved by this quantization of the model.

It is interesting to remark that the interference phenomenon occurs in the "outer" region of the configuration space, the $(\tau - x)$ plane. These features are explained in Fig. 5. As we will discuss later on, such a behavior is complementary to that observed in the case of a generalized uncertainty principle.

3.2.5 Semiclassical Limit

We end up our analysis by obtaining the correct semiclassical limit of the model. Within this framework, to obtain the proper continuum limit of the polymer representation, the value of $k_0$ is not arbitrary, but has to be chosen according to the request $k_0a \ll 1$. Since the range of the variable conjugated to the anisotropy variable is compactified, then $k_0$ has to be small with respect to the length of the interval\(^6\).

As a result, differently from the other cases, the value of $k_0$ around which the wave packet is peaked is not arbitrary, but constrained by the characteristic scale $a$ we are investigating. The ordinary WDW behavior is therefore recast, as plotted in Fig. 6. Even though taking $ap \ll 1$ is enough to reproduce the ordinary Hamiltonian (as a general feature of the polymer representation because of relation (3.36)), the fact that the correct semiclassical limit for the polymer quantum Taub Universe is obtained for a wave packet peaked at $k_0 \ll 1/a$ is a non-trivial feature of the model.

\(^6\) We recall that the length of the integration interval $L$ of (3.68) is $L \propto 1/a$, so that $k_0 \ll L$. 
3.2.6 Comparison with other approaches

We can deeper understand the physical implications of this model by comparing it with other applications of the polymer representation in cosmology and with the implementation of a generalized uncertainty principle for the Taub Universe. In fact, in our model, the cosmological singularity is not probabilistically suppressed, as one could expect from other models. Let us now discuss the main differences from those models.

Isotropic Polymer Cosmology.

The fact that the cosmological singularity is not removed within this framework could look apparently in contrast with other models, such as [206, 195, 196]: in the cosmological isotropic sector of General Relativity, i.e. the FRW models, the singularity is removed by loop quantum effects. In particular, the wave function of the universe exhibits a non-singular behavior at the classical singularity, and the Big Bang is replaced by a Big Bounce, when a free scalar field is taken as the relational time [71]. There are however at least two fundamental differences with respect to our model.

- Within our scheme, the variable $\tau$, which describes the isotropic expansion of the Universe, is not discretized, but treated in the ordinary way. In fact, in the ADM reduction of the model, this variable emerges as the time coordinate, and cannot be discretized in a polymer framework. More precisely, the phase space

Figure 3.6: The semiclassical limit of the polymer wave packet $|\Psi(x, \tau)|$ for the Taub model, with $k_0a = 1/20$ ($a = 50$, $k_0 = 0.001$, $\sigma = 0.01$).
of this model is four-dimensional, but we naturally select a two-dimensional submanifold of it, i.e. the \((x - p_x)\) plane, where we implement the polymer paradigm. In other words, we must discretize the anisotropy variable only, without modifying the volume (time) one. On the other hand, in the FRW case, the scale factor of the universe is directly quantized by the use of the polymer (loop) techniques. So far, the evolution itself of the wave packet of the universe is deeply modified by such an approach.

- The solution of the equations of motion is radically different in the two cases. In fact, in our case, the variable \(p\), conjugated to the anisotropy, is a constant of motion, and, from the Schroedinger equation, it describes also \(k\), the energy of the system. According to the polymer substitution (3.36), it is always possible to choose a scale \(a\) for which the polymer effects are negligible during the whole dynamics, at classical level. On the other hand, the Hamiltonian constraint in the FRW case does not allow for a constant solution of the variable conjugated to the scale factor. For this reason, it is not possible to choose a scale, such that the polymer modifications are negligible throughout the whole evolution.

