General Relativistic
Electrodynamical Processes in
Neutron Stars and Black Holes

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General introduction

Indubitably, the cosmos is a very rich, enthralling, alluring and captious, beguiling environment. It has inspired poets, paramours and it has been puzzling the researchers since its commence, if anything, and it seems it shall keep doing so for a long time, perchance eternally. It has been the locus of the creation of all the matter we know, and it appears its “travail” continues unrelentingly. Because of its richness, it keeps always teaching and challenging us, both with philosophical questions and experimental observations. We do not need to go so far to get aware of its mysteries; it could actually baffle us in all of its scales. Even in local ones (the focus of this thesis) we still have a lot to understand. Which are the constituents of compact objects, such as neutron stars, strange stars, if anything? Are there black holes in the central regions of galaxies? Do they really exist? If so, could they be characterized simply by three parameters? Which are the quantum aspects they should display? Why does general relativity seem to have a so profound relationship with thermodynamics? Even when one is more conservative, not asking himself/herself such sort of questions, other mysteries still remain. Which are the good observational tools to approach these issues? These are just some of the multitudinous deep questions yet unanswered. An astrophysicist should always keep them in mind irrespective of the issue he/she is investigating.

The scope of this thesis is infinitely more modest than the one drawn before. Here we scrutinize some aspects of one facet of the (astrophysical) problem: electrodynamical processes in neutron stars and black holes. We are interested in investigating some of their possible origins, as well as by-products. This issue is only per se colossal, since generically they mingle with electro-magneto-hydrodynamical, purely gravitational and microscopical effects, amongst many others, broadening very much their precise description. We shall be content in examining some of the topics more connected with conceptual issues of systems endowed with electrodynamical structures, as well to attempt to search for some
tools to possibly probe them.

Some of the queries apropos of black holes that we try to address here are: if there are charged black holes, which are the basic ingredients one should consider in order to understand them? Should they be described by which theory? Why would it be interesting to analyze black holes described by nonlinear theories of the electromagnetism? How could one probe them?

Now, w.r.t. neutron stars, the matter seems less problematic at least under the point of view of their electrodynamical structure. It is observationally known that such systems are endowed with electromagnetic aspects [1]. What generates such fields is still under debate. Nevertheless, they also have another aspect: they are really compact. So, phase transitions should possibly take place, with the presence of surface degrees of freedom in the interface of two phases. Questions we shall be interested here are: which is the role of surface degrees of freedom in the system? Which are their properties? How do they “couple” to electromagnetic fields? Could their dynamics be ignored? How to pose the problem of stability for stratified systems? How to probe such aspects?

Naturally, even within this small window of analysis, we are far from being complete. Many important issues were left as perspectives of future works, that can be found in each chapter of the thesis. We tried to write independent chapters, for the benefit of the reader. The order of the chapters does not necessarily reflect the chronology they were conceived and investigated. They were chosen to give the reader a logical sense of evolution of the concepts. This work could basically be divided into two main categories: black holes and neutron stars. The first three chapters only deal with charged black holes minimally coupled to generalizations of the Maxwellian Lagrangian dependent upon its two invariants [2], that we call nonlinear electrodynamics. The two next ones deal with stratified compact objects. The last chapter could be seen as a mixture of both aforesaid issues, since there we want to analyze the dynamics of slowly rotating thin shells that could be present in compact objects (as surfaces of discontinuity, where some of the physical parameters of the system, such as the pressure and the energy density, are generically discontinuous at the shell’s positions), could interact with black holes (for instance the process of gravitational collapse of a thin-shell onto a black hole), or could even define external black hole solutions with no singularities.

The plan of this thesis is the following. In Chapter 1 we elaborate upon the amount of energy that could be extracted (we shall not be concerned about the
possible mechanisms) out of a spherically symmetric and charged black hole
described by nonlinear electrodynamics. As an automatic consequence of it, we
explore some of the issues apropos of its thermodynamics. In special, we show
that it is possible to obtain a formula that naturally leads to the first law of black
hole mechanics [3, 4], the mass formula, or the “equation of state for a nonlinear
black hole”.

In Chapter 2 we use the mass formula obtained in Chapter 1 to talk about
the black holes themselves. This is so since this constraint equation is the only
one in agreement with “inalienable” conservation laws (energy and charge), and
therefore would single out only the physically relevant cases. In order to have a
material case to analyze, we choose the celebrated Born-Infeld Lagrangian to the
electromagnetism [5]. We show there that the analysis is simplified tremendously
when the mass formula is taken into account, even when one wants to have
insights about the interactions of black holes.

Chapter 3 is devoted to give a further step into the description of nonlinear
charged black holes: we attempt to work in the axially symmetric case. Natu-
rally we consider the approximation of slow rotation. There we work up to first
order on the “rotational parameters” of the black holes (or spacetimes). This de-
scription is not the one that would allow us to contemplate the issue of energy
budget of the systems, but we show that there are many other facets that could
exhibit the fingerprints of slowly rotating and nonlinear charged black holes. We
use neutrinos to probe such intrinsic aspects nonlinear black holes have. They
are chosen due to their special properties, that we scrutinize in detail there, be-
ing naturally produced during the process of gravitational collapse, the believed
progenitors of black holes. Besides, we analyze other tools for assessing these
spacetimes, such as the response gyroscopes would give in the neighborhoods
of such black holes.

In Chapter 4 we commence our analyses for compact bodies. We focus on
stratified systems with electromagnetic aspects. This first analysis is devoted
to comprehend the dynamics of the surface degrees of freedom that could be
present between any two phases of a stratified system. We attempt to under-
stand their nature first, evidencing their subtleties and assets (for instance the
fingerprints they could have on astrophysical systems, such as the stability of
compact stars). Afterwards, we show their relevance into the description of strat-
ified stars. In order to have realizations of our generic analyses, we apply the
formalism into some classes of compact stars, such as globally neutral neutron
Chapter 5 is the development apropos of Chapter 4, because there we analyze the stability against radial displacements of stratified stars endowed with surface degrees of freedom. The main goal of this chapter is not to scrutinize particular systems, but to pose the problem clearly and to propose a solution to it. This is deemed important, because many subtleties will rise that could be important for any general relativistic approach to layered systems. It is clear that when the stratification is present, further additional boundary conditions should be taken, incorporating on them the peculiarities of the surface degrees of freedom. Naturally further boundary conditions change the nature of the solutions, but the important query here is: which are these boundary conditions? How to get them? We show that all of these queries could be answered with a generalized distributional formalism, in searching for distributional solutions to Einstein’s equations, of cardinal importance when one deals with combinations of general relativistic solutions for physical systems, given their nonlinearities.

In chapter 6 we develop the formalism for matching slowly rotating space-times. Here we work up to second order of approximation on the rotational parameters. Our goal is to analyze the nature of the surface degrees of freedom when rotations are taken into account, as well as their subtleties when matches are done. Here we are also interested in posing the problem clearly, before applying it to particular cases. We show that there are always situations where the surface quantities of a shell (energy and tension) decrease for some of its points. This could point to interesting cases of possible violations of some energy conditions [6] in nonperturbative cases. There, kinematical issues are also investigated, that could shed some light into possible observational and philosophical aspects that glued objects could display.

Obviously all of the details analyzed in this thesis were not elucidated in this general introduction. Each chapter shall also have a particular introduction and conclusion, where we shall deal with other more particular details, such as motivations for the study under question, connections with other areas and possible observational approaches for probing them. We apologize in advance for important references that were not cited, as well as important issues that were not touched nor further elaborated. By any means they were done purposely. In a certain sense this shows the vastness of the areas under analysis and we hope we could convey their main ideas and the possible new insights we have contributed to some of them.
Chapter 1

Black hole mass decomposition in nonlinear electrodynamics

1.1 Introduction

Black hole solutions to the Einstein equations have always attracted the attention of researchers, not only due to their unusual properties, but also from the discovery that they could be one of the most plenteous sources of energy in the Universe.

From conservation laws, R. Penrose [7] showed how energy could be extracted from a black hole [8]. D. Christodoulou [9] and D. Christodoulou and R. Ruffini [10], through the study of test particles in Kerr and Kerr-Newman spacetimes [11], quantified the maximum amount of energy that can be extracted from a black hole. These works deserve some comments. First, this maximum amount of energy can be obtained only by means of the there introduced, reversible processes. Such processes are the only ones in which a black hole can be brought back to its initial state, after convenient interactions with test particles. Therefore, reversible transformations (or reversible processes) constitute the most efficient processes of energy extraction from a black hole, because energy is not “lost” in any intermediate step. Furthermore, it was also introduced in Refs. [9, 10] the concept of irreducible mass. This mass can never be diminished by any sort of processes and hence would constitute an intrinsic property of the system, namely the fundamental energy state of a black hole. This is exactly the case of Schwarzschild black holes. From this irreducible mass, one can immediately verify that the area of a black hole never decreases after any infinitesimal trans-
1. Black hole mass decomposition in nonlinear electrodynamics

formation performed on it. Moreover, one can write down the total energy of a black hole in terms of this quantity [10].

Turning to effective nonlinear theories of electromagnetism, their conceptual quality is that they allow for the insertion of effects such as quantum-mechanical, avoidance of singular solutions and others via classical analyses [12]. As a first approach, all of these theories are built up in terms of the two local invariants constructed out of the electromagnetic fields [2, 13]. Notice that the field equations of nonlinear theories have the generic problem of not satisfying their hyperbolic conditions for some physical situations (see e.g. [14, 15]). The aforementioned invariants are assumed to be functions of a four-vector potential in the same functional way as their classic counterparts, being therefore also gauge independent invariants. We quote for instance the Born-Infeld Lagrangian [5], conceived with the purpose of solving the problem of the infinite self-energy of an electron in the classic theory of electromagnetism. The Born-Infeld Lagrangian has regained interest since the effective Lagrangian of string theory in its low energy limit has an analog form to it [4]. It has also been minimally coupled to general relativity, leading to an exact solution [16, 17] and this coupling has been studied in a variety of problems [18–20]. Another worthwhile example of nonlinear electromagnetic theory is the Euler-Heisenberg Lagrangian [21, 22]. This Lagrangian allows one to take effectively into account one-loop corrections from the Maxwellian Lagrangian coming from Quantum Electrodynamics (QED), and it has been extensively studied in the literature [12]. Nonlinear theories of the electromagnetism have also been investigated in the context of astrophysics. For instance, they could play an important role in the description of the motion of particles in the neighborhood of some astrophysical systems [23] and as a simulacrum to dark energy [24].

In connection with the above discussion, the thermodynamics of black holes [3] in the presence of nonlinear theories of the electromagnetism has also been investigated. The zeroth and first laws [see below Eq. (1.40) in section 1.7] have been studied in detail [4, 25], allowing for the raise of other important issues. We quote for example the difficulty in generalizing the so-called Smarr mass [26, 27] (a parametrization of the Christodoulou-Ruffini mass [10]) for nonlinear theories [4]. Many efforts have been pursued in this direction, through the suggestion of systematic ways to write down this mass, which has led to some inconsistencies (see e.g. Ref. [28]). For some specific nonlinear Lagrangians, this problem has been circumvented [29].
1.1. Introduction

We first deal with static spherically symmetric electrovacuum solutions to the Einstein equations minimally coupled to Abelian nonlinear theories of the electromagnetism, i.e. nonlinear charged black holes, for electric fields that are much smaller than the scale fields introduced by the nonlinearities, i.e. *weak field* nonlinear Lagrangians. We decompose the total mass-energy of a charged black hole in terms of its characteristic parameters: charge, irreducible mass, and nonlinear scale parameter. We also show the constancy of the black hole outer horizon in the case of reversible transformations. We then generalize the previous results for a generic nonlinear theory leading to an asymptotically flat black hole solution for any range of the electric field. As an immediate consequence of this general result, we show that the first law of black hole thermodynamics (or mechanics) in the context of nonlinear electrodynamics [4] is a by-product of this mass decomposition. These results also allow us to investigate the extraction of energy from charged black holes in the framework of nonlinear theories of electromagnetism.

One of the purposes of this chapter is to see the pertinence of the nonlinearities of the electromagnetism into the energy budget of charged black holes. Besides, it ensues a series of important conceptual issues, such as thermodynamics of nonlinear black holes and their “equation of state”. The motivations for studying such a case are manifold. Firstly, it is known that neutron stars, the believed progenitors of black holes, are endowed with an electromagnetic structure [1]. Therefore, one could reasonably state and assume that, at least for some instants of time, these electrodynamical aspects in neutron stars could lead to the formation of charged black holes with fields that could even be higher than certain scales ones (due to the cataclysmic effects present during the process of gravitational collapse and the smallness of the characteristic distance scales involved in the problem, where even small charges lead to high fields). This is actually even an ingredient for a model for gamma ray bursts [12]. Secondly, with the insertion of nonlinearities, one is tacitly introducing desirable physical aspects into the description of black holes (e.g. the nontrivial vacuum aspects from QED [30] and QCD [31]), even without having complete theories where they would be automatically present (e.g. quantum gravity), having therefore insights for black hole descriptions based on first principles. Thirdly, when applied to certain astrophysical scenarios, it could address their puzzles (e.g. black hole singularities [32] and cosmological ones [31, 33]), which encourages further applications.
We use geometric units and metric signature $-2$.

### 1.2 Field equations

The minimal coupling between gravity and nonlinear theories to the electromagnetism that depends only on the local invariant $F$, $\em F$, can be stated mathematically through the action

$$S = \int d^4 x \sqrt{-g} \left( - \frac{R}{16\pi} + \frac{\em F}{4\pi} \right) = S_g + \frac{\em S}{4\pi}, \quad (1.1)$$

where $F = F^{\mu\nu} F_{\mu\nu}$, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A_\mu$ is the electromagnetic four-potential, $R$ is the Ricci scalar, $S_g$ is the action for the gravitational field, $\em S$ is the action of the electromagnetic theory under interest, and $g$ the determinant of the metric $g_{\mu\nu}$ of the spacetime. Under the variation of Eq. (1.1) with respect to $g^{\mu\nu}$, and applying the principle of least action, one obtains (see e.g. Ref. [2])

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T^{(em)}_{\mu\nu}, \quad (1.2)$$

with $R_{\mu\nu}$ the Ricci tensor and $T^{(em)}_{\mu\nu}$ the energy-momentum tensor of the electromagnetic field, defined as

$$4\pi T^{(em)}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \em S}{\delta g^{\mu\nu}} = 4 \em L F_{\mu\alpha} F_{\nu\rho} g^{\alpha\rho} - \em L g_{\mu\nu}, \quad (1.3)$$

where $\em L = \partial L^{(em)} / \partial F$.

Application of the Principle of least action in Eq. (1.1) with respect to $A_\mu(x^\beta)$ gives

$$\nabla_\mu (\em L F^{\mu\nu}) = 0, \quad (1.4)$$

since we are interested in solutions to general relativity in the absence of sources.

In the static and spherically symmetric case, it is possible to solve Einstein’s equations minimally coupled to nonlinear electromagnetism theories [see Eqs. (1.2) and (1.4)] and due to the energy-momentum tensor in this case the metric must
be of the form
\[ ds^2 = e^{\nu(r)} dt^2 - e^{-\nu(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (1.5) \]
with [2, 25]
\[ e^{\nu(r)} = 1 - \frac{2M}{r} + \frac{8\pi}{r} \int_r^\infty r'^2 T_{00}^{(0)}(r') dr', \quad (1.6) \]
where the integration constant \( M \) stands for the total mass-energy of the black hole as measured by observers at infinity.

The Eqs. (1.4) in this special spherically symmetric case reduce to
\[ L_{F}^{(em)} E_r r^2 = -\frac{Q}{4}, \quad (1.7) \]
where \( Q \) is an arbitrary constant representing the total charge of the black hole.

If one defines
\[ E_r(r) = -\frac{\partial A_0}{\partial r} \quad \text{and} \quad \frac{\partial N}{\partial r} = -L_{F}^{(em)} r^2, \quad (1.8) \]
and take into account Eqs. (1.4), (1.5) and (1.7), then Eq. (1.6) can be rewritten as
\[ e^{\nu(r)} = 1 - \frac{2M}{r} + \frac{2QA_0}{r} - \frac{2N}{r}, \quad (1.9) \]
where it has been imposed a gauge such that the scalar potential \( A_0 \) goes to zero when the radial coordinate goes to infinity, which also holds to \( N \). These conditions ensure that the associated nonlinear black holes are asymptotically flat (Minkowskian). We are not interested in Lagrangian densities which do not fulfill this condition; see for instance [34].

The black hole horizons are given by the solutions to
\[ g_{00}(r_h) = e^{\nu(r_h)} = 0. \quad (1.10) \]
1.3 Reversible and irreversible transformations

A way to investigate the motion of test particles in a static spherically symmetric spacetime is through the solution to the Hamilton-Jacobi equation. The trajectories of the test particles can be obtained in the traditional way (see e.g. Ref. [35]) through the particle constants of motion (energy $E$, orbital angular momentum $l$, and the Carter constant) [11, 36]. The energy of the test particle is given by [9–11, 36]

$$E = q A_0 + \sqrt{\frac{q^2}{r^2} \left[ r^4 (p^\theta)^2 + \frac{l^2}{\sin^2 \theta} + m^2 r^2 \right] + (p^r)^2},$$

(1.11)

where $p^\mu = m dx^\mu / d\tau$, $\tau$ an affine parameter along the worldline of the particle and $q$ its charge. The “+” sign has been chosen in Eq. (1.11), because we are interested just in particles traveling to the future [36, 37].

If the worldline of an arbitrary test particle intersects the outer horizon, i.e. the largest solution to Eq. (1.10), then the changes in the energy and charge of the black hole (which lead to another black hole configuration infinitesimally close to the initial one) read: $\delta M = E$ and $\delta Q = q$ [36], respectively.

From Eq. (1.11), one can see that the only way to apply a reversible transformation in the sense of Christodoulou-Ruffini [9, 10, 12] to a black hole interacting with a test particle is by demanding that its square root term is null. This guarantees that a nonlinear black hole can always be restored to its initial configuration, as required by reversible transformations, see section 1.1, after a test particle has crossed the horizon. Hence, from Eq. (1.11) and the aforementioned conservation laws, reversible transformations select geodesics whose changes to the black hole masses are minima and are given by

$$\delta M_{\text{min}} = q A_0(r_+) = \delta Q A_0(r_+).$$

(1.12)

Clearly, Eq. (1.12) is the mathematical expression for the physical case where $|p^r(r_+)|$ is much smaller than $|q A_0(r_+)|$, that is, when irreversible processes are negligible. Reversible transformations are important processes since like the internal energy of a thermodynamical system, the energy $M$ of a black hole is assumed to be an exact differential. This therefore allows one to describe intrinsic properties of the spacetime by using test particles; see Eq. (1.12).
For the sake of completeness, in the case of general black hole transformations one has

$$\delta M \geq qA_0(r_+).$$

\[1.13\]

### 1.4 Weak field nonlinear Lagrangians

An interesting and convenient limit for investigating nonlinear properties of Lagrangians is when their electric fields are small compared to their scale or threshold fields, automatically introduced by the nonlinearities [24]. In this limit, one expects that their leading term be the Maxwell Lagrangian [25]. In this line, assuming the nonexistence of magnetic charges, let us first investigate Lagrangian densities given by

$$L^{(em)} = -\frac{F^4}{4} + \frac{\mu}{4} F^2,$$

\[1.14\]

where $\mu$ is related to the scale field of the theory under interest, and as a necessary condition to avoid any violation of the most experimentally tested physical theory, the Maxwell theory, this nonlinear term is assumed such that it must be much smaller than the Maxwell one. This means we are generically interested in electric fields that satisfy

$$E_r \ll \frac{1}{\sqrt{\mu}}.$$  

\[1.15\]

Physically speaking, the second term of Eq. (1.14) is a first order correction to the Maxwell theory. For instance, in the case of the Euler-Heisenberg Lagrangian, the nonlinearities are related to quantum corrections, whose scale field is $E_c = m_e^2 c^3 / (e\hbar) \approx 10^{18}$ V/m, where $m_e$ is the electron rest-mass, $e$ is the fundamental charge, and $\hbar$ is the reduced Planck constant (see e.g. [12], and references therein). Hence, in virtue of this limit a perturbative analysis could be carried out. The sign of $\mu$ in principle could be arbitrary. Nevertheless, from the inspection of the Euler-Heisenberg Lagrangian, for instance, one sees in this case that this constant turns out to be positive [22]. The same behavior happens if one expands perturbatively the Born-Infeld Lagrangian [4, 5, 12, 16–18, 24]. The two aforesaid Lagrangian densities are very well-known ones, coming from more
fundamental analyses, such as quantum field theory and string theory. This suggests that physically interesting Lagrangian densities to look at are those whose scale parameters $\mu$ are positive. It is worth highlighting that the weak field analysis is however not very restrictive in terms of the strength of the fields. Note for instance that for the Euler-Heisenberg and Born-Infeld theories, our analysis is meaningful for electric fields $E_r \sim 10^{18}$ V/m (see section 1.8).

When one interprets nonlinear Lagrangians as the ones related to effective media [5], then one expects that their associated electric field solutions should be reduced. This constrains the sign of $\mu$, as we shall show below. Nevertheless, it is not ruled out in principle Lagrangians where the associated electric field could increase.

By substituting Eq. (1.14) into Eq. (1.7) and the first term of Eq. (1.8), solving exactly and then expanding perturbatively (or by directly working perturbatively), one can easily show that

$$E_r(r) = \frac{Q}{r^2} \left( 1 - \frac{4\mu Q^2}{r^4} \right), \quad A_0(r) = \frac{Q}{r} \left( 1 - \frac{4\mu Q^2}{5r^4} \right), \quad (1.16)$$

Expressions (1.16) are just meaningful if the characteristic distances of the system are such that

$$r \gg r_c, \quad r_c^4 = |\mu| M^2 \xi^2, \quad \xi = \frac{Q}{M}. \quad (1.17)$$

As we pointed out before, when $\mu > 0$, the electric field diminishes in comparison to the pure Maxwellian case, while the opposite happens when $\mu < 0$. The former case is exactly what happens in usual media [38], while the latter could happen in the so-called metamaterials (see e.g. Ref. [39, 40]).

From Eq. (1.14), the second term of Eq. (1.8) and Eq. (1.16), and assuming that the constraint in Eq. (1.17) is fulfilled, it is also readily shown that

$$N = \frac{Q^2}{2r} \left( 1 - \frac{6\mu Q^2}{5r^4} \right). \quad (1.18)$$

When Eqs. (1.16) and (1.18) are substituted in the expression for the time-time component of the metric, Eq. (1.9), one obtains after simple manipulations
that

\[ e^\nu(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{2\mu Q^4}{5r^6}. \]  

(1.19)

The above result agrees with the one obtained in Ref. [41], for the Euler-Heisenberg Lagrangian density, in the corresponding units. Notice that when \( \mu = 0 \), i.e., for the Maxwell Lagrangian [see Eq. (1.14)], Eq. (1.19) gives the well-known Reissner-Nordström solution (see e.g. Ref. [36]).

The outer horizon can be found perturbatively from Eqs. (1.10) and (1.19) and the result is

\[ r_+ = \mathcal{R}_+ \left( 1 + \frac{\mu Q^4}{5(\mathcal{R}_+)^5 \sqrt{M^2 - Q^2}} \right), \]  

(1.20)

where we defined

\[ \mathcal{R}_+ = M + \sqrt{M^2 - Q^2}, \]  

(1.21)

functionally the same as the outer horizon in the Reissner-Nordström solution. Besides, in Eq. (1.20), it is understood that the second term inside the parenthesis is much smaller than the unity. These latter equations are not valid in the case \( Q = M \), and up to what extent the above perturbative analysis is meaningful in the proximity of this limit is dictated by the value \( \mu/M^2 \). Namely, the smaller the \( \mu/M^2 \), the closer one can approach \( Q = M \) using perturbative theory. For the sake of reference, in Euler-Heisenberg and standard Born-Infeld theories, \( \mu \sim 10^{-33}\text{(e.s.u)}^{-2} \) [5, 12], hence for objects of masses around \( M \sim 10^5M_\odot \), \( \mu/M^2 \) when brought to geometrical units \( (\mu[\text{cm}] = \mu[(\text{e.s.u})^{-2}]c^4/G \) and \( M[\text{cm}] = M[\text{g}]/G/c^2 \), would be approximatively \( 10^{-4} \). In this specific example, the limit \( Q = M \) can be therefore approached with a precision of up to four decimals within the perturbative analysis presented here.

For the classic extreme value \( Q = M \), the perturbative solution to the outer horizon is

\[ r_+^{Q=M} = M \left\{ 1 + \sqrt{\frac{2\mu}{5M^2}} - \frac{4\mu}{5M^2} + O \left[ \left( \frac{\mu}{M^2} \right)^{\frac{3}{2}} \right] \right\}, \]  

(1.22)
and there exist inner horizons given by

\[ r_{Q=M} = M \left\{ 1 - \sqrt{\frac{2\mu}{5M^2}} - \frac{4\mu}{5M^2} + O \left( \left( \frac{\mu}{M^2} \right)^{\frac{3}{2}} \right) \right\} \]  

(1.23)

\[ r_{ncl}^{Q=M} = M \left\{ \left( \frac{2\mu}{5M^2} \right)^{\frac{1}{4}} + \left( \frac{\mu}{10M^2} \right)^{\frac{1}{2}} + O \left( \left( \frac{\mu}{M^2} \right)^{\frac{3}{4}} \right) \right\}, \]  

(1.24)

where \( r_{Q=M} \) in Eq. (1.23) is the nonlinear version of the inner horizon in Reissner-Nordström solution, and the solution given by Eq. (1.24) has no classical (ncl) counterpart, being intrinsically due to corrections to the Maxwell theory, e.g. quantum. Notice that when \( \mu \neq 0 \) the inner and outer horizons are never equal in an arbitrary nonlinear theory given by Eq. (1.14) in the case \( Q = M \). Hence, as we expect, when corrections are added to Maxwell’s theory, the degeneracy in the case \( Q = M \) is broken. Nevertheless, due to the continuity of the metric, there always exists a value of \( |\xi| \) where the horizons degenerate, depending now on \( \mu/M^2 \). We stress that Eq. (1.24) is just a mathematical solution to Eqs. (1.19) and (1.10), being physically meaningless, as the following analysis shows. Assume that the charge of the black hole is comparable with its mass (minimum value for being relevant the nonclassical horizon), that is \( Q^2 \sim M^2 \). From Eq. (1.24), however, one has \( r_{ncl}^{Q=M} \sim (\mu M^2)^{1/4} = r_c \). Since just distances much larger than \( r_c \) are physically meaningful in the realm of our perturbative calculations; see Eq. (1.17), it is proved that \( r_{ncl}^{Q=M} \) is not physically relevant. The above ratiocination implies that perturbative changes in the Maxwell Lagrangian just lead to corrections in the Reissner-Nordström horizons. This means that naked singularities still rise in such theories, but now for values of \( |\xi| \) slightly larger or smaller than one, depending upon the sign and absolute value of \( \mu/M^2 \).

1.5 The weak field black hole mass decomposition

Assume a test particle being captured by a black hole under a reversible transformation. In mathematical terms, this means that the equality in Eq. (1.13) is to be taken into account and the changes can be considered as infinitesimals. By considering the second term in Eq. (1.16) and Eq. (1.20), one ends up to first
order approximation with

\[
\frac{dM}{dQ} = \frac{Q}{\mathcal{R}_+} - \frac{\mu Q^3}{5(\mathcal{R}_+)^5} \left[ \frac{Q^2}{\mathcal{R}_+\sqrt{M^2 - Q^2}} + 4 \right]. \tag{1.25}
\]

Since we are supposing that the second term of the above equation is much smaller than the first one, the method of successive approximations can be used. We shall assume that

\[
M(Q) = M^{(0)}(Q) + \mu M^{(1)}(Q), \tag{1.26}
\]

where the second term of the above expression is thought of as a perturbation. At the zeroth order approximation, \(M^{(0)}\) satisfies the differential equation

\[
\frac{dM^{(0)}}{dQ} = \frac{Q}{M^{(0)} + \sqrt{(M^{(0)})^2 - Q^2}}. \tag{1.27}
\]

As it is known, the solution to the above equation is [10]

\[
M^{(0)}(Q) = M_{irr} + \frac{Q^2}{4M_{irr}}, \tag{1.28}
\]

where \(M_{irr}\) is a constant of integration known as the irreducible mass and it accounts for the total energy of the system when the charge of the black hole is zero. Expression (1.28) is the Christodoulou-Ruffini black hole mass formula valid for a classical spherically symmetric charged black hole. By substituting this expression into Eq. (1.21) one obtains \(\mathcal{R}_+ = 2M_{irr}\) and then it follows that \(Q^2/(2\mathcal{R}_+) \leq M/2\), where the equality is valid in the case \(Q = M\). Hence, up to 50\% of the total mass of a black hole is due to the electromagnetic energy contribution \(Q^2/(4M_{irr})\).

Substituting Eq. (1.26) into Eq. (1.25) and working now up to first order approximation, using Eqs. (1.27) and (1.28) we have

\[
\frac{dM^{(1)}}{dQ} = -\frac{Q}{2M_{irr}[M_{irr} - Q^2/(4M_{irr})]} \left[ M^{(1)} + \frac{Q^4}{160M_{irr}^3} \right] - \frac{Q^3}{40M_{irr}^2}. \tag{1.29}
\]
from which we obtain

\[ M^{(1)}(Q) = -\frac{Q^4}{160M_{irr}^5}. \]  

(1.30)

The above equation is obtained by imposing \( M^{(1)}(0) = 0 \), which is physically clear from our previous considerations. Since energy could be extracted from black holes only when they are charged [see Eq. (1.13)], the extractable energy, \( M_{ext} \), or the blackholic energy [12], in weak fields nonlinear theories of electromagnetism given by Eq. (1.14) is

\[ M_{ext}(Q) = \frac{Q^2}{4M_{irr}} - \frac{\mu Q^4}{160M_{irr}^5}. \]  

(1.31)

It is not difficult to see that Eq. (1.31) for the Euler-Heisenberg Lagrangian reproduces some of the results of Ref. [30]. As it can be checked easily, the above equation is exactly the electromagnetic energy \( E^{(em)} \) [42, 43] stored in the electric field in the spacetime given by Eq. (1.19) viz.,

\[ E^{(em)} = 4\pi \int_{r^+}^{\infty} T_{0}^{0} r^2 dr = \int_{r^+}^{\infty} \int_{2\pi}^{0} \int_{\pi}^{0} \sqrt{g} d\theta d\varphi dr, \]  

(1.32)

where \( g \) is the determinant of the metric, that in Schwarzschild coordinates is given by \( r^2 \sin^2 \theta \); see Eq. (1.5). Notice that even in the case where corrections to the Maxwell Lagrangian are present (e.g. quantum), \( r^+ = 2M_{irr} \), as is clear from Eqs. (1.20), (1.21), (1.26), (1.28) and (1.30).

From Eq. (1.31), one distinctly sees that the total amount energy that can be extracted from a nonlinear charged black hole is reduced if \( \mu > 0 \), in relation to the Maxwell counterpart. The positiveness of \( \mu \) is valid both to the Euler-Heisenberg effective nonlinear Lagrangian to one-loop QED as well as to the standard Born-Infeld Lagrangian, as we pointed out previously. Hence, in these theories, the extractable energy is always smaller than 50% of the total energy. More precisely, from Eqs. (1.20), (1.21), (1.26), (1.28) and (1.30),

\[ M_{ext} \leq \frac{M}{2} - \frac{\mu Q^4}{320M_{irr}^4 \sqrt{M^2 - Q^2}}, \]  

(1.33)

the equality in this case being true only when \( \mu = 0 \).
1.6 Transformations in the outer horizon

Under the capture of a test particle of energy $E$ and charge $q$, the black hole undergoes the (infinitesimal) changes $\delta M = E$ and $\delta Q = q$, satisfying Eq. (1.13). Since the outer horizon of this black hole is dependent upon $M$ and $Q$, it would also undergo a change. Such a change can be obtained in the scope of the perturbative description we are carrying out and the basic equation for doing so is Eq. (1.20).

By using Eqs. (1.20), (1.21), (1.13) and the second term of Eq. (1.16), one can easily show that

$$\delta r_+ \geq -\frac{\mu Q^4 \delta R_+}{5(R_+)^3(M^2 - Q^2)}[R_+ + 3\sqrt{M^2 - Q^2}].$$  \hspace{1cm} (1.34)

As it can be seen from Eqs. (1.21), (1.26), (1.28) and (1.30), $\delta R_+ \sim O(\mu)$, then, up to first order in $\mu$, we have $\delta r_+ \geq 0$. This result is uncomplicated if one notices that up to first order approximation in $\mu$, based on the two last sections, $r_+ = 2M_{irr}$. Under irreversible transformations, however, Eq. (1.34) shows that the outer horizon increases. Notice that the above results are just valid for $Q/M < 1$.

Another way of realizing whether or not there is an increase of the outer horizon due to the capture of a test particle is to search for the solutions to Eqs. (1.10) and (1.19) when one performs the changes $M \rightarrow M + \delta M$ and $Q \rightarrow Q + \delta Q$, satisfying Eq. (1.13). If one defines generally $r_+$ as the largest solution to Eqs. (1.10) and (1.19), then it is simple to verify that $\delta r_+ = 0$ for reversible transformations. For irreversible transformations, $\delta r_+ > 0$. Hence, generically, one has $\delta r_+ \geq 0$ for an arbitrary infinitesimal transformation undergone by the black hole in nonlinear weak field electromagnetism.

1.7 Energy decomposition for asymptotically flat nonlinear black holes

Weak field nonlinear Lagrangians propound that the outer horizon of spherically symmetric $L(F)$ theories are $r_+ = 2M_{irr}$ when reversible transformations are considered, for any range of the electric field, and not only for the one where $E_r \ll 1/\sqrt{\mu}$. Now we shall show that assuredly this is the case. This means that
it is possible to obtain the total mass-energy of spherically symmetric, asymptotically flat, nonlinear black holes in an algebraic way, subduing problems in solving differential equations coming from the thermodynamical approach. Also, it automatically gives us the extractable energy from nonlinear black holes.

Assume that the invariant $F = -2E_r^2$ can be seen as $F = F(r, Q)$. From Eqs. (1.7)-(1.9), it follows that

$$Q \frac{\partial A_0}{\partial Q} = \frac{\partial N}{\partial Q}. \quad (1.35)$$

Assume now that $r_+ = C =$constant, that is, the outer horizon is an intrinsic property of the system. From Eqs. (1.10) and (1.35), one shows immediately that

$$\delta M = \delta Q A_0 \big|_{r_+ = C}. \quad (1.36)$$

It can be checked that the above equation is valid only when $r_+ = C$. We recall that we postulated Eq. (1.36) as the law for reversible transformations (energy conservation). Thereby, we showed that reversible transformations are fully equivalent to having constant horizons in spherically symmetric black hole solutions to general relativity. Since Eq. (1.36) is valid for any stage of the sequence of reversible transformations for any theory satisfying the conditions mentioned before, it is even so when $Q = 0$ and hence, $C = 2M_{irr}$. So, horizons for reversible transformations are dependent just upon the black hole fundamental energy states, $2M_{irr}$. Even more remarkable is that we already know the solution to Eq. (1.36), which from Eqs. (1.6), (1.10) and (1.9) is

$$M = M_{irr} + Q A_0 \big|_{r = 2M_{irr}} - N \big|_{r = 2M_{irr}}$$

$$= M_{irr} + 4\pi \int_{2M_{irr}}^{\infty} r'^2 T_0^0 (r') dr'.$$ \quad (1.37)

The above equation is the generalization of the Christodoulou-Ruffini black hole mass decomposition formula to $L(F)$ theories that do not depend upon $M$. If this is not the case, one then has an algebraic equation to solve. The extractable energy $M - M_{irr}$ from $L(F)$ can be read off immediately from Eq. (1.37) and as we expect, it is the same as Eq. (1.32); it can also be checked that Eq. (1.37) is in total agreement with the results for the weak field Lagrangians in terms of the differential approach.
It is worth noting that Eq. (1.37) could be also obtained from Eq. (30) of Ref. [25], by replacing there the relation $r_h = 2M_{irr}$. However, following the purely mathematical approach in [25], this latter assumption does not find a clear physical justification. Our approach in this work is completely different from [25]: it is based on physical requirements of energy and charge conservation laws and reversible transformations. As a consequence of these physical requirements, we actually demonstrated that the horizon is indeed a constant of integration, hence an independent quantity.