**Homogeneous Loop Cosmology.**

Also the Bianchi cosmological models have been analyzed in the framework of Loop Quantum Cosmology, according to the ADM reduction of the dynamics. The main difference between these works and our approach consists in the fact that in \([207, 208, 209]\) all the degrees of freedom are quantized by Loop techniques. In particular, also the time variable, i.e. the Universe volume, is treated at the same level as the others. In most cases, the time variable is defined by a phase space variable, i.e. it is an internal one. As a result, also the Bianchi Universes are singularity-free \([210, 211, 212]\). In this respect, our analysis is based on considering the time variable as an ordinary Heisenberg variable.

**GUP Cosmology.**

The Taub universe, in this ADM reduction, has also been described within the framework of GUP \([213]\). In that case, the conjugate variables \(x - p_x\) are quantized by
3.2 Cosmology

means of a deformed Heisenberg algebra. As a result, the cosmological singularity is probabilistically suppressed, since the deformation parameter helps localize the wave function of the universe far away from it. This way, comparing the GUP approach and the polymer one allows us to infer that it is not always sufficient to "deform" the anisotropy variable to obtain significant modifications on the universe evolution. However, the polymer paradigm is a Weyl representation of the commutation relations, while, as explained in [214, 215], a generalization of the commutation relations cannot be obtained by a canonical transformation of the Poisson brackets of the system.

Moreover, it is possible to show how the effective framework of loop cosmological dynamics can be obtained by the opposite sign of the deformation term of the modified Heisenberg algebra [216]. This feature is phenomenologically in agreement with our analysis.

3.2.7 Discussion

In this work, we have analyzed the polymer quantization of the Taub Universe. The Taub model admits a four-dimensional phase space, and its ADM reduction allows for an emerging time variable. So far, the energy variable and its conjugate momenta are treated canonically, while the anisotropy variable and its conjugate momenta are quantized according to the polymer paradigm. In particular, the anisotropy variable is assumed as discrete, while its conjugate momenta is replaced by its exponentiated version on a compactified space.

This investigation has been developed at both classical and quantum levels. In the first case, trajectories are illustrated to flatten, with respect to the standard case. However, the most interesting result appears at the quantum level, when the evolution of wave packets is discussed. In fact, an interference phenomenon is illustrated to occur between the potential wall and the incoming particle (Universe), described as a localized wave packet. Nevertheless, the interference is not strong enough for the wave packet evolution to be localized. As a result, the corresponding outgoing particle (Universe) appears, whose evolution towards the cosmological singularity is not probabilistically avoided.

The features of the polymer Taub Universe enhance the comparison with other approaches. On the one hand, the polymer quantization technique has been also applied
to isotropic models. In this case, the choice of the scale factor as the polymer-discrete variable involves the singularity directly. This way, a non-singular quantum cosmology arises. On the other hand, the GUP approach to the Taub model leads to a singularity-free Universe. In particular, from an effective point of view, the consequences of the polymer scheme are complementary to those predicted by the GUP framework.

3.3 Compactification scenarios

The polymer representation of quantum mechanics, and, in particular, the possibility to use it as a valuable tool for the description of discretized features of the spacetime, can find interesting applications in 5D scenarios. In fact, as it will be clearer in the following, the implementation of this representation of the canonical commutation relations to the extraD manifold can lead to a natural truncation of the KK modes, which is suggested by both theoretical and phenomenological motivations. Furthermore, it is worth recalling that this approach offers a description analogous to that of fuzzy spheres in higher dimensional theories. The original part of the work appeared on (PL14).

3.3.1 Kaluza-Klein models

The mathematical formalism of GR allows for a geometrical description of the gravitational interaction. Nevertheless, other kinds of interactions cannot be described via this kind of formalism in 4 dimensions. It is however possible to hypothesize a higher number of dimensions, such that the 4D expression of interactions can be interpreted as the result of different manifestations of a spacetime, which is composed of a higher number of dimensions, where particle move along geodesics, which appear as trajectories imposed both by gravity and other interactions.