Since in the current case the horizon area is $A = 4\pi r_+^2$, Eq. (1.37) can as well be written in terms of it. As we showed above, for reversible transformations the outer horizon must be kept constant and the mass change must be given by Eq. (1.36). Nevertheless, intuitively, one would expect the total mass of a given black hole to have a definite meaning. In this sense, Eq. (1.37) in terms of the black hole area should be the expression for the mass even in the case $A$ changes. Such a general statement is reinforced by the fact that it is true for black holes described by the Maxwell Lagrangian \(^*\). As we show now, this is precisely the case also in nonlinear electrodynamics. Initially we recall that the surface gravity [3] in spherically symmetric solutions in the form [44]

$$\kappa = \frac{(e^{\nu})'|_{r_+}}{2}$$  \hspace{1cm} (1.38)

where the prime means differentiation with respect to the radial coordinate and from Eqs. (1.9) and (1.10) the above equation can be cast as

$$\kappa = \frac{1}{2r_+} \left[ 1 + 2Q \frac{\partial A_0}{\partial r_+} - 2 \frac{\partial F}{\partial r_+} \right].$$  \hspace{1cm} (1.39)

From Eqs. (1.10) and (1.35), one can see in the general case that

$$\delta M = A_0 \delta Q + \frac{\kappa}{8\pi} \delta A,$$  \hspace{1cm} (1.40)

where Eq. (1.39) was used. Nevertheless, this is nothing but the generalized first law of black hole mechanics for nonlinear electrodynamics [4]. Since $M$ as given in Eq. (1.37) was derived from Eqs. (1.10) and (1.35), it is assured its variation

\(^*\)This can be seen in Ref. [26, 27] when one works with its final mass expression, $M$, and check it is exactly the same as Eq. (2) of Ref. [10] in the context of reversible transformations
satisfies Eq. (1.40). Hence, it is the generalization under the physical approach of the parametrization done by Smarr [26, 27] for the classical Christodoulou-Ruffini black hole mass formula in the context of nonlinear electrodynamics. We would like to stress that all the previous ratiocination is a direct consequence of having \( M \) as an exact differential. Besides, Eq. (1.37) can be written in the suggestive way as

\[
M = Q A_0 (r_+) + \frac{A}{8 \pi r_+} \left[ 1 - 2 \frac{\mathcal{N}(r_+)}{r_+} \right].
\]  

(1.41)

From Eq. (1.39), we see that in general the term in the square brackets of the above equation does not coincide with \( 2 \kappa r_+ \). This could be easily seen in the scope of weak field nonlinear theories described by Eq. (1.14) analyzed previously. Nevertheless, for the case of the Maxwell Lagrangian, the term inside the square brackets of Eq. (1.41) is exactly \( 2 \kappa r_+ \). It implies that the generalized Christodoulou-Ruffini black hole mass formula does not keep the same functional form in nonlinear electrodynamics as in the classic Maxwellian case.

### 1.8 Conclusions

As we have shown above, in the weak field limit of nonlinear Lagrangians, a generalization of the Christodoulou-Ruffini black hole mass decomposition formula can always be obtained; see Eqs. (1.26), (1.28) and (1.30). Indeed, also in the weak field limit of nonlinear theories of electromagnetism one obtains the constancy of the outer horizon when reversible transformations are taken into account (\( 2M_{irr} \), exactly as the horizon in the Schwarzschild case). For irreversible transformations, it always increases. We have also shown that these results actually are valid for nonlinear asymptotically flat black holes, once it is the only way to lead to the equation coming from the laws of energy and charge conservation for reversible transformations; see the equality in Eq. (1.13). As a by-product, it allowed us to write down the total mass and the extractable energy (upper limit) of nonlinear spherically symmetric black holes in terms of their charge, outer horizon areas and the scale parameters coming from the electrodynamic theory under interest. When irreversible transformations are present, for each transformation, \( \delta r_+ > 0 \) iff \( (1 - 8 \pi T^0_0 |_{r_+} r_+^2) > 0 \), as it can be seen from Eq. (1.6). From the same equation, it can be checked that this is always valid when there exists
an outer horizon. Hence, for $L(F)$, the areas of the outer horizons never diminish for irreversible processes. With this generalized Christodoulou-Ruffini black hole mass formula, one can notice that the known first law of black hole mechanics, see Eq. (1.40), is just its direct consequence and thus one could say that it defines such a law. In general such a mass is not functionally the same as the one obtained in the case of the classic Maxwell electrodynamics. If the entropy of a black hole is proportional to its horizon area, the approach of reversible and irreversible transformations lead to the conclusion it can never decrease even in the context of nonlinear electrodynamics.

Turning to astrophysics, it is important to discuss the specific sign of the nonlinear correction parameter, $\mu$. Its positiveness is indeed in agreement with very well-founded nonlinear Lagrangians, such as the Euler-Heisenberg and the Born-Infeld Lagrangians. We have shown that $\mu > 0$ implies that the extractable energy of a black hole described by weak field nonlinear Lagrangian is always smaller than the one associated with the Maxwell Lagrangian; see Eqs. (1.31) and (1.33). Hence, due to a continuity argument, we are led to a most important corollary of this work: nonlinear theories of the electromagnetism reduce the amount of extractable energy from a black hole with respect to the classical Einstein-Maxwell case. It signifies that the extractable energy from nonlinear black holes are always smaller than half of their total mass, which is the largest amount of extractable energy obtained from the Christodoulou-Ruffini black hole mass formula. This result might, in principle, be relevant in the context of gamma-ray bursts (see e.g. [45] and references therein) since their energy budget, as shown by Damour & Ruffini [46], comes from the electromagnetic energy of the black hole extractable by the electron-positron pair creation process à la Sauter-Heisenberg-Euler-Schwinger. Therefore, in this model, stages emerge where charged black holes would be formed and their energy would be extracted from an electron-positron plasma that subsequently would interact with the interstellar medium producing the radiation that we observe.

We finally stress that it is important to keep in mind that for quantitative estimates the perturbative analysis presented in this work is valid only if the condition (1.15) is satisfied. In the case of the Euler-Heisenberg Lagrangian $(1/\sqrt{\mu} \approx 200E_c)$ and for a black hole mass $M \sim 3M_\odot$, as expected from the gravitational collapse of a neutron star to a black hole, $\mu/M^2 \approx 8.2 \times 10^5$, so for a charge to mass ratio $\xi = 5 \times 10^{-4}$ (at the outer horizon $R = r_+, E_r/E_c \approx 21$), the reduction of the extractable energy is of only 0.5% with respect to the Maxwell
case. For supermassive black holes in active galactic nuclei, e.g. $M \sim 10^9 M_\odot$ ($\mu/M^2 \approx 7.4 \times 10^{-12}$), we obtain for $\xi = 0.9999$ (at $r_+$, $E_r/E_c \approx 5 \times 10^{-4}$) the extractable energy is reduced only by $10^{-8}\%$ with respect to the Maxwell case.

1.9 Perspectives

This chapter dealt more with generalizations of concepts and we have had as their outcome the possibility of always finding the mass formula in an algebraic way for spherically symmetric nonlinear charged black holes. Such an expression encompasses all of their thermodynamical facets, rising in our approach from the laws of conservation of energy and charge. Therefore, it should be applied in all black hole scenarios, since it would single out just the physically relevant cases. It would be of interest to scrutinize it in the scope of the so-called "regular black holes" [32, 47], intriguing objects that possess outer horizons but not singularities. Black hole interactions could also be qualitatively investigated with the mass formula, allowing therefore one to have insights about the relevance of certain conceived scenarios. It seems also interesting to make use of it in the context of black hole radiation, for trying to enlighten further its understanding. Some of these issues are addressed in the next chapter for the Born-Infeld Lagrangian.

Possibly the most relevant point of charged black holes would be finding mechanisms and observational evidences for their existence, or at least stages where they could raise. In this regard, gamma ray bursts could possibly be their evidences [12] and charge asymmetries in brane worlds could be one of the possible mechanisms to form charged black holes [48]. To the best of our knowledge, it is still missing a definite answer to the issue of charged black holes. If positive replies to the aforesaid points raise, then it would be another encouragement to extend the results obtained in chapter for axially symmetric spacetimes.

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Chapter 2

Energy decomposition within Einstein-Born-Infeld black holes

2.1 Introduction

The first concept of a charged black hole raised after the solution of spherically symmetric Einstein-Maxwell equations, due to Reissner and Nordström [36]. As it is very well-known, the constant $Q$ coming from Maxwell equations, to be identified with the charge of the black hole, modifies in a fundamental manner the event horizon of the associated black hole. It competes with another constant coming from the Einstein equations, $M$, identified with the mass (energy) of the black hole for distant observers. Whenever $|Q| > M$, the associated singularity is not even clothed by event horizons. What is not evident from the standard approach (solving Einstein-Maxwell equations), though, is the relationship between $M$ and $Q$. Intuitively this must be the case, since electromagnetic energies have their origin in charges. This relationship can be found in a variety of ways. We cite for instance the approach done by Bardeen, Carter and Hawking [3], by means of the symmetries of the spacetime. Another manner, notably physical, was put forward by Christodoulou [9] and Christodoulou and Ruffini [10], by means of reversible transformations [9]. Such transformations are the only ones that could bring back the black hole parameters to their original values after any transformation processed by a test particle with parameters $m$ and $q$ (where $M \gg m$ and $Q \gg q$).

It has been shown in the context of spherically symmetric spacetimes that reversible transformations are fully equivalent to the constancy of the event hori-
2. Energy decomposition within Einstein-Born-Infeld black holes

zon upon such changes for any nonlinear theory of the electromagnetism \( L(F) \) that leads to asymptotically flat general relativistic solutions [49]. Due to the generality of the analysis, such a constant must be \( 2M_{irr} \), where \( M_{irr} \) is the total energy of the system when \( Q = 0 \). Due to this fact, the constant of integration \( M_{irr} \) must be always positive. The above mentioned equivalence allows one to exchange the problem of solving nonlinear differential equations for nonlinear theories by the trivial problem of solving algebraic equations. It goes without saying that such a procedure just works for the cases where event horizons are present. Just for completeness, whenever irreversible processes take place for a given configuration, the irreducible mass varies, always increasing for each process, being proportional to the amount of radial momentum carried by the test particles when evaluated by local observers ideally on the event horizon [10].

The Born-Infeld theory [5] has regained interest due to its analogous emergence as an effective theory to String Theory [4]. Such a theory was constructed with the purpose of remedying the singular behavior in terms of energy of a pointlike charged particle. It just requires a parameter \( b \), identified with the absolute upper limit for the electric field of a system where just electric aspects are present [5]. Motivated by the unitarian viewpoint [5], Born and Infeld fixed this parameter by imposing that in the Minkowski spacetime the associated electromagnetic energy coming from a pointlike electron equals to its rest mass. Nevertheless, the dualistic viewpoint [5] could have been equally well assumed and the parameter \( b \) should now be determined by a theory relying on it. A known theory where this aspect is present is quantum mechanics [5]. Actually, the Born-Infeld theory has been applied to the hydrogen atom, both the non-relativistic and relativistic one [50, 51]. The outcome of the numerical analyses shows that \( b \) must be much larger than the value initially proposed by Born and Infeld [50, 51]. Notwithstanding, a definite value for it has not been obtained.

In this chapter we wish to elaborate on the consequences of taking into account the black hole energy decomposition (black hole equation of state, obtained out of their thermodynamics) in the nonlinear context for their own description in light of such a constraint equation, as well as in their interactions. For so, we chose to work with Einstein-Born-Infeld black holes, due to their long list of assets, as stated previously. The consideration of an equation of state for black holes will automatically single out the physically relevant cases out of the “space of all possible black hole parameters”, corresponding to analyses directly from solutions to general relativity. We demonstrate here that Einstein-Born-Infeld
black holes could be separated basically into two families based only upon their charges. The ones corresponding to Reissner-Nordström generalizations would have their irreducible masses within a given range, for one of the aforesaid families, whilst all the others would display just a single (non-degenerated) horizon.

As another application, we further analyze the total radiation (electromagnetic and gravitational) emitted in a process of coalescence of two Einstein-Born-Infeld black holes and show the relevance played by the parameter $b$ into the results by means of a toy model.

Geometric units are used and the signature of the spacetime is $-2$.

### 2.2 Black hole mass decomposition for nonlinear theories

In the context of spherically symmetric solutions to general relativity minimally coupled to nonlinear Lagrangians of the electromagnetism, it can be shown that the general solution to the metric is 

\[
    ds^2 = e^{\nu(r)} dt^2 - e^{-\nu(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where

\[
    e^{\nu(r)} = 1 - \frac{2M}{r} + \frac{8\pi}{r} \int_r^\infty r'^2 T_{00} dr'
\]

\[
    = 1 - \frac{2M}{r} + \frac{2QA_0}{r} - \frac{2N}{r},
\]

and

\[
    E_r(r) = -\frac{\partial A_0}{\partial r}, \quad T^\mu_\nu = \frac{4LF^\mu_\beta F_{\nu\beta} - L\delta^\mu_\nu}{4\pi}, \quad \frac{\partial N}{\partial r} = -Lr^2.
\]
is the radial component of the electric field and $A_0$ is its associated potential. In the expressions for $A_0$ and $\mathcal{N}$, it has been chosen a gauge where they are null at infinity. We stress that for obtaining $A_0(r)$ and $\mathcal{N}(r)$ from given $E_r(r)$ and $L(E_r)$, it is tacit one has to integrate from an arbitrary $r$ to infinity, since we are interested in black hole solutions [18]. The radial electric field satisfies the equation

$$L_F E_r r^2 = -\frac{Q}{4} \text{ or } \frac{\partial L}{\partial E_r} = \frac{Q}{r^2}. \quad (2.5)$$

In a spherically symmetric spacetime, infinitesimal reversible transformations are defined by

$$\delta M = \delta Q A_0(r_+), \quad (2.6)$$

where $r_+$ is the outermost horizon from a given black hole theory, defined as the largest root of Eq. (2.3). For a general transformation, one has the formal replacement “$\rightarrow \geq$” in the above equation. Actually, Eq. (2.6) encompasses the law of conservation of energy and charge in the case of reversible transformations for test particles interacting with a black hole. This is evidenced by the identifications: $\delta Q = q$ and $\delta M = E_{t.p.}$, where $E_{t.p.}$ is the conserved energy of a test particle of mass $m$ and charge $q$ in a spacetime described by Eqs. (2.1), (2.3), (2.4) and (2.5). Energy could always be extracted from a black hole whenever the right hand side of Eq. (2.6) is negative. Notice that albeit we have commenced with a test particle, in the end reversible transformations just evidence the properties of the spacetime itself when conservation laws are taken into account. It signifies that the integration of Eq. (2.6) will give us intrinsic information about the spacetime. For an irreversible transformation, one unavoidably has a description dependent upon the test particle aspects [10]. We emphasize that $\delta M$ is assumed to be an exact differential, like the internal energy of a thermodynamical system. Hence, by means of convenient processes one could obtain the general expression for the energy decomposition for nonlinear black holes, i.e., their equation of state. From what we have discussed previously, reversible processes are manifestly such ones.

The customary approach for obtaining the mass formula (energy decomposition) would be integrating Eq. (2.6), given the outer horizon in terms of the parameters coming from the electromagnetic theory under interest and the
2.2. Black hole mass decomposition for nonlinear theories

spacetime. In general, it turns out to be impossible to work analytically for $L(F)$ theories in such a case. Since one knows that there is a correlation between black holes and thermodynamics [3, 44], one would suspect that Eq. (2.6) (thermodynamics) is somehow inside the equations of general relativity (or vice-versa). It can be shown easily that this is indeed the case, provided that the outer horizon keeps constant under reversible transformations [49]. Since it is so, it follows that the outer horizon must be identified with its associated Schwarzschild horizon (where $Q = 0$), and it will be denoted by $r_+ = 2M_{irr}$.

For the nonlinear theories where the electric potential $A_0$ is independent of the parameter $M$, it follows from the above reasoning and Eq. (2.3) that

$$M = M_{irr} + QA_0|_{r=2M_{irr}} - N|_{r=2M_{irr}} = M_{irr} + 4\pi \int_{2M_{irr}}^{\infty} r^2 T^0_0(r')dr'. \quad (2.7)$$

The above equation is the way of decomposing the total energy in terms of intrinsic and extractable quantities. It can be shown with ease [49] that it implies the so-called generalized first law of black hole mechanics for nonlinear electrodynamics [4], thus superseding it. Notice from the above equation that one could not associate all $M_{irr}$ (given $M$ and $T^0_0$) with the outer horizon. The reason for this is simple: Eq. (2.7) was defined by $e^\nu(2M_{irr}) = 0$, which encompasses also $M_{irr}$ related to the inner horizon, not relevant in the discussion of reversible transformations for black holes. Nevertheless, it is uncomplicated to single out the set of $M_{irr}$ corresponding to the outer horizon. One knows that the condition that leads to the degeneracy of the horizons is the common solution to $e^\nu(2M_{irr}) = 0$ and $de^\nu/dr|_{r=2M_{irr}} = 0$. These requirements and Eq. (2.7) imply that the horizons are degenerated at the critic points of $M$ as a function of $M_{irr}$. Hence, since outer horizons are larger than inner ones, it follows that the set of irreducible masses relevant in our analysis are the ones that always give $dM/dM_{irr} \geq 0$. We highlight that just the outer horizon is of physical importance to the energy decomposition expression, Eq. (2.7). This is so due to the fact that the outer horizon is the separatrix of regions where the $r$-coordinate is time-like and space-like. In the region between the inner and outer horizons, particles can not remain still and are mandatorily impelled towards the singularity. Due this unidirectional property, irrespective of their fate inside the inner horizon, conservation laws guarantee that just their crossing at the outer horizon will impinge changes to
2. Energy decomposition within Einstein-Born-Infeld black holes

the black hole mass and charge in the way we described previously for external observers.

2.3 Born-Infeld Lagrangian

The Born–Infeld Lagrangian, \( L_{BI} \), can be written as (compatible with our previous definitions)

\[
L_{BI} = b^2 \left( 1 - \sqrt{1 + \frac{F}{2b^2}} \right),
\]

where \( b \) is the fundamental parameter of the theory and counts for the maximum electric field exhibited by an electrically charged and at still particle in flat spacetime [5]. This parameter naturally defines a scale to the Born-Infeld theory and thence it has a non-vanishing energy–momentum tensor trace, as it could be checked from the energy momentum tensor associated with Eq. (2.8) (see Ref. [24] for more details).

Putting Eq. (2.8) into Eqs. (2.3) and (2.4) and performing the integral from a given arbitrary radial coordinate \( r \) up to infinity, one gets (see for instance Ref. [18])

\[
e^{\nu(r)} = 1 - \frac{2M}{r} - \frac{2}{3} b^2 y^2 + \frac{2Q^2}{3 \sqrt{|\beta|} \sqrt{r^2 - |\beta|}} F \left[ x, \frac{1}{\sqrt{2}} \right],
\]

where we defined

\[
x(x^2) = \arccos \left( \frac{r^2 - |\beta|}{r^2 + |\beta|} \right), \quad y^2 = \sqrt{r^4 + \beta^2 - r^2},
\]

\[
\beta^2 = \frac{Q^2}{b^2}, \quad F \left[ x, \frac{1}{\sqrt{2}} \right] = 2 \int_{\sqrt{|\beta|}}^{\infty} \frac{du}{\sqrt{1 + u^4}},
\]

where \( F[x, 1/\sqrt{2}] \) is the elliptic function of first kind [53].

The modulus of the radial electric field and its scalar potential in this case, as
given by the first term of Eq. (2.4) and Eq. (2.5), are

\[ E_r(r) = \frac{Q}{\sqrt{r^4 + \beta^2}}, \quad A_0(r) = \frac{Q}{2 \sqrt{\beta}} \mathcal{F} \left[ x, \frac{1}{\sqrt{2}} \right]. \]  

(2.12)

As it is clear from Eq. (2.12), the electric field of an unmoving pointlike charged particle is always finite, as well as its associated scalar potential and they are positive monotonically decreasing functions of the radial coordinate. Hence, from Eq. (2.6), it implies that the necessary and sufficient condition for extracting energy from a Einstein-Born-Infeld black hole is to use test particles with an opposite charge to the hole.

### 2.3.1 Reissner-Nordström limit

Note that in the limit \( b \to \infty \) (i.e., \( \beta \to 0 \)) we recover the RN solution

\[ e^\nu(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \]  

(2.13)

since

\[ -\frac{2}{3} b^2 y^2 \to -\frac{1}{3} \frac{Q^2}{r^2}, \]  

(2.14)

and

\[ \mathcal{F} \left[ x, \frac{1}{\sqrt{2}} \right] = x + O(x^3), \]  

(2.15)

with

\[ x = \arccos \left( \frac{r^2 - |\beta|}{r^2 + |\beta|} \right) \sim \frac{2 \sqrt{|\beta|}}{r} \]  

(2.16)

so that

\[ \frac{2Q^2}{3 \sqrt{|\beta|} r} \mathcal{F} \left[ x, \frac{1}{\sqrt{2}} \right] \to \frac{4}{3} \frac{Q^2}{r^2}. \]  

(2.17)
Furthermore,

\[ A_0(r) = \frac{Q}{r}, \quad E_r(r) = \frac{Q}{r^2}. \tag{2.18} \]

### 2.3.2 Singularity structure

The metric given by Eqs. (2.9), (2.10) and (2.11) has been studied in detail in Ref. [18]. It has been pointed out there that the dimensionless quantities \( \tilde{M} \equiv b M \), \( \alpha \equiv Q / M \) and \( u \equiv r / M \) are convenient to scrutinize the properties of such a metric. Nevertheless, apparently some interesting properties of Eq. (2.9) have not been stressed. Under the above definitions, Eq. (2.9) may be written as

\[
e^{-\nu} = 1 - \frac{2}{u} + \frac{2}{3} \tilde{M}^2 u^2 \left( 1 - \sqrt{1 + \frac{\alpha^2}{\tilde{M}^2 u^4}} \right) + \frac{2\alpha^2}{3u} \sqrt{\frac{\tilde{M}}{|\alpha|}} \mathcal{F} \left[ \arccos \left( \frac{\tilde{M}u^2 - |\alpha|}{\tilde{M}u^2 + |\alpha|} \right), \frac{1}{\sqrt{2}} \right].
\tag{2.19}
\]

The horizons are obtained as the zeros of the above equation. As a result, one can verify that Eq. (2.19) has no minimum, and hence it is a monotonic function iff

\[
b < \frac{9 M^2}{|Q|^3 \mathcal{F}^2 \left[ \pi, \frac{1}{\sqrt{2}} \right]} \approx \frac{0.654 M^2}{|Q|^3},
\tag{2.20}
\]

which can also be cast as

\[
M > M_0, \quad M_0 \equiv \frac{\sqrt{b |Q|^3}}{3} \mathcal{F} \left[ \pi, \frac{1}{\sqrt{2}} \right].
\tag{2.21}
\]

As the limit of \( u \) going to zero in Eq. (2.19) shows us, Eq. (2.20) also guarantees that the associated spacetime will always exhibit just one horizon (not a degenerated one). The above inequality has no classical counterpart, since it can be formally obtained by taking the limit of \( b \) going to infinity. Equation (2.20) sets a fundamental inequality concerning the parameters \( Q, b \) and \( M \). Whenever Eq. (2.20) is not verified, it does automatically imply the existence of a minimum. Nevertheless, as it is easy to be realized, it does not imply the existence of horizons. Hence another more restrictive condition must exist assuring it. A simple
2.3. Born-Infeld Lagrangian

analysis shows us that such a requirement can be cast as

\[ u_+ \leq \frac{\sqrt{4M^2\alpha^2 - 1}}{2M}, \quad \frac{d}{du}(e^\nu)\big|_{u=u_+} = 0, \tag{2.22} \]

which is just the consequence of imposing that \( e^{\nu(u_+)} \leq 0 \), \( u_+ \) being the critical point of \( e^\nu \), thus guaranteeing the existence of an outer horizon. Just as a reference, in the limit when \( \tilde{M} \) goes to infinity, the above condition reduces to \( |\alpha| \leq 1 \), as it is well-known from the Reissner-Nordström solution for assuring the existence of horizons. As the above inequality suggests, the term \((4b^2Q^2 - 1)\) plays a fundamental role into the horizon description. We shall see that this is also the case in the approach related to the energy decomposition, as one would expect from our previous reasoning. Specialized to the Born-Infeld Lagrangian, Eq. (2.8), the total mass [see Eq. (2.7)] of an Einstein-Born-Infeld black hole can be decomposed as

\[
M = M_{irr} - \frac{8}{3}b^2M_{irr}^3 \left( \sqrt{1 + \frac{\beta^2}{16M_{irr}^4}} - 1 \right) \\
+ \frac{\sqrt{b|Q|^3}}{3} F \left[ \arccos \left( \frac{4M_{irr}^2 - |\beta|}{4M_{irr}^2 + |\beta|} \right), \frac{1}{\sqrt{2}} \right]. \tag{2.23} \]

We recall that the validity of the above energy expression automatically ensures the existence of an outer horizon. One should though keep in mind that just a subset of all possible \( M_{irr} \) are related to the aforementioned horizon and they are the values that always lead to \( dM/dM_{irr} \geq 0 \). The nonexistence of a horizon here means the impossibility of finding an irreducible mass for a given energy. In the next section we shall take a closer look to Eq. (2.23).

It is well-known that any nonlinear theory of the electromagnetism has a nonlinear medium associated with it [5] in the scope of the Maxwellian electromagnetism (the converse is not necessarily true!). Thence, one expects the extractable energy of a charged nonlinear black hole described by Born-Infeld theory to be smaller than its classical counterpart. This is corroborated by the case the fields are much smaller than the scale field \( b \) [49]. It can be verified this is indeed always the case.
2. Energy decomposition within Einstein-Born-Infeld black holes

2.4 Clothed singularities

From now on we shall assume that Eq. (2.23) is a valid decomposition to the total energy of a Einstein-Born-Infeld black hole. A simple analysis tells us that whenever

\[ 2b|Q| > 1 \]

(2.24)

is valid for the parameter \( Q \), given \( b \), Eq. (2.23) does have a minimum with respect to \( M_{\text{irr}} \), associated with the critical irreducible mass

\[ M_{\text{irr}}^c \equiv M_{\text{irr}}^{\text{min}} = \frac{\sqrt{4b^2Q^2 - 1}}{4b}. \]

(2.25)

Note that \( M_{\text{irr}}^c \) is always related to the case where the horizons are degenerated (extreme black holes), as we have pointed out in section 2.2, and it is always smaller than its classical counterpart, \( |Q|/2 \) (where \( M = |Q| \)). From our previous discussions, the relevant irreducible masses to the analysis for reversible transformations for black holes are \( M_{\text{irr}} \geq M_{\text{irr}}^c \). Substituting the above critical irreducible mass on Eq. (2.23), one has that such a minimum total energy is

\[
M_{\text{min}} = \frac{\sqrt{4b^2Q^2 - 1}}{6b} + \frac{b|Q|^3}{3} \mathcal{F} \left[ x \left( \frac{4b^2Q^2 - 1}{4b^2} \right), \frac{1}{\sqrt{2}} \right],
\]

(2.26)

which is naturally positive and it can be verified to be smaller than \( M_0 \) defined by Eq. (2.21). For the case \( 2b|Q| > 1 \), one can check that an immediate solution to \( M = M_0 \) is \( M_{\text{irr}} = 0 \) (not of relevance for us). There also is a nontrivial solution that cannot be expressed analytically in general, that we shall denote by \( M_{\text{irr}}^t \). This solution is very important since it will delimit the transition from space-like singularities to time-like ones with respect to the radial coordinate [see Eqs. (2.20) and (2.21)]. This signifies that clothed black hole solutions which generalizes Reissner-Nordström ones have irreducible masses \( 0 < M_{\text{irr}} < M_{\text{irr}}^t \).

An arbitrary black hole with \( M_{\text{irr}} \geq M_{\text{irr}}^t \) shall present a sole horizon and hence when test particles have crossed it, their fate is unavoidably its associated singularity. Figure 2.1 exemplifies the analysis from the previous sentences for a selected value of the parameter \( b|Q| \) for the case \( 2b|Q| > 1 \).

We consider now the case where Eq. (2.24) is violated. In this case, \( M \), as
2.5 Hawking radiation from Einstein-Born-Infeld black holes

Figure 2.1: Mass formula (thick plus dotted curves), Eq. (2.23), when the parameter \(|b|Q|\) satisfies Eq. (2.24), chosen here as 2. The dashed curve represents \(M_0\), as given by Eq. (2.21). The dot-dashed curve is the asymptote to \(M, M_{\text{irr}}\). Besides, \(M\) exhibits a minimum at the critical point \(M_{\text{irr}}^b \approx 0.97\) (where the horizons become degenerated) and for \(M_{\text{irr}} \leq M_{\text{irr}}^b, b \approx 3.18\) (the range of irreducible masses that generalize Reissner-Nordström black holes), \(M \leq M_0\). For \(M_{\text{irr}} > M_{\text{irr}}^b\), there is a sole horizon (not degenerated) inside with the radial coordinate is always timelike. The irreducible masses associated with the outer horizon are \(M_{\text{irr}} \geq M_{\text{irr}}^c\).

The dotted curve is related to the inner horizon solutions (for given configurations) and are not relevant to the analyses concerning the black hole mass decomposition.

given by Eq. (2.23), is a monotonic function of \(M_{\text{irr}}\). Since it is given by Eq. (2.21) when \(M_{\text{irr}} = 0\) and it increases linearly in \(M_{\text{irr}}\) for large \(M_{\text{irr}}\), we conclude that Eq. (2.20) is always satisfied. Hence, whenever Eq. (2.24) is not satisfied, one is led unavoidably to a singularity with just one horizon, for any value the associated irreducible mass of the system may have. This means that once test particles enter the horizon the singularity is unavoidable for them. Just for completeness, Fig. 2.2 compactifies the above mentioned properties for a selected value of the parameter \(|b|Q|\) such that \(2b|Q| \leq 1\). Besides, in Fig 2.3 we depicted all the different classes associated with the parameter \(|b|Q|\), assuming that the charge in all cases is fixed.

2.5 Hawking radiation from Einstein-Born-Infeld black holes

Subsequent to the work of Hawking on the semiclassical quantization of a scalar
2. Energy decomposition within Einstein-Born-Infeld black holes

Figure 2.2: Mass decomposition when the parameter $b|Q|$ does not satisfy Eq. (2.24) and is chosen to be 0.4. The curves have the same meaning as the ones in Fig. 2.1. From the solid curve we see that $M$ is a monotonic function and always larger than $M_0$. This means that such a case characterizes a scenario where there is always a sole event horizon and there is no classical analogue to it.

Figure 2.3: Mass formula for selected values of the parameter $b|Q|$ (numbers appearing on the curves) that encompasses all physically distinct classes of black holes for the Born-Infeld Lagrangian. The dotted curve represents the mass formula for the Maxwell Lagrangian. The dot-dashed curve demarcates the transition from two horizon solutions (as given by the thick curve) to a single one (as given by the dashed curve), where its associated inner horizon is null. The branches related to the inner horizons were removed.
field in some curved spacetimes [54], it is widely accepted that black holes radiate thermally, although this view has still some criticisms [55, 56]. Motivated by the first law of blackhole thermodynamics, which is a direct consequence of the mass decomposition expression given by Eq. (2.7) [49], and the results from the aforesaid semiclassical quantization, we shall now study the consequences of conjecturing that clothed black holes should behave like blackbodies to observers at infinity, radiating at temperatures proportional to their surface gravity [54]. In the spherically symmetric case, such a quantity is proportional to \( \frac{de^\nu}{dr}|_{r=r_+} \) [6, 44]. From Eq. (2.19) and preceding definitions, one has

\[
T \propto \frac{1 + 8b^2M^2_{irr} - 2b\sqrt{16b^2M^4_{irr} + Q^2}}{M_{irr}}. \tag{2.27}
\]

We notice some particularities of the insertion of the parameter \( b \) into the description of the electromagnetic fields. As in the classical case, \( b \to \infty \), it is possible to attain \( T = 0 \), but now as far as \( M(T = 0)_{irr} = \sqrt{\frac{4b^2Q^2 - 1}{4b}} \). (2.28)

Notice that \( M_{irr}^c = M_{irr}^{(T=0)} \). This is not surprising, since from our previous comments, the condition for null temperature of a black hole with charge \( Q \) occurs exactly at the critical points of the energy with respect to its irreducible mass. When Eq. (2.24) holds, one sees that the temperatures of the associated clothed black holes must decrease with the decrease of their irreducible masses until they eventually reach zero, for \( M_{irr} = M_{irr}^{(T=0)} \). This would mean that black holes where Eq. (2.24) is valid should radiate off finite amounts of energy, namely \( M(M_{irr}) - M(M_{irr}^{(T=0)}) \). Besides, from the analyses of the energy decomposition, black holes could never have negative temperatures. For the case Eq. (2.24) does not hold, it is impossible to have \( T = 0 \) and the temperature increases with the decrease of the irreducible mass. Fig 2.4 compactifies the dependence of the temperature upon the irreducible mass for selected values of \( b|Q| \), assuming the charge for all the cases is fixed.

We elaborate now on the temperature evolution of evaporating black bodies. For an arbitrary black hole case where \( 2b|Q| > 1 \), as we know, the temperature decreases as the irreducible mass of the system does so [see Fig. 2.4]. Hence,
it would allow us to conceive a situation where just the emission of uncharged scalar particles are present. For this simplified case, the charge of a hole would remain constant. Given that the black holes would behave like blackbodies for observers located at infinity (where there is a meaning to talk about the total energy of a black hole), their energy loss could be estimated by Stefan’s law [57]

\[
\frac{dM}{d\lambda} \propto -M_{irr}^2 T^4, \tag{2.29}
\]

where \(\lambda\) is related to the observer’s time receiving the radiation. For the emission of uncharged scalar particles, the above equation and Eq. (2.23) imply that

\[
\frac{d\tilde{M}_{irr}}{d\lambda} \propto - \frac{\left(1 + 8\tilde{M}_{irr}^2 - 2\sqrt{16\tilde{M}_{irr}^4 + \tilde{Q}^2}\right)^3}{\tilde{M}_{irr}^2}. \tag{2.30}
\]

In the above equation, for an arbitrary quantity \(A\), \(\tilde{A} \equiv bA\). We show now that
for this case the temperature never reaches the absolute zero. Since the irreducible mass can decrease until \( M_{\text{irr}}^{\text{min}} \), after a convenient transient time interval, the right hand side of Eq. (2.30) can always be expanded around \( M_{\text{irr}}^{\text{min}} \), leading to

\[
\frac{d\tilde{M}_{\text{irr}}}{d\lambda} \propto -\left( \frac{M_{\text{irr}}^{\text{min}} - \frac{1}{4}\sqrt{4\tilde{Q}^2 - 1}}{\sqrt{4\tilde{Q}^2 - 1}(1 + 4\tilde{Q}^2)} \right)^3.
\] (2.31)

The above equation has an analytic solution and when the limit of \( \lambda \) going to infinity is taken, one obtains \( \tilde{M}_{\text{irr}}(\infty) = \tilde{M}_{\text{irr}}^{\text{min}} \). This means an associated black hole never reaches the absolute zero and tends asymptotically to have just one horizon. The previous description naturally generalizes the analyses for Reissner-Nordström black holes [57, 58].

For an arbitrary black hole satisfying \( 2b|Q| \leq 1 \), it seems that a juncture shall arrive where its thermal energy will be sufficient to create pairs that could even neutralize the hole. This would befall since in this case the thermal energy of a black hole would augment with the diminution of its irreducible mass [see Fig 2.4]. Hence its description would be much more elaborated than the former one. Black holes with \( 2b|Q| \leq 1 \) are expected to evaporate after finite amounts of time, as corroborated by numerical analyses from Eq. (2.30). We shall not pursue the issues commented in the antecedent sentences in this chapter.