The reason why these extraD’s have not been experimentally detected yet are still an open issue. It is however possible to estimate the upper and the lower bound for the length of such dimensions, if a compactification mechanism is assumed. The upper bound follows from the position-energy uncertainty relation

$$\Delta x \geq \frac{\hbar c}{E}.$$  

(3.69)
according to which it is impossible to detect lengths $\leq 10^{-17} \text{cm}$, since modern experimental devices can reach energy scales up to the Tev. On the other hand, the lower bound can be taken as the Planck length $\sim 10^{-33} \text{cm}$, at which quantum effects and relativistic ones are comparable.

The kind of interaction that can be geometrized depend on the number of the extraD’s and on their topology.

In the simplest case, one considers and extraD ring, such that the 5D manifold consists of a 4D manifold and a ring, i.e. $V^5 = V^4 \oplus S^1$. The 5D metric $j_{AB}$ defines the line element $ds^2 = j_{AB}dx^A dx^B$. Because the extraD is a ring, the metric tensor has to be an $L$-period function of the extraD coordinate $x^5$, where $L$ is the length of the ring, i.e. $j_{AB}(x^\mu, x^5 + L) = j_{AB}(x^\mu, x^5)$. Furthermore, all objects that acquire physical relevance must be expressed as periodic functions of $x^5$ as well, and can be Fourier decomposed with respect of this variable as the sum of an infinite number of Fourier modes. For the metric tensor, we have

$$j_{AB}(x^\mu, x^5) = \sum_{k=1}^{+\infty} j_{AB}^{(k)}(x^\mu) e^{i\frac{2\pi k}{L}x^5}. \quad (3.70)$$

The cylindricity hypothesis states that the metric tensor must not depend on $x^5$, so that the the Fourier expansion has to be truncated at the zero-th mode, i.e.

$$j_{AB} = j_{AB}(x^\sigma, x^5) = j_{AB}(x^\sigma). \quad (3.71)$$

The transformation law of the metric tensor under a general coordinate transformation and the analysis of the degrees of freedom allows for the interpretation of the 5D metric tensor as

$$j_{\phi\phi} = \phi^2 \quad (3.72a)$$

$$j_{\phi\mu} = \phi^2 qk A_\mu \quad (3.72b)$$

$$j_{\mu\nu} = (qk \phi)^2 A_\mu A_\nu = g_{\mu\nu}, \quad (3.72c)$$

where $A_\mu$ is a $U(1)$ Abelian gauge field, $\phi$ is a scalar field, $g_{\mu\nu}$ is the 4D metric, and $q$ and $k$ are the dimensionfull constants that allow for such an identification.

In the following, we will concentrate on the structure of the extra D rather than on the geometrization of gauge interactions.
3.3.2 Scalar field

In the simplest toy model, i.e., a scalar field \( \Psi \) in a 5-dimensional (5D) spacetime, described as the direct product of a 4D manifold plus a ring, \( M^5 = M^4 \otimes S^1 \), the Kaluza-Klein (KK) tower is defined as

\[
\Psi(x^\mu, x^5) = \sum_{-\infty}^{+\infty} \psi_m(x^\mu)e^{i x^5 m/L}, \quad L \equiv 2\pi R, \tag{3.73}
\]

that is the infinite sum of the Fourier harmonics, labeled by \( m \). In this compactification scheme, because of the periodic (boundary) condition on the modes of the tower,

\[
\Psi(x^\mu, x^5) = \Psi(x^\mu, x^5 + L), \quad L = 2\pi R,
\]

i.e., of the identification of the points \( 0 \leftrightarrow 2\pi R \), \( \psi(x_5) \) is defined on \( S_1/Z \sim \mathbb{R} \).

If we set \( A_m u \equiv 0 \) and \( \phi \equiv 1 \), the scalar-field wavefunction obeys the Klein-Gordon equation

\[
\sum_{-\infty}^{+\infty} \left( \partial^\mu \partial_\mu \psi_m + 5 M_m^2 \right) \psi_m e^{i x^5 m/L} = 0, \quad 5 M_m^2 \equiv 4 M^2 + (m/L)^2, \tag{3.74}
\]

and the expression of the extraD component in the momentum representation reads

\[
\tilde{\psi}^m(P_5) = \delta(P_5 - m/L).
\]

From 3.73, it is easy to understand the the structure of the extraD geometry can be described by means of the extraD projection of physical objects, and truncating the expansion would correspond to modifying the underlying geometry and algebra\[217, 218\]. The most dramatic truncation is considering \( m = 0 \) only\[219, 220\]. Otherwise, if \( N < \infty \), \( N \) is the maximum number of particles allowed in the tower, which corresponds to possible \( P_5 \) discrete values.