We further point out that from Eq. (2.26) for \( 1/(2|Q|) \leq b \leq \infty \), one has that

\[
\frac{1}{3\sqrt{2}} \mathcal{F} \left[ \pi, \frac{1}{\sqrt{2}} \right] (\approx 0.874) \leq \frac{M_{\text{min}}}{|Q|} \leq 1.
\] (2.32)

This means that the electron, with \( |Q| = |e| \approx 10^{-34} cm \) and \( M \approx 10^{-55} cm \), when seen as a fundamental clothed black hole, would necessarily imply \( b < 1/(2|e|) \approx 10^{33} cm^{-1} \). As it can be verified, in order to obtain the ratio \( M/|Q| \) compatible with the electron, one should assume \( b < b_0 \), \( b_0 \approx 10^{-9} cm^{-1} \) the value obtained by Born and Infeld under the unitarian viewpoint. This is disagreement with the results from the hydrogen atom being described by the Born-Infeld Lagrangian, where the description of its energy spectrum requires a value for \( b \) larger than \( b_0 \) [50, 51]. Hence, the electron should still be seen as a naked singularity and it would be meaningless to try to ascribe a temperature to it.
2. Energy decomposition within Einstein-Born-Infeld black holes

2.6 Superradiance

Now we briefly describe the case where the scalar field is charged and massive. Its perturbations in a charged BI black hole background are governed by the Klein-Gordon equation

\[
[(\nabla^\mu - iqA^\mu)(\nabla_\mu - iqA_\mu) - \mu^2]\Phi = 0,
\]

where \( \mu \) and \( q \) denote the mass and charge of the scalar field, respectively. The above equation admits separable solutions of the form

\[
\Phi(t, r, \theta, \phi) = e^{-i\omega t}e^{im\phi}R(r)S(\theta),
\]

where \( \omega > 0 \) is the wave frequency and \( m \) is the azimuthal separation constant. The angular equation can be reduced to the form of spherical harmonics equation with separation constant \( K = l(l+1) \), so that \( S(\theta) = Y_{lm}(\theta) \). The radial equation instead writes as

\[
\frac{d}{dr} \left( r^2 N^2 \frac{dR(r)}{dr} \right) + V_{(rad)}(r)R(r) = 0,
\]

with \( N^2 = e^\nu \) and

\[
V_{(rad)}(r) = \frac{r^2}{N^2} \left[ \omega - qA_0(r) \right]^2 - \mu^2 r^2 - l(l+1).
\]

By introducing the scaling \( \tilde{R} = rR \) and the “tortoise” coordinate transformation \( r \to r_* \) defined as

\[
\frac{dr_*}{dr} = \frac{1}{N^2},
\]

the radial equation can be transformed into the one-dimensional Schrödinger-like equation

\[
\frac{d^2}{dr_*^2} \tilde{R}(r) + \tilde{V} \tilde{R}(r) = 0,
\]
2.6. Superradiance

with the potential

\[ \tilde{V} = [\omega - qA_0(r)]^2 - \frac{N^2}{r^2} [\mu^2 r^2 + l(l + 1) + 2rNN'] . \]  

(2.39)

The asymptotic form of the radial equation as \( r \to \infty \) \((r_* \to \infty)\) is

\[ \frac{d^2}{dr_*^2} \tilde{R}(r) + (\omega^2 - \mu^2) \tilde{R}(r) \approx 0 , \]  

(2.40)

with solution \( \tilde{R} \sim e^{i\sqrt{\omega^2 - \mu^2} r_*}, \) i.e., \( R \sim e^{i\sqrt{\omega^2 - \mu^2} r_*} / r \) (decaying field at spatial infinity).

On the other hand close to the horizon \( r \to r_+ \) \((r_* \to -\infty)\), the asymptotic form of the radial equation becomes

\[ \frac{d^2}{dr_+^2} \tilde{R}(r) + [\omega - qA_0(r_+)]^2 \tilde{R}(r) \approx 0 , \]  

(2.41)

with solution \( \tilde{R} \sim R \sim e^{-i[\omega - qA_0(r_+)] r_+} \) (purely ingoing waves at the horizon). Therefore, if \( \mu < \omega < qA_0(r_+) \) energy flows out from the hole, i.e. one has superradiant scattering.

For the derivation of the Hawking temperature, the limit of geometric optics is assumed \[54\]. For this case, the relevant wavelengths must be smaller than the outer horizon of the system, \( r_+ \). Therefore, in natural units, \( \omega \gg 1/r_+ \). For the charge of the scalar field, it is reasonable to assume that \( q/Q \ll 1 \). For the case \( 2b|Q| > 1 \), it follows from Eq. (2.25) and \( r_+ = 2M_{irr} \) that \( r_+^2 > \beta \).

Therefore, it generalizes the results coming from Einstein-Maxwell black holes. In this case, assuming that \( \omega = n/r_+ \) and \( q = Q/n \), with \( n \gg 1 \), the condition for superradiance can be approximated to \( n < Q[cm]/l_p = \alpha (M[g]/m_p) \), where \( l_p \) and \( m_p \) are the Planck length and mass, respectively. Taking \( M \) of the order of the sun’s mass, \( \alpha > 10^{-38} n \) would lead to superradiance. For the case \( 2b|Q| \leq 1 \), the superradiance condition can be cast generically as \( n^2 < F[x|_{r_+}, 1/\sqrt{2}] r_+ \). Whenever \( r_+^2 < \beta \), which is always possible for this case, the right hand side of the previous inequality is very small and therefore superradiance does not take place. Else, similar conclusions as for the case \( 2b|Q| > 1 \) apply.
2. Energy decomposition within Einstein-Born-Infeld black holes

2.7 Energy loss of interacting Einstein-Born-Infeld black holes

Until the present time, no gravitational waves were directly detected [59]; in this regard, improvements on the detectors are on their way in the coming years [59]. Nevertheless, several detailed studies characterizing their possible behavior as due to conceived physical scenarios are already available (see for instance Ref. [60] and references therein), being future catalogues for direct gravitational waves observations. As it is very intuitive, accurate studies can just be performed numerically [60]. One can also imagine the tremendous increase of difficulty in numerically taking into account nonlinear effects of electrodynamics imprinted on the gravitational radiation of any physical system. Thereby, estimates would be the most natural first step. It is possible to find an estimate for the total amount of gravitational radiation that could be emitted by interacting black holes by relying on the so-called Second Law of black hole mechanics [3] and the concept of the mass decomposition. This would allow one to assess the relevance of nonlinear effects into gravitational waves, pointing or not for the need of more detailed studies, as well as using gravitational waves as probes for sifting electromagnetic phenomena in the cosmos.

In this section we shall make use of the energy decomposition given by Eq. (2.23) to find the imprint the parameter \( b \) has on the energy radiated off by two interacting Einstein-Born-Infeld black holes. For accomplishing such a goal, we shall also utilize the aforementioned second law of black hole mechanics. Such a theorem implies that the area of the resultant black hole can never be smaller than the sum of the areas of the initially (far away) interacting black holes [3, 36]. For simplifying the reasoning, we will assume that all the black holes involved are spherically symmetric Einstein-Born-Infeld ones. This problem can easily be solved for Einstein-Maxwell black holes (Einstein theory minimally coupled to the Maxwell Lagrangian), because their outer horizons are analytical. For nonlinear black holes, in general just numerical solutions are possible. In the mass decomposition approach, it is possible to carry out the analytical investigations further. The key for this is that whenever the mass formula is taken into account, the outer horizon must be always proportional to its associated irreducible mass for any theory.

Assume that the two initially interacting black holes have irreducible masses
2.7. Energy loss of interacting Einstein-Born-Infeld black holes

$M_{i1}$ and $M_{i2}$, respectively, giving rise to another (final) one of the same kind with irreducible mass $M_{if}$. Concerning its final charge, if one assumes that just radiation is allowed to leave the system (carried away by neutral particles), it must be the sum of the charges of the two initial black holes. Since the irreducible masses are proportional to the horizon areas, Hawking’s theorem (or the second law of black hole mechanics) implies that

$$M_{if}^2 \geq M_{i1}^2 + M_{i2}^2,$$  \hspace{1cm} (2.42)

A comment here is in order. Since we assumed that the initial black holes are very far away, the total energy of the system is ideally $E_t = M_1 + M_2$, where $M_a$, $a = 1, 2, f$, is the total energy of the $a$th black hole. Invoking the first law of black hole mechanics for an isolated system [36], the final energy of the two interacting black holes, $M_f$, can never be larger than $E_t$. The difference in the energy balance is due to the emission of gravitational and electromagnetic radiation, hence, $W_{rad} = M_1 + M_2 - M_f \geq 0$. By the cognizance of the minimum final energy of the system, it is even possible to obtain its maximum radiated off energy.

For fixing ideas, let us analyze first a particular case, namely two Reissner-Nordström black holes interacting in a way to lead to another Reissner-Nordström black hole. We know that the total energy of each black hole can be written as [10]

$$M_a = M_{ia} + \frac{Q_a^2}{4M_{ia}},$$  \hspace{1cm} (2.43)

where we have defined $Q_a$ as the charge of the $a$th black hole. It is easy to see that just $M_{if^-} \leq M_{if} \leq M_{if^+}$, with

$$M_{if}^\pm = \frac{M_1 + M_2 \pm \sqrt{(M_1 + M_2)^2 - (Q_1 + Q_2)^2}}{2}$$  \hspace{1cm} (2.44)

is in agreement with the above mentioned positivity of $W_{rad}$. Naturally, choices for $M_{if}$ must satisfy simultaneously Eqs. (2.42) and (2.44). When nonlinear theories are present, it is clear that in general the above range of final irreducible masses will not agree with the classical (Einstein-Maxwell black holes) case. It means that many possible classical situations will not exist in the nonlinear case and vice-versa even in the simple case of symmetry conserved binary interac-
tions. This could possibly lead to significant deviations for the amounts of radiation emitted by some systems when they are treated classically or not.

In the Einstein-Born-Infeld theory, the physical interval for $M_{if}$ can not be determined (numerically) unless the fundamental parameter $b$ is given. What is known [50] is that $b > b_0 \approx 10^{-9}$ cm$^{-1}$, where $b_0$ is the value for the scale field determined by Born and Infeld using the unitarian viewpoint [5].

Let us take a closer look at the Einstein-Born-Infeld black holes when compared to their classical counterparts. Assume just for simplicity that $M_{i1} = M_{i2}$ and $Q_1 = Q_2 \equiv Q > 0$. For this choice, Eq. (2.44) gives us $-\sqrt{1/\alpha^2 - 1} \leq M_{if}/Q - 1/\alpha \leq \sqrt{1/\alpha^2 - 1}$, where $\alpha$ is here defined as the charge-to-mass ratio of the initially interacting black holes. Let us choose, just for simplicity, $M_{if}/Q = 1/\alpha$. From the Einstein-Maxwell case, one can check easily that for the above analysis $W_{rad}^{(clas)}/Q = (1 - \alpha^2)/\alpha$. For the above particular choice of parameters, one can show that Eq. (2.42) is just satisfied if $\alpha \geq \sqrt{2(\sqrt{2} - 1)} \approx 0.91$. Such cases are of theoretical interest since they would evidence the departures of the Born-Infeld theory from the Maxwell theory. For investigating smaller values of $\alpha$, one should select different final irreducible masses for the black holes.

Figure 2.5 compactifies the possibilities for the above chosen $M_{if}$ for $\alpha = 0.95$, due to miscellaneous values of $bQ$. One sees in this case that nonlinear and linear black holes may radiate off very different amounts of energy. Besides, the energy released for interacting Born-Infeld black holes is always larger than its Maxwellian counterpart. Notice finally that $Q = \alpha M$, $M$ being the mass of any of the black holes when they are far apart, which would also allow one to compare the energies radiated off by the black holes during their process of interaction with the total initial energy of the system.

2.8 Conclusions

Foremost, it is clear that the approach of analyzing a given black hole solution just from its metric and the one from its metric and energy decomposition expression must be consistent since both approaches use intrinsic properties of the spacetime. Nevertheless, the latter approach is much more restrictive than the former one. This is in full analogy with all possible equations of state for thermodynamic systems and the ones that satisfy the first law. It must be stressed that the energy decomposition (black hole thermodynamics) is mandatory for
2.8. Conclusions

Figure 2.5: Total radiation (gravitational plus electromagnetic) $W_{\text{rad}}/Q$ released in the process of coalescence of two identical Einstein-Born-Infeld black holes with $\kappa = 0.95$ under the assumption it leads to another one of the same type with the same parameters as their classical counterparts. The thick curve represents such a case. The dashed curve stands for the radiation encountered in the Einstein-Maxwell theory, $W_{\text{rad(clas)}}/Q$. The associated gravitational radiation tends to its classical counterpart when $bQ$ goes to infinity. The energy released in the case of nonlinear black hole interaction is always larger than the one coming from its classical counterpart, for a given charge $Q$. Recall that $Q = aM$.

the proper description of any (clothed) black hole phenomenon, since it singles out the configurations (via a constraint equation) that are in accord with conservation laws. Such an equation of state automatically evidences the physically relevant cases in black hole physics, hence leading to a better and pellucid penetration of them.

The energy decomposition analysis within Einstein-Born-Infeld black holes leads us to their split into two fundamental families of black holes. Whenever $2b|Q| \leq 1$, independent of their irreducible masses, one is led to an associated black hole whose singularity can not be forestalled after test particles cross its sole, non degenerated horizon. Naturally this befalls since due to Eq. (2.23) all the associated masses for the black holes must always be larger than $M_0$, defined by Eq. (2.21), which automatically warrants the validity of Eq. (2.20). Besides, the previous inequality naturally leads to an absolute upper limit to the charge of approximately $10^8 \text{cm} \approx 10^3 M_\odot$, given that $b > 10^{-9} \text{cm}^{-1}$ [50]. We stress that the previous conclusions are strictly nonclassical consequences of the finiteness of $b$. The second family of black holes is defined by those satisfying $2b|Q| > 1$ and
2. Energy decomposition within Einstein-Born-Infeld black holes

constitutes the generalization of Einstein-Reissner-Nordström black holes just for irreducible masses smaller than transitional values, the nontrivial solutions to \( M = M_0 \). Above such values, again due to the finiteness of \( b \), single black hole horizons are also present for this family.

It is often stated that the charges of the black holes should be much smaller than their associated total masses, due to natural neutralization processes steered by the electric forces [57]. As a simple look at Eq. (2.12) shows us in the scope of Born-Infeld theory, its electric force strength could diminish significantly when compared to its Maxwellian counterpart, which could make the neutralization process less efficient in principle. This could possibly lead to larger charge parameters to the black holes. If this is not the case, it is reasonable the existence of an upper limit to the charge, possibly with single clothed singularities only (with \( 2b|Q| \leq 1 \)). Whenever \( b \neq \infty \) this could always be the case. If it is large enough, it would lead automatically just to small values of the charge. Nevertheless, issues apropos of their evaporation should be properly addressed for assessing their possible lingering until present.

Our analyses showed that the minimum irreducible masses of black holes satisfying \( 2b|Q| > 1 \) should decrease concerning their classical counterparts. Such black holes should radiate off (suppose by emitting uncharged scalar particles) until their temperatures reach \( T = 0 \), taking for doing so an infinite amount of time, settling down exactly at their smallest energy states. Further energy could even be extracted from them (obviously by means of other processes) since they still have an ergosphere. The previous results are in full analogy with the unattainability of null temperatures for thermodynamic systems and their tendency to be in their lowest energy states. For the case \( 2b|Q| \leq 1 \), it is impossible to have \( T = 0 \) and they are expected to keep radiating, with a much more complex dynamics, until their total evaporation likely after a finite amount of time as measured by the observer who receives the radiation. Whenever charged scalar fields are taken into account, the phenomenon of superradiance could also take place for almost all physical configurations, rendering their dynamics even more cumbersome. Nevertheless, superradiance is of interest for charged non-linear black holes, since it is another energy extraction mechanism for them and would couple to the nonlinearities of the electromagnetic field.

Concerning the issue of energies radiated off due to the interaction of black holes, as we showed here with a toy model, the changes imprinted by the Einstein-Born-Infeld black holes w.r.t. their classical counterparts may be signifi-
2.9. Perspectives

cant, depending on $\alpha$ for a range of values of the fundamental parameter $b$. This could be important for gravitational wave detectors calibrated based on classical results. Besides, if it is possible to identify sources of radiation, then measurements upon such a quantity could give us information about electromagnetic interactions. We analyzed the radiated energies due to charged black hole interactions. This means that also electromagnetic radiation is always present in such processes. Identifying and analyzing this part of the radiation would give direct information about astrophysical electrodynamical processes.

We further point out that all the above conclusions remain valid even in the case the systems present a slow rotation (when the rotational parameter $a \equiv J/M$, $J$ being the total angular momentum of the system as seen by distant observers, is much smaller than the outer horizon area or the mass of hole). This is the case since the energy decomposition must be an even power of $a$, due to invariance requirements. Thereby, the previous analyses are in a sense stable against rotational perturbations.

Finally, in the astrophysical context, it does not seem possible to extract the value of the parameter $b$ by means of test particles [36] (contrasting with the case where gravitational waves were measured). The reason for this is the following. The physically relevant orbits are just outside the outer horizon, which does not change sensitively with the changes of $b$, that can be checked by Eq. (2.19) for black holes with fixed masses and charges. Nevertheless, for a fixed value of $b$ (found by non-gravitational experiments), with the use of test particles the two families of Einstein-Born-Infeld black holes discussed here could be experimentally probed, because depending on $2b|Q|$ orbits could or could not extend up to more central regions, which can play a role into the observed spectra of such systems.

2.9 Perspectives

In this chapter we tried to emphasize the need of also taking into account the mass decomposition of a charged black hole for talking about the physical aspects it could display. Conceptually speaking this is of relevance since it could give us acumen of where and how to search experimentally for charged black holes. In this regard, it would be also of interest to investigate the aspects of the electromagnetic radiation coming from the coalescence of charged black holes, because it could be much more easily observed, it would give us direct infor-
2. Energy decomposition within Einstein-Born-Infeld black holes

Further studies of the phenomenon of superradiance are of interest for charged nonlinear black holes, since it is another energy extraction mechanism for them and would couple to the nonlinearities of the electromagnetic field. It also seems that QPOs could shed a light on the illusion of black hole charges, since they talk about phenomena that take place in the innermost regions of black holes (see [1] and references therein). For simulations assessing the germaneness of nonlinear charged black holes into gravitational wave spectra, it would be of interest to scrutinize the energy decomposition for axially symmetric spacetimes, though this a toil.

2.10 Acknowledgements

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Chapter 3

Nonlinear electrodynamics in slowly rotating spacetimes and their probe through the physics of neutrinos

3.1 Introduction

In this chapter we shall take a closer look at slowly rotating and charged black holes described by nonlinear theories of the electromagnetism dependent upon both of its invariants, $L(F, G)$. The presence of rotation shall be accounted here by the search of axially symmetric solutions to the (minimally) coupled system of equations coming from General Relativity and nonlinear electromagnetism. By slow rotation we signify that our analysis shall be carried out up to first order of approximation on “$a$”, the angular momentum of the system per unit mass, assumed to be much smaller than the outer horizon ($r_+$) (or the mass) of the associated black hole. As we are going to show in the sequel, there is a generic solution to the aforementioned case. Naturally this is so due to the still high symmetry present in the system. Analyses up to the first order on “$a$” obviously shall not evidence changes in the energy budget of the spacetime, though they may have an important imprint concerning its kinematical aspects, such as the dragging of inertial frames, that we shall also analyze in detail here, and the motion of test particles in it.

It is well-known that particles endowed with spin also interact with gravity [61]. In the astrophysical scenario, perchance the most common ones are photons, though neutrinos are also produced bountifully in coalescing systems (see
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[62] and references therein), due to nuclear fusion reactions, as well as in the nucleosynthesis of heavy elements [63], playing a very important role in both cases. We shall be mainly interested here with neutrinos as seen individually as test particles in a given (charged) spacetime. These particles subsist just on superpositions of mass eigenstates: flavor states [64]. This aspect is noteworthy since it implies flavors can oscillate under convenient conditions, which may lead to observable effects [64]. This aspect was exactly the early reason for the introduction of the flavor states, in order to lead to the neutrino oscillations that could explain the theretofore anomalous neutrinos coming from the sun [65, 66], as well as the ones present in material media [67, 68]. Due to the improvements in detecting neutrinos, e.g. ice cube experiments [69, 70], detailed analyses where they take place become more pertinent. More importantly, due to the unavoidable interaction of neutrinos with gravity, such particles could give us invaluable and precise information about various astrophysical objects.

Several observations and very reasonable examples corroborate that electrodynamics take place also in the astrophysical scenario [1, 61], even possibly in gamma ray burst events [12]. Maybe the most evident one is in the early universe, where matter was just ionized or even in the form of more fundamental particles [61]. However, in the local universe, it also plays a role into the description of neutron stars, for instance, where it is known that they host surface magnetic fields in the range of $10^{10} - 10^{13}$ G [71, 72]. For white dwarves, surface magnetic fields should be in the range between $10^{6}$ and $10^{9}$ G (see e.g. [73] and references therein). Therefore, extrapolating that even black holes (one of the believed byproducts of the stellar evolution) may also be endowed with electromagnetic aspects, at least for some instants of time, seems very reasonable.

This is our main scope in this chapter: to surmise the existence of axially symmetric charged black holes and study their properties. In order to do so, we shall mainly use neutrinos to probe intrinsic spacetime aspects, specially concerning their electromagnetic fingerprints. We focus here on generalizations of the Maxwell Lagrangian, known as nonlinear electrodynamics. Such a description has as an inherent asset the insertion of desired aspects (quantum vacuum, string aspects, etc) under the classical point of view, giving insights about more fundamental analyses. The inclusion of nonlinear effects could be justified, for instance, by the very cataclysmic events and short length scales involved that may take place in the astrophysical scenario, such as the gravitational collapse of stars, that would render the electromagnetic fields, at least for some instants
of time, larger than some scales ones, where the Maxwellian Lagrangian would not lead to a very precise description to the system anymore.

Our analyses here could also be seen under the following perspective: of finding astrophysical entities and environments that could be used as tools to probe electrodynamical processes in the cosmos, quite similarly, conceptually speaking, to Crispino and collaborators’ investigations, who used scattered electromagnetic radiation to probe the charge of a black hole [74].

We work in this chapter with Gaussian units, such that the speed of the light in vacuum \((c)\) and the gravitational constant \((G_N)\) are equal to the unit. The metric signature is here chosen to be \(-2\).

### 3.2 Field equation for slowly rotating black holes

When one considers that the norm of the angular momentum per unit of mass, \(a\), of a black hole is constrained to be much smaller than its outer horizon \(r_+\) (which implies \(a/r \ll 1\), as well as \(a/M \ll 1\), \(M\) its mass), then, based on the Kerr-Newman solution [36], the Ansatz to the metric to account for nonlinear Lagrangians of the electromagnetism can be written in Schwarzschild coordinates \((t, r, \theta, \phi)\) as

\[
ds^2 = g_{00}(r)dt^2 - \frac{1}{g_{00}(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 - 2a\sin^2\theta A(r)dt d\phi, \tag{3.1}
\]

where \(g_{00}(r)\) is the solution for the associated static and spherically symmetric black hole to the theory under interest. In Eq. (3.1), \(A(r)\) is an arbitrary function to be determined from the nonlinear electromagnetic field equations.

From the minimal coupling of gravity (Einstein-Hilbert action [2]) with nonlinear theories of the electromagnetism \(L(F, G)\), and by assuming that the independent fields are \(g_{\mu\nu}\) (metric of the underlying spacetime) and \(A_\mu\) (the electromagnetic four-potential, defined by means of \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\), with \(F_{\mu\nu}\) the electromagnetic field tensor [2]), it follows that the field equation coming from the principle of least action (see, for instance, Ref. [2]) for this case reads

\[
G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \partial_\mu[\sqrt{-g}(L_F F^{\mu\nu} + L_G G^{\mu\nu})] = 0, \quad \partial_\mu(\sqrt{-g} F^{\mu\nu}) = 0, \tag{3.2}
\]

with an energy-momentum tensor built only on the nonlinear electromagnetic
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fields given by

\[ 4\pi T_{\mu\nu} = 4L_F F_{\mu\alpha} F_{\nu\beta} \tilde{g}^{\alpha\beta} - (L - G L) g_{\mu\nu}. \]  \hspace{1cm} (3.3)

We have defined in the above equations that \( L_X \) is the derivative of the Lagrangian \( L \) with respect to the invariant \( X, F = F^{\mu\nu} F_{\mu\nu}, G = F^{\mu\nu} F_{\mu\nu}, \tilde{F}^{\mu\nu} = \eta^{\mu\nu\alpha\beta} F_{\alpha\beta} / (2\sqrt{-\det g_{\mu\nu}}), \eta^{0123} = +1, \) is a totally antisymmetric tensor, \( \partial_{\mu} = \partial / \partial x^\mu \) and \( \tilde{F}^{\mu\nu} \) is its associated dual [2]. Besides, \( \det g_{\mu\nu} \) has been defined as the determinant of the metric given by Eq. (3.1). Finally, let us define the electromagnetic fields by means of: \( F_{tr} \equiv E_r, F_{t\theta} \equiv E_\theta, F_{r\phi} \equiv B_\theta \) and \( F_{\phi\theta} \equiv B_r \). Local fields are to be obtained by means of a tetrad decomposition of \( F_{\mu\nu} \) following the above-mentioned definitions. Notice from the second term of Eq. (3.2) that we are assuming our system is such that its (external) current four-vector is null.

In the spherically symmetric case, the equations for \( L(F) \) (\( G \) is zero in this case, since we are assuming the nonexistence of magnetic charges) with asymptotically flat black hole solutions are [49]

\[ g_{00}(r) = 1 - \frac{2M}{r} + \frac{2QA_0}{r} - \frac{2N}{r} \hspace{1cm} (3.4) \]

and

\[ \frac{\partial L}{\partial E_{r0}} = \frac{Q}{r^2} \hspace{1cm} (3.5) \]

with

\[ E_{r0} \equiv -\frac{\partial A_0}{\partial r} \hspace{0.5cm} \text{and} \hspace{0.5cm} \frac{\partial N}{\partial r} \equiv -Nr^2. \hspace{1cm} (3.6) \]

The constants \( M \) and \( Q \) are the total mass (total energy) and charge of the system, respectively, and are formally constants of integration. In Eq. (3.4), a gauge has been imposed such that \( A_0(r) \) is null at infinity, the same as for \( N \), guaranteeing the flatness of the solutions. Given \( E_{r0} \) and \( L = L(E_{r0}), A_0(r) \) and \( N(r) \) can be obtained by means of integration from an arbitrary radial coordinate \( r \) up to infinity.
Let us assume that the fields for the slowly rotating black holes are

\[ E_r = E_{r0} + \mathcal{O}(a^2), \quad B_r = B_{ra} a + \mathcal{O}(a^2), \]
\[ E_\theta = \mathcal{O}(a^2), \quad B_\theta = B_{\theta a} a + \mathcal{O}(a^2). \] (3.7)

By putting Eqs. (3.1) and (3.7) into Eq. (3.2), one can easily show that the only new equation arising, apart from the one in the spherically symmetric case reads

\[ 8B_{\theta a} E_{r0} g_{00} L_F + 2L A(r) \sin^2 \theta = \sin^2 \theta \left\{ \frac{A(r)(g_{00} r)'}{r^2} + \frac{1}{2} [g_{00}'' A(r) - g_{00} A''(r)] \right\}, \] (3.8)

where the prime symbol stands for the derivative with respect to the \( r \) coordinate. Since the Lagrangian \( L(F,G) \) is an at least quadratic function of the fields, then in the above equation it is implicit that \( L \) and \( L_F \) are evaluated at \( a = 0 \). From Eq.(3.8), one can immediately check that it is meaningful just if

\[ B_{\theta a} = f(r) \sin^2 \theta, \] (3.9)

where \( f(r) \) is an arbitrary function of the radial coordinate. Hence, putting the above equation into Eq. (3.8), one finally obtains

\[ 8f(r) E_{r0} g_{00} L_F + 2L A(r) = \frac{A(r)(g_{00} r)'}{r^2} + \frac{1}{2} [g_{00}'' A(r) - g_{00} A''(r)]. \] (3.10)

The equation governing the field components \( B_{ra} \) and \( B_{\theta a} \) can be obtained by the second and third terms of Eq. (3.2) and are

\[ \frac{\partial B_{ra}}{\partial r} + \frac{\partial B_{\theta a}}{\partial \theta} = 0 \] (3.11)

and

\[ \sin \theta \frac{\partial}{\partial r} \left( L_F \left[ -A(r) E_{r0} + g_{00} \frac{B_{\theta a}}{\sin^2 \theta} \right] \right) - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{L_F B_{ra}}{\sin \theta} - \frac{r^2 L_G E_{r0}}{a} \right) = 0. \] (3.12)

From Eqs. (3.9) and (3.11), it follows that

\[ B_{ra} = g(r) \sin 2\theta, \] (3.13)
which leads to the very simple relation

$$f(r) = -g'(r).$$

(3.14)

Since the Lagrangian must be an even power of the invariant $G$, it is straightforward to see that

$$L_G = -\frac{8aL_YE_{r0}B_{rd}}{r^2 \sin \theta} + O\left(\frac{a^2}{r^2}\right), \quad Y \equiv G^2.$$

(3.15)

Finally, gathering Eqs. (3.9), (3.13) and (3.15), Eq. (3.12) can be cast in the form

$$-\left\{L_F[A(r)E_{r0} + g_{00}g'(r)]\right\}' + \frac{2g(r)}{r^2}[L_F + 8L_YE_{r0}²] = 0.$$

(3.16)

Up to zeroth order, one can also put Eq. (3.4), with the help of Eq. (3.5), to the form

$$(g_{00})' = 2Lr² - 2QE_{r0} + 1.$$

(3.17)

Then, from Eqs. (3.10), (3.14) and the above one, we simply have

$$2Qg'(r)g_{00} = (-2QE_{r0} + 1)A(r) + \frac{r²}{2}[g''_{00}A(r) - g_{00}A''(r)].$$

(3.18)

Hence, we have two undetermined functions $g(r)$ and $A(r)$ and two coupled equations, Eqs. (3.16) and (3.18). As a boundary condition, for large $r$, the functions should approach their classical (clas) counterparts,

$$A(r) \rightarrow A(r)_{clas} = g_{00} - 1 = \frac{Q²}{r²} - \frac{2M}{r}, \quad g(r) \rightarrow g(r)_{clas} = \frac{Q}{r},$$

(3.19)

as it can be shown by choosing $L(F, G) = -F/4$ and solving the equations for the metric $g_{00}$, $E_{r0}$ and Eqs. (3.16) and (3.18). Observe that the aforementioned equations in general do not admit the solution $f(r) = E_{r0}(r)$, as it eventuates in the classic case [see Eq. (3.19)].

Since in general the problem set out above can just be solved numerically, it turns out to be that the $r$ variable is not convenient for this end. Numerically, it is much more suitable to use the dimensionless variable $u$, defined as $u = M/r$. 

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In terms of this variable, Eqs. (3.16) and (3.18) become

\[
\frac{d}{du} \left\{ L_F(u) \left[ A(u) \tilde{E}_{r0}(u) - u^2 g_{00}(u) \frac{d}{du} g(u) \right] \right\} + 2 g(u) [\tilde{L}_Y(u) \tilde{E}_{r0}^2(u)] = 0 \quad (3.20)
\]

and

\[
-2u^2 a g_{00}(u) \frac{d}{du} g(u) = \left[ -2a \tilde{E}_{r0}(u) + 1 \right] A(u) + \frac{1}{2} \left[ A(u) \frac{D}{du} g_{00}(u) - g_{00}(u) \frac{D}{du} A(u) \right], \quad (3.21)
\]

where we defined

\[
\tilde{E}_{r0}(u) \equiv E_{r0}(u) M, \quad \tilde{L}_Y(u) \equiv \frac{L_Y(u)}{M^2}, \quad \alpha \equiv \frac{Q}{M}, \quad \frac{D}{du} \equiv 2u \frac{d}{du} + u^2 \frac{d^2}{du^2}. \quad (3.22)
\]

In terms of \( u \), the boundary conditions for the variables \( A(u) \) and \( g(u) \) can now be readily implemented numerically as origin conditions. Concerning the integration, the variable \( u \) should run from 0 until \( u_h \), where the latter is given as the smallest solution to \( g_{00}(u_h) = 0 \).

### 3.3 Geodesics in slowly rotating nonlinear spacetimes

Forasmuch as we are interested in describing neutrinos in spacetimes related to axially symmetric nonlinear black holes, the general study of geodesics is necessary, given that these particles have no charge and hence are not sensitive to forces of electromagnetic origin. This is also so since the Dirac equation in the limit of the WKB approximation * assures that the phase part of neutrino spinors satisfy a Hamilton-Jacobi-like equation [75] and by assuming that their amplitudes vary slowly, they do not play a role for convenient spacetime distances, revealing therefore the test particle aspects of the neutrinos (quite similar to the notion of rays in optics). This approximation is assumed in our analyses in order to allow neutrinos to be treated as test particles (see Ref. [75] for further details) in the axial spacetime we have just found. We underline from the previous sentences that we are overlooking for the nonce the interaction of the B-field with the neutrino anomalous magnetic moment and spin. Upon this premise, there

*For us it means searching for oscillatory solutions to the Dirac equation whose amplitudes vary slowly when compared to their phases, the ones that we shall be interested in here
3. Nonlinear electrodynamics in slowly rotating spacetimes and their probe through the physics of neutrinos

are several ways of accomplishing such a goal. An elegant approach would be solving the associated Hamilton-Jacobi equation [2] for the spacetime given by Eq. (3.1). Nevertheless, we will follow the Lagrangian approach. We will refrain ourselves from analyzing the geodesics lying in the plane $\theta = \theta_0 = \text{const}$. For this case, the proper Lagrangian for test particles (t.p.) is [76]

$$L^{(t.p.)} = \frac{1}{2} \left[ g_{00} \dot{t}^2 - \frac{\dot{r}^2}{g_{00}} - r^2 \sin^2 \theta_0 \dot{\theta}^2 - 2aA(r) \sin^2 \theta_0 \dot{\phi}^2 \right], \quad (3.23)$$

where $\dot{x}^\mu = dx^\mu/d\lambda$, where $\lambda$ is an affine parameter along the curve followed by the test particle. From Eq. (3.1), the coordinates $t$ and $\phi$ are cyclic ones for the above Lagrangian. Hence, the quantities $p_t = E$ and $p_\phi = -l$, with $p_\mu = g_{\mu\nu} p^\nu$, $p^\mu = m \dot{x}^\mu$, $m$ the rest mass of the test particle of interest, are constants along the geodesics. This is nothing but a consequence of the existence of two killing vectors, $\zeta_0^\mu = \delta_0^\mu$ and $\zeta_3^\mu = \delta_3^\mu$, associated with Eq. (3.1), related to its time ($t$) and angular ($\phi$) symmetries, such that $p_t = \zeta_0^\mu p_\mu$ and $p_\phi = \zeta_3^\mu p_\mu$.

From the constants of motion, and the previous definitions, we have

$$\dot{r}^2 = \tilde{E}^2 - g_{00}(r) \left[ \frac{\tilde{l}^2}{r^2 \sin^2 \theta_0} - \frac{2\tilde{E} \bar{a}A(r)}{g_{00}(r) r^2} + \epsilon \right], \quad (3.25)$$

where $\epsilon = 0, 1$, according to which the geodesic is light-like or time-like, respectively. The above equation is obtained by means of the line element given by Eq. (3.1). Just for the sake of completeness, the last first integral of our analysis is $\dot{\theta} = 0$. From Eq. (3.25), one can even define an effective potential by means of $\tilde{V}^2 = \tilde{E}^2$, for $\dot{r} = 0$, [36, 77–79] which then reads

$$\tilde{V}_\pm = \frac{\bar{a}A(r)}{\sin^2 \theta_0 r^2} \pm \sqrt{g_{00}(r) \left[ \frac{\tilde{l}^2}{\sin^2 \theta_0 r^2} + \epsilon \right]} \quad (3.26)$$

One could just work with $\tilde{V}_+$, since the “symmetry rule” $\tilde{V}_- (\bar{l}) = -\tilde{V}_+ (-\bar{l})$ holds [77–79]. All the features characterizing the motion of neutral test particles
3.4 Neutrino flavor oscillation

As stated previously, neutrino flavor oscillations would take place due to the fact that neutrino flavor eigenstates $|\nu_\alpha\rangle$ are linear combinations of neutrino mass eigenstates $|\nu_j\rangle$ as (see e.g. [80] and references therein)

$$|\nu_\alpha\rangle = U_{\alpha j} \exp[-i\Phi_j] |\nu_j\rangle,$$  \hspace{1cm} (3.27)

where repeated indexes are summed over. In the above equation, the $\alpha$ index stands for the neutrino flavor eigenstates, while the $j$ one stands for the masses eigenstates. The matrix $U_{\alpha j}$ is a unitary matrix that gives the mixing between the flavor eigenstates and the mass eigenstates. Besides, $\Phi_j$ is the phase associated with the $j$th mass eigenstate. Due to the predicted small masses of all neutrinos species when compared to their relativistic energies, it is customary to approximate the neutrino energy eigenstates by the neutrino mass eigenstates and assume that the eigenvalues to the former eigenstates are all equal [80]. For curved spacetimes, $\Phi_j$ reads [80]

$$\Phi_j = \int p_{(j)\mu}dx^\mu, \hspace{1cm} (3.28)$$

where $p_{(j)\mu}$ is just to indicate the four momentum of the mass eigenstate $j$. We shall assume in the sequel that there are just two spin flavors in the system. It is well-known [81] for this case that one can introduce a mixing angle, $\Theta$, such that the transition probability from one flavor eigenstate $\alpha$ to another $\beta$ reads

$$P(\nu_\alpha \rightarrow \nu_\beta) = \sin^2(2\Theta) \sin^2\left(\frac{\Phi_{jk}}{2}\right), \hspace{1cm} (3.29)$$
where $\Phi_{jk} = \Phi_j - \Phi_k$. Whenever one is interested in neutrino propagation in spacetimes given by Eq. (3.1), after Eqs. (3.24) and (3.25) are taken into account, Eq. (3.28) can be cast into the form

$$
\Phi_j = \int dr \frac{m_j e}{r} = m_j e \int \frac{dr}{\sqrt{E^2 - g_{00}(r) \left[ \frac{l^2}{\sin^2 \theta_0 r^2} - \frac{2E l a A(r)}{g_{00}(r) r^2} + m_j^2 e \right]}}.
$$  (3.30)

Notice that Eq. (3.30) is exact and is zero for null geodesics. This is easily understood by the fact that $p_\mu dx^\mu = g_{\mu\beta} p^\beta dx^\mu \propto ds^2$, which is zero for null paths. Hence, when it is stated that null paths are taken into consideration, approximations are done such that in parts of the Eq. (3.30) properties of null geodesics are utilized. In the nonlinear case, it is momentous to bear in mind that photons do not follow null geodesics in their background spacetimes, but in the so-called effective geometries (see e.g. [82] and references therein). Therefore, the distinction between massive particles and photons in our case is paramount.

From the second expression in Eq. (3.24) we see that in general it is impossible to have $\dot{\phi} = 0$. Hence, pure radial geodesics do not exist in axially symmetric spacetimes. The origin for so is the dragging of inertial frames, also known as Lense-Thirring effect [36]. Nevertheless, approximating $\dot{\phi} \sim O(a)$ is always possible if one assumes that $l \sim O(a)$. For these particular (radial) geodesics, the effects introduced by the nonlinearities of the Lagrangians are completely washed away, since $E \gg m$, and $0 < g_{00} \leq 1$ outside the horizon. Hence, although the neutrino oscillation expression is that from special relativity, the general relativistic effect of frame dragging still persists on their trajectories.

It is instructive to find the corrections coming from the nonlinear rotation of charged black holes on the neutrino oscillations. For making it evident, let us assume that $l/r \ll E$, which is always valid for long enough distances from the holes. For this case, Eq. (3.30) reads

$$
d.\Phi_j \simeq \frac{m_j^2}{E} \left( 1 + \frac{\dot{l} a A(r)}{r^2 E} - \frac{g_{00} l^2}{2E^2 r^2} \right) dr.
$$  (3.31)

The third term of the right-hand side of the above equation is always negative. From Eq. (3.31), then one sees that the nonlinearities of the electromagnetism play a double role into the neutrino oscillations, which may lead to an increase or decrease on the amplitude of the flavor oscillation phase as compared to its
3.5. Neutrino spin precession

special relativistic counterpart.

### 3.4.1 Neutrino oscillation length

Another fundamental quantity that arises in the description of neutrino oscillations is the oscillation length [83]. Basically it estimates the length over which a given neutrino has to travel for \( \Phi_{jk} \) to change by \( 2\pi \). For the mixing angle \( \Theta = \pi/4 \), the oscillation length would be directly related to the flavor change. Therefore, for talking about oscillation lengths, the proper relativistic covariant distance is of importance [84]. For the case of the spacetimes described by Eq. (3.1), it can be cast in the form [83] (assuming that the particles involved have the same energy \( E \))

\[
L_{osc} \equiv \frac{dl_{pr}}{d\Phi_{jk}/(2\pi)} = \frac{2\pi E}{\sqrt{g_{00}(m_j^2 - m_k^2)}}, \tag{3.32}
\]

where we assumed that \( dl_{pr} \) is the infinitesimal proper distance, given by [2]

\[
dl_{pr}^2 = \left( -g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}} \right) dx^i dx^j, \tag{3.33}
\]

with \( i, j = 1, 2, 3 \). If one wants to restore the conventional units, the right-hand side of the equality in Eq.(3.32) must be multiplied by \( \hbar/c^3 \).

Hence, from Eq. (3.32) we learn that oscillation length decreases whenever \( g_{00} \) increases. This is exactly the case for nonlinear charged black holes, when compared to a Schwarzschild black hole. This means that when the black hole is charged, neutrinos will tend to oscillate more than they would when it is not charged, for each spacetime point (location). One should bear in mind, though, that out of the horizon the metric time-time component, \( 0 < g_{00} \leq 1 \), generally.

### 3.5 Neutrino spin precession

In this section we summarize the main points about neutrino flavor spin precession, also named neutrino flavor spin-flip, or neutrino-antineutrino oscillations [85], and study them in the framework of the metric given by Eq. (3.1). For point-like particles, the equations governing the spin \( S^\mu \) coupling of test particles with
3. Nonlinear electrodynamics in slowly rotating spacetimes and their probe through the physics of neutrinos

The gravitational field are \[ [61] \]

\[
\frac{DS^\mu}{d\lambda} = 0, \quad \frac{Du^\mu}{d\lambda} = 0, \tag{3.34}
\]

where \( D / d\lambda \) stands for the absolute derivative [2]. From the definition of the absolute derivative, one sees that the spin does change whenever spin connections are not null as contrary to the case of an intrinsically flat Minkowski spacetime.

A proper analysis about the spin evolution of a test particle by a local observer is done with the use of the tetrad \( (e^a_\mu) \) formalism [2]. These quantities are defined such that the local spacetime geometry is projected onto a flat spacetime via the following rules [2]

\[
g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu, \quad e^a_\mu e^b_a = \delta^a_\nu, \quad e^a_\mu e^a_b = \delta^a_b, \tag{3.35}
\]

where

\[
\eta_{ab} = \text{diag}(1, -1, -1, -1), \quad e^{\mu a} = g^{\mu\nu}e^a_\nu, \quad e^a_\mu = \eta^{ab}e^b_\mu. \tag{3.36}
\]

The tetrad decomposition of a four-vector \( C^\mu \) is defined as \( C^a = e^a_\mu C^\mu \), with the derivative of \( C_b, C_{b,c} \), instead is defined by [2]

\[
C_{b,c} = e^c_\mu \partial C_b \frac{d}{d \chi^\mu}. \tag{3.37}
\]

From \( u^a = e^a_\mu u^\mu \), with \( u^\mu = dx^\mu / d\tau \) (the four velocity of the test particle), it follows that \( S_{a,b} u^b = dS^a / d\tau \), which allows to conclude that \( S_{a,b} = \partial S^a / \partial y^b \) and \( u^a = dy^a / d\tau \), with \( y^a \) the coordinates utilized by the local observers. Hence, \( e^a_\mu = \partial y^a / \partial x^\mu \) and \( ds^2 = \eta_{ab}dy^a dy^b = g_{\mu\nu}dx^\mu dx^\nu \).

From Ref. [2]

\[
A_{\mu;\nu} e^\nu_a e^\mu_b = A_{a,b} + \eta^{cd} \gamma_{cab} A_d, \tag{3.38}
\]

where \( \gamma_{abc} \) are the Ricci rotation coefficients, defined as [2]

\[
\gamma_{abc} = e_{a\mu;\nu} e^\mu_b e^\nu_c, \tag{3.39}
\]
and by using Eqs. (3.38) and (3.39), Eq. (3.34) can be recast as

\[
\frac{dS^a}{d\tau} = G^{ab} S_b, \quad \frac{du^a}{d\tau} = G^{ab} u_b, \tag{3.40}
\]

with

\[
G^{ab} \equiv \eta^{ac} \eta^{bd} \gamma_{cde} u^e. \tag{3.41}
\]

In virtue of the antisymmetry of $\gamma_{abc}$ on its first two indexes [2], it follows that $G^{ab}$ is an antisymmetric tensor. Hence, it can be decomposed into “electric” and “magnetic” parts, $E^G$ and $B^G$, respectively, quite in a similar way as it is done with the electromagnetic tensor. In other words,

\[
E^G_i \equiv G^0_{0i}, \quad G_{ij} \equiv -\epsilon_{ijk} B^G_k, \tag{3.42}
\]

with $\epsilon_{ijk}$ being a totally antisymmetric tensor such that $\epsilon_{123} = 1$. We have used the convention that the Latin indexes that run from zero to three are the ones from the beginning of the alphabet ($a, b, c, ...$), while those that run from one to three are the ones from the middle of the alphabet ($i, j, k, ...$).

In general, neutrinos have a nonzero velocity with respect to a tetrad defined at a given point of the spacetime. Thence, it can always be defined a “locally comoving frame”, where in the latter the particle is instantaneously unmoving. In this frame, $s^a = L^i \delta^a_i$ and $\bar{u}^a = \delta^a_0$. Thus, by using the Lorentz transformations to connect both systems, one ends up with the relation [85]

\[
s^a = \left[ L \cdot u, L + \frac{u (L \cdot u)}{1 + u^0} \right], \tag{3.43}
\]

where $u^0$ and $u$ are the temporal and the spatial components, respectively, of the comoving frame with respect to the inertial one, or the four-velocity of the particle in this reference system. By substituting Eq. (3.43) in Eq. (3.40) [it is important to use both equations], and taking it into account Eq. (3.42), one arrives at

\[
\frac{dL}{d\tau} = [L \times G], \quad G \equiv \left( B^G + \frac{E^G \times u}{1 + u^0} \right). \tag{3.44}
\]
3. Nonlinear electrodynamics in slowly rotating spacetimes and their probe through the physics of neutrinos

Then, it is an elementary task to verify that the spin \( \mathbf{L} \) of the particle precesses about the vector \( \mathbf{G} \).

From now on, we shall be interested in applying the above formalism for the case of the intrinsic (quantum) spin of neutrinos moving in spacetimes given by Eq. (3.1). To start with, as suggested by Eq. (3.1), for local measurements, we shall choose the tetrad

\[
\begin{align*}
e^0_\mu &= \left( \sqrt{g_{00}}, 0, 0, -\frac{aA(r) \sin^2 \theta}{\sqrt{g_{00}}} \right), \
e^1_\mu &= \left( 0, \frac{1}{\sqrt{g_{00}}}, 0, 0 \right), \\
e^2_\mu &= (0, 0, r, 0), \
e^3_\mu &= (0, 0, r \sin \theta).
\end{align*}
\]  

(3.45)

Just for the sake of completeness, the corresponding inverse tetrad reads

\[
\begin{align*}
e^0_\mu &= \left( \frac{1}{\sqrt{g_{00}}}, 0, 0, 0 \right), \
e^1_\mu &= (0, \sqrt{g_{00}}, 0, 0), \\
e^2_\mu &= (0, 0, \frac{1}{r}, 0), \
e^3_\mu &= \left( \frac{aA(r) \sin \theta}{rg_{00}(r)}, 0, 0, \frac{1}{r \sin \theta} \right).
\end{align*}
\]  

(3.46)

It can be easily checked that the properties given by Eq. (3.35) hold for the above tetrad up to the first order on “\(a/r\)”, as internal consistency demands. For the aforesaid tetrad, we now present the nonvanishing Ricci rotation coefficients for Eq. (3.1). They follow from Eq. (3.39) as

\[
\begin{align*}
\gamma^{010} &= -\frac{g_{00,r}}{2\sqrt{g_{00}}}, & \quad \gamma^{013} &= -a \sin \theta [A(r)g_{00,r} - g_{00}A(r), r] \\
\gamma^{023} &= aA(r) \cos \theta \frac{1}{r^2 \sqrt{g_{00}}}, & \quad \gamma^{031} &= -\gamma^{013}, \\
\gamma^{032} &= -\gamma^{023}, & \quad \gamma^{122} &= -\frac{\sqrt{g_{00}}}{r}, \\
\gamma^{130} &= -\gamma^{013}, & \quad \gamma^{133} &= \gamma^{122}, \\
\gamma^{230} &= -\gamma^{023}, & \quad \gamma^{233} &= -\frac{\cos \theta}{r \sin \theta}.
\end{align*}
\]  

(3.47)

In the above equations we have defined \(C, r \equiv \partial C/\partial r\) for a given function \(C(r)\). From the geodesic motion of test particles, it follows that

\[
\dot{u}^a = \left[ \sqrt{g_{00}} \dot{t} - \frac{aA(r) \sin^2 \theta}{\sqrt{g_{00}}} \dot{\phi}, \frac{\dot{r}}{\sqrt{g_{00}}}, \dot{\theta}, \dot{r} \sin \theta \dot{\phi} \right].
\]  

(3.48)
3.5. Neutrino spin precession

From Eqs. (3.44), (3.42), (3.41) and (3.47), and by assuming that the orbits lie in the plane \( \theta = \text{const.} \), so that \( \dot{\theta} = 0 \), one obtains the “electric” component of tensor \( G_{ab} \) in the form

\[
E^G = \left[ -\frac{g_{00,r}}{2} + \frac{aA(r)}{2} \frac{r^2}{\sqrt{g_{00}}} \sin^2 \theta_0 \phi + \frac{aA(r)}{r \sqrt{g_{00}}} \sin \theta_0 \cos \theta_0 \phi \right. \\
\left. \frac{a \sin \theta_0 \left[ A(r) g_{00,r} - g_{00} A(r), r \right]}{2rg_{00}^3} \right],
\]

(3.49)

and the “magnetic” component

\[
B^G = \left[ \cos \theta_0 \phi + \frac{aA(r)}{r^2 \sin \theta_0} - \sqrt{g_{00}} \sin \theta_0 \phi \right. \\
\left. - \frac{a \sin \theta_0 \left[ A(r) g_{00,r} - g_{00} A(r), r \right]}{2r \sqrt{g_{00}}} \right].
\]

(3.50)

Let us investigate now circular orbits under the condition that \( \dot{r} = 0 \) and \( \partial \tilde{V} / \partial r = 0 \). From the critical points of the effective potential \( \tilde{V} \), one has that it implies

\[
P^2 = \frac{g_{00,r}^2 r^2 \sin^2 \theta_0}{2g_{00} - g_{00}, r}.
\]

(3.51)

From Eq. (3.25), taking into account the above equation, the \( p^0 \) component of the 4-vector momentum

\[
\tilde{E}_+ = \frac{1}{\sqrt{2g_{00} - g_{00}, r}} \left[ \sqrt{2g_{00}} - aA(r) \sin \theta_0 \sqrt{g_{00}, r} \sqrt{r} \right].
\]

(3.52)

In the above expression for the energy we just took its positive root, since again \( \tilde{E}_+ (\tilde{I}, r) = -\tilde{E}_- (\tilde{I}, r) \) (see e.g. [78]). For the remainder first integrals, we have

\[
\dot{i} = \sqrt{\frac{2}{2g_{00} - g_{00}, r}} \dot{\phi} = \frac{1}{\sqrt{2g_{00} - g_{00}, r}} \left[ \sqrt{\frac{g_{00}, r}{r}} \frac{1}{\sin \theta_0} - \frac{\sqrt{2} aA(r)}{r^2} \right].
\]

(3.53)

For circular orbits in the equatorial plane, it is not difficult to verify that from Eqs. (3.44), (3.49) and (3.50) the angular velocity of the precession is \(|G| \dot{\theta} \).

Therefore, due to the symmetries of the spacetime, we have that the probability of having a spin-flip (s.f.) for neutrinos in this slowly rotating and charged
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spacetime is then

\[ P_{s.f.}(\tau) = \sin^2(|G|\tau), \tag{3.54} \]

where \( \tau \) is the proper time associated with local observers. We remember that in the above equation it is assumed that the spin of the neutrino is initially anti-parallel to its momentum vector.

3.6 Neutrino oscillations and spin-flip for the Born-Infeld Lagrangian

We now study neutrino spin-flip and neutrino oscillations for Lagrangian put forward by Born and Infeld in the 1930’s. It can be written as [5]

\[ L_{B,I} = b^2 \left[ 1 - \sqrt{1 + \frac{F^2}{2b^2} - \frac{G^2}{16b^4}} \right]. \tag{3.55} \]

In the above Lagrangian, \( b \) represents the scale field and it sets out the upper limit for the electric field when magnetic aspects are absent from the problem under interest. At this point it is noteworthy to make the following comment: It was recently shown [50, 51] that the \( b \) proposed by Born and Infeld is not able to reproduce the energy spectrum of the hydrogen atom, both in nonrelativistic and relativistic quantum mechanics frameworks. A value much larger than that one predicted under the unitary viewpoint is required, although a definite value has not been obtained. This fact makes the direct probe of the Born-Infeld Lagrangian even subtler, due to the present difficulty in getting hyper-high electromagnetic fields in laboratory. In an attempt to possibly overcoming such a difficulty, an analog model for assessing Eq. (3.55) has been put forward in the realm of fields that exceed the scale field [86]. There, such analog model was constructed by means of nonlinear media that exhibit unidirectional light propagation aspects. Apart from the aforementioned problematic issue, hereafter we treat such a scale field as a free parameter.

We start our analyzes with the behavior of the metric given by Eq. (3.1) and the electromagnetic fields for a slowly rotating axially symmetric spacetime in the scope of the Born-Infeld Lagrangian. Such an analysis is important for it
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gives the range of the parameters where considerable departures from the static nonlinear counterpart takes place. To this end, we note that it is already known that the Born-Infeld Lagrangian leads to an exact solution to Einstein’s equations in spherically symmetric case [18] (the seed for slowly rotating analyses), and it can be cast as

\[
g_{00} = 1 - 2u + \frac{2}{3u^2} (bM)^2 \left( 1 - \sqrt{1 + \frac{\alpha^2 u^4}{(bM)^2}} \right) \\
+ \frac{2\alpha^2 u}{3} \sqrt{\frac{bM}{|\alpha|}} F \left[ \arccos \left( \frac{bM - |\alpha| u^2}{bM + |\alpha| u^2} \right), 1/\sqrt{2} \right],
\]  

(3.56)

where \( F[\ldots, 1/\sqrt{2}] \) is the elliptic function of first kind [53]. In Figs. 3.1, 3.2 and 3.3 we show the numerical integration of Eqs. (3.20) and (3.21), for some selected values of \( \alpha = Q/M \), with \( bM = 0.017 \), for the components of the polar and radial magnetic fields and the metric functions \( A(r) \), vis-à-vis the Maxwellian Lagrangian. One can perceive from the plots that the distinctness start to become more accentuated the closer the horizon is approached. Near that border there seems to exist a region where the magnetic field experiences a sharp deviation w.r.t. Maxwellian one. The selected value of \( bM \) for the above mentioned plots was picked out such that

\[
bM < \frac{9}{|\alpha|^3 F^2 \left[ \pi, \frac{1}{\sqrt{2}} \right]} \approx 0.654 \frac{0.654 \cdot |\alpha|}{|\alpha|^3}.
\]  

(3.57)

This means that the associated black holes just exhibits one horizon, not a degenerated one [87, 88]. Consequently, \( g_{00} \) for this case is a monotonic function of the radial coordinate. We ought to recall that Eq. (3.57) does not have a classical limit, which could be obtained formally when \( b \) tends to infinity. Whenever the inequality in Eq. (3.57) occurs, one should expect significant deviations from the standard classical solution, as it can be seen again in Figs. 3.1, 3.2 and 3.3 for some values of the parameter \( \alpha \). For the case where Eq. (3.57) is not valid, Einstein-Born-Infeld black holes are the generalization of their Einstein-Maxwell counterparts. Naturally, when naked singularities are present, the aforementioned solutions can be considerably different, especially close to the singularity. There the fields coming from Born-Infeld Lagrangian are minute when compared to their associated classical ones, due to the regularity of the former
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Figure 3.1: Ratio of the off-diagonal term $A(r)$ in Eq. (3.1) coming from Born-Infeld (BI) Lagrangian and the Maxwell (Ma) Lagrangian for selected values of the parameter $\alpha$ (each number associated with a curve) with $bM = 0.017$. The value of $bM$ was chosen such that Eq. (3.57) is satisfied for all selected $\alpha$. In this case, the associated black holes do have just one horizon and do not have classical counterpart.

Figure 3.2: Ratio of the the polar magnetic fields for the same theories, selected values of $\alpha$ and meaning of the curves as in Fig. 3.1.

Lagrangian at the singularity.

We should emphasize that on the light of the black hole energy decomposition scheme in nonlinear electrodynamics [49], when does it apply, that in comparing nonlinear black holes with their linear counterparts (Maxwellian La-
3.6. Neutrino oscillations and spin-flip for the Born-Infeld Lagrangian

We now progress with the analyses of the neutrino oscillations within the Born-Infeld Lagrangian. We primarily want to compute the oscillation length in the case where just two neutrino flavors are considered. This is easily accomplished by the use of Eqs. (3.32) and (3.56). In Fig. 3.4, we plot the ratio of the oscillation lengths for selected values of $\alpha$, the charge to mass ratio, with a fixed value of $bM$ that satisfies Eq. (3.57), assuming that the oscillating particles do have the same energy $E$. Notice that in all the cases the neutrino oscillation lengths are smaller in the scope of the Maxwellian Lagrangian than in the Born-Infeld one. This could be understood physically by the effective screening of the total charge due to the nonlinearities. Either theory, though, leads to smaller neutrino oscillation lengths than the ones for Schwarzschild black holes. Let us now take a closer look into the spin-flip for the Born-Infeld Lagrangian. We restrain ourselves to the spherically symmetric spacetimes, by analyzing circular orbits [we postpone our studies for axially symmetric spacetimes given by the line-element of Eq. (3.1) to the section about the gyroscope precessions, since there the changes brought by the axial symmetry of the spacetime become manifest].

For the spherically symmetric case, from Eqs. (3.48), (3.49), (3.50) and (3.53), the angular velocity of precession of the neutrino spin, Eq. (3.44), can be simplified.
3. Nonlinear electrodynamics in slowly rotating spacetimes and their probe through the physics of neutrinos

\[
(L^{Ma}/L^{BI})^2
\]

Figure 3.4: Maxwell to Born-Infeld black holes oscillation lengths ratio for selected values of \( \alpha \) for a fixed value of \( bM \) satisfying Eq. (3.57).

to the form

\[
G = \Omega_{s.f.} = \hat{\theta} \sqrt{\frac{g_{00},r}{2r}}.
\]  

(3.58)

We highlight that the main facets of the frequency of spin-flip depend upon the choice of the parameter \( bM \). Whenever \( bM \gg 1 \), Eq. (3.56) gives us

\[
g_{00} = 1 - 2u + \alpha^2 u^2 - \frac{\alpha^4 u^6}{20(bM)^2} + \mathcal{O}[(bM)^3].
\]  

(3.59)

This means that the Einstein-Born-Infeld theory leads to the lessening of the metric when compared to the Reissner-Nordström metric. Therefore, \( bM \gg 1 \) leads to an augment of the frequency of spin-flip, Eq. (3.58), when compared to the classical case. One also perceives from Eq. (3.59) that, like in the classical case, \( \Omega_{s.f.} \) diminishes with the increase of \( \alpha \).

Whenever \( bM \ll 1 \), we have that Eq. (3.56) can be approximated to

\[
g_{00} = 1 - 2u + \frac{2}{3} \alpha^{3/2} \sqrt{bM} \mathcal{F} \left[ \pi, \frac{1}{\sqrt{2}} \right] u + \mathcal{O}(bM).
\]  

(3.60)

The comparison of the case \( bM \ll 1 \) with the Reissner-Nordström solution (same \( \alpha \)) is not immediate, though. For a given \( \alpha \), if \( u \leq \mathcal{F}[\pi, 1/\sqrt{2}] \sqrt{bM/\alpha}/3 \), then it can be shown that \( \Omega_{s.f.}^{bM \ll 1} \geq \Omega_{Max}^{s.f.} \). For a given \( u \), the frequency of spin-flip
3.6. Neutrino oscillations and spin-flip for the Born-Infeld Lagrangian

increases with the decrease of $\alpha$. Notwithstanding, in both cases apropos of $bM$, the frequency of spin-flip for the case the charge is absent is larger than the case it is not. We exemplify some of the aforementioned scenarios in Figs. 3.5 and 3.6.

![Figure 3.5](image1)

Figure 3.5: (color online). Transition probability of neutrino spin-flip, Eq. (3.54), for selected values of $\alpha$ and $bM$, for circular orbits at $u = 0.24$.

![Figure 3.6](image2)

Figure 3.6: (color online). Transition probability of neutrino spin-flip, Eq. (3.54), for selected values of $\alpha$ and $bM$, for circular orbits at $u = 0.24$. Notice that in this case, $u \leq \mathcal{F}[\pi, 1/\sqrt{2}]\sqrt{bM/\alpha}/3$, and so the spin-flip frequency in the Born-Infeld theory is larger than its Maxwellian counterpart.

In Fig. 3.7 we plot the Born-Infeld frequency as a function of the radial distance for selected values of the parameter $\alpha$ and a fixed $bM$ that violates the
inequality given by Eq. (3.57). It can be verified that the Born-Infeld frequency
departures from the Maxwell case are too small to be appreciated numerically.
Nevertheless, in terms of the spin-flip probability, the deviations become clear,
due to cumulative effects. Naturally, the values of the radial coordinate where
the square of the spin-flip frequency becomes negative means that such a pre-
cession does not take place. It must be borne in mind that when one compares
the spin-flip frequencies in Born-Infeld theory with the Maxwell one, one is im-
plcitly comparing black holes with the same total mass, but different irreducible
masses [86].

![Figure 3.7: Neutrino spin-flip frequency for the Born-Infeld Lagrangian for selected values of $a$ and a fixed $bM$ that does not satisfy Eq. (3.57). The departures from the the Maxwell Lagrangian in the aforementioned case are too small to be appreciated numerically. They are evident just when cumulative effects are present, such as the spin-flip probability.](image)

### 3.7 Nonlinear precessions

Next, we shall deduce the angular velocity of precession of gyroscopes placed
at a point of the spacetime defined by Eq. (3.1). The formalism is the same as
the one for spin-flip described previously. Now, however, we place particles at
rest (with respect to local observers) at given spacetime points and sift their spin
precession uniquely due to the “rotation of the spacetime”. This is nothing more
but the effect of dragging of inertial frames [36]. It could be seen as another effect
to probe Eq. (3.1) in the context of nonlinear theories of the electromagnetism.
3.7. Nonlinear precessions

In the spherically symmetric and static case such an analysis does not take place and spin-flip effects can only be made explicit for moving particles. For axially symmetric spacetimes this is not the case, for in a certain sense the spacetimes themselves do so. As it can be seen in Refs. [36, 85] and from Eq. (3.34) when it is expanded in terms of connections, the components of the angular velocity $\Omega^k$ of precession of a gyroscope with respect to a given tetrad can generically be calculated by means of the relation [36]

$$\epsilon_{ijk} \Omega^k = -\Gamma_{ij0}$$

(3.61)

where the tetrad decomposition of the Christoffel symbol is defined by the expression

$$\Gamma_{ijk} = \epsilon^\mu_i \epsilon^\nu_j \epsilon^\beta_k \Gamma_{\mu\nu\beta}$$

and

$$\Gamma^\mu_{\nu\beta} = \frac{1}{2}(\partial_\beta g_{\mu\nu} + \partial_\nu g_{\mu\beta} - \partial_\mu g_{\nu\beta}).$$

(3.62)

Notice that the sign present in Eq. (3.61) does not show up in Ref. [36] due to fact that we chose a different signature to the metric.

Subsequent to uninvolved calculations, one obtains the following results for the $\Omega^k$ components of the metric related to Eq. (3.1) and the tetrad given by Eq. (3.45)

$$\Omega^\rho = -\frac{aA(r) \cos \theta}{\sqrt{g_{00}r^2}},$$

$$\Omega^\theta = \frac{a[g_{00}\partial_r A(r) - A(r)\partial_r g_{00}] \sin \theta}{2g_{00}r},$$

$$\Omega^\phi = 0.$$

(3.63)

As we have already advanced, the above local angular precession can also be got (apart from a sign due to the vector product order chosen in Ref. [36]) from the spin-flip formalism by assuming there $u^a = \delta^a_0$. It can be easily perceived that the known classical result for the angular velocity of precession of a gyroscope at large distances from the black hole [36] is compatible with the above prescription. For checking so, one should take $g_{00} = 1$ and $A(r) = A(r)_{clas}$ [see Eq. (3.19)] in Eqs. (3.63) and choose the orientation of the coordinate axes such that the angular momentum of the black hole $J$ is given by $J = |J|\hat{z}$.

In Fig. 3.8 we plot the Born-Infeld to Maxwell ratio of the radial precession
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frequency, as appearing in Eq. (3.63), for various values of $\alpha$ with $bM = 0.1$ and an arbitrary $\theta$. One sees that there is a minute change of $\Omega'$ coming from the aforementioned theories. In Fig. 3.9, the plot illustrates the polar angular frequency component in Eq. (3.63). For this case the Born-Infeld theory could deviate considerably from the Maxwell one. This is particularly the case for large values of $\alpha$ and distances close to their associated outermost horizons.

Figure 3.8: Born-Infeld to Maxwell ratio of $\Omega'$ appearing in Eq. (3.63) for various values of $\alpha$ (numbers on the curves) and $\theta$, with $bM = 0.1$.

Figure 3.9: Born-Infeld to Maxwell ratio of the polar frequency component $\Omega^\theta$ appearing in Eq. (3.63) for various values of $\alpha$ (a number associated with each curve) and $\theta$, with $bM = 0.1$. 
Hence, if measurements could be done concerning the polar component of the precession of gyroscopes in the environs of the horizon a slowly rotating black hole (where we expect the precession should be more relevant), then one would be directly probing intrinsic properties of such spacetime, as well as of electromagnetism, this way overcoming the current experimental difficulty of probing it on terrestrial and atmospheric laboratories.

### 3.8 Some simple estimations

In the calculations of this work, we assumed that $a/r_+ \ll 1$, which is equivalent to considering $a/r \ll 1$ or $a/M \ll 1$. In order to give an astrophysical application to this approximation, consider the following system: a star (e.g. neutron star) whose charged nucleus is ongoing a gravitational collapse which has an oppositely charged crust that is left behind. Conjecture that such a charged core spins with constant angular velocity, whose norm we take as $\Omega$. Besides, take its mass to be $M$. If the system rotates slowly, in first order of approximation we could take it as spherically symmetric. Therefore, its angular momentum could be estimated as proportional to $MR^2\Omega$, where we took its radius (in general dynamic) to be $R$. Let us check the meaning of $a/r \ll 1$ for the above mentioned astrophysical star. Such a condition, taking into account general relativistic requirements, can be cast as

$$\Omega R \ll c,$$

which is naturally the same as in Ref. [89]. Close to its horizon, here taken to be of the order of the Schwarzschild one, $R \approx 2MG/c^2$. Therefore, $a/r \ll 1$ in such a case is equivalent to $\Omega \ll c^3/(2MG) = 10^5(M_\odot/M)$. Let us assume that a neutron star has an angular velocity around $10^3$Hz. Actually, so far, the fastest pulsar measured has around 720Hz ([90] and references therein). Hence, our description could have a relevance even for some neutron stars. For an ordinary stable Neutron Star, $M \approx M_\odot$ and $R_\star \approx 10^6 cm$, thence its Schwarzschild horizon is $R_{schw} \approx 10^5 cm$. Let us posit that during the dynamical collapse process (triggered by reasons we are not interested in investigating here) of the core of an star satisfying Eq. (3.64) that its crust has remained at $R_\star$. Then the latest neutrinos emitted by the star could travel up to $10R_{schw}$ before interacting with the crust. In this region, nonlinear effects could play a role. Assume, just as an
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example, that for the charged core $\alpha = 0.5$ and $bM = 0.0017$. Take for the radial coordinate the value $u = M/r = 0.45$ for the neutrino emission. Therefore, for this case,

$$g^{BI}_{00}(0.45) \approx 0.11 \quad \text{and} \quad g^{Ma}_{00}(0.45) \approx 0.15.$$  (3.65)

From Eq. (3.32), when brought to usual units, we have

$$L_{osc}(cm) = \frac{123 E/MeV}{\sqrt{g^{00}(\Delta m/eV)^2}}.$$  (3.66)

For the case $\Delta m \approx 0.01eV$, and $E \approx MeV$, we have that $L_{osc}^{BI} \approx 3.7 \times 10^6cm$, while $L_{osc}^{Ma} \approx 3.2 \times 10^6cm$. Thereby, $(L_{osc}^{Ma}/L_{osc}^{BI})^2 \approx 0.7$, as it can be checked in Fig. 3.4. For this case, there is change of around 15% in the oscillation lengths concerning the Born-Infeld and Maxwell Lagrangians. Notice that this example gives an oscillation length of the same order of distance as the charged crust lies. Therefore, the different theories chosen could dramatically change the fate of the charged crust left behind, as well as the envelope surrounding such star. Since the number of neutrinos emitted in a neutron star system is colossal, even small changes on the neutrino oscillation lengths impinged by nonlinear Lagrangians could play an important role into the evolution of the aforementioned system.

3.9 Conclusions

In this chapter we first solved generically Einstein’s equations for slowly rotating black holes minimally coupled to nonlinear lagrangians of the electromagnetism dependent upon its two local invariants. We used neutrinos (in the WKB approximation, where the sought solutions for the Dirac equation are oscillatory with amplitudes varying slowly when compared to the their associated phases) to probe some of the aspects of these spacetimes, which may be invaluable tools to discern charged and uncharged black holes. The major departures from the classical case concerning the magnetic fields, the off-diagonal metric term, the precession of gyroscopes, the spin-flip, the neutrino flavor oscillation, etc. will occur just near the outer horizon of a nonlinear black hole because it is there that the spacetime properties alter more pronouncedly. Besides, kinematical effects such as precessions (to be measured with gyroscopes) could be of relevance
in order to distinguish nonlinearities present in charged black holes, as well as experiments that take into account magnetic fields (asymptotically dipolar ones).

3.10 Perspectives

It would be of interest to check if the effects we analyzed here could have a relevance in the neutrino physics associated with supernova events and all of its byproducts. The reason for this is simple. During the process of gravitational collapse of a star, neutrinos are produced due to nuclear fusion reactions and released. If there is a stage during the collapse where the system becomes charged, especially near its gravitational radius, then the neutrino dynamics should be affected by it, what would also change their interaction with the star remnants, related to what a supernova is intimately related to. If the answer to the above query is positive, then it would also be very interesting to solve Einstein’s equations at least to second order on the rotational parameter, in order to scrutinize the energy budget of the system. The ultimate goal is naturally to solve the problem nonperturbatively. Our calculations suggest that magnetic fields from nonlinear electrodynamics should deviate more pronouncedly apropos of their Maxwellian counterparts. Therefore, subsequent investigations on the probe and nature of charged black holes should focus more closely on this aspect.