The analysis of truncated Kaluza-Klein (KK) tower can be performed on the ground of several considerations.

In fact, as it can be easily seen in (3.73), the label of the mode is deeply connected both with mass and extraD momentum, which can also be identified with the quantum number of a geometrized interaction, thus allowing one to suppose a strict connection between the extraD and the internal structure. From a theoretical point of view, the truncation of the tower would correspond to the introduction of a cutoff in the extra

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D, based on the fact that it would make little sense to specify the localization of a particle below its Compton length. The exact localization of a particle in the extraD geometry would yield interpretative difficulties, such that a more general description of the internal structure, which does not automatically allow for an exact notion of point, should be looked for. Furthermore, an infinite spectrum of particles brings field-theoretical as well as algebraic difficulties. From a phenomenological point of view, possible indications of the existence of an extraD would be provided both by geodesic deviation and scattering amplitudes. In the second case, the truncation would simplify the calculation of scattering amplitudes and would anyhow account for the impossibility of reaching an infinite energy in experiments.

As a result, the symmetries that characterize KK theories can be compared in the cases of infinite and truncated series. Since, in this toy model, the extraD expansion of the wavefunction of the scalar field is the only feature that accounts for the extraD, the truncation of the series would correspond to some modifications of the extraD geometry, as remarked for 3.73. This way, it will be possible to analyze KK symmetries in both cases and possible compactification scenarios.

It has already been proposed to gain insight into the geometrical interpretation of truncated harmonics expansion on a circle by considering it as worked out from a higher-dimensional structure, thus obtaining a "fuzzy circle", in a "matrix'manifold" scheme. As a result, the ultraviolet cutoff of the model implies a minimal wavelength [221].

We will propose three alternative strategies:
- a) propose a finite set of approximating wave functions, whose finite sum should reproduce the periodicity on the extraD coordinate. This preliminary speculation will be aimed at pointing out the main difficulties of the problem.
- consider the truncated wavefunction as a quasi-periodic function, projected on a finite set of Fourier modes. For this purpose, we will analyze different representations of the standard operator algebra, given by the canonical commutation relations of the extraD operators $\hat{x}_5$ and $\hat{P}_5$;
- go the other way round and establish generalized commutation relations.

The investigation of the role of the operators $\hat{x}_5$ and $\hat{P}_5$ in the extra-D symmetry and the different compactification mechanisms that arise from these scheme will motivate the comparison between the different approaches [222].

A phenomenological comparison[223] of the two scenarios is however not possible,
because the Stone-von Neumann theorem cannot apply for discontinuous operators (in the first case) and for non-unitary transformations (in the second case).

**Modified eigenfunctions**

If the series is truncated, then

\[ \Psi(x, x^5) = \sum_{-N}^{+N} \psi_n(x)e^{ixm/L}, \quad N < \infty \]

where \( N \) is the maximum number of particles allowed in the tower.

If the number of particles is arbitrarily large but finite, then it is reasonable to hypothesize that position and momentum wave functions, i.e., plane waves and delta’s functions, resp., are different from those of the infinite case, which must be recovered in the limiting process. In fact, if the expansion is truncated, the extraD full periodicity can be reproduced only if the extraD expansions is performed by projecting on "deformed" functions, which will not be, in general, eigenfunctions of the position or momentum operators, unless special hypotheses are formulated, and the correspondent Hilbert space will be affected by some modifications. Furthermore, in this preliminary analysis, it will be useful to specify the nature of the "occupation-number" space. In fact, form an intuitive point of view, it should be decided whether this Fock space is

- i) the same as that of a theory predicting an infinite spectrum, where an infinite number of particles is allowed but not considered, or
- ii) that of a model allowing only a finite number of particles.