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Chapter 4

Thin shell stability in stars with phase transitions

4.1 Introduction

In the Newtonian theory of gravity, it is well-known the procedure to deal with a given surface of discontinuity. One should simply impose the continuity of the gravitational potential across it and the discontinuity of the gravitational field comes from its surface mass. Such boundary conditions can be easily concluded from the field equation (linear), given the precise notion of reference systems. Nevertheless, in general relativity the problem is much more involved, due to the nonlinearity of the field equations and also due to the principle of general covariance [91, 92]. For the elucidation of the problem and references, see Refs. [91–93]. Its solution consists of imposing specific boundary conditions to the induced metric tensor and the extrinsic curvature [6, 36] on a hypersurface splitting spacetimes in a manifestly covariant way. Such a procedure is generically called either the thin-shell formalism or the Darmois-Israel formalism. It can be applied to a variety of astrophysical scenarios, such as thin-shells (with and without charge) interacting with black holes even to the elucidation of nontrivial effects present in quantum field theory in curved spacetimes (see e.g. [88, 93–101]). It was already shown that such a formalism is equivalent to searching for distributional solutions to Einstein’s equations [102]. For details about the derivation of the hypersurface conditions in this case, see Ref. [6].

The thin shell formalism would be meaningful for stars that are expected to display transitional core-crust layers much smaller than their characteristic
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sizes [1], endowed generically with nontrivial quantities, such as surface tensions and surface energy densities. Common examples of believed stratified systems would be Quark Stars [1] and Neutron stars with all fundamental interactions [1, 103]. We highlight that the Darmois-Israel formalism gives the nontrivial properties of transitional layers fully taking into account general relativity, but just under the macroscopic point of view, as one would indeed expect in the light of the intimate relationship of general relativity with thermodynamics [57]. More generally, the thin shell formalism would be the proper formalism for approaching any gravitational system that presents discontinuous behaviors in their physical parameters.

For us a star with discontinuities in its parameters is seen as the match of given spacetimes split by hypersurfaces that possess nontrivial properties, which in turn lead to nontrivial dynamics, imputable to general relativity. The dynamics of a shell does not imply compression or dilution of the matter contents in the spacetimes, since it is induced by fixed geometries associated with its adjacent spacetimes. When a shell moves, one phase (thick layer) tends to “swallow” the other, leading the shell to “absorb” degrees of freedom to it [104], in order to compensate the changes of the energy-momentum in the adjacent phases and on itself. All of these aspects shall be clarified subsequently.

Dully taking into account surface degrees of freedom in a stratified astrophysical system is of cardinal importance because they are the quantities that guarantee that combinations of solutions to general relativity are themselves also solutions to this theory. These combinations of solutions are tacitly assumed when one tries to match solutions. This is far from trivial, since the theory is nonlinear.

Our intention in this chapter is to analyze under the scope of the thin shell formalism the stability of transitional thin layers for some models of stars against radial displacements of the aforesaid layers. These latter perturbations are the outcome of surface ones only. They are assumed to occur adiabatically and therefore will propagate with the speed of the sound [105, 106], to be properly inspected for these continua systems. In order to evidence the intrinsic role the surface degrees of freedom (surfaces tensions and energy densities) play in systems with discontinuities, we shall work in the context where just surface perturbations are present. This analysis is deemed to be important since instabilities of the transitional shells in stars (triggered by physical mechanisms we shall not investigate) may be a direct sign of instabilities of the whole systems, even in
4.2 Thin shell formalism in the spherically symmetric case

the absence of perturbations in the glued spacetimes. Besides, it would pave the way for stability analyses where the phases of the system are also perturbed.

Units in this chapter are such that \( G = c = 1 \). Unless it is not otherwise stated, we work with a signature +2. Greek indexes run from zero to three, while Latin ones run from zero to two.

4.2 Thin shell formalism in the spherically symmetric case

The Darmois-Israel formalism can be enunciated as follows [93]. Consider two pseudo-Riemannian manifolds, \( M_+ \) and \( M_- \), endowed with metric fields \( g^+_{\alpha\beta}(x^\mu_+) \) and \( g^-_{\alpha\beta}(x^\mu_-) \), with respect to two independent coordinate systems \( x^\mu_+ \) and \( x^\mu_- \). Assume that such manifolds have boundaries \( \Sigma_+ \) and \( \Sigma_- \). If such boundaries are identified, then a natural match of manifolds can be done, where the resultant manifold, \( M \), is the union of the aforementioned ones. Call such a common hypersurface as \( \Sigma \). Assume that a coordinate system \( y^a \) is adapted to it. As any hypersurface, it represents a constraint of the spacetime coordinates, here defined as \( \Psi^\pm(x^\mu_\pm) = 0 \). It can also be written in the parametric form \( x^\mu_\pm = x^\mu_\pm(y^a) \).

A natural basis can be defined on \( \Sigma \) by means of tangent vectors to its coordinate curves. Define the components of it as \( e^\mu_{\pm a} = \partial x^\mu_\pm / \partial y^a \). With these basis vectors, one can easily find the induced metric on \( \Sigma \), \( h_{ab} \), when the spacetime line element is constrained to such a hypersurface. From our previous reasoning, it is clear that such an induced geometry must be unique. De facto, this is the first boundary conditions one has to impose in order to have a well-defined pseudo-Riemannian manifold made out of the glue of two other ones. This can be viewed as the general relativistic generalization of the continuity of the gravitational potential across a surface in the Newtonian theory of gravity. The geometry of \( \Sigma \) is

\[
h_{ab} = g^\pm_{\mu\nu} e^\mu_{a\pm} e^\nu_{b\pm}. \tag{4.1}\]

Observe that such an induced geometry is independent of the coordinate systems \( x^\mu_\pm \) utilized.
4. Thin shell stability in stars with phase transitions

The normal unit four-vector to $\Sigma$ is defined such that [6]

$$n_{\mu} \doteq \frac{\epsilon \partial_{\mu} \Psi^\pm}{|g_{\alpha \beta} \partial_{\alpha} \Psi^\pm \partial_{\beta} \Psi^\pm|^2}, \quad \text{(4.2)}$$

where $\partial_{\mu} \doteq \partial/\partial x^\mu$ and it is tacit that $x^\mu$ is actually a shortcut to $x_\pm^\mu$. Besides, $n^\alpha n_\alpha = \epsilon = \pm 1$, depending on the nature of the hypersurface. Notice that the case where $n_\alpha$ is null is not contemplated here. Equation (4.2) also guarantees that $n^\mu \partial_{\mu} \Psi > 0$.

Another important quantity for characterizing a hypersurface is its extrinsic curvature, defined as

$$K_{ab} \doteq n_\alpha e^\alpha_a e^\beta_b, \quad \text{(4.3)}$$

where we did not put the “$\pm$” labels just not to overload the notation. One sees that the extrinsic curvature components are the tetrad components of the tensor $n_{\mu\nu}$, thence a tangent vector [6].

Let us define the jump of a given tensorial quantity across $\Sigma$ as $[A]^\pm \doteq A(x^+)_{|\Sigma} - A(x^-)_{|\Sigma}$. It is tacit in the previous definition that $A(x^\pm)_{|\Sigma}$ stand for a given quantity $A$ being evaluated in arbitrary points belonging to the disjoint regions implied by $\Sigma$ and then taken the limit when they tend to an arbitrary point on $\Sigma$. The energy-momentum tensor on $\Sigma$ coming from general relativity, $S^{\mu\nu}$, can be expressed in terms of the jump of the extrinsic curvature by means of the Lanczos equation [91, 102]

$$S_{\mu\nu} = S_{ij} e^i_\mu e^j_\nu, \quad S_{ij} = -\frac{\epsilon}{8\pi} ([K_{ij}]^+ - h_{ij}[K]^+), \quad \text{(4.4)}$$

where $K = h^{lm} K_{lm}$. This is the second boundary conditions one should impose. Clearly, it is the (manifestly covariant) generalization of the jump the gravitational field experiences in the classical theory. Note that $S^{\mu\nu}$ also contributes to the energy-momentum tensor of the matched manifold as $S^{\mu\nu} \delta(\Psi)$, $\delta$ the Dirac delta, and this is essential to guarantee the Bianchi identities for $M$ [102]. The evolution equation to $S_{ij}$ is [6]

$$S_{a b}^\mu = -\epsilon [T_{a\beta} e^a_\mu h^\beta]_{|\Sigma}, \quad \text{(4.5)}$$
4.2. Thin shell formalism in the spherically symmetric case

where \( A_{ab|c} \equiv A_{\mu
u\alpha}\epsilon_{\alpha}^{\mu}e_{\beta}^{\nu}e_{\gamma}^{\alpha} \) [6]. One sees from the above equation that thin shells in continuous systems are naturally subjected to current fluxes, generically given by \( j_a = T_{a\beta}e^\alpha_\beta n^\beta \) [6]. Notice that vacuum systems do not present such currents.

For spherically symmetric spacetimes, following Ref. [93], we take the line element of the glued spacetimes as

\[
    ds_\pm^2 = -e^{2\kappa_\pm(r_\pm)}dt_\pm^2 + e^{2\beta_\pm(r_\pm)}dr_\pm^2 + r_\pm^2d\Omega_\pm^2, \quad (4.6)
\]

where

\[
    d\Omega_\pm^2 = d\theta_\pm^2 + \sin^2\theta_\pm d\phi_\pm^2. \quad (4.7)
\]

We consider that the hypersurface \( \Sigma \) is described by the equation \( \Psi_\pm = r_\pm - R(\tau) = 0 \), where \( \tau \) is the proper time of an observer on it. Besides, we take \( \theta_\pm = \theta \) and \( \phi_\pm = \phi \) for the remaining coordinates on \( \Sigma \). In other words, we are selecting a geodetic observer for describing the geometry of the thin shell. For the above choice of coordinates, it is clear that \( e^\mu_0 \pm = U^\mu_\pm \), the four-velocity of \( \Sigma \). Hence, \( n_\mu U^\mu = 0 \). The well-definiteness of the geometry of \( \Sigma \) is translated into the condition

\[
    \dot{t}_\pm = e^{\beta_\pm - \alpha_\pm}R^2 + e^{-2\beta_\pm}, \quad (4.8)
\]

as it can be easily shown. We are defining the dot operation as the derivative with respect to \( \tau \). Taking into account the previous points, the geometry of \( \Sigma \) is therefore

\[
    ds_\Sigma^2 = -d\tau^2 + R^2(\tau)(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.9)
\]

The Lanczos equation, Eq. (4.4), for the spherically symmetric case implies that

\[
    S^a_b = \text{diag}(-\sigma, \mathcal{P}, \mathcal{P}) \quad (4.10)
\]
with

\[ \sigma = -\frac{1}{4\pi R} \left[ \sqrt{e^{-2\beta} + \dot{R}^2} \right]^+, \quad (4.11) \]

\[ \mathcal{P} = -\frac{\sigma^2}{2} + \frac{1}{8\pi R} \left[ \frac{Ra'(e^{-2\beta} + \dot{R}^2) + \dot{R}R + \beta'R^2}{\sqrt{e^{-2\beta} + \dot{R}^2}} \right]^+, \quad (4.12) \]

where the prime was defined as the derivative with respect to the radial coordinate defined in each region of the glued manifold \( M \). Equation (4.4) for this case gives [93]

\[ \dot{\sigma} = -\frac{2\dot{R}}{R} (\sigma + \mathcal{P}) + \Delta \dot{R}, \quad (4.13) \]

with

\[ \Delta = \frac{1}{4\pi R} \left[ (\alpha' + \beta') \sqrt{e^{-2\beta} + \dot{R}^2} \right]^+. \quad (4.14) \]

Equation (4.13) can be rewritten in a much more appealing form as

\[ \frac{d}{d\tau} (4\pi R^2 \sigma) = -\left( \mathcal{P} - \frac{\Delta R}{2} \right) \frac{d}{d\tau} (4\pi R^2). \quad (4.15) \]

We would like to emphasize that the thin shell formalism in the spherically symmetric case leads to Eqs. (4.11) and (4.12) [or Eq. (4.15)], while the unknown variables to the problem are \( \sigma, \mathcal{P} \) and \( R \). This means that an equation of state \( \mathcal{P} = \mathcal{P}(\sigma) \), must be given for closing the system of equations. Such an equation of state would embrace the microphysics of the fluid on the shell. Otherwise, a free parameter will be present into the formalism.

From the components of the ideal surface energy-momentum tensor \( S^\alpha_{\beta\gamma} \), one sees that Eq. (4.15) is analogous to the first law of thermodynamics adapted to a spherically symmetric surface. It then lead us indeed to the interpretation of \( \sigma \) as the energy density on \( \Sigma \), while \( \mathcal{P} \) as the pressure (surface tension) connected with the work done by the internal forces in the shell. Besides, \( 4\pi R^2 \dot{R} \Delta \) is the work done by the nonzero normal flux of momentum \( T_{\mu\beta} U^\beta n^\alpha \) across \( \Sigma \). That a thermodynamical relation would raise out of the thin shell formalism is by far a
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surprise, given the profound connection between it and general relativity, as we stressed previously.

Now we proceed with the stability analyses for the thin shell against radial perturbations of the latter (the description where also the adjacent spacetimes are perturbed will be analyzed in another chapter). Equation (4.11) can be cast in the suggestive form [93]

\[ V(R) + \dot{R}^2 = 0, \quad (4.16) \]

with [93]

\[ V(R) = \frac{1}{2}(e^{-2\beta_+} + e^{-2\beta_-}) - \frac{1}{4}(4\pi R\sigma)^2 - \frac{1}{4}\left(\frac{e^{-2\beta}}{4\pi R\sigma}\right)^2. \quad (4.17) \]

One thus interprets \( V(R) \) as the effective potential the shell is subjected to. The solution \( R(\tau) = R_0 = \text{Const} \) implies that \( V(R_0) = V'(R_0) = 0 \), that in turn leads \( R_0 \) to be automatically a critical point to the effective potential. Assume now small radial displacements from this solution. As it is well-known, just in the case \( V''(R_0) > 0 \) one has a stable behavior of the system. From Eqs. (4.17), (4.13) and the identity \( A'|_\Sigma = (A/\dot{R})|_\Sigma \), after some simple calculations, one shows that the stability condition, for the case \( \sigma > 0 \), can be written as

\[ \tilde{V}'' \doteq -\left[ e^{-\beta} \left\{ (2\eta + 1)(1 + R_0\beta') - R_0^2(a'' - \beta'a') \right\} \right]_+ > 0, \quad (4.18) \]

with \( \eta \doteq \partial P / \partial \sigma = P'/\sigma' \). The above equation is a pivotal one for the analyses we shall carry out. We are not interested here in exploring the microphysics of the shell. Thence, we will allow \( \eta \) to be a free parameter in our description. Nevertheless, it must be borne in mind that a physical system is ascribed solely to an equation of state (therefore a single value of \( \eta \) for a given point) and our analyses with free \( \eta \) can be seen as the construction of a generic stability catalogue for given matched spacetimes. It is worth mentioning that the issue of equations of state for surfaces remains thus far knotty even in the forefront investigations of material sciences, where there are yet phenomenological models awaiting theoretical frameworks [107].

We shall be content in this chapter with constraining \( \eta \), as well as other parameters appearing in \( \beta_\pm \) and \( \alpha_\pm \), particularized the configuration, so that they
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lead to stable solutions to the thin shell to that case. For doing so, we shall only impose that the speed of the sound is smaller than the speed of light. The former speed in the scope of continuous systems will be deduced in the next section. The search for a microphysics that satisfies the constraints we shall investigate is out of the scope of this chapter.

4.3 Speed of the sound for shells embedded in continuous systems

In this section we use the $-2$ metric signature. In order to describe the stability of a dynamic thin shell, one has to study the properties of perturbations (sound) propagating on $\Sigma$, as we have just shown by Eq. (4.18). Since our shell is embedded in a continuous medium, the normal flux of momentum, the right-hand side of Eq. (4.5), must be properly taken into account. For the spherically symmetric case this can be easily done, as we shall show in this section. The important points to be realized are that the geometry of $\Sigma$, given by Eq. (4.9), for each instant of time $\tau$, is flat and that the energy momentum of $\Sigma$ is that of a perfect fluid. Hence, we can work with Cartesian coordinates, come back to spherical ones at the end of the calculation, and then suppress the radial coordinate for finding the dynamics of perturbations on $\Sigma$. From the above comments, we can pose the problem in the following form. Given

\[ T^{\mu\nu}, \mu = f^\mu(x^\nu), \quad T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p\eta^{\mu\nu}, \]  

with $f^\mu$ a given four-vector dependent upon the spacetime coordinates, we want to find the equation governing the evolution of the perturbations on the pressure $p$, the energy density $\rho$ and the velocity of the fluid $v$, when the unperturbed solution for the latter is zero. We have that Eq. (4.19) can be split into

\[ (\rho u^\mu_{,\mu} + p u^\mu_{,\mu} = f_{\mu} u^\mu \]  

and

\[ (\rho + p)v^\nu_{,\mu} u^\mu + (p_{,\mu} + f_{\mu})(u^\mu u^\nu - \eta^{\mu\nu}) = 0. \]
4.3. Speed of the sound for shells embedded in continuous systems

The above equations admit a solution with \( u^\mu = (1, 0) \) iff

\[
\rho, t = f_0, \quad p, j = -f_j, \quad (4.22)
\]

where here \( i = 1, 2, 3 \). Now, let us suppose that \( \tilde{u}^\mu = (1, \delta v_i) \), \( \tilde{p} = p + \delta p \), and \( \tilde{\rho} = \rho + \delta \rho \), where \( p \) and \( \rho \) are given by the solutions of Eqs. (4.22) and \( \delta \rho \) and \( \delta p \) are functions of the spacetime coordinates. By putting \( \rho, \tilde{p} \) and \( \tilde{u}^\mu \) into Eqs. (4.20) and (4.21), taking into account Eq. (4.22) and working up to first order of approximation in the \( \delta \) terms, one has

\[
(\delta \rho)_{,t} + [(\rho + p)\delta v^i]_{,i} = 0 \quad (4.23)
\]

and

\[
[(\rho + p)\delta v^i]_{,t} + (\delta p)^i = 0, \quad (4.24)
\]

where we defined \( (\delta p)^i \equiv \partial(\delta p)/\partial x^i \). Equations (4.23) and (4.24) lead us to

\[
(\delta \rho)_{,tt} - [(v_c^2 \delta \rho)^i]_{,i} = 0, \quad (4.25)
\]

where are assumed that

\[
\delta p = \frac{\partial p}{\partial \rho} \delta \rho \approx v_c^2 \delta \rho. \quad (4.26)
\]

In other words, we considered the system to be adiabatic. By assuming that \( v_c = v_c(x^\mu) \), we have that Eq. (4.25) reads

\[
\frac{\partial^2 \delta \rho}{\partial t^2} - (\nabla^2 v_c^2) \delta \rho - v_c^2\nabla^2 \delta \rho - \delta^{ij} \partial_i v_c^2 \partial_j \delta \rho = 0, \quad (4.27)
\]

with \( \nabla^2 \equiv \delta^{ij} \partial_i \partial_j \) and \( \delta^{ij} = -\eta^{ij} \). We see from the above equation that in general there will be a damping factor for the propagation of disturbances.

From Eq. (4.18), we need to analyze the speed of the sound at the equilibrium point of the shell [critical point of the effective potential in Eq. (4.17)] in order to assess its stability. In order to analyze these propagations on the surface of \( \Sigma \), we should leave out the \( r \) coordinate of Eq. (4.27), taking it as a constant, keeping
just the spherical ones. By the spherical symmetry of the system, it is clear that \( v_c \) is neither dependent on \( \theta \) nor \( \phi \). Hence, from Eq. (4.27), we conclude that the usual expression for the speed of the sound is the one to be used in our stability analyses.

Before we move onward, we would like to elaborate better on the physical substance of the parameters \( \sigma \) and \( P \). Our thermodynamical analysis has already shown us that they are the surface energy density and the surface tension, respectively, entering into the energy-momentum tensor of the shell, induced by the distributional solution to Einstein’s equations. We would like to substantiate this by analyzing the classical counterparts of Eqs. (4.11) and (4.12). We shall do so in the forthcoming section.

### 4.4 Physical interpretation of the thin shell parameters

In this section we essay to interpret \( \sigma \) and \( P \) arising from the thin shell formalism for the spherically symmetric case. We recall that Einstein’s equations in this scenario can be written as [2]

\[
e^{-2\beta(r)} = 1 - \frac{2m(r)}{r}, \quad m(r) = 4\pi \int_0^r \rho(\bar{r})\bar{r}^2 d\bar{r}, \tag{4.28}
\]

\[
\alpha' = \frac{8\pi pr^3 + 2m(r)}{2r \{ r - 2m(r) \}^r}, \tag{4.29}
\]

where we defined \( \rho \) and \( p \) as the energy density and the radial pressure, respectively, measured by local Lorentz observers. Here we suppressed the \( \pm \) labels once more just not to overload the notation. The energy density \( \sigma \) and the pressure \( P \) must naturally be relativistic generalizations to classical quantities. In this regard, we proceed with their weak field analysis for the equilibrium configuration. This is done by assuming that \( 2m_\pm(R_0)/R_0 \ll 1 \). For this case, it is easy to show that Eqs. (4.11) and (4.12), on account of Eqs. (4.28) and (4.29), read

\[
\sigma_0 = \frac{1}{4\pi} [g(R_0)]_+, \quad g(R_0) = \frac{m(R_0)}{R_0^3}, \tag{4.30}
\]
We see from Eq. (4.30) that \( \sigma \) is indeed the generalization of the surface energy density (mass density) of the shell. Equation (4.31) is the well-known Young-Laplace equation for a spherically symmetric bubble-like system [108] and hence \( \mathcal{P} \) is the generalization of the surface tension. For the latter case, we notice that in the smallest order of approximation, the surface tension is just obtained by geometric considerations. For higher order corrections, once more from Eqs. (4.28) and (4.12) in the static case, we have

\[
\mathcal{P}_0 = \frac{1}{2} R_0 [p(R_0)]^\pm + \frac{G}{16 \pi R_0^3} [m^2]^\pm + \frac{G [p m]^\pm}{2c^2} + \frac{G^2 [m^3]^\pm}{8 \pi c^2 R_0^4} + ..., \tag{4.32}
\]

where we have restored the units just for completeness. From the above expression, it is clear classical (geometric and gravitational) and general relativistic terms to the surface tension, as evidenced by the speed of the light in vacuum, \( c \).

We stress that the above well-known classical results for \( \sigma \) and \( \mathcal{P} \) are a direct consequence of the Israel-Darmois formalism. Hence, any general relativistic system endowed with nontrivial surface quantities must be described by the aforementioned method.

### 4.5 Newtonian thin shell stability

Now we analyze Eq. (4.18) in the classical limit. The decisive equations to be taken into account here are Eqs. (4.28) and (4.29) in the weak field limit, where one can make the approximations

\[
\beta \simeq \frac{m(r)}{r} \quad \text{and} \quad \alpha' \simeq \phi' = \frac{m(r)}{r^2}, \tag{4.33}
\]

where \( \phi \) is the gravitational potential, such that the gravitational field is \( g = -\nabla \phi \). We are also assuming that \( m(r)/r \ll 1 \). When one substitutes Eq. (4.33) into Eq. (4.18), one concludes that the stability at \( R_0 \) is translated into

\[
2 \eta \left[ m'(R_0) - 2 \frac{m(R_0)}{R_0} \right]_+ = 2 \eta R_0^2 [\phi'']_+ < 0. \tag{4.34}
\]
That the stability condition be related to the second derivative of a potential is self-explanatory. Its jump being negative at a surface of discontinuity signifies that the norm of the gravitational force density must decrease, exactly what one excepts from stable systems. Poisson’s equation for the potential given by Eq. (4.33) reads

$$\frac{m'(r)}{r^2} = 4\pi \rho(r),$$  \hspace{1cm} (4.35)

with \(\rho(r)\) the mass density at \(r\). Putting Eq. (4.35) into Eq. (4.34), the latter can be cast as

$$4\eta \left[ 2\pi \rho(R_0)R_0^2 - \frac{m(R_0)}{R_0} \right]_+^− < 0. \hspace{1cm} (4.36)$$

From the above equation and previous considerations on \(\eta\), one sees that thin shells with no surface mass densities are stable if \([\rho(R_0)]^+ < 0\). This is exactly what one expects for stars, specially if the mechanism to the stratification are nuclear fusion reactions. For vacuum systems, the stability simplifies to \([m(R_0)]^+ > 0\), i.e., the surface mass on \(R_0\) should be positive. The physical reason for having stability even in the vacuum case is due to the induced gravitational surface tension [see Eq. (4.32)].

### 4.6 Hypersurfaces under constant pressures

In this section we show that the thin shell formalism guarantees that for a physically reasonable astrophysical body an interface splitting two phases under a given pressure is stable against radial displacements whenever the mass it nests is much smaller than its total mass. So, we shall hypothesize that \([p]^+ = 0\) and \(\sigma = 0\), which means that \([e^\beta]^+ = 0\), and show that the associated surface of discontinuity is always stable. For this case, Einstein’s equations on the hypersurface \(\Sigma\) lead us to

$$\left[\beta'\right]^+ = 4\pi e^{2\beta} R[\rho]^+ \quad \text{and} \quad [\alpha']^+ = 0. \hspace{1cm} (4.37)$$

In the static case, from Eqs. (4.12) and (4.37), we have that \(P = 0\). The aforesaid hypotheses do not render the system continuous since \([\rho]^+\) is yet unspecified. In
the dynamic case generally $P \neq 0$. From Eq. (4.18) and the above equation, the stability condition simplifies to

$$2\eta R_0 [\beta']^\prime < 0.$$ \hspace{1cm} (4.38)

We have shown that $\eta$ is the square velocity of the sound and therefore must be positive. Hence, from Eqs. (4.37) and (4.38) we see that the hypersurface is stable iff $[\rho] < 0$. The aforesaid result is reasonable, since the energy density should always be a monotonically decreasing function of the radial distance [1] (for continuous systems actually $[\rho] = 0$, which is a local aspect and does not reveal its decreasing monotonocity; nevertheless, for stratified ones, $[\rho] = 0$ is generically the case).

Therefore, we have generically shown that for physically reasonable systems interfaces with phase transitions at constant pressure and negligible masses are always stable against radial perturbations. These results could justify the usual way of dealing with neutron stars with phase transitions as we explain in the sequel. The constancy of the pressure is commonly a requirement either in the scope of the Maxwell or Gibbs constructions for dealing with phase transitions in neutron stars [1], associated with their mechanical equilibria. It is important to recall, however, that in systems with more than one conserved charges (e.g. baryon and electric) the Gibbs construction leads to the appearance of mixed phases, in between of the pure phases, with an equilibrium pressure that varies with the density. This leads, in turn, to a spatially extended phase-transition of non-negligible thickness with respect to the star radius (see, e.g., [1], and references therein). It is clear that such an extended mixed phase region, separating the two pure phases, cannot be treated within a thin-shell approach. Nevertheless, its interfaces demarcating the onset and the termination of a mixed phase always can. In the aforementioned sense, the thin-shell treatment is intrinsically more suitable to model configurations in which the phase-transition follows a Maxwell construction, where the phases are in “contact” each other. Though under the thermodynamical point of view the Gibbs construction is the correct one, the use of the Maxwell or the Gibbs constructions in the treatment of thermodynamic phase-transitions in multi-component systems such as the ones present in compact stars is still a matter of debate in the literature, especially because both treatments lead to approximately the same macroscopic parameters (mass and radius) there.
4. Thin shell stability in stars with phase transitions

From either one of the aforesaid constructions and other supplementary conditions one can find just under the microscopic point of view all of the characteristics apropos of phase transitions of given fluids in a star. This procedure also allows one to “construct” an equation of state for the whole system, where different phases are glued at the position where their pressures and chemical potential equal. With this, one could solve (numerically) Einstein’s equations for the system in equilibrium. Due to the discontinuity generally implied by the phase transitions in the equation of state, “cusps” arise in their byproducts. Nevertheless, such an approach has the inconvenience that there are points where the pressure gradient is not well-defined, a conceptual problem (though not numerical) for nondistributional solutions to general relativity.

Another way of integrating the system of equations would be solving them for each phase present. For a system with two phases, it is normally assumed that the output of the integration for the inner part of the star is used as the input for the integration of the outer part, apart from the discontinuity of the energy density. Proceeding in this way, one is tacitly assuming that the mass and the surface tension on the interface of the two phases are negligible in the static case. Though reasonable under the intuitive point of view, such procedure is rather restrictive. For example, the system could be equally static in the presence of pressure discontinuities whenever surface tensions are permitted to exist. In the subsequent sections we shall analyze cases in this direction, where more degrees of freedom are present on the interface splitting two phases, obtained as a consequence of more general equilibrium conditions.

4.7 Thin shell stability for constant energy density stars

With the purpose of obtaining more intuition about the main aspects of the thin shell formalism, it is instructive to work with an exact fully relativistic case. In this regard in this section we analyze the stability of stars with constant densities that present “phase transitions”. In order to simplify the reasoning, let us assume that they are locally neutral. Let us assume a star that has a constant density \( \rho^- \) from the origin to a radius \( R \) (that could be even dynamic), and from \( R \) until its surface \( R_s \) it has another constant density \( \rho^+ \). Assume further that its associated discontinuity surface \( \Sigma \) has a negligible energy density when
4.7. Thin shell stability for constant energy density stars

compared to either regions it defines. We shall seek solutions that are regular at $r = 0$. The integration of the general relativistic system of equations with the aforesaid assumptions lead us to the following solutions. For $r < R$:

$$e^{-2\beta} = 1 - \frac{8\pi r^2 \rho^-}{3}, \quad e^{2\alpha} = A_0^- \left( \frac{\rho^- + p^-}{\rho^- + p_0^-} \right)^{-2}, \quad (4.39)$$

where $A_0^-$ and $p_0^-$ are arbitrary constants of integration and the inner pressure $p^-$ is

$$p^- = \rho^- \left\{ (\rho^- + 3p_0^-) \left( 1 - \frac{8\pi \rho^- r^2}{3} \right)^{\frac{1}{2}} - (\rho^- + p_0^-) \right\} \frac{3(\rho^- + p_0^-) - (\rho^- + 3p_0^-) \left( 1 - \frac{8\pi \rho^- r^2}{3} \right)^{\frac{1}{2}}}{3(\rho^- + p_0^-) - (\rho^- + 3p_0^-) \left( 1 - \frac{8\pi \rho^- r^2}{3} \right)^{\frac{1}{2}}}. \quad (4.40)$$

For $R < r \leq R_s$,

$$e^{-2\beta^+} = 1 - \frac{8\pi}{3r} \left\{ R^3(\rho^- - \rho^+) + \rho^+ r^3 \right\}, \quad (4.41)$$

$$e^{2\alpha^+} = A_0^+ \left( \frac{\rho^+ + p^+}{\rho^+ + p_0^+} \right)^{-2}, \quad (4.42)$$

with

$$A_0^+ = \left( 1 - \frac{2M}{R_s} \right) \left( \frac{\rho^+ + p_0^+}{\rho^+ + p_0^+} \right)^2, \quad (4.43)$$

$$p^+ = \rho^+ \left( \rho^+ + 3p_0^+ \right) \left( 3 - 8\pi \rho^+ r^2 \right)^{\frac{1}{2}} - (\rho^+ + p_0^+) \left( 3 - 8\pi \rho^+ R^2 \right)^{\frac{1}{2}} \frac{3(\rho^+ + p_0^+) (3 - 8\pi \rho^+ R^2)^{\frac{1}{2}}}{3(\rho^+ + p_0^+) (3 - 8\pi \rho^+ R^2)^{\frac{1}{2}} - (\rho^+ + 3p_0^+) (3 - 8\pi \rho^+ R^2)^{\frac{1}{2}}}, \quad (4.44)$$

$$p_0^+ = \rho^+ \left( 3 - 8\pi \rho^+ R_s^2 \right)^{\frac{1}{2}} - (3 - 8\pi \rho^+ R^2)^{\frac{1}{2}} \frac{(3 - 8\pi \rho^+ R^2)^{\frac{1}{2}} - 3(3 - 8\pi \rho^+ R_s^2)^{\frac{1}{2}}}{3(3 - 8\pi \rho^+ R_s^2)^{\frac{1}{2}}}, \quad (4.45)$$
The total mass of the system was defined such that

\[ M = \frac{4\pi}{3} \left\{ R^3(\rho^- - \rho^+) + \rho^+ R_s^3 \right\}. \]  

(4.46)

In the above equations, it was assumed that the outer pressure at \( R_s \) is null. Equation (4.43) guarantees the outside match of the star with the Schwarzschild metric. In the scope of the stability of a thin shell immersed in a continuous system, the constant multiplicative factor on the time-time metric component is not of importance. This is related to the freedom of scaling the time coordinate for the metric in Eq. (4.6). Therefore, \( A^\pm_0 \) will not play any relevance to our stability analyses. Notice further that we can always have \( p^+(R) = p^-(R) \), due to the arbitrary constant of integration \( p_0^- \). As we already know from the preceding section, this case is stable iff \( [\rho]^+_0 < 0 \). Let us analyze another case, where \( p_0^- \) is a free parameter.

It is convenient to relate \( R \) and \( R_s \), as well as \( \rho^\pm \). Let us assume that \( R_s = C_1 R \) and \( \rho^- = C_2 \rho^+ \). We stress the fact that the solution for constant densities led to the constraint of \( \rho R_s^2 \) to be of the order of unity. Assuming that the radii of our systems are similar to the ones expected to neutron stars, reasonable values for the \( r^- \) coordinate would be of the order of \( 10^6 \text{cm} \). Therefore, the maximum densities allowed for constant density stars would be \( \rho_{max} \approx 10^{-12} \text{cm}^{-2} = 10^{16} \text{g/cm}^3 \). Such densities are well above the nuclear one, of the order of \( 10^{14} \text{g/cm}^3 \approx 10^{-14} \text{cm}^{-2} \). The pressure at the origin \( p_0^- \) for this stratified system is arbitrary. Nevertheless, reasonable values for it are of the order of (or higher than) \( p_0^- \approx 10^{35} \text{dyn/cm}^2 \approx 10^{-13} \text{cm}^{-2} \).

By replacing Eqs. (4.39), (4.41) and (4.42) in Eq. (4.18), we would have a very involved expression. Numerical analyses are much more enlightening. Figures 4.1, 4.2 and 4.3 show some aspects from the numerical evaluation of the right-hand side of Eq. (4.18) for some scenarios.

One can see from Figs. 4.1 and 4.2 that \( C_2 \) and \( R \) play a relevant role into the stability of the system. It is uninvolved to verify that \( C_1 \) does not. These results are expected since any change in \( C_1 \) would lead to physically similar configurations, while a modification of \( C_2 \) and \( R \) would lead to an alteration in the normal flux of momentum through \( \Sigma \), that clearly affects the stability. Besides, also the central pressure influences the stability of the system. This can be checked in Fig. 4.3. The above-mentioned figures also show that fully general relativistic analyses may considerably change the classical picture, where the
stability condition would merely read $[\rho]^+ < 0$. The reasons for that are related to the important role played by the pressure into the system, as well as general relativistic corrections to the classical case.

Figure 4.1: Stability of thin shells as a function of the square of the speed of sound $\eta$, for various values of the parameter $C_2$. One sees here that for the choice of the remaining parameters close to the ones of ordinary neutron stars, not any configuration for the system would lead to stable solutions. In special, the case $[\rho]^+ > 0$ (dashed curve), for almost all equations of state is unstable. Classical stars are stable for the case we are interested in whenever the jump of the mass density is negative.

### 4.8 Numerical analyses for the thin shell stability in neutron stars with all fundamental interactions

Recently, it was proposed a model for globally neutral (zero electric charge) neutron stars endowed with all fundamental interactions in thermodynamical equilibrium [103]. The motivations for this new description rely on previous findings [109, 110] concerning the incompatibility of local charge neutrality with the constancy of the generalized Klein potentials [109–111], related to what the thermodynamic equilibrium is defined. Since the central densities of neutron stars are expected to be larger than the nuclear one ($\rho_{\text{nuc}} \approx 3.10^{14} \text{g/cm}^3$) [103], it is manifest that strong forces shall play an important role in their description. In the aforementioned work, strong interactions were taken to be described by
4. Thin shell stability in stars with phase transitions

Figure 4.2: Stability of thin shells as a function of the square of the speed of sound $\eta$, for various values of the parameter $C_2$. One sees here that the change of the choice of the remaining parameters as related to Fig. 4.1 modifies the stability of the system.