The second hypothesis looks the most natural, and will influence the interpretation of some proposed calculations, because of the strict relation between number of particles and extraD momentum.

A possible strategy is studying the delta-approximating functions in the momentum space and their FT’s.

Three classes of delta-approximating functions can be considered:

1. Bessel or Airy functions;
2. finite-width peaked oscillating functions, such as \( \tilde{\psi}_N(P_5) = \frac{1}{2\pi} \frac{\sin(N(P_5 - m/L))}{P_5 - m/L} \).
3. finite-width peaked functions such as \( \tilde{\psi}_N(P_5^m) = \sqrt{\frac{N}{\pi}} e^{-N(P_5-m/L)^2} \),
that is, those functions that are somehow peaked around the value selected by a delta function, but have a different behavior elsewhere.
The qualitative analysis of their properties suggests us to eliminate the first two and consider just the third possibility.
In fact, Bessel or Airy functions do approximate the delta function in the vicinity of the peak, but exhibit a behavior than cannot fit our purposes outside the considered region.
On the other hand, such functions as \( \frac{1}{2\pi} \frac{\sin(N(P_5-m/L))}{P_5-m/L} \) are peaked around the desired value, but then exhibit secondary peaks around all the other values, and, for this reason, do not allow for any proper definition of scalar product, even in the momentum representation.
For the choice of a Gaussian function, i.e.,
\[
\tilde{\psi}_N^m(P_5) = \sqrt{\frac{N}{\pi}} e^{-N(P_5-m/L)^2},
\] (3.75)
for which the scalar product reads
\[
\frac{N}{\pi} \int_{\text{range}} dP_5 e^{-N(P_5+r/L)^2} e^{-N(P_5-m/L)^2} = \frac{1}{2} \left[ \frac{\sqrt{N}}{2\pi} e^{-N(P_5)} \right]_\text{range}.
\] (3.76a)

The momentum representation of the differential operator \( \hat{x}_5 \) is ill-defined, and the introduction of a non-trivial measure can no way overcome this problem, since a different measure for every mode of the tower would be required. Nevertheless, its mean value between two states is still well-behaved on any symmetric interval, according to the scalar product (3.76a). In particular, an \( i \)-type Fock space would give the exact delta behavior, because the integration range would run from \( -\infty \) to \( \infty \), while case \( ii \) appears non-viable. Anyhow, even in case \( i \), the distributional identification of (3.76a) as the proper delta function could give rise to some difficulties.
The position wavefunction can be obtained by Fourier transforming 3.75, i.e.,
\[
\frac{1}{2\pi} \frac{\sqrt{N}}{\pi} \int_{\text{range}} e^{-N(P_5-m/L)^2} e^{ix_5P_5} = -\frac{1}{4\pi} e^{ix_5P_5} e^{-\frac{x_5^2}{4\pi}} \left[ \text{Erf} \left( -\sqrt{N}(P_5 - \frac{m}{L}) + i\frac{x_5}{2\sqrt{N}} \right) \right]_\text{range},
\] (3.77)
where the result would still be affected by the choice of the occupation-number space.
The differential operator \( \hat{P}_5 \) is ill-defined, as well as its mean value.
A non-trivial measure could be introduced "by hand" to simplify the calculation:
- \( dx_5/e^{-x_5^2/2N} \) would little improve the scalar product in the position representation;
- \( dP_5/e^{-NP_5^2} \) would bring finite-width peaked oscillating wave functions in the position space.

Nonetheless, the possibility of introducing a non-trivial measure for these Hilbert spaces naturally suggests us to abandon these attempts and to turn at other models, where the Hilbert space is endowed with a non-trivial measure.