Figure 4.3: Stability of thin shells as a function of the central pressure $p_0^-$, for various values of the parameter $C_2$. For the parameters chosen, the change of $C_2$ and $R$ influence the stability for a given equation of state, as exemplified here by $\eta = 0.8$.

the $\sigma - \omega - \rho$ model [112]. Weak interactions were modeled by the assumption of $\beta$-equilibrium. Electromagnetic interactions also take place, due to the global neutrality and were modeled by the Maxwell-Thomas-Fermi equations [109]. The global charge neutrality is implemented by the Thomas-Fermi formalism.
4.8. Numerical analyses for the thin shell stability in neutron stars with all fundamental interactions

through the Wigner-Seitz cells [113]. Just numerical solutions to the associated system of equations are possible, even in the case the system is at zero temperature. Phase transitions (discontinuity of physical quantities) are also present in this model. They lead to a core-crust system. It is then expected that a transitional shell (or layer) of thickness about the Compton wavelength of the electron, \( \lambda_e = \hbar/(m_e c) \sim 100 \text{fm} \), would emerge. The most prominent fact that ensues from the constancy of the generalized Klein potentials and the nontrivial electrodynamic structure is the presence of overcritical electric fields in the transitional shell. Electron-positron pairs are not produced due to the Pauli blocking mechanism. For counterbalancing the pressure discontinuity in the core-crust interface, the transitional shell must possess a surface density. Some aspects of such a quantity were studied in Ref. [114] in the context of nuclear physics.

We now proceed with the numerical analyses of the set of equations for neutron stars endowed with all fundamental forces and globally neutral, as appearing in Ref. [103]. The numerical integration undergone in such a paper ensues by giving a central baryon density and regulatory conditions at the origin. The coordinate radius of the core is defined as the radius where the density equals the nuclear density. For describing the crust, its mass and density at its base must be given. In general, for the outer crust, its density is supposed to be of the order of the dripping density, \( \rho_{\text{drip}} \approx 4 \times 10^{11} \text{g/cm}^3 \). For the inner crust, the densities are much bigger, being even of the order of the nuclear one. Due to the expected smallness of the transitional shell, numerically speaking, one considers the aforementioned initial conditions at the core radius. The integration terminates when the pressure is zero, construe as the surface of the star. Both pressure and density of the system are discontinuous at the core-crust radius, due to the utilization of a prescribed equation of state for the crust, the BPP-BPS ones [115], and the microphysical approach to the core. The crust can be described with a high degree of accuracy by the TOV system of equations [103]. The core, on the other hand, has a more tortuous structure, with all fundamental interactions being taken into account [103]. Due to the nonlocal aspect of the global charge neutrality, the density at the base of the crust impinges upon the thickness of the shell [103].

We recall that the thin shell formalism is an outstanding approximation to the analyses to be carried out. This is so since the transitional layer is expected to have a thickness of \( \lambda_e = \hbar/(m_e c) \sim 100 \text{fm} \), while the radius of the core is of the order of some kilometers [103]. Besides, the microscopic information of
the layer is embraced in the macroscopic parameters $P$ and $\sigma$ by means of an equation of state. Further, the shell is supposed to be initially in equilibrium and at the core’s radius; see Eq. (4.17) for $R = \text{const}$. Departures from equilibrium are due to radial displacements. Their description depends upon perturbations on the shell; see in Eq. (4.18) the presence of $\eta$. We do not assume that perturbations take place in the phases making up the star. Hence, it would allow us to assess directly the role played by the surface degrees of freedom to the system.

One can appraise the energy density on the shell by dint of electric fields present there. Once they are unfailingly a few thousand times larger than the critical ones coming from QED [103], the total energy of the shell is expected to be much smaller than the one of the core ($M \sim M_\odot$) for each central and at the base of the crust densities the system may have. For all numerical analyses, following [103], we shall work with the NL3 nuclear parameters [116]. Our numerical results show that, for central densities smaller than $2.19\rho_\text{nuc}$ ($M \sim 1.97M_\odot$), there always exists a minimum density at the base of the crust, of the order of the nuclear density, above which Eq. (4.18) could be satisfied for convenient equations of state. Besides, for central densities smaller than $2.19\rho_\text{nuc}$, there always exists also a second minimum density (as well of the order of the nuclear density) at the base of the crust above which any equation of state for the fluid on the thin shell leads to stable configurations. We observe further that the neutron stars that have inner and outer crusts in our description should have masses smaller than $1.97M_\odot$. In Fig. 4.6, such points are summarized for selected values of the central densities.

The case where a neutron star core is clothed just by a thin shell can also be analyzed within the Darmois-Israel formalism. In this case, the core solution must be matched with the external Schwarzschild solution to the star. Figure 4.5 shows that for all physically possible central densities, stable solutions could abide. Each curve spans some masses the system could have for a given central density, due to changes of the surface degrees of freedom on the core-vacuum interface. From the aforementioned figure, one sees that for densities above $2.83\rho_\text{nuc}$ there is always a range of masses the system may have that lead to stable configurations, irrespective of the EoS of the fluid on the shell. In any case where there exist minimum velocities of the sound for the surface fluid, they are always small. Hence, any hard equation of state would lead to stable “thin crust” configurations in the scope of the proposal of [103]. We point out that the very small mass variations with respect to the ones coming from the core inte-
4.9 Stability analyses for thin shells within strange stars

In the core of astrophysical compact objects, like neutron stars or protoneutron stars, nuclear matter is expected to reach densities which are several times the nuclear saturation density. Calculations based on microscopic equation of states (EoS), which include only nucleonic degrees of freedom, show that the central densities of the most massive neutron stars can be $\sim 7 - 10$ times the nuclear saturation density $\rho_{\text{nuc}}$. In prevalent models, hadrons are assumed to be the true ground state of the strong interaction. However, it has been pled [117–121] that strange quark matter (SQM) may be the authentic ground state of all matter.
Figure 4.5: $v_{\text{min}}^2$ “minimum square velocity” as a function of the total mass of the system (in units of solar masses $M_\odot$) for cores just with a thin layer. Negative values for $v_{\text{min}}^2$ signify that any equation of state leads to stable solutions. Each curve is related to a given central density, in units of the nuclear density. The small mass variations with respect to the core integrations could correspond physically to thin crusts clothing the hadronic cores.

Thus, according to the strange matter hypothesis strange quark matter could be more stable than nuclear matter. Therefore, the interior of neutron stars should be predominantly composed of up, down, and strange quarks, plus leptons to ensure charge neutrality. Models in which the interior is assumed to be composed of strange matter are often called strange stars. Here, we are also investigating whether or not it is possible that a strange star be clothed within the thin shell formalism.

In order to have some insights about this important case, we start our analyses with its simplest model, the MIT bag model [122]. Such a model assumes that quarks are free inside a “bag” whose width is related to the vanishing of its pressure. The associated equation of state reads

$$p_{\text{bm}} = \frac{1}{3}(\rho - 4B), \quad (4.47)$$

where $B$ is a constant and measures the energy density at with the quark pressure is null. As in Ref. [121], we shall assume in our calculations that $B = (145\text{MeV})^4$. Actually, its precise value will not change the main qualitative conclusions to be drawn. Also in order to be as simple as possible, we shall assume
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For this case that there is local charge neutrality, which shall permit us to use TOV equations for such a star.

We are interested here in investigating whether or not it is possible for a strange star to have a crust on top of its core’s surface (with zero pressure). The density of the crust at the edge with the electronic layer has to be necessarily lower than the neutron-drip value, \( \rho_{\text{drip}} \approx 4.3 \times 10^{11} \text{ g cm}^{-3} \), since having zero electric charge, any free neutron created in the crust would flow to the core where it would be converted into strange matter. For the crust we therefore use the Baym-Pethick-Sutherland, commonly referred to as BPS, equation of state (EoS) [115].

We solve the Einstein’s equations (4.28)-(4.29) for the above described equation of state, for selected values of the central density and different densities at the base of the crust in each case. Then, we seek for values of the parameter \( \eta \) that satisfy the stability condition (4.18) of the shell’s effective potential. Our numerical results show that thin-shells splitting the quarkionic phases from crusts are always unstable for densities at the base of the crust of the order of the neutron drip density. In Fig. 4.6 one can indeed verify that a quark star would be stable just if its density at the base of the crust was a hundred times the neutron drip one, which is clearly not permissible, as we commented before.

<table>
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<tr>
<th>Core ( \rho_c ) = ( 2.22 \rho_{\text{nuc}} )</th>
<th>( \rho_{\text{bcr}} / \rho_{\text{drip}} )</th>
<th>( \Delta R_{\text{cr}} / R_c )</th>
<th>( M_{\text{cr}} / M_\odot )</th>
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Table 4.1: Widths of the crusts (\( \Delta R_{\text{cr}} \)) for various strange stars configurations, densities at the base of the crusts (\( \rho_{\text{bcr}} \)) and associated masses of the crust (\( M_{\text{cr}} \)).
4. Thin shell stability in stars with phase transitions

The previous result suggests us that strange stars should just have a tiny a crust cloaking the quark’s core surface. Therefore, we should seek for solutions where the quark core would be matched directly with the Schwarzschild exterior spacetime, hence treating the crust itself as part of the thin-shell. We recall that the quark stars we are analyzing here have at their core’s edge null pressures and \( \rho = 4B \). For this case and matching with Schwarzschild’s solutions, we already know from section 4.6 that the associated thin-shells are stable, irrespective of the fluids they host. Physically speaking this result means that extremely thin crusts could always be taken as parts of thin-shells. Table 4.2 suggests that, indeed, very low densities at the base of the crusts would allow us to interpret the crusts as constituents of thin-shells. Such a table is constructed by solving the system of equations from general relativity with the core’s EoS given by Eq. (4.47) and the BPS EoS [115] for the crust, when central densities and densities at the base of the crust are given.

![Figure 4.6: \( v^2_{\min} \) “minimum square velocity” as a function of the “density at the base of the crust”, \( \rho_{\text{bcr}} \), normalized by \( \rho_{\text{drip}} \approx 4.10^{11} \text{g/cm}^3 \). Negative values for \( v^2_{\min} \) mean that any equation of state leads to stable solutions. Each curve is related to a given central density, in units of the nuclear density, \( \rho_{\text{nuc}} \approx 628 \rho_{\text{drip}} \). From Eq. (4.47), one sees that the pressure vanishes at the density \( 4B \approx 1.51 \rho_{\text{nuc}} \approx 948 \rho_{\text{drip}} \). The crust was matched to the core exactly when the aforesaid density is reached for each case analyzed. The densities at the base of the crust that would render stable solutions to the thin-shells are only of the order of hundreds of the neutron drip density. Such densities are not admissible for crusts on quarkionic cores.](image-url)
4.10 Thin shell formalism for slowly rotating layers

Now we are going to show that our results remain unchanged even in the case where the shell is allowed to have a small rotation. This is indeed what one expects and we show it here just for self-consistency. For this case, one supposes that

\[ ds^2 = ds^2_{ss} - 2f(r_\pm) r_\pm^2 \sin^2 \theta \pm a_\pm dt_\pm d\varphi_\pm, \] (4.48)

where \( a_\pm \) are the rotation parameters in the regions \( M_\pm \). Besides, the first term on the right hand side of Eq. (4.48) is simply a shortcut for the spherically symmetric line element given by Eq. (4.6). We take \( \bar{\Sigma} \), the hypersurface for this slow rotation case, in first order of approximation to be also spherically symmetric \( [r_\pm = R(\tau)] \), but at this time we adapt on it the coordinates \( \bar{y}^a = (\tau, \bar{\theta}, \bar{\varphi}) \), such that

\[ t_\pm = T_\pm(\tau), \quad \varphi_\pm = \bar{\varphi} + C_\pm(\tau), \quad \theta_\pm = \bar{\theta}. \] (4.49)

<table>
<thead>
<tr>
<th>Core</th>
<th>( \rho_{bcr} / \rho_{drip} )</th>
<th>( \Delta R_{cr} / R_c )</th>
<th>( M_{cr} / M_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_c = 3.5 \rho_{nuc} )</td>
<td>( 10^{-9} )</td>
<td>( 2.88 \times 10^{-7} )</td>
<td>( 1.25 \times 10^{-16} )</td>
</tr>
<tr>
<td>( R_c = 11.42 \text{ Km} )</td>
<td>( 10^{-5} )</td>
<td>( 3.49 \times 10^{-4} )</td>
<td>( 2.97 \times 10^{-12} )</td>
</tr>
<tr>
<td>( M_c = 1.77 \text{ M}_⊙ )</td>
<td>( 10^{-2} )</td>
<td>( 6.06 \times 10^{-3} )</td>
<td>( 3.50 \times 10^{-8} )</td>
</tr>
<tr>
<td></td>
<td>( 10 )</td>
<td>( 4.56 \times 10^{-3} )</td>
<td>( 4.65 \times 10^{-5} )</td>
</tr>
<tr>
<td>( \rho_c = 5 \rho_{nuc} )</td>
<td>( 10^{-9} )</td>
<td>( 2.32 \times 10^{-7} )</td>
<td>( 1.13 \times 10^{-16} )</td>
</tr>
<tr>
<td>( R_c = 11.34 \text{ Km} )</td>
<td>( 10^{-5} )</td>
<td>( 2.81 \times 10^{-4} )</td>
<td>( 2.10 \times 10^{-12} )</td>
</tr>
<tr>
<td>( M_c = 1.97 \text{ M}_⊙ )</td>
<td>( 10^{-2} )</td>
<td>( 4.87 \times 10^{-3} )</td>
<td>( 2.48 \times 10^{-8} )</td>
</tr>
<tr>
<td></td>
<td>( 10 )</td>
<td>( 3.67 \times 10^{-3} )</td>
<td>( 3.28 \times 10^{-5} )</td>
</tr>
<tr>
<td>( \rho_c = 8 \rho_{nuc} )</td>
<td>( 10^{-9} )</td>
<td>( 2.01 \times 10^{-7} )</td>
<td>( 2.20 \times 10^{-16} )</td>
</tr>
<tr>
<td>( R_c = 10.85 \text{ Km} )</td>
<td>( 10^{-5} )</td>
<td>( 2.43 \times 10^{-4} )</td>
<td>( 1.56 \times 10^{-12} )</td>
</tr>
<tr>
<td>( M_c = 2.02 \text{ M}_⊙ )</td>
<td>( 10^{-2} )</td>
<td>( 4.21 \times 10^{-3} )</td>
<td>( 1.84 \times 10^{-8} )</td>
</tr>
<tr>
<td></td>
<td>( 10 )</td>
<td>( 3.17 \times 10^{-3} )</td>
<td>( 2.44 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

Table 4.2: Widths of the crusts (\( \Delta R_{cr} \)) for various strange stars configurations, densities at the base of the crusts (\( \rho_{bcr} \)) and associated masses of the crust (\( M_{cr} \)).
Then, it can be readily demonstrated that the well-definiteness of the geometry of $\tilde{\Sigma}$ can be obtained if

$$\frac{d}{d \tau} C_\pm = f_\pm (R) \dot{T}_\pm a_{\pm}, \quad \dot{T}_\pm = \sqrt{e^{-2\beta_\pm} + R^2 e^{\beta_\pm - a_{\pm}}}.$$  (4.50)

The above conditions guarantee that the geometry of $\tilde{\Sigma}$ is

$$d\tilde{s}^2|_{\Sigma} = \tilde{h}_{ab} d\tilde{y}^a d\tilde{y}^b = -d\tau^2 + R^2(\tau)(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2).$$  (4.51)

Undemanding calculations lead to the additional extrinsic curvature

$$K^\pm_{\tau\tilde{\phi}} = e^{-\beta_\pm - a_{\pm}} R^2 f'_\pm \sin^2 \tilde{\theta} a_{\pm}.$$  (4.52)

The diagonalization of the surface energy-momentum tensor in this case is done by solving the eigenvalue equation $S^a_b \bar{u}^b = -\tilde{\sigma} \bar{u}^a$. The unknown quantities here are $\tilde{\sigma}$ and $\bar{u}^a$, complemented with the normalization condition $\bar{u}^a \bar{u}_a = -1$. Besides, $\bar{u}^a$ are the components of the tetrad decomposition of the four-velocity of the shell with respect to the coordinate system $\tilde{y}^a$. The pressure in the surface energy-momentum tensor in the $\tilde{y}^a$ coordinate system is simply $2\tilde{P} = S^{ab}(\bar{u}_a \bar{u}_b + \tilde{h}_{ab})$. The solution to the above eigenvalue problem is

$$\tilde{\sigma} = \sigma, \quad \tilde{P} = P, \quad \bar{u}^a = (1,0,\omega), \quad \omega = -\frac{S^\phi_{\tau}}{\sigma + \tilde{P}}.$$  (4.53)

Notice that $\omega$ is polar angle dependent. Equations (4.53) and (4.49) tell us that inertial observers inside the shell are rotating with angular velocity proportional to $dC_+ / d\tau$ with respect to the fixed stars when the inner spacetime is spherically symmetric. For the shell itself, $\Omega_{\text{shell}} \propto \omega + dC_+/d\tau$. At first order, there is no change in the parameters of the shell. As we commented formerly, this is already expected since the corrections imprinted by the rotation to the shell parameters must be independent of its direction of rotation. Nevertheless, up to first order of correction on the rotational parameters $a_{\pm}$, a frame dragging effect is present, whose associated angular velocity gives a direct information about the surface energy-momentum tensor parameters. We shall elaborate upon these issues in a forthcoming chapter.
4.11 Conclusions

First off, the dynamics of perturbations on the core and on crust (and their relevance to the stability of the whole system) are just material when surfaces characterizing discontinuities in the physical parameters are stable. This is precisely the reason of why we just analyzed the perturbation of thin shells in this chapter. Whenever $V''(R_0) < 0$, small radial shell displacements would grow without limits. In the scope of the thin shell formalism, this means that one phase would “swallow” the other [104], making one of them evanesce.

In the scope of the thin-shell formalism for continuous stars, we have showed that whenever one considers phase transitions at constant pressures and with a negligible masses on the surfaces splitting them, the latter ones are always stable. This is relevant for commonly implemented phase transitions based on either the Maxwell or Gibbs constructions, since it would justify their use in the scope of the thin-shell formalism. In this case the degrees of freedom on the surface of discontinuity present in the dynamic case would always lead to stable configurations. Our analyses also show that strange stars with zero pressure at surfaces of their cores could only have tiny crusts, associated with thin-shells enveloping the quark core. Nevertheless, it seems also possible to have strange stars harboring thick crusts. Obviously now the thin shell splitting the core and the crust should not linger on where the strange star’s core has a zero pressure. It is very likely that such a strange core will start to be absorbed by the thin shell, coming to a halt when the latter reaches a position that equals the pressure at the base of the associated crust (generally very near the original core’s surface, owning to the steep increase of the pressure due to the strong force), since this configuration is stable.

It should be pointed out that our results from [103] show that when the densities at the base of the crust are allowed to vary up to the order of the nuclear one, for $\rho_{central} \leq 2.19\rho_{nuc}$, a critical crust base density is reached above which, depending on the equation of state, the thin shell could be stable against radial displacements. A second critical density at the base of the crust also raises, above which the system is stable for any causal equation of state.

We also showed that a neutron star with a core being encased by thin shells are stable for almost all configurations they may have. Our analyses suggest that the stars proposed in [103] with masses larger than $1.97M_\odot$ could be clothed just by extremely thin crusts. For masses below $1.97M_\odot$, the thin shell formalism...
suggests that the aforementioned stars should have inner and outer crusts. There is also the possibility of having neutron stars with masses smaller than $1.97M_\odot$ with just thin crusts. Notice that this should be the case already when there are not perturbations in the phases constituting neutron stars, modeled here as a match of spacetimes, due uniquely to the nontrivial dynamics of the surface degrees of freedom.

When perturbations on volumetric star elements are present, in principle they would also be dependent upon surface degrees of freedom by means of additional boundary conditions, what would also change the eigenfrequencies of the system. This will be investigated in the next chapter. At this first approach, the aforementioned subtleties were not taken into account and we restricted ourselves to finding constraints where the surface perturbative analyses give a definite answer to the stability. This is due to the fact that the scenario where the phases are not perturbed evidences directly the consequences of the dynamics of the degrees of freedom of a shell, giving us thence insights for more elaborate analyses.

Another important point for the macroscopic investigations carried out here is related to the energy conditions for the shell parameters. It is believed they should hold, although this issue is still controversial [6]. From our numerical analyses, we have that not all shell energy conditions hold, since the energy densities were purposely (by reasonable physical arguments; see section 4.8) assumed to be small. From the physical interpretation $\mathcal{P}$ has, this last inequality is qualitatively reasonable, since a surface tension needs to be present to counterbalance the pressure discontinuity, even in the case the shell does not have a mass, as in the classical analysis with the Young-Laplace equation.

Concerning second and higher (even) order corrections to the rotation parameters of the shell in the stability analyses, a more detailed study is in order, to be attained in a forthcoming chapter. Such an investigation is also motivated by the fact it is already known [90] the effect of second order corrections to rotation in the astrophysical systems we investigated here. For the first order corrections to the rotational parameters, just dragging effects are of pertinence. If they were measured, then one could obtain a direct information of the shell parameters, that could shed some light on the issues raised in the previous paragraphs.

An interesting case for thin-shell stability investigations would be the one where conformal degrees of freedom take place, as related to mechanisms such as color superconductivity (see, e.g. [123] and references therein) and quantum
4.12. Perspectives

Hall effect (see for instance [124] and references therein) that could be present in the interior of compact stars, leading to to trace-free surface energy momentum tensors, which in the spherically symmetric case would imply that $P = \sigma / 2$ [see Eq. (4.10)]. Detailed stability analyses here could only be done when the phases associated with a given hypersurface are given (we stress that the stability analyses done in Ref. [104] are not correct since flux terms [Eq. (4.14)], that are always present in continuous systems, were not taken into account there). Nevertheless, as we showed in section 4.6, whenever the hypersurface is at constant pressures, stable hypersurfaces can always rise. This case contrasts with the impossibility of having stable thin shells for linear thin-shell equations of state in the spherically symmetric case [96], as it can also be seen from Eqs. (4.13), (4.14) [with $\Delta = 0$ here], (4.16) and (4.17).

A very thought-provoking case that is possible to be analyzed in the Darmois-Israel formalism is the one where $\sigma < 0$. If this is valid, irrespective of its magnitude, then the inequality in Eq. (4.18) should be reversed. Such a case would in principle render stable unstable configurations for the case $\sigma > 0$. Due to the cast of the neutron stars we considered here, it is tempting to state that effects such as the Casimir one should possibly be relevant in the astrophysical scenario. This is so because as a simple calculation (using the expression for the energy density to the Casimir effect for two concentric spheres [125]) shows, its energy density is of the same order of magnitude as the one coming from electromagnetic interactions, when the thickness of the shell is of some hundreds of femtometers. This should be better scrutinized. We intend to shed some light into this issue in an upcoming chapter.

4.12 Perspectives

Since there are good reasons for stars being stratified, surfaces degrees of freedom on surfaces of discontinuity could play a role there. It is then necessary to search for their observational fingerprints. In this regard, the “glitches” observed in astrophysical systems [1] could be a sign of the stratification of a system and deserve a closer look in the light of the description of this chapter. The effect of precession could also be invaluable for giving us information about surface quantities, especially the ones near the surface of a star, presumably related to a thin crust cloaking the nucleus. Connecting this with QPOs [1, 126, 127] would be of fundamental importance. Especially in order to try to falsify our analyses
4. Thin shell stability in stars with phase transitions

based on Ref. [103], it would be of interest to search for signatures of heavy elements in stars exceeding $1.97 M_{\odot}$.

Another crucial point to very precise predictions would be knowledge of the equation of state on a surface of discontinuity. Therefore, studies attempting to unveil the microphysics of a thin shell imbedded in a star are also in order.
Chapter 5

Radial stability of stratified stars

5.1 Introduction

In this chapter we shall take a look at perturbations in systems such as stars constituted of various phases that are split by surfaces that host nontrivial degrees of freedom. This analysis is purported to generalize the ordinary ones valid for continuous systems (for a comprehensive analysis of properties, types and stability of continuous anisotropic fluids, see Ref. [128] and references therein). By investigating the dynamics of perturbations, we are automatically probing the stability of systems. We shall trammel ourselves to the simplest case possible: spherically symmetric extended bodies where radial perturbations take place. In order to model the problem, we shall assume these surfaces of discontinuity are very thin and a generalized distributional approach * [129] shall be utilized. We commence our analyses by the Newtonian case, in order to gain intuition of the relevant aspects of the problem and finally generalize them to general relativity in the presence of electrodynamical interactions.

The main motivation for this study lies on the fact that astrophysical bodies, such as neutron stars, should indeed be made up of several phases [1]. This is a consequence related to the astonishingly high densities they could attain in their inner regions, even bigger than the nuclear one, where QCD (quantum chromodynamics) should play an important role [1]. This latter theory allows for phase transitions and this is one of the reasons neutron stars should be stratified. Between any two phases, which can be very different, it is reasonable to investigate the situation where some quantities are discontinuous (characterizing what we

*We shall often call it by distributional approach, or solely distributions, just for simplicity.
5. Radial stability of stratified stars

would call phase transitions), e.g. the pressure. Such discontinuities can be easily harnessed by appropriate surface tensions. Nevertheless, these surface quantities influence the stability of a system, since as being singular, they add up new boundary conditions to the problem, which ultimately would modify the set of eigenfrequencies and eigenmodes of a star. Here we address such points for various star configurations. Our purpose is solely to expound the problem and seek to solve it as generically as we can. Our analysis is far from being complete. It is thought to be the first step towards deeper investigations. Scrutinies of specific cases shall be the objects of posterior studies.

Usually the issue of stratified systems is analyzed in contexts where the influence of surface degrees of freedom is ignored. This is related to interesting and relevant scenarios, but certainly is not the most general one. With the insertion of surface degrees of freedom we could appreciate broader situations, that could give us insights about processes taking place in astrophysics. There are no doubts extended systems are endowed with electrodynamical structures. For instance, magnetic fields must take place in order to explain the radiation observed from rotating neutron stars (e.g. cyclotron and synchrotron radiations) [130]. Actually, it should be present whenever densities are high enough, as it can be the case for neutron stars. Indeed, the complete equilibrium of the multi-component fluid in the cores of compact stars needs the presence of an electric charge separation caused by gravito-polarization effects [109, 110], favoring a sharp core-crust transition that ensures the global, but not the local, charge neutrality [90, 103]. This would also justify our analyses in this chapter in this direction.

We are going to show in this chapter that phase transitions in the presence of surface degrees of freedom can be enclosed into additional boundary conditions to the problem. Besides, our formalism gives us as a direct consequence that such boundary conditions are only self-consistent when set of eigenfrequencies of the perturbational modes is related to the global system, not with individual phases. This is in congruence with the very well-known results from coupled springs, where there are global frequencies. The presence of further boundary conditions naturally modify the possible set of eigenfrequencies, since we are inserting further restrictive aspects to the physical modes. Therefore, measurements on the pulsational modes in a star could tell us very precisely about its internal structure, being a sort of its fingerprint. This could be invaluable to understand better the nature of these systems and constitutes another very important motivation.
to the analyses ensuing in this chapter.

Units here are such that $c = G = 1$, unless otherwise stated. The signature we utilize in this chapter is $-2$.

5.2 Stability of classical layered systems

Assume a continuous classical astrophysical system with spherical symmetry. When its volume elements are perturbed radially, it is well-known [131] that the evolution of perturbations of the form $\tilde{\xi}(r, t) = \xi(r) e^{i\omega t}$, with $\omega$ an arbitrary constant, is described by

$$
\frac{d}{dr}\left[\Gamma \frac{1}{r^2} \frac{d(r^2 \xi)}{dr}\right] - 4 \frac{dP}{dr} \xi + \omega^2 \rho \xi = 0, 
$$

(5.1)

where $P(r)$ is the pressure of the background system under hydrostatic equilibrium and

$$
\Gamma = \frac{\rho}{P} \frac{\partial P}{\partial \rho},
$$

(5.2)

with $\rho(r)$ the mass density of the background system. Put in this way, we have at hands an eigenvalue problem. For the case of continuous systems, the boundary conditions to be supplemented to Eq. (5.1) are quite simple. They are directly related to the spherical symmetry of the system, as well as to the vanishing of its pressure on its border even in the presence of perturbations (other situations where a surface tension would be present could also be envisaged and we shall attempt to elaborate on them in the sequel). In other words, we impose that

$$
\xi(0) = 0, \quad \text{and} \quad \xi(R_s) = \text{finite},
$$

(5.3)

where $R_s$ was defined as the the radius of the star, such that $P(R_s) = 0$. For further details about these boundary conditions, see Ref.[131]. Equation (5.1) supplemented with Eq. (5.3) constitutes a Sturm-Liouville problem, where the aspects of its solutions are already known [131]. Concerning the eigenfrequencies $\omega^2$, they are all real and form a discreet hierarchical set. When one seeks for stable solutions to Eq. (5.1), one seeks for solutions with positive $\omega^2$, specially for its fundamental mode. Negative values for $\omega^2$ indicate instabilities in the as-
5. Radial stability of stratified stars

sumed background system, which leads to the conclusion that they do not linger on in time. They would either implode or explode.

Now we turn to the more involved problem of permitting the system to be stratified and harboring surface degrees of freedom on the interface of two given phases. Such degrees of freedom have themselves a dynamics, described generically by the thin shell formalism or Darmois-Israel formalism [6, 91–93]. Here, we are interested in another aspect of the problem, though. We want to understand the role such degrees of freedom play on the stability of the system when its parts (defined naturally by the hypersurface that hosts the aforesaid degrees of freedom) are perturbed. Therefore, before anything, it is assumed that for being meaningful to talk about this scenario, one has that the hypersurfaces of discontinuity themselves are stable. If this is not the case, any displacement of the hypersurface of discontinuity would trigger a cataclysmic set of events that would have as a result its disruption.

One expects that the stratified problem could be accounted for additional boundary conditions to the system. The reason for this is simple. The perturbations in the upper and lower regions with respect to a given surface of discontinuity would be described by the same physics [e.g. Eq. (5.1)], as well as the outcome of the match (stratified star as a whole). The only missing points would be connection of the phases (the looked for boundary conditions, allowing combinations of solutions to be also solutions to the physical equations involved) and its generalization by means of surface quantities. In this regard, the existence of a surface tension would account to an extra surface force term. The same ensues with the presence of a surface mass [enclosed by a surface mass density] and the associated presence of a surface gravitational force. Clearly these forces play a role into the dynamics of a system and must be taken into account for its correct description. Therefore, in order to properly describe stratified systems, one must make use of distributions [6, 129], which automatically give their boundary conditions.

Now we proceed with the distributional generalization to the equations describing (continuous) fluids under gravitational fields. Assume that a surface harboring surface degrees of freedom in a continuous system is at \( r = R \), being the latter one of its equilibrium points. The first equation to be generalized in terms of distributions in this case is the equation of hydrostatic equilibrium. This
5.2. Stability of classical layered systems

should now read

\[
\frac{dP}{dr} + \rho g(r) - \frac{2\mathcal{P}}{R} \delta(r - R) = 0, \quad (5.4)
\]

where \( g(r) \) has been defined as the norm of the gravitational field, solution to (the distributional) Poisson’s equation \( \nabla \cdot g = -4\pi G\rho \), \( g = -g(r)\hat{r} \), that can always be written as

\[
g(r) = \frac{GM(r)}{r^2}, \quad M(r) = 4\pi \int_{0}^{r} \rho(\bar{r})\bar{r}^2 d\bar{r}. \quad (5.5)
\]

Besides, \( \mathcal{P} \) in Eq. (5.4) stands for the surface tension \([132]\) on the surface of discontinuity at \( r = R \). The expression on the aforesaid equation is the result of restoring surface forces (this is the reason they have an opposite direction to the pressure gradient) at a small surface area (the factor 2 comes from the principal curvatures in a surface element, which in the spherically symmetric case are equal). The gravitational force from the sheet of mass at \( r = R \) is naturally incorporated on the distributional definitions of \( \rho \) and \( g(r) \) as we shall show in the sequel. The mass density, on the other hand, must be expressed as

\[
\rho(r) = \rho^-(r)\theta(R - r) + \rho^+(r)\theta(r - R) + \sigma \delta(r - R), \quad (5.6)
\]

with \( \theta(r - R) \) the Heaviside function, whose derivative is the Dirac delta function \( \delta(r - R) \) in the sense of distributions \([100]\). From the existence of a surface mass density, it is trivial to see that the gravitational field is discontinuous at a given surface of discontinuity of the system (at \( R \)) and it is

\[
[g(R)]^+_\mathcal{=} = 4\pi G\sigma, \quad (5.7)
\]

where we have used the convention \([A]_\mathcal{=}^+ = A^+ - A^-\), the jump of \( A(r) \) across \( r = R \). Therefore, \( g(r) \) must be represented distributionally as

\[
g = g^+(r)\theta(r - R) + g^-(r)\theta(R - r). \quad (5.8)
\]

The above equation means that the associated distributional gravitational potential \( \phi(r) \) (\( g = -\nabla\phi, \ g = -g\hat{r} \)) is always a continuous function, though not
differentiable at a surface of discontinuity.

Besides, we will assume that also the pressure is discontinuous at $r = R$, being hence written as

$$P(r) = P^-(r)\theta(R - r) + P^+(r)\theta(r - R). \quad (5.9)$$

The reason of why this is so will be clarified when we deal with our original problem in the scope of general relativity. The heuristic argument corroborating the validity of Eq. (5.9) is that it is meaningless to colligate a surface term to the radial pressure, since its associated force would necessarily be normal to it and therefore would not lie on the surface. Just tangential pressures should have surface terms related to them. From Eqs. (5.9), (5.6), (5.5) (5.7) and (5.8), we have (for the equilibrium case)

$$\mathcal{P} = \frac{R}{2}[P(R)]^+ + \frac{G}{16\pi R^3}[M^2(R)]^- \quad (5.10)$$

and

$$\frac{dP^\pm}{dr} + \rho^\pm g^\pm = 0. \quad (5.11)$$

The arithmetic average present in Eq. (5.10) [see the definition of $g(r)$ and the value of $\sigma$] is a general consequence of the product of delta functions with Heaviside ones in the sense of generalized distributions [129, 133]. Notice from Eq. (5.10) that its first term is the known Young-Laplace equation for spherical surfaces at equilibrium [108, 132], where just geometric aspects are taken into account for the surface tension. Its second term, though, is the gravitational surface tension, uniquely due to the non-zero surface mass. If the surface tension were null, the pressure jump could not be arbitrary, but proportional to the surface mass density and must be a monotonically decreasing function of the radial coordinate [see Eqs. (5.7) and (5.10)]. From Eq. (5.10), one sees further that the force per unit area associated with the surface tension is exactly the one necessary to counterbalance both the forces coming from the pressure gradient at $R$ and the surface gravitational force, as it should be.

One sees that the above procedure generalizes our notion of hydrostatic equilibrium in each phase the stratified system has [see Eq. (5.11)] and automatically gives the surface tension at $R$ that guarantees the hydrostatic equilibrium for
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arbitrary pressure jumps and surface masses. We will keep the same philosophy now concerning the generalization of Eq. (5.1). From our generalized hydrostatic equilibrium equation, we have that an important term for deduction of the equation governing radial perturbations would be the application of the Lagrangian operator \( \Delta [\Delta A = A(t, r + \xi) - A_0(t, r)] \), \( A_0 \) and \( A \) being a physical quantity in the equilibrium and perturbed cases, respectively] on the surface force in Eq. (5.4), i.e. [see Eq. (5.19)]

\[
\Delta \left[ \frac{\mathcal{P}}{R} \delta (r - R) \right] = \frac{\Delta \mathcal{P}}{R} \delta (r - R) - \frac{\mathcal{P} \xi}{R^2} \delta (r - R),
\]

(5.12)

since \( \Delta \delta (r - R) = 0 \) and \( \Delta R = \xi \). Now we assume that \( \mathcal{P} = \mathcal{P}(\sigma) \). This means that we are endowing the fluid at the surface of discontinuity with adiabatic properties and the underlying microphysics is not contemplated in this procedure. For continuous media, the total mass in the interface of two phases is generally not a constant. This means that mass fluxes are allowed to take place. This generically would render the mass of each phase not constant, an aspect not taken into account in Eq. (5.1). Nevertheless, if the displacements of the surface of discontinuity are small and oscillatory, we have that on average the masses on each phase are conserved (here it becomes clear why the surface of discontinuity should be stable). For adiabatic processes, we have

\[
\Delta \mathcal{P} = \eta^2 \Delta \sigma, \quad \eta^2 = \frac{\partial \mathcal{P}}{\partial \sigma},
\]

(5.13)

with \( \eta^2 \) the square of the speed of the sound in the fluid at the surface of discontinuity. The missing term \( \Delta \sigma \) can be easily found with the thin shell formalism when the classical limit is taken there. Generically, in the static and spherically symmetric case, \( \sigma \) can be written as [93]

\[
\sigma = -\frac{c^2}{4\pi GR} \left[ e^{-\beta(R)} \right]_+, \quad \beta(R) \approx GM(R) / (rc^2),
\]

(5.14)

with the classical limit \( \beta(r) \approx GM(r) / (rc^2) \ll 1 \). It is very easy to see that Eq. (5.14) reduces to Eq. (5.7) in the aforementioned limit. When perturbed, it can be showed that \( \beta \to \beta + \delta \beta, \) with \( \delta \beta = -4\pi Gr_0 \xi / (c^2) \) [36], \( r_0 \) here meaning
the mass density in the hydrostatic background solution. Hence,

$$\Delta \sigma = \delta \sigma + \sigma'_0 \tilde{\xi} = -[\rho_0 \tilde{\xi}]^+ + c'_0 \tilde{\xi},$$

(5.15)

where we also considered $\sigma_0$ the background solution [Eq. (5.7)]. Another simpler way of obtaining Eq. (5.15) would be by means of the dynamics of $\delta g$ [$\delta g = -(\delta g)\hat{r}$]. It is simple to show that in the spherically symmetric case we have $\delta g = -4\pi G \rho \tilde{\xi}$ [131]. Since $\delta g \equiv g - g_0$, with $g_0$ the norm of the gravitational field without the perturbation $\tilde{\xi}$, from Eq. (5.7), we finally have that

$$\Delta \sigma = \left[\frac{\delta g(R)}{4\pi G}\right]^+ + \sigma'_0(R) \tilde{\xi} = c'_0(R) \tilde{\xi} - [\rho_0 \tilde{\xi}]^+,$$

(5.16)

as we have found previously. Besides, from Eqs. (5.5) and (5.7), for the case when the jump of $\tilde{\xi}$ is null at a surface of discontinuity (that shall be justified in the sequel), one shows that the above equation can be further simplified to

$$\Delta \sigma = -\frac{2}{r} \sigma \tilde{\xi}.$$

(5.17)

Now we exhibit the general equation governing the propagation of radial perturbations. For $P$ defined as in Eq. (5.9), $\rho$ in terms of Eq. (5.6), $g$ as given by Eq. (5.8) and finally

$$\tilde{\xi}(r, t) = \tilde{\xi}^+(r, t)\theta(r - R) + \tilde{\xi}^-(r, t)\theta(R - r),$$

(5.18)

the equation governing the evolution of perturbations on a given volume element of the fluid is

$$\Delta \left[\rho \frac{dv_r}{dt} + \frac{\partial P}{\partial r} + \rho g(r) - \left\{\frac{2P}{R} + \frac{\sigma}{2}(\dot{v}_r^+ + \dot{v}_r^-)\right\}\delta(r - R)\right] = 0.$$

(5.19)

Notice that we assumed that $v_r$ is a distribution like $\tilde{\xi}$. Physically this must be taken because the phases are always localizable and do not mix. In other words, this constraint reflects the intuitive fact that the surface of discontinuity should be well-defined. The mathematical reason of what this is so shall be given in the sequel and it is related to the well-posedness of the problem.

When developed, by taking into account the hydrostatic equilibrium equation
5.2. Stability of classical layered systems

[see Eq. (5.4)] and Eqs. (5.13) and (5.17), Eq. (5.19) can be simplified to

\[
\frac{\partial}{\partial r} \left[ \frac{\Gamma P}{r^2} \frac{\partial (r^2 \xi)}{\partial r} \right] - 4 \frac{\partial P}{\partial r} \xi - \frac{d^2 \xi}{dt^2} \rho = - \left[ 2 \xi \overline{R}^2 \right] (3 \mathcal{P} - 2 \eta^2 \sigma) \\
+ \frac{\sigma}{2} \left( \frac{d^2 \xi^+}{dt^2} + \frac{d^2 \xi^-}{dt^2} \right) \delta (r - R).
\]

(5.20)

First note that coefficients multiplied by \( \delta^2 (r - R) \) or \( \delta' (r - R) \) in Eq. (5.20) must be all null. Taking into account Eq. (5.18), it means that \( \xi^+ = 0 \). This automatically warrants \( \xi^+ \) as a distribution without Dirac deltas, as we have advanced previously. In order to obtain Eq. (5.20), we used the results that for a distribution \( A(r) = A^+(r) \theta (r - R) + A^-(r) \theta (R - r) \),

\[
\Delta A = \delta A + \frac{\partial A}{\partial \xi^+} \xi - [A]^+ \xi^+ \delta (r - R),
\]

(5.21)

\[
\Delta \left( \frac{\partial A}{\partial r} \right) = \frac{\partial}{\partial r} (\Delta A) - \frac{\partial \xi}{\partial r} \frac{\partial A}{\partial r} + \frac{1}{2} \left[ \frac{\partial \xi^+}{\partial r} + \frac{\partial \xi^-}{\partial r} \right] [A(R)]^\pm \delta (r - R),
\]

(5.22)

besides having made use of

\[
\Delta \rho = - \rho \frac{\partial}{\partial r} \left( r^2 \xi \right) + \frac{\sigma}{2} \left( \frac{\partial \xi^+}{\partial r} + \frac{\partial \xi^-}{\partial r} \right) \delta (r - R),
\]

(5.23)

which is a direct consequence of assuming that the total mass in each phase is constant, even in the presence of perturbations. This is just guaranteed if the surface of discontinuity is stable, a prime hypothesis for having a well-posed stability problem. We also have assumed that \( P = P(\rho) \), which implies that \( \Delta P^\pm = \Gamma^\pm \rho^\pm \Delta \rho^\pm / \rho^\pm \). For deriving Eq. (5.20) we further took into account Eq. (5.10). We finally stress that a simpler way to obtain Eq. (5.20) is to recall that \( \Delta M = 0 \), which guarantees that \( \Delta g = - 2 \xi \). The fact that \( \Delta M = 0 \) means that comoving observers with the fluid do not notice a mass change. The aforementioned result can also be directly showed by Eqs. (5.5), (5.22) and (5.23).

It is simple to see that solutions of the type \( \xi^\pm (r, t) = e^{i \omega^\pm t} \xi^\pm (r) \) for Eq. (5.26) are just meaningful if \( \omega^+ = - \omega^- = \omega \). This is the only way to eliminate the time dependence in Eq. (5.26), and also to guarantee that the jump of \( \xi^\pm \) be null for any surface of discontinuity at any time. Therefore, we arrive at the important
5. Radial stability of stratified stars

conclusion that even a stratified system where oscillatory radial perturbations take place should be described by a sole set of frequencies. Each member of this set describes the eigenfrequency of the whole system, instead of one or another phase. Nevertheless, we recall that at the surface of discontinuity the frequencies are in principle not defined. Bearing in mind the above conclusions, we have that

$$\tilde{\xi}(r, t) = e^{i\omega t}\tilde{\xi}(r),$$ (5.24)

and boundary condition $[\tilde{\xi}(R)]^+_- = 0$, or $\tilde{\xi}^+(R) = \tilde{\xi}^-(R) \equiv \tilde{\xi}(R)$. In this scope, the only meaningful $\Gamma$ are given by

$$\Gamma(r) = \Gamma^-(r)\theta(R - r) + \Gamma^+(r)\theta(r - R).$$ (5.25)

Gathering the previous equations on Eq. (5.20), we arrive at

$$\frac{d}{dr} \left[ \Gamma P \frac{\partial}{\partial r}(r^2\xi) \right] - \frac{4}{r} P \frac{dP}{dr} \xi + \omega^2 \rho \xi = -\tilde{\xi}(R) \left[ \frac{2}{R^2} \left( 3\mathcal{P} - 2\eta^2\sigma \right) - \omega^2 \sigma \right] \delta(r - R).$$ (5.26)

Ones sees from Eq. (5.26) that in the case where $\mathcal{P}$ and $\sigma$ are null, the classical expression [Eq. (5.1)] is recovered.

Summing up, by substituting Eqs. (5.24) and (5.6) into Eq. (5.26), one sees that the only way to satisfy such an equation is by imposing that

$$\frac{d}{dr} \left[ \Gamma P \frac{dP}{dr}(r^2\xi) \right] \pm - \frac{4}{r} P \frac{dP}{dr} \xi^\pm + \omega^2 \rho^\pm \xi^\pm = 0, \quad (5.27)$$

$$\left[ \Gamma P \frac{d}{dr}(r^2\xi) \right]^\pm - 4R\tilde{\xi}(R)[P(R)]^\pm + 2\tilde{\xi}(R)[3\mathcal{P} - 2\eta^2\sigma] = 0, \quad (5.28)$$

and for completeness,

$$\tilde{\xi}^+(R) = \tilde{\xi}^-(R) = \tilde{\xi}(R).$$ (5.29)

Equations (5.27) are obtained here as a consequence of our distributional search for solutions to the radial Lagrangian displacements. These are exactly what one
5.3 Homogeneous stars

Let us now investigate a star made out of two phases whose mass densities are constants. For these stars, assume also that the associated $\Gamma$ for each region is an arbitrary constant. Naturally this is just an academic example. Nevertheless, it already evidences some aspects stratified systems should have. For this case it is straightforward to solve Poisson’s equation and the equation of hydrostatic equilibrium (see [131] for further details) and we have for $r < R$

$$P^-(r) = \frac{2\pi G \rho^2}{3} (R^2 - r^2), \quad \mathcal{R} = \frac{3 p_0^-}{2\pi G \rho^2},$$

(5.30)

where $p_0^-$ is an arbitrary constant that corresponds to the pressure of system at the origin. For $r > R$, instead

$$P^+(r) = \frac{2\pi G \rho^2}{3} (R^2 - r^2).$$

(5.31)
5. Radial stability of stratified stars

The constant mass density in the inner (outer) region was defined as $\rho^- (\rho^+)$. The pressure at the origin $p_0^-$ could always be chosen such that it matches the pressure at the base of the outer phase. One sees from Eq. (5.31) that the pressure at the surface of the star is null. Substituting Eqs. (5.30) and (5.31) on Eq. (5.27), we are led to

$$
(1 - x_{\pm}^2) \frac{d^2}{dx_{\pm}^2} \xi_{\pm} + \left( \frac{2}{x_{\pm}} - 4x_{\pm} \right) \frac{d}{dx_{\pm}} \xi_{\pm} + \left( A_{\pm} - \frac{2}{x_{\pm}^2} \right) \xi_{\pm} = 0, \quad (5.32)
$$

where we assumed that $x_{\pm} \equiv r/R_s$ and $x_{\pm} \equiv r/\mathcal{R}$ and

$$
A_{\pm} \equiv \frac{3\omega_{\pm}^2}{2\pi G \rho \mp \Gamma_{\pm}} + \frac{8}{\Gamma_{\pm}} - 2. \quad (5.33)
$$

We try to solve Eq. (5.32) by the method of Frobenius. Just for the sake of simplicity, we shall drop the $\pm$ notation. We therefore assume that its solutions are of the form

$$
\xi = \sum_{n=0}^{\infty} a_n x^{n+s}, \quad (5.34)
$$

where $a_n$ and $s$ are arbitrary constants to be fixed by primarily demanding that the first condition of Eq. (5.3) is satisfied, as well as $\xi(x)$ is always finite. By substituting Eq. (5.34) into Eq. (5.32), it is easy to see that the solutions to $s$ are either $s = 1$ or $s = -2$. The associated recurrence relation obtained generally is

$$
a_{m+2} = \frac{(m + s)(m + s + 3) - A}{(m + s + 2)(m + s + 3) - 2} a_m, \quad m = 0, 2, 4..., \quad a_1 = a_3 = a_5 = ... = 0. \quad (5.35)
$$

Let us analyze first the inner region. It is clear in this case that the associated $a_0$ for $s = -2$ must be null, as a consequence of one of our boundary conditions. From Eq. (5.35), one clearly sees that the power series given by Eq. (5.34) does not converge actually for any $s$. Therefore, in order to satisfy the finiteness anywhere of $\xi$, we have to impose that the series be truncated somewhere, rendering it actually a polynomial. Hence

$$
A_{m,s=1}^- = (m + 1)(m + 4). \quad (5.36)
$$
5.3. Homogeneous stars

From Eq. (5.33), one sees that just discreet frequencies [given by Eq. (5.36)] are possible to this region. From Eqs. (5.33) and (5.36), for having the frequency of the fundamental mode \((m = 0)\) positive, one should have \(\Gamma^+ \geq 4/3\). Summing up, the physically relevant solution to this case just leaves out an arbitrary constant of integration, as required due to the scaling law present to \(\xi\) from Eq. (5.27).

Let us now analyze the outer region. This is the most physically interesting region since origin problems are absent and therefore in principle one could even have two linearly independent solutions to \(\xi\). Due to the finiteness of \(\xi\) in this region, the outer counterpart of Eq. (5.36) must again take place. Nevertheless, for \(s = -2\), one should also impose

\[
A^+_{m,s=-2} = (m - 2)(m + 1). \tag{5.37}
\]

From Eq. (5.33), one clearly sees from this case that its associated fundamental mode \((m = 0)\) is unstable. This means in principle that this solution to the outer region should be excluded, leaving out just the one from the case \(s = 1\), where we should consider that

\[
A^+_{m,s=1} = (m + 1)(m + 4). \tag{5.38}
\]

Notwithstanding, our previous analysis exhibits clear problems: there are not enough arbitrary constants to fix Eqs. (5.28) and (5.29) and the eigenfrequencies in each region are different. However we shall show that the condition of a same eigenfrequency for the whole system, as required by our formalism, addresses all the problems. From what we have showed before, obviously the stable eigenfrequencies to the star are only related to the solution \(s = 1\). However, here they could rise either from aspects of the inner or the outer phases of the star. Let us see how this conclusion ensues. Assume initially that the only possible \(\omega\)'s are given by Eq. (5.36), associated with the modes \(m^-_{s=1}\). So, for having finite \(\bar{\xi}^+\) related to \(s = 1\), one must impose that there exists a \(m^+_{s=1}\) to the outer phase such that the numerator of the associated recurrence relation be null. It is easy to show that this is just the case if

\[
m^+_{s=1} = \frac{-5 + \sqrt{9 + 4A^+_{s=1}(m^-_{s=1})}}{2}. \tag{5.39}
\]
5. Radial stability of stratified stars

Therefore, Eq. (5.39) demands that

\[ 9 + 4A_{s=1}^+ = (2p + 1)^2, \quad p \geq 2, p \in N. \]  \hspace{1cm} (5.40)

For the case \( s = -2 \) to \( \xi^+ \), it is simple to show that the condition for the existence of a \( m_{s=-2}^+ \) related to \( a_{m_{s=-2}^+} = 0 \) is exactly given by Eq. (5.40). The mode itself is

\[ m_{s=-2}^+ = \frac{1 + \sqrt{9 + 4A_{s=1}^+(m_{s=1}^-)}}{2}. \] \hspace{1cm} (5.41)

Summing up: if Eq. (5.40) is satisfied for any natural \( p \geq 2 \), there always exist modes [characterized by Eqs. (5.39) and (5.41)] that guarantee the finiteness of \( \xi^+ \) as a linear combination of solutions for \( s = 1 \) and \( s = -2 \), associated with a given eigenfrequency \( \omega_{m_{s=1}^-} \) that just takes into account aspects of the inner phase of the system. In this case, one is able to come out with two arbitrary constants of integration, that would then guarantee that the additional boundary conditions raised by the stratification [see Eqs. (5.29) and (5.28)] be satisfied. It is immediate to see that a similar reasoning as above ensues if one chooses now \( \omega_{m_{s=1}^+} \) as coming from aspects of the outer region [given now by Eqs. (5.33) and (5.38)].

For this case, we will find now a \( m_{s=1}^- \) and a \( m_{s=-2}^+ \) associated with \( \omega_{m_{s=1}^+} \), as given by Eqs. (5.39) and (5.41) [with the condition given by Eq. (5.40)], replacing \( A_{s=1}^-(m_{s=1}^-) \) by \( A_{s=1}^+(m_{s=1}^+) \). Since \( \Gamma^\pm \) and \( \rho^\pm \) are given quantities, one sees that the only possible eigenfrequencies to system should satisfy \( 9 + 4A_{s=1}^\pm = (2p + 1)^2 \). This constraint is uniquely imposed due to the extra boundary conditions to the problem and is very restrictive.

This simple example shows how stratification with surface degrees of freedom render the physical systems more complex. Now we study another simple example which is motivated by the profile dark matter should have (at least in some regions around a galaxy): isothermal spheres.
5.4 Isothermal spheres

Now we assume that there is a region of a given system whose mass density given by

\[ \rho(r) = \frac{k}{r^2}, \]  

(5.42)

where \( k \) is a positive arbitrary constant. The solution to the Poisson equation with this density profile in the spherically symmetric case is simple and its associated gravitational potential is

\[ \phi(r) = 4\pi k \ln r. \]  

(5.43)

Notice that this potential only leads to bound trajectories. The corresponding gravitational field decreases with the inverse of the distance and is therefore a long range force. The hydrostatic equilibrium to this isothermal (is) phase is guaranteed by the pressure

\[ P_{is}(r) = A \left( \frac{1}{C} + \frac{1}{r^2} \right), \quad A = 2\pi k^2, \]  

(5.44)

where \( C \) is an arbitrary constant of integration. From Eqs. (5.42) and (5.44), one sees that the adiabatic index \( \Gamma \) (see Eq. (5.2)) for this phase is

\[ \Gamma = \frac{C}{C + r^2}, \]  

(5.45)

and its associated equation of state is linear.

We observe that for a single phase solution the pressure should be zero at infinity, what would mean that \( \Gamma = 1 \) and \( P = A/r^2 \). The stability analysis to this case (see Ref. [131]) shows that this system is unstable. Nevertheless, if it is finite and one assumes that at its boundary, say at \( r = R_0 \), the pressure is null, being positive for any \( r < R_0 \), then \( C = -R_0^2 \) and this case would always be stable against radial perturbations. We are not interested here in scrutinizing further such aspects, but just in analyzing whether or not enough arbitrary constants could appear for allowing the match with other phases.

The other regions constituting the system in this moment are not important
5. Radial stability of stratified stars

to be specified. We proceed now with the radial Lagrangian perturbations of isothermal spheres. We take in the region where it is defined, just by convenience, the equation of radial perturbations in the form [134]

\[
\frac{d}{dr} \left( \Gamma r^4 \frac{d}{dr} \xi \right) + \xi \left[ \omega^2 r^4 + r^3 \frac{d}{dr} (3 \Gamma - 4) P \right] = 0, \tag{5.46}
\]

which is equivalent to Eq. (5.1) by the identification \( \xi = r \tilde{\xi} \). By substituting Eqs. (5.42), (5.44) and (5.45) into Eq. (5.46), it is easy to show that it can be cast as

\[
\frac{d}{dr} \left( r^2 \frac{d}{dr} \tilde{\xi} \right) + \tilde{\xi} \left[ \frac{\omega^2 r^2}{2\pi k} + 2 \right] = 0. \tag{5.47}
\]

Notice that the above equation is independent of \( C \), as it must be, since it is an arbitrary constant.

It can be checked that solutions to Eq. (5.47) exist for every value of \( \omega^2 > 0 \), given in terms of the spherical Bessel functions of imaginary arguments. In special, for the case \( \omega^2 = 0 \) they are

\[
\xi = C_1 \sqrt{r} \cos \left( \frac{\sqrt{7}}{2} \ln r \right) + C_2 \sqrt{r} \sin \left( \frac{\sqrt{7}}{2} \ln r \right), \tag{5.48}
\]

where \( C_1 \) and \( C_2 \) are real constants. Notice that Eq. (5.48) is null at the origin, as requested by the first condition of Eq. (5.3), if the isothermal sphere phase happens to be there. It is also finite for limited regions. From Eq. (5.48), we have two arbitrary constants of integration, fundamental for the proper match [see Eqs. (5.29) and (5.28)] of the isothermal phase with adjacent regions. Therefore, a system with an isothermal sphere could easily be glued to any other phases.

We have just shown two simple examples where some of the aspects imprinted by stratification raise. Whenever there are two arbitrary solutions to \( \xi \) in a given phase, it will be always possible to satisfy the constraints (5.29) and (5.28).

5.5 Classical layered electromagnetic systems

Now we attempt to give a further step in our distributional generalizations of continuous classical fluids interacting with gravity, by endowing the phases (as
well as the surface of discontinuity) with an electromagnetic structure. Just for clarity, let us work with a system that exhibits just an electric field. The first point to be taken into account is the additional electric force present in the system. This would have the same structure as the gravitational force and therefore its generalization is trivial. Now one should define also a distributional solution to the charge density. The surface force associated with the surface tension should have the same form as previously, but now should also take into account the present electric aspects. The pressure in this case would also change due to the presence of the electric field and its jump over a surface of discontinuity could still be kept free by the insertion of convenient surface tensions.

From the (distributional) Maxwell equations in the spherically symmetric case one knows that

$$ E(r) = \frac{Q(r)}{r^2}, \quad Q(r) = 4\pi \int_0^r \rho_c(\bar{r}) \bar{r}^2 d\bar{r}, \quad \text{(5.49)} $$

where $\rho_c(r)$ is the charge density at $r$. The associated “force density” is $d\mathbf{F}_e/dv = \rho_c E(r) \hat{r}$. Therefore, the distributional equation of hydrostatic equilibrium now reads

$$ \frac{dP}{dr} + \rho(r)g(r) - \rho_c(r)E(r) - \frac{2PQ}{R} \delta(r - R) = 0. \quad \text{(5.50)} $$

Like the gravitational field, the electric one also presents a jump at any surface of discontinuity (at $r = R$) endowed with surface charges. We write the charge density as

$$ \rho_c(r) = \rho_c^-(r)\theta(R - r) + \rho_c^+(r)\theta(r - R) + \sigma_c \delta(r - R), \quad \text{(5.51)} $$

while the distributional electric field is

$$ E(r) = E^-(r)\theta(R - r) + E^+(r)\theta(r - R), \quad [E(R)]^+ = 4\pi \sigma_c. \quad \text{(5.52)} $$

By substituting now Eqs. (5.9), (5.49), (5.51) and (5.52) into Eq. (5.50), we have that the surface tension at equilibrium should read

$$ P_Q = \frac{R}{2}[P(R)]^+ + \frac{G}{16\pi R^3}[M^2(R)]^+ - \frac{1}{16\pi R^3}[Q^2(R)]^+. \quad \text{(5.53)} $$
5. Radial stability of stratified stars

Notice that the existence of a surface mass would lead to $[M^2(R)]^+ > 0$, while $[Q^2(R)]^\pm$ could in principle be any.

Since in the presence of an electric field the hydrostatic equilibrium equation and the surface tension changes, it is simple to verify that Eq. (5.26) keeps the same functional form. In drawing this conclusion, it was also assumed that the total charge of the system is a constant. This also means that $\Delta Q = 0$. One also sees immediately that the main results concerning the stability of the stratified charged case are totally analogous to the neutral one, obtained by simply making the replacement $P \rightarrow P_Q$.

Now we touch the point of the consistency of the above approach. Just for simplicity, assume that there is just one type of fluid in the system. In order to render the analysis unambiguous, we have to consider the particle density (number of particles per unit volume $n$) instead of mass or charge densities to solve the coupled system of equations coming from classical gravitation, electromagnetism and the hydrostatic equilibrium. In order to do so, for adiabatic processes, an equation of state connecting the pressure with the particle number should be given in each phase [i.e. $p = p(n)$]. The densities are themselves given by $\rho = m_b n$ and $\rho_c = q_b n$, where $m_b$ and $q_b$ are the mass and the charge of the particles, respectively. For this case, $\Delta P = \partial P / \partial n \Delta n$. We shall not be concerned with multi-fluids here.

5.6 Stratified systems in general relativity

Now we attempt to generalize the analysis of stratified systems to general relativity. From the classical analysis, we have learned that surface quantities must also be inserted into the generalized equation of hydrostatic equilibrium. Therefore, in a certain sense, we must find the proper generalization of the surface forces in general relativity. This will not be difficult bearing in mind the thin shell formalism, as we shall see in the sequel. Such a formalism states that in order to search for distributional solutions to general relativity, one has to consider an energy-momentum tensor at a surface of discontinuity, that we shall name $\Sigma$. It is precisely this surface content that leads to the jump of quantities that are related to physical observables, such as the extrinsic curvature. We now outline the formalism succinctly. Let us work just in the spherically symmetric case, where $\Sigma$ is defined as $\Phi = r - R(\tau) = 0$, with $\tau$ the the proper time of an observer on the aforesaid hypersurface. Assume that the metrics in the regions
Stratified systems in general relativity

above and below $\Sigma$ (w.r.t. to the normal vector to it), described by the coordinate systems $x^\mu_\pm = (t_\pm, r_\pm, \theta_\pm, \varphi_\pm)$, respectively, are given by

$$
\begin{align*}
\text{ds}^2_\pm &= e^{2\alpha_\pm(r_\pm)} dt^2_\pm - e^{2\beta_\pm(r_\pm)} dr^2_\pm - r^2_\pm d\Omega^2_\pm, \\
&\quad \text{where} \\
\text{d}\Omega^2_\pm &= \text{d}\theta^2_\pm + \sin^2 \theta_\pm \text{d}\phi^2_\pm.
\end{align*}
$$

(5.54)

Assume that the (three dimensional) hypersurface $\Sigma$ be described by the (intrinsic) coordinates $y^a = (\tau, \theta, \varphi)$ such that at the hypersurface $t_\pm = t_\pm(\tau)$, $\theta_\pm = \theta$ and $\varphi_\pm = \varphi$, besides obviously $r_\pm = R(\tau)$. In order to render the procedure consistent, one has to impose primarily that the intrinsic metric to $\Sigma$ is unique. This fixes the coordinate transformations $x^\mu_\pm = x^\mu_\pm(y^a)$. This is the generalization of the continuity of the gravitational potential across a surface harboring surface degrees of freedom. Now, if the jump of the extrinsic curvature is non-null, it is automatically guaranteed the existence of a surface energy-momentum tensor [6] that in the spherically symmetric case can always be cast as $S^a_b = \text{diag}(\sigma, -P, -P)$, with (see for instance [6])

$$
\sigma = -\frac{1}{4\pi R} \left[ \sqrt{e^{-2\beta} + R^2} \right]^+, \quad \text{(5.56)}
$$

$$
P = -\sigma^2 + \frac{1}{8\pi R} \left[ \frac{R\alpha'(e^{-2\beta} + R^2) + \dot{R} R + \beta' R^2}{\sqrt{e^{-2\beta} + R^2}} \right]^+, \quad \text{(5.57)}
$$

where generically $A' = \partial A / \partial r$ and $\dot{A} = dA / d\tau$. Finally, the discontinuity of the extrinsic curvature is the generalization of the discontinuity of the gravitational field across a surface with nontrivial degrees of freedom. The case of interest to be analyzed here is the static and stable (upon radial displacements of $\Sigma$) one $\dot{R} = \ddot{R} = 0$, i.e. an equilibrium point. Only in this case the system lingers on in the way it initially was defined.

Let us see now how the generalization of the surface forces appear in this formalism. First of all, we know that

$$
T_{\mu\nu} = T^+_{\mu\nu}(r - R) + T^-_{\mu\nu}(R - r) + e^a_\mu e^b_\nu S_{ab}\delta(r - R),
$$

(5.58)
where $e^a_\mu \equiv \partial y^a / \partial x^\mu$ and $S^a_{\mu} = h_{ac} S^c_\mu$. Notice that $e^a_\mu$ is itself defined as a distribution. For $h_{ab}$, it does not matter the side of $\Sigma$ one takes to evaluate it, since it must be unique. Let us constrain ourselves first to the case of perfect fluids (locally neutral) on each side of $\Sigma$. One sees from Eq. (5.58) and the coordinate transformations at $\Sigma$ that

$$T^0_0 = \rho = \rho^+ \theta (r - R) + \rho^- \theta (R - r) + \sigma \delta (r - R), \quad (5.59)$$

$$T^1_1 = -P = -P^+ \theta (r - R) - P^- \theta (R - r) \quad (5.60)$$

and

$$T^2_2 = T^3_3 = -P_t = -P - P \delta (r - R). \quad (5.61)$$

From Eq. (5.60), we notice that there are not associated surface stresses for radial pressures. This is exactly what we advanced in the classical case with heuristic arguments and obtained here as a general consequence of distributional solutions to general relativity. Let us search formally for solutions to Einstein’s equations with the energy-momentum given by Eqs. (5.59)–(5.61) with the Ansatz

$$ds^2 = e^{2 \alpha} dt^2 - e^{2 \beta} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (5.62)$$

The distributional nature of Eq. (5.62) will be evidenced by Eq. (5.58). As the solution, we have

$$e^{-2 \beta} = 1 - \frac{2 m(r)}{r}, \quad m(r) \equiv 4 \pi \int_0^r \rho (\bar{r}) \bar{r}^2 d\bar{r}. \quad (5.63)$$

Notice from the above equation that $\rho$ is given by Eq. (5.59) and therefore $e^{-2 \beta}$ is a distribution generally discontinuous at $R$. For $\alpha$ we have, though

$$\alpha' = \frac{e^{2 \beta}}{r^2} [4 \pi P r^3 + m(r)], \quad (5.64)$$

where $P$ is given by Eq. (5.60). From Eqs. (5.63) and (5.64), we see that $\alpha'$ is a distribution with no Dirac delta functions terms. Therefore, it implies that $[\alpha]_\pm = 0.$
In other words, the function $\alpha$ is generally continuous though not differentiable at $R$. Nevertheless, from the conservation law of the energy-momentum tensor given by Eq. (5.58), we also have (for perfect fluids)

$$\alpha'_{\pm} (\rho_{\pm} + P_{\pm}) = -P'_{\pm}.$$  \hfill (5.65)

We notice that $T^{\mu\nu} = 0$, taking into account Eq. (5.58), would give us in principle terms dependent upon Heaviside functions, Dirac delta functions and their derivatives. The terms associated with the Heaviside functions are null due to the validity of Einstein’s equations on each side of $\Sigma$. The nullity of the remaining terms is associated with identities that the surface energy-momentum tensor has to satisfy (see for instance Ref. [102]). It is not difficult to show that such identities are automatically satisfied when one takes into account Eqs. (5.56) and (5.57) (see e.g. Ref. [93]). This shows that the thin shell formalism is consistent and the surface terms must indeed be taken as the aforesaid equations.

In order to put Eq. (5.65) in the form that would allow us to consider Eqs. (5.59) and (5.60), we mandatorily should add surface terms. It is simple to realize that the correct way of doing it is

$$\alpha' (\rho + P) = -P' + \left\{ [P(R)]_{\pm} + \frac{\sigma(R)}{2} [\alpha'_{\pm} (R) + \alpha'_{-} (R)] \right\} \delta (r - R),$$  \hfill (5.66)

where $\sigma$ is given by Eq. (5.56). Combining Eqs. (5.64) and (5.66), we finally arrive at the distributional equation of hydrostatic equilibrium in general relativity:

$$\frac{dP}{dr} = -\frac{(\rho + P)}{r^2} e^{2\beta} [4\pi Pr^3 + m(r)] + \frac{2P}{R} \delta (r - R)$$

$$+ \left\{ [P(R)]_{\pm} + \frac{\sigma}{2} [\alpha'_{\pm} (R) + \alpha'_{-} (R)] + \frac{\sigma}{R} - \frac{[\alpha' e^{-\beta}]_{\pm}}{4\pi R} \right\} \delta (r - R).$$  \hfill (5.67)

Now we show that in the classical limit, Eq. (5.67) reduces exactly to Eq. (5.4) and therefore it is its proper generalization. First of all, notice that in such a limit, $\alpha = \phi (r)$, $\phi (r)$ the gravitational potential. Besides, it is simple to show that in such a case

$$\sigma = \frac{1}{4\pi R^2} [M(R)]_{\pm},$$  \hfill (5.68)
where the above quantities are in cgs units. In Eq. (5.68) one recognizes the jump of the gravitational field \( g(r) = \phi' = GM(r)/r^2 \) at \( r = R \), as exactly given by Eq. (5.7). Therefore,

\[
\frac{\sigma}{2} \{a'_+(R) + a'_-(R) \} \simeq \frac{G}{8\pi R^4} [M^2(R)]^\pm. \tag{5.69}
\]

Substituting Eq. (5.69) into Eq. (5.67), we see that the term inside the curly brackets of the latter equation is null [see Eqs. (5.57) and (5.10)]. Hence, the remaining term in front of the delta function is exactly \( 2\mathcal{P}/R \) as we already advanced and expected [see Eq. (5.4)].

Now we are in a position to talk about perturbations in the general relativistic scenario. When they take place, metric and fluid quantities change at a given spacetime point from their static counterparts. It is customary to assume that such departures are small, what allows us to work perturbatively. The primary task is to find such changes from the system of equations coming from relativistic hydrodynamics and general relativity. Nevertheless, these solutions are already very well-known [36]. Our ultimate task is simply to generalize them to the distributional case.

The equation governing the evolution of the fluid displacements on each side of \( \Sigma \) is the general relativistic Euler equation, related to the orthogonal projection of \( T^{\mu\nu};\nu \) onto \( u^{\mu} \), viz. (here for perfect fluids)

\[
(\rho + P)u^{v};\mu u^{\mu} \doteq (\rho + P)u^{v} = (g^{\mu\nu} - u^{\mu}u^{\nu})P_{;\mu}, \tag{5.70}
\]

where the labels “\( \pm \)” for each term in the above equation were omitted just not to overload the notation.

In the hydrostatic case we have only that \( u^{t\pm}_\pm = e^{-\alpha^\pm_0} \). When perturbations are present [36]

\[
u^{t\pm}_\pm = e^{-\alpha^\pm} = e^{-\alpha^\pm_0} (1 - \delta\alpha^\pm), \tag{5.71}
\]

where \( \delta\alpha \) is the change of the static solution \( \alpha_0 \) in the presence of perturbations at a given spacetime point. For the \( u^{r\pm}_\pm \) component, using the normalization condition \( u^{\mu\pm}_\pm u^{\mu\pm}_\pm = 1 \), one shows that [36]

\[
u^{r\pm}_\pm = e^{-\alpha^\pm_0} \frac{\xi^\pm_0}{e^\pm}, \tag{5.72}
\]
5.6. Stratified systems in general relativity

with $\ddot{\xi}^\pm = \partial \xi^\pm / \partial t^\pm$. Just for completeness, $u_\pm^0 = u_\pm^R = 0$. For the components of $u_r^\pm$ given by Eqs. (5.71) and (5.72), the left-hand side of Eq. (5.70) gives as the only nontrivial component $a_r^\pm = -e^{-2\beta^\pm} a_r^\pm$, with

$$-a_r^\pm = a_0^\pm + \delta a_r^\pm + e^{2(\beta_0^\pm - a_0^\pm)} \ddot{\xi}^\pm.$$  (5.73)

and the associated equation of motion

$$(\rho^\pm + P^\pm)(-a_r^\pm) = -\frac{\partial P^\pm}{\partial r^\pm}. \quad (5.74)$$

From Eq. (5.73), Eq. (5.74) can be cast as

$$(\rho_0^\pm + P_0^\pm)e^{2(\beta_0^\pm - a_0^\pm)} \ddot{\xi}^\pm = -\frac{\partial P^\pm}{\partial r^\pm} - (\rho^\pm + P^\pm) a_r^\pm. \quad (5.75)$$

Therefore, in terms of distributions Eq. (5.75) reads

$$(\rho + P)e^{2(\beta - a)} \ddot{\xi} = -\frac{\partial P}{\partial r} - (\rho + P) a_r + \frac{2P}{R} \delta(r - R) + \left[ \frac{\sigma}{2R} - \frac{[\alpha' e^{-\beta}]^\pm}{4\pi R} \right] \delta(r - R) + \left[ \frac{P(R)}{2} \{ e^{2(\beta^+ - a^+)} \ddot{\xi}^+ + e^{2(\beta^- - a^-)} \ddot{\xi}^- \} + \alpha'_+ + \alpha'_- \right] \delta(r - R), \quad (5.76)$$

where $\rho$ and $P$ are given by Eqs. (5.59) and (5.60), respectively, and now

$$\ddot{\xi}(r, t) = \ddot{\xi}^+(r^+, t^+) \theta(r^+ - R) + \ddot{\xi}^-(r^-, t^-) \theta(R - r^-). \quad (5.77)$$

Notice that in Eq. (5.76) we are considering jumps and symmetrizations of quantities defined in the presence of perturbations. As we stated previously, such perturbations change slightly the value of the physical quantities w.r.t. their hydrostatic values. The square brackets term of Eq. (5.76) is the proper generalization of the curly brackets term in Eq. (5.19). Naturally the reasoning for the eigenfrequencies of $\ddot{\xi}$ in the general relativistic case will keep concerning the classical one. The same can be said about the continuity, though not differentiability, of $\ddot{\xi}$ at any surface of discontinuity.