**Modified commutation relations: generalized uncertainty principle**

It is however possible to modify the canonical commutation relations between the extraD operators \( \hat{x}_5 \) and \( \hat{P}_5 \). Within the framework of a higher number of extraD, compactification on non-commutative spaces has been widely investigated [224]. In5D, nonetheless, it is obviously impossible to implement non-commutative coordinates; contrastingly, modified commutation relations are the tool that can allow for a similar procedure [214, 225]. Since unitary transformations preserve commutation relations, prediction based on the GUP approach cannot be compared with ordinary ones.

The most general commutation relation that can be considered reads [214]

\[
[\hat{x}_5, \hat{P}_5] = i\hbar(1 + \alpha \hat{x}_5^2 + \beta \hat{P}_5^2).
\]  

(3.78)

Nonetheless, because of the problem we are investigating, it would sound sensible to set \( \beta \equiv 0 \): The most direct modification of the commutation relation, which suite the previous analysis, is

\[
[x_5, P_5] = i(1 + \alpha x_5^2).
\]  

(3.79)

According to [214], no momentum eigenstate can be found, since it would have zero uncertainty. The differential operator \( \hat{x}_5 \) in the position representation reads

\[
\hat{x}_5 \psi(x_5) = x_5 \psi(x_5), \quad \hat{P}_5 \psi(x_5) = -i\partial_{x_5} \psi(x_5).
\]

Information about momentum can be recovered by:

- investigating "maximal-momentum-localization" states

\[
\tilde{\psi}_m(x_5) = \sqrt{\frac{2\sqrt{\alpha}}{\pi \sqrt{1 + \alpha x_5^2}}} e^{-i\frac{\text{atan}(\sqrt{\alpha}x_5)}{\sqrt{\alpha}}},
\]

which generalize plane waves in the coordinate representation,

- and then using the projection on "quasi- momentum" wave functions for general states.
3.3 Compactification scenarios

So far, this description is dual to that presented in [214], but an interpretative mismatch occurs, when the dispersion relation corresponding to this modified scheme are looked for. In fact, although the Fourier duality between position and eigenfunction is widely accepted, the Fourier dual of a dispersion relation is not so clearly understood. Furthermore, the interpretative difficulties of Fourier dualism arising from non-standard geometries [226] is due to the fact that Born reciprocity is somehow deformed by (3.79).

So far, we can however retain the main idea of considering modified commutation relations to investigate the implications of such a model.

We can adopt the following commutation relation,

$$\left[\hat{x}_5, \hat{P}_5\right] = i \left(1 + \frac{P_5^2}{N^2}\right), \quad (3.80)$$

where the parameter $\alpha$ in 3.78 has been set equal to zero, and $\beta$ has been connected with the maximum number of particles, i.e., $\beta \equiv 1/N^2$. This choice will be discussed throughout this section.

This way,

$$\hat{P}_5\psi(P_5) = P_5\psi(P_5) \quad \hat{x}_5\psi(P_5) = i \left(1 + \frac{P_5^2}{N^2}\right) \partial P_5\psi(P_5) : \quad (3.81)$$

The maximal-localization wave functions read

$$\hat{\psi}_\xi(P_5) = \sqrt{\frac{2}{\pi \sqrt{N}}} \frac{1}{\sqrt{1 + \frac{P_5^2}{N^2}}} e^{-i\sqrt{N} \arctan\left(\frac{P_5}{\sqrt{N}}\right)} \quad (3.82)$$

and the quasi-position wave functions read

$$\psi(\xi) = \sqrt{\frac{2}{\pi \sqrt{N}}} \int_{-\infty}^{+\infty} \frac{dP_5}{\left(1 + \frac{P_5^2}{N^2}\right)^{3/2}} e^{i\sqrt{N} \arctan\left(\frac{P_5}{\sqrt{N}}\right)} \psi(P_5). \quad (3.83)$$

The momentum and position operators act on quasi-position wavefunction as

$$\hat{P}5\psi(\xi) = N \tan\left(-\frac{i}{N} \partial_\xi\right) \psi(\xi), \quad (3.84)$$

$$\hat{x}5\psi(\xi) = \left(\xi + \frac{1}{N} \tan\left(-\frac{i}{N} \partial_\xi\right)\right) \psi(\xi), \quad (3.85)$$

where the function of an operator has to be understood as its formal Taylor expansion.

In this representation, we can find the modified dispersion relation

$$\lambda = \frac{2\pi}{\sqrt{N}} \frac{1}{\arctan\left(\frac{m}{N^2}\right)}. \quad (3.86)$$

If we consider $N$ as an independent parameter of the theory and let $m = \infty$, we obtain a minimal wavelength $\lambda_N = 4/N$, vanishes for $N \rightarrow \infty$, when canonical commutation relation are recovered. Contrastingly, we can consider $N$ as the maximum number of particles
allowed in the model, and consequently, \( m_{\text{max}} = N \): in this case, the minimal length that characterizes this scheme is \( \lambda_{N,L} = 2\pi/(N \arctan \frac{1}{L}) \). In both cases, a minimal length is predicted, and the comparison of the two length illustrates that the error in considering \( \lambda_{N,L} \) instead of \( \lambda_N \) can be made negligible even by shrinking the compactification scale [227], \( L \), while keeping \( N \) large but finite.

**Modified operators: the polymer representation**

The introduction of a non-trivial measure suggests one to hypothesize that we are dealing with a representation of the Weyl algebra of the operators \( \{x_5, P_5\} \), whose Schroedinger representation does not admit self-adjoint operators unless an extra term is appended, i.e., whose states belong to a Hilbert space equipped with a non-trivial measure, \( H = L^2(\mathbb{R}, d\mu) \).

The truncated expansion

\[
\Psi^\varepsilon(x^\mu, x^5) = \sum_{-N}^{+N} \psi_m(x^\mu)e^{ix^5m/L}, \quad N < \infty
\]

can be interpreted as consisting of true plane waves, \( e^{ix^5m/L} \), and almost-periodic functions of \( x^5 \), \( \Psi(x^\mu, x^5) \). This way, it is interesting to analyze the polymer representation of such a problem and to specify the previous formalism for this problem. We then choose \( N \sim 1/d \), so that the limit \( N \to \infty \) is recovered in the limit \( d \to 0 \). The complex structure of the model will therefore depend on \( N \), so that

\[
J_N : (x_5, P_5) \to \left( -\frac{P_5}{N^2}, x_5, N^2 \right)
\]

We then choose a discrete momentum variable, so that the operator \( \hat{x}_5 \) does not exist in the limit \( N \to \infty \), while its exponentiated version still does. This way, the representation of the algebra will be inequivalent to the usual one, as discussed previously. Starting from the canonical commutation relation \([x_5, P_5] = i\), one can take the exponentiated version of the operators,

\[
U(a) = e^{iax_5}, \quad V(b) = e^{ibP_5},
\]

acting on the set of wave functions \( \{\phi\} \), which are mapped to the Schroedinger representation through the isometric isomorphism

\[
\psi(P_5) = \frac{\sqrt{N}e^{-P_5^2N^2/2}}{\pi^{1/4}}\phi_a(P_5) \in L^2(\mathbb{R}, dP_5),
\]

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where $\phi_0(P_5) = 1$.

If $P_5$ is discrete, states can be decomposed as

$$|\psi> = \sum_{0}^{N} a_i |m_i>, \ m \in \mathbb{Z}.$$ 

$\hat{x}_5$ is defined through its exponentiated version, i.e.,

$$\hat{U}|m> = \frac{1}{2L} (|m + 1> - |m - 1>).$$

In the $p$ polarization

$$\psi(P_5) = \delta_{P_5,m}.$$ 

The position space is compactified because $m \in \mathbb{Z}$, and the wave functions are plane waves, i.e.,

$$\Psi(x_5) = e^{ix_5\frac{m}{L}}.$$ 

As a result, the model can be interpreted as compactified on a "fuzzy" circle[228], because $m < \infty$. Since non-commutative geometries do not apply in 1 dimension, this result can be looked to as an alternative construction of the fuzzy circle, which can be obtained from a fuzzy 2-sphere[221], with similar cut-off properties

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