Now, in order to have the proper generalization of Eq. (5.26), we should evaluate Eq. (5.76) at $r + \ddot{\xi}$ and then subtract it from its evaluation at $r$ concerning the static solution [or simply apply the Lagrangian displacement operator $\Delta$, ...
\[ \Delta A = A(r + \xi, t) - A_0(r, t), \]

where \( A_0 \) concerns the quantity \( A \) at equilibrium, in both sides of Eq. (5.76). In order to do it properly, one should take into account the general results for the case of Lagrangian displacements coming from the standard procedure (see [36]), but now in the sense of distributions. It is not difficult to see that we have the following results in the general relativistic distributional case (see [36] for the treatment of continuous systems) and for the classical results, we have

\[ \Delta \beta = -a' \xi, \quad (5.78) \]

\[ \Delta a' = 4 \pi r e^{2 \beta_0} [\delta P + 2 \delta \beta P_0] + \frac{e^{2 \beta_0}}{r} \delta \beta + a'_0 \xi, \quad (5.79) \]

\[ \Delta P = -\gamma P_0 \left[ \frac{e^{-\beta_0} (r^2 e^{\beta_0} \xi)'}{r^2} + \delta \beta \right], \quad (5.80) \]

\[ \Delta \rho = -(\rho_0 + P_0) \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \xi) \right] - \frac{d P_0}{d \xi} \xi + \frac{\sigma_0}{2} \left\{ \frac{\partial \xi^+}{\partial r} + \frac{\partial \xi^-}{\partial r} \right\} \\
- \xi \left[ e^{-\beta_0} P_0 \right]^+ + \frac{[\cosh \beta_0]^+}{4 \pi R^2} \left[ P_0 \right]^+ \right\} \delta (r - R), \quad (5.81) \]

\[ \gamma = \frac{\rho_0 + P_0}{P_0} \frac{\partial P}{\partial \rho}_{s=\text{const}}, \quad (5.82) \]

We stress that Eq. (5.80) assumes adiabatic processes, in which one considers \( P = P(\rho) \). For Eq. (5.81), we used that

\[ \Delta \sigma = -\frac{2 \sigma_0 \xi}{R} - \xi \left[ e^{\beta_0} P_0 \right]^+ + \frac{[\cosh \beta_0]^+}{4 \pi R^2} \right\}, \quad (5.83) \]

where Eq. (5.78) was taken into account, as well as \( [\xi]^+ = 0 \). We shall search for solutions to the perturbations as \( \xi = e^{i \omega t} \xi(r) \) with \( \omega \) the same for all the phases the system may have. We just need to concern about the Dirac delta function.
term, since it gives us the desired boundary condition valid for the separation of each two phases. The terms in front of the Heaviside functions by default shall be the ones found in continuous media. It is not difficult to see that the surface terms at the end should satisfy the condition

\[
\frac{\Delta \sigma}{R} \left(2\eta^2 + 1\right) - \frac{\Delta \left(\alpha' \epsilon^\beta\right)^+}{4\pi R} = 0,
\]

where we have used Eq. (5.22). When the last term on the left-hand side of above equation is expanded, by using Eqs. (5.64) and (5.78), it can be further simplified to

\[
\frac{2\eta^2 \Delta \sigma}{R} - \frac{2P}{R^2 \xi} - [\Delta P e^{\beta_0}]^+ + \xi [Pe^\beta \alpha']^+ - \Delta \left(\frac{[\cosh \beta]^+}{4\pi R^2}\right) = 0.
\]

One sees from Eq. (5.85) that Eq. (5.28) is recovered in the classical limit by simply recalling that \( \beta = M(r)/r \), which implies that the last term of the above equation is \( 4\sigma (g^+ + g^-)/(2R) \). Besides, in this limit we take \( P \to 0 \) and \( e^\beta \to 1 \) for the remaining terms.

The case where electromagnetic interactions are also present is also of interest since its associated energy-momentum tensor is anisotropic. This naturally influences the equation of hydrostatic equilibrium, since it now becomes

\[
\alpha' (P + \rho) = -P' - \frac{2}{r} (P_1 - P),
\]

where \( P, P_1 \) and \( \rho \) are the resultant radial pressure, tangential pressure and energy density of the fluid, respectively. For the electromagnetic fields, clearly \( (P_1 - P) \) is solely related to them. Due to the aforementioned aspects, the latter should also influence the dynamics of the radial perturbations, as we shall show in the next section.

### 5.7 General relativistic electrodynamical layered systems

We consider now the inclusion of electromagnetic interactions within the scope of stratified systems in general relativity. An important comment at this level
is in order. Since we are dealing with electromagnetic fields in stars, it would be more reasonable to assume the Maxwell equations in material media. Nevertheless, since the knowledge of the structures constituting the stars is yet not so precise, it seems difficult to assess their realistic dielectric coefficients. Since working with Maxwell equations in the absence of material media gives us upper limits to the fields under normal circumstances, this seems to be a good first tool to evaluate the relevance and effects of electromagnetism in stars. We shall be contented for the time being to follow this approach. The energy-momentum tensor of each layer of the system we are now interested should also have the electromagnetic one, viz.

\[ 4\pi T_{\mu\nu}^{(em)} = -F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} + g_{\mu\nu} \frac{F_\alpha F_\beta}{4}, \]  

(5.87)

where we defined \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Solving Einstein’s equations on each layer of a stratified system leads us to the following equilibrium condition (see [135])

\[ \frac{\partial P}{\partial r} = \frac{Q(r)Q'(r)}{4\pi r^4} - \alpha'_Q (\rho + P), \]

(5.88)

where

\[ Q(r) = \int_0^r 4\pi r^2 \rho e^{\beta Q} dr, \quad E(r) = e^{\alpha_Q + \beta_Q} \frac{Q(r)}{r^2}, \]

(5.89)

\[ e^{-2\beta_Q} = 1 - \frac{2m_Q(r)}{r} + \frac{Q^2(r)}{r^2} \]

(5.90)

\[ \alpha'_Q = \frac{e^{2\beta_Q}}{r^2} \left[ 4\pi r^3 P + m_Q(r) - \frac{Q^2(r)}{r} \right] \]

(5.91)

and

\[ m_Q(r) = \int_0^r 4\pi r^2 \rho dr + \frac{Q^2}{2r} + \frac{1}{2} \int_0^r \frac{Q^2}{r^2} dr. \]

(5.92)

\[ ^* \text{We shall restrict our analyses to the Maxwell Lagrangian, } -\frac{F_{\mu\nu} F^{\mu\nu}}{4} \Leftrightarrow -\frac{F}{4}. \]
Equations (5.89), (5.90), (5.91) and (5.92) are the charge, the radial and time components of the metric and the energy of the system up to a radial coordinate \( r \), respectively. Besides, in Eq. (5.89), we also exhibited the electric field \( E(r) \) in the context of general relativity, obtained by means of the definition \( E_{tr} = E(r) = -\partial_r A_0 \). We stress that \( \rho_c \) is the physical charge density of the system, defined in terms of the four-current by \( j^\mu = e^{-\alpha} u^\mu \rho_c \), where \( u^\mu \) is the four-velocity of the fluid w.r.t. the coordinate system \((t, r, \theta, \phi)\) (for further details, see Ref. [2]).

From Eq. (5.88) one sees that in the scope of general relativity the effect of the charge is not merely to counterbalance the gravitational pull. For certain cases, it could even contribute to it. The reason for that is due to the contribution of the electromagnetic energy to the final mass of the system, as clearly given by Eq. (5.92). Notice from Eq. (5.92) that we have assumed that the mass at the origin is null, in order to avoid singularities there. More generically, one could assume point or surface mass contributions in Eq. (5.92) by conveniently adding Dirac delta functions in \( \rho \). Finally, we stress that Eq. (5.89) can indeed be seen as the generalization of the charge in general relativity, since it takes into account the nontrivial contribution coming from the spacetime warp due to its energy-momentum content.

Notice that the classical limit to Eq. (5.88) can easily be shown to coincide with Eq. (5.50), by recalling that \( E_{\text{clas}}(r) = Q_{\text{clas}}(r)/r^2 \) and from Eq. (5.89), \( Q'_{\text{clas}}(r) = 4\pi r^2 \rho_c \). Besides, we recall that when brought to cgs units, the term \( Q^2/r \) (here in geometric units) goes to \( Q^2/(c^2 r) \), which is null in the classic limit, as well as any pressure term on the right-hand side of the aforementioned equation (since the latter is obtained formally by taking the speed of light going to infinity).

Now, consider the analysis of a charged system constituted of two parts, connected by a surface of discontinuity (at \( r^\pm = R \)) which hosts surface degrees of freedom, such as an energy density, charge density and a surface tension. Its generalization to an arbitrary number of layers is immediate since each surface of discontinuity is only split by two phases. The proper description of the charge density in this case would be given by the generalization of Eq. (5.51). Therefore, one would have at equilibrium that

\[
Q(r) = Q^-(r^-)\theta(R - r^-) + Q^+(r^+)\theta(r^+ - R).
\] (5.93)
5. Radial stability of stratified stars

For $Q'(r)$, a Dirac delta shall rise, due to $\rho_c$.

Let us define the distribution

$$\bar{\rho}_c \equiv \rho_c e^{\beta_Q} \equiv \bar{\rho}_c^+(r^+) \theta(r - R) + \bar{\rho}_c^-(r^-) \theta(R - r) + \sigma_c \delta(r - R). \quad (5.94)$$

From the above definition, we have that $Q' = 4\pi r^2 \bar{\rho}_c$. It implies that the total charge is the same as the one associated with $\bar{\rho}_c$ in an Euclidean space. Therefore, all classical results apropos of the charge densities and total charges that we deduced in the previous sections ensue here for $\bar{\rho}_c$.

We seek now for the distributional generalization of Eq. (5.88). This can be easily done by following the same reasoning from the previous section, which finally leads us to

$$\frac{dP}{dr} = \frac{Q(r)\bar{\rho}_c}{r^2} - (\rho + P)\bar{\rho}_c' + \frac{2PQ}{R} \delta(r - R)$$

$$+ \left\{ [P(R)]^+ + \frac{\sigma_Q}{2} [\alpha_Q^+(R) + \alpha_Q^-(R)] + \frac{\sigma_Q}{R} - \frac{[\alpha_Q^e - \beta_Q]^+}{4\pi R} \right. - \frac{\sigma_c}{2R^2} [Q_+(R) + Q_-(R)] \left. \right\} \delta(r - R), \quad (5.95)$$

where we are assuming surface quantities with the subindex “$Q$” are related to the charged versions of Eqs. (5.56) and (5.57) [see also Eqs. (5.90) and (5.91)]. It is easy to show that in the classical limit Eq. (5.53) naturally raises, implying that in such a limit the curly brackets in Eq. (5.95) is null.

We consider now the case where radial perturbations take place in our charged system. This case is more involved than the neutral case since the charged particles also feel an electric force. The equation describing the evolution of the displacements can be showed to be generalized to (see [136] for the dynamics of the radial perturbations in a given phase)

$$(\rho_0 + P_0) e^{2(\beta_{Q0} - \alpha_{Q0})} \tilde{\chi}_b = - \frac{\partial P}{\partial r} - (\rho + P)\alpha_Q' + \frac{Q(r)\bar{\rho}_c}{r^2} + \frac{2PQ}{R} \delta(r - R)$$

$$+ \left[ \frac{\sigma_Q}{R} - \frac{\alpha_Q^e - \beta_Q}{4\pi R} \right] + \frac{\sigma_Q}{2} \left\{ e^{2(\beta_{Q0} - \alpha_{Q0})} \tilde{\chi}_b^+ + e^{2(\beta_{Q0} - \alpha_{Q0})} \tilde{\chi}_b^- + \alpha_Q^+ + \alpha_Q^- \right\}$$

$$- \frac{\sigma_c}{2R^2} [Q_+(R) + Q_-(R)] + [P(R)]^+ \right\} \delta(r - R). \quad (5.96)$$

For the change in $Q'(r)$, it is easy to show (see [135]) that in the comoving frame
5.7. General relativistic electrodynamic layered systems

there are no currents. This means that the Lagrangian displacements of \( Q \) are null, \( \Delta Q = 0 \). Equation (5.96) takes into account the values of the physical quantities in the presence of perturbations at \( r \). In order to obtain the generalization of Eq. (5.26), we should evaluate Eq. (5.96) at \( r + \tilde{\xi} \) and subtract it from Eq. (5.95). This is due to the definition of the Lagrangian displacement \( [\Delta A = A(r + \tilde{\xi}) - A_0(r)] \) of a given physical quantity, intrinsically related to the notion of comoving observers with the fluid, who naturally could describe its thermodynamics.

In order to simplify Eq. (5.96), we have that in the generalized charged case (see Ref. [136] for the treatment in a phase of a charged system)

\[
\Delta \beta_Q = -\alpha'_Q \tilde{\xi},
\]

\[
\Delta \alpha'_Q = 4\pi e^{2\beta Q_0} \left[ \frac{\delta P + 2\beta P_0 \left( P_0 - \frac{Q_0^2}{8\pi r^4} \right)}{r} + \frac{e^{2\beta Q_0}}{r} \delta \beta Q \right] + \frac{4\pi e^{2\beta Q_0} \bar{\rho}_c Q_0 \tilde{\xi}}{r} - \frac{2\pi \tilde{\sigma}c}{R} \left( Q_0^+ e^{2\beta_0^+} + Q_0^- e^{2\beta_0^-} \right) \tilde{\xi} \delta (r - R) + \alpha'_Q \tilde{\xi},
\]

\[
\Delta P = -\gamma P_0 \left[ \frac{e^{-\beta Q_0} (r^2 e^{2\beta Q_0} \tilde{\xi})'}{r^2} + \delta \beta Q \right],
\]

\[
\Delta \rho = -\left( \rho_0 + P_0 \right) \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \tilde{\xi} \right) - \frac{dP_0}{dr} \tilde{\xi} + \frac{Q_0^c \tilde{\xi}}{r^2} \tilde{\xi} + \left[ \frac{\sigma Q_0}{2} \left\{ \frac{\partial \tilde{\xi}^+}{\partial r} + \frac{\partial \tilde{\xi}^-}{\partial r} \right\} - \frac{2\pi R^2}{Q_0} \right] \delta (r - R) \right],
\]

\[
\gamma = \frac{\rho_0 + P_0}{P_0} \left( \frac{\partial P}{\partial \rho} \right)_{s=const}.
\]

The additional “0” subindex in a physical quantity means that its value at equilibrium was taken. We just stress that Eq. (5.101) is the general relativistic definition of the adiabatic index and assumes the existence of an equation of state linking the pressure and the density of the system, \( P = P(\rho) \). In this sense, it generalizes \( \Gamma \) as defined by Eq. (5.2).
By seeking for solutions $\hat{\xi} = e^{i\omega t}\xi(r)$, one can see that Eq. (5.96) just gives meaningful boundary conditions when $\omega$ is the same for all the phases present in the system. We emphasize this is an universal property of the approach developed here, due to the surface degrees of freedom and the well-posedness of the problem of radial perturbations in stratified systems. The associated boundary condition rising from this analysis leads us to the conclusion that generically $\xi(r)$ is not differentiable at a surface of discontinuity, though continuous. It can be shown that the associated boundary condition to be taken into account here is functionally the same as Eq. (5.84) [or Eq. (5.85)], where now the metric and surface quantities should be related to the charged case.

### 5.8 Conclusions

In this chapter we developed a formalism for dealing with various stratified systems in general relativity in the scope of radial oscillations. It makes use of the generalized theory of distributions, since we assumed that the surfaces of discontinuity are thin, host surface degrees of freedom and the phases separated by them do not mix. We showed that though the phases may be very different amongst themselves, when perturbations take place, they lead to the notion of a set of eigenfrequencies describing the whole system, instead of an independent set for each phase. Besides, our formalism gave us as a consequence the proper additional boundary conditions to be taken into account when working with stratified systems. Such boundary conditions encompass surface degrees of freedom in the surfaces of discontinuity and generically modify the set of eigenfrequencies with respect to their continuous counterpart. This should be a generic fingerprint of stratified systems with nontrivial surface degrees of freedom. Our analyses could be relevant for numerical computations of the stability of more realistic star models, since there integrations just ensue with the precise notion of boundary conditions. It was not our objective to systematically apply our formalism here, but simply to derive and expound it.

The radial instabilities shown in our analyses should be interpreted analogously as for continuous stars since, even in the stratified case, a global set of eigenfrequencies raises. Thus, stratified stars would either implode or explode when they are radially unstable.
5.9 Perspectives

Practical analyses taking into account realistic systems are in order. This could give us the departures stratified systems exhibit when compared to continuous systems, especially for their set of eigenfrequencies. It would be of interest to extend our analyses to axially symmetric cases, as well as nonradial perturbations in stratified systems. From the practical point of view, it would be important to find the translation of stability in the stratified case where only numerical analyses are possible. Conceptually speaking, one should also scrutinize the formulation of the problem under a variational viewpoint. Electrodynamical analyses in the scope material media (emulating stars) could also be enlightening, though certain issues there still need to be clarified [137]. Glitches and QPOs [1] should be further scrutinized for layered systems, since they are direct observable quantities from compact stars and their neighborhoods and could be invaluable to probe the analyses of this chapter. Finally, for very precise and realistic numerical stability calculations apropos of stratified stars, the micro-physics of the surfaces of discontinuity must also be better investigated.

Though we were concerned only with radial perturbations, it is of interest to investigate the additional oscillation modes owing to nonradial perturbations. The only point to be added, with respect to continuous stars, is the proper re-definition of the surfaces of discontinuity when such perturbations take place. This is clearly a richer scenario that insert additional degrees of freedom into the system, leading to the appearance of additional modes such as the gravitational ‘g-modes’ [see, e.g., 138]. Such an analysis, however, is as a second step that goes beyond the goal of the present chapter, and that we are planning to investigate elsewhere.
5. Radial stability of stratified stars
Chapter 6

Matching arbitrary slowly rotating spacetimes

6.1 Introduction

In this chapter we shall investigate the match of two arbitrary slowly rotating spacetimes whose surface of discontinuity separating them has an arbitrary dynamics. Such an analysis is supposed to give us some insights about the ultimate goal of solving the same problem nonperturbatively, as well as allowing us to see the influence and subtleties of rotation on the surface degrees of freedom in a wholly general relativistic framework. Besides, it would let us cognise the correct degrees of freedom to be taken into account for slowly rotating systems in order to lead to physically (or observationally) motivated solutions to general relativity, being this a tool to test their relevance. These analyses are believed to shed some light on the description of rotating astrophysical bodies endowed with thick layers, being split by thin ones, as it could be the case of neutron stars. In this case, additionally, our description would help start paving the way for more precise stability analyses when the phases of the system themselves (that are not spherically symmetric) are perturbed.

To the best of our knowledge the general analysis we are proposing here has not yet been done before, but just some more restrictive ones, which though were applied to a multitude of physical phenomena, e.g. [139–146]. In this regard, it is worth citing the work of de La Cruz and Israel [139]. These authors have joined the Minkowski spacetime with the Kerr one by means of prescribed axially symmetric hypersurfaces when the latter are in equilibrium. It was shown
that there is an infinitude of ways of joining the flat spacetime with the Kerr one, each of them characterized by a spinning shell with a given ellipticity. Besides, rigid rotation of observers inside the shell with the shell itself does not take place generally. Another very recent work we cite is due to Uchikata and Yoshida [140], who investigated the matching of a slowly rotating Kerr-Newman spacetime with the de Sitter one by means of a thin shell at equilibrium. As expected, they also found that the surface degrees of freedom on the thin shell are also polar angle dependent. Cases were found where the surface energies could be negative, but the authors deemed them to be unphysical.

A clear asset of gluing spacetimes is the possibleness of having manifolds sans singularities. There is also an astrophysical consensus that physically appealing solutions to general relativity should be axially symmetric. The reason for this is simple. It is highly improbable that the whole evolution of an astrophysical system be spherically symmetric since it is rather chaotic. The aforementioned aspects naturally constitute motivations to the analyses that shall carry out here. Besides, in axially symmetric spacetimes, the dragging of inertial frames is inherent. We show here how this quantity would allow us to scrutinize even the existence of layers in physical systems. The possibility of exploring such a kinematical aspect (the gravity probe B experiment [147] would be an example for testing general relativity with the Earth) in order to talk about the constitution of physical systems is another motivation for our analyses.

Our interest in this chapter is to solve generically the problem posed, examining its subtleties. We show, for instance, that there are situations where the corrections to the energy density and surface tension concerning their spherically symmetric counterparts are not everywhere positive. This seems to be of interest, since it would point to the likelihood of violating some energy conditions in some regions of some surfaces of discontinuity in the nonperturbative ambit.

Geometric units are used in this chapter. The metric signature is $-2$. 

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6. Matching arbitrary slowly rotating spacetimes

A clear asset of gluing spacetimes is the possibleness of having manifolds sans singularities. There is also an astrophysical consensus that physically appealing solutions to general relativity should be axially symmetric. The reason for this is simple. It is highly improbable that the whole evolution of an astrophysical system be spherically symmetric since it is rather chaotic. The aforementioned aspects naturally constitute motivations to the analyses that shall carry out here. Besides, in axially symmetric spacetimes, the dragging of inertial frames is inherent. We show here how this quantity would allow us to scrutinize even the existence of layers in physical systems. The possibility of exploring such a kinematical aspect (the gravity probe B experiment [147] would be an example for testing general relativity with the Earth) in order to talk about the constitution of physical systems is another motivation for our analyses.

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Geometric units are used in this chapter. The metric signature is $-2$. 

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6.2 Gluing two slowly rotating spacetimes

In the slowly rotating case, following Hartle [89], we shall assume that each spacetime that we shall marry can be written as

\[ ds^2 = e^{\nu(r)}[1 + 2\epsilon^2 h(r, \theta)]dt^2 - e^{\lambda(r)}[1 + 2\epsilon^2 j(r, \theta)]dr^2 - r^2[1 + 2\epsilon^2 k(r, \theta)](d\theta^2 + \sin^2 \theta\{d\phi - \epsilon\omega(r)dt\}^2). \]  

(6.1)

In our approximations, we are going until “second order on the rotational parameter”, fingerprinted by the quantity “\(\epsilon\)”. It was put in the above equation just as an indicator of the order of the rotational expansion taken into account. At the end of the calculation, it will be dropped out. In its absence, one has the metric of a spherically symmetric spacetime. The functions \(h, j, k, \omega\) are to be found by Einstein’s equations in the scope of axially symmetric solutions. The corrections to \(e^\nu\) and \(e^\lambda\) from the spherically symmetric case just start appearing at second order corrections on “\(\epsilon\)” simply because they are connected with energy terms, which do not depend upon the direction the spacetime “turns”, but solely on its “turn”.

In order to match two spacetimes, one must also know their common hypersurface (that is to say, it must be well-defined), as well as the coordinates of each manifold are related to the ones defined on such a hypersurface. Once this is procured, finding the surface energy-momentum tensor that would allow such a glue is more of an algebraic exercise, made it possible principally by Lanczos’ seminal work [6, 91, 93, 148].

To begin with, let us assume that the intrinsic coordinates to the splitting hypersurface, defined as \(\Sigma\) are \(y^a = (\tilde{\tau}, \tilde{\theta}, \tilde{\phi})\). For the nonce, \(\tilde{\tau}\) is just a label for a time-like coordinate on \(\Sigma\). Further ahead such a notation shall be justified when we shall try to related it with the proper time as measured by observers on \(\Sigma\), at least to some order on “\(\epsilon\)”. We shall assume that the equation for \(\Sigma\) is given by \(\Psi(r, \tilde{\tau}, \tilde{\theta}) = 0\), with

\[ \Psi(r, \tilde{\tau}, \tilde{\theta}) = r - R(\tilde{\tau}) - \epsilon^2 B(\tilde{\tau}, \tilde{\theta}). \]  

(6.2)

Up to this point, \(R\) and \(B\) are unknown functions. The above notation for \(\Psi\) is just to bespeak that the function \(B\) is related to a second order correction on the
6. Matching arbitrary slowly rotating spacetimes

rotational parameter to the spherical case. For clarity, we shall keep doing in this way, to evidence the order of each correction analyzed.

Besides, let us surmise that the spacetime coordinates relate to the ones of the hypersurface as

\[ t = T(\tilde{\tau}) + \epsilon^2 A(\tilde{\tau}, \tilde{\theta}), \quad \theta = \tilde{\theta}, \quad d\phi = d\tilde{\phi} + \epsilon C(\tilde{\tau}) d\tilde{\tau}. \]  \hspace{1cm} (6.3)

Just not to overload the notation, we dropped the “±” indexes that should be present on the coordinates of the spacetimes to be matched. Such labels would be related to a region above (“+”) and below (“−”) \( \Sigma \), naturally defined by its normal vector.

Given the Ansatz, Eqs (6.2) and (6.3), our task is to find the functions \( A, B, C, R, T \) that lead the metrics given by Eq. (6.1) to have a zero discontinuity at \( \Sigma \). This is the first boundary condition to be imposed when one is gluing spacetimes whose resultant one shall lead to a distributional solution to Einstein’s equations [6]. From Eqs. (6.2) and (6.3) and our previous assumptions, we have

\[ dt^2 = \dot{T}^2 d\tilde{\tau}^2 + 2\epsilon^2 \dot{T} dA d\tilde{\tau}, \]  \hspace{1cm} (6.4)

\[ dr^2 = \dot{R}^2 d\tilde{\tau}^2 + 2\epsilon^2 \dot{R} dB d\tilde{\tau}, \]  \hspace{1cm} (6.5)

\[ d\phi^2 = d\tilde{\phi}^2 + \epsilon^2 \dot{C}^2 d\tilde{\tau}^2 + 2\epsilon C d\tilde{\tau} d\tilde{\phi}, \]  \hspace{1cm} (6.6)

where by consistency we have just kept terms up to the second order on “\( \epsilon \)” and we have defined \( \dot{f} \equiv \partial f / \partial \tilde{\tau} \).

By substituting Eqs. (6.4)–(6.6) into Eq. (6.1), we have to impose that

\[ C = \omega(R) \dot{T}. \]  \hspace{1cm} (6.7)

and

\[ e^{v(R)} \dot{T}^2 - e^{\lambda(R)} \dot{R}^2 = 1 \]  \hspace{1cm} (6.8)

in order to eliminate the first order on the rotational parameter of the induced
6.2. Gluing two slowly rotating spacetimes

metric, as well as to retrieve our spherically symmetric solution with \( \bar{\tau} \) as the proper time there on \( \Sigma \). We recall that all calculations are being performed on \( \Sigma \), since we are seeking to find solutions to Eqs. (6.2) and (6.3). From Eq. (6.8), we see that \( R(\bar{\tau}) \) is not constrained from the well-definiteness of the induced metric. Up to this point, \( R(\bar{\tau}) \) is a free function whose dynamics will be related to the spherically symmetric configuration. We know from such a case that its determination is just possible when an equation of state for the always perfect-like fluid on \( \Sigma \) is given [88, 93].

Since we are working perturbatively on “\( \varepsilon \)”, we are permitted to expand the functions \( A \) and \( B \) on first orders Legendre polynomials, in the same fashion as it is done for \( h, j, k \). This is the only way of rendering the problem “tractable”. Therefore, let us assume that

\[
B(\bar{\tau}, \bar{\theta}) = B_0(\bar{\tau}) + B_2(\bar{\tau})P_2(\cos \bar{\theta}),
\]

(6.9)

and

\[
A(\bar{\tau}, \bar{\theta}) = A_0(\bar{\tau}) + A_2(\bar{\tau})P_2(\cos \bar{\theta}),
\]

(6.10)

where we are defining \( P_2(\cos \bar{\theta}) \) the second order Legendre polynomial, given by

\[
P_2(\cos \bar{\theta}) = \frac{1}{2}(3\cos^2 \bar{\theta} - 1).
\]

(6.11)

We recollect that

\[
h(r, \theta) = h_0(r) + h_2(r)P_2(\cos \theta),
\]

(6.12)

\[
j(r, \theta) = j_0(r) + j_2(r)P_2(\cos \theta),
\]

(6.13)

\[
k(r, \theta) = k_0(r) + k_2(r)P_2(\cos \theta).
\]

(6.14)

Due to the freedom in performing the coordinate change \( r \to f(r') \), without modifying the form of the metrics given by Eq. (6.1), one can always assume...
6. Matching arbitrary slowly rotating spacetimes

that $k_0 = 0$. On the hypersurface $\Sigma$, the functions $h_0, j_0, h_2, j_2$ and $k_2$ have their radial dependence just replaced by $R$, given that they are already second order functions on the rotational parameter. Notwithstanding, the spherically symmetric function $e^\nu$ on the hypersurface $\Sigma$, on account of Eqs. (6.2) and (6.9), changes to

$$e^{\nu(\tau)} \approx e^{\nu(R)} \left( 1 + \frac{\partial \nu}{\partial R} R^2 B \right), \quad (6.15)$$

with $e^\lambda$ akin to the above equation. Therefore, further corrections appear when one works up to $\epsilon^2$. The one given by Eq. (6.15) is due to the change of the shape of the $\Sigma$, impinged solely by the rotation.

After having taken into account Eqs. (6.4), (6.5) and (6.6), as well as Eqs. (6.7), (6.8) and (6.15) in Eq. (6.1), the induced metric on $\Sigma$ can be cast as

$$ds^2_{\Sigma} = d\tilde{\tau}^2 \left[ 1 + e^{2 \nu(R)} \left[ 2\dot{T}A + 2h\dot{T}^2 + \frac{\partial \nu}{\partial R} B\dot{T}^2 \right] - e^{\lambda(R)} \left[ 2\dot{R}B + 2j\dot{R}^2 + \frac{\partial \lambda}{\partial R} B\dot{R}^2 \right] \right]$$

$$+ 2\epsilon^2 \left[ e^{\nu(R)} \frac{\partial A}{\partial \tilde{\theta}} - e^{\lambda(R)} \frac{\partial B}{\partial \tilde{\theta}} \right] d\tilde{\tau} d\tilde{\theta} - (R^2 + 2RB + 2kR^2) [d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2]. \quad (6.16)$$

Now, for reasons that shall become clear subsequently, in order to guarantee the well-definiteness of the geometry on $\Sigma$, if we now substitute Eqs. (6.10), (6.9), (6.12), (6.13) and (6.14) into Eq. (6.16), we are led to the system of equations

$$e^{\nu(R)} \left[ \dot{T}A_0 + h_0 \dot{T}^2 \right] - e^{\lambda(R)} \left[ \dot{R}B_0 + j_0 \dot{R}^2 \right] + \frac{B_0}{2} \left[ e^{\nu(R)} \dot{T}^2 \frac{\partial \nu}{\partial R} - e^{\lambda(R)} \dot{R}^2 \frac{\partial \lambda}{\partial R} \right] = 0, \quad (6.17)$$

$$e^{\nu(R)} \dot{T}A_2 = e^{\lambda(R)} \dot{R}B_2, \quad (6.18)$$

$$[B_2 + k_2 R]^+ = 0, \quad (6.19)$$

$$[B_0]^+ = 0 \quad (6.20)$$
6.2. Gluing two slowly rotating spacetimes

and

\[ [\alpha_2]^\pm = 0, \quad (6.21) \]

with

\[ \alpha_2 = e^{\nu(R)} \left[ T A_2 + h_2 T^2 + \frac{1}{2} \frac{d\nu}{dR} B_2^2 \right] - e^{\lambda(R)} \left[ \dot{R} B_2 + j_2 \dot{R}^2 + \frac{1}{2} \frac{d\lambda}{dR} B_2 \dot{R}^2 \right]. \quad (6.22) \]

In the previous equations, we have defined \([A]^\pm = A^+|_\Sigma - A^-|_\Sigma\) as the “jump of \(A\) across \(\Sigma\)”. We stress that each of Eqs. (6.17) and (6.18) actually are two equations, related to each region \((\pm)\) defined by \(\Sigma\). We have ergo eight unknown functions, \((A_0^\pm, A_2^\pm, B_2^\pm, B_0^\pm)\), to seven equations, Eqs. (6.17)–(6.21). The missing equation is related to the latitude concerning transformations of the type \(\tilde{\tau} \rightarrow g(\tilde{\tau})\) on the hypersurface metric. Without any loss of generality, we can take \(B_0^\pm = 0\), which will render Eq. (6.20) an identity and we will be left with a system of six equations to six variables. Notice that Eq. (6.17) is trivially solved for \(A_0\).

Equations (6.17)–(6.22) warrant that the induced metric has a zero discontinuity over \(\Sigma\) and can be cast as

\[
\left. ds^2 \right|_{\Sigma} = d\tilde{\tau}^2 \left( 1 + 2e^2 \{\alpha_2\}^\pm P_2(\cos \tilde{\theta}) \right) - \left[ R^2 + 2e^2 R \{B_0\}^\pm + 2e^2 R \{B_2 + k_2 R\}^\pm P_2(\cos \tilde{\theta}) \right] [d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2],
\]

where we are defining for a given quantity \(F\), \(\{F\}^\pm = (F^+ + F^-)/2\). Notice that the coordinate systems given by Eq. (6.3), as well as Eqs. (6.17)–(6.22) do not conduce to an induced metric that resembles its spherically symmetric counterpart. After having solved the system of equations given by Eqs. (6.17)–(6.22), one can obtain the relationship betwixt the coordinates \((t, \varphi)\) in terms of \((\tilde{\tau}, \tilde{\varphi})\), needed for the calculations apropos of the surface energy momentum tensor on \(\Sigma\) that allows the match of two slowly rotating spacetimes. Out of this process, one will conclude that the presence of terms up to second order on “\(\epsilon\)” will alter the physical quantities on \(\Sigma\), its energy density and surface tension, changing hence their dynamics.

We stress that all of our previous reasoning remains immutable if we perform the change \(P_2(\cos \theta) \rightarrow CP_2(\cos \theta)\), with \(C\) an arbitrary constant. This is related to the freedom we always have in choosing the ellipticity of \(\Sigma\) when matching two
slowly rotating spacetimes. We finally underline that in the slowly rotating case, $\tilde{\tau}$ is not the proper time on $\Sigma$ if one takes into account Eqs. (6.17)–(6.22). This is not a problem, since the coordinates of $\Sigma$ can be chosen without hindrance.

### 6.3 Hypersurfaces at equilibrium

We attempt to solve now Eqs. (6.17)–(6.22) in the case $\Sigma$ is not endowed with a dynamics, i.e. $\dot{R} = 0$, or $R(\tau) = R_0 = \text{Const}$. This case is very important since it tests the consistency of the system of equations obtained from the imposition of zero-discontinuity of the induced metric on $\Sigma$. Besides, it gives us the requisites for constructing static structures around slowly rotating spacetimes. For this case, Eq. (6.8) gives us that $\dot{T} = e^{-\nu}$. It is not difficult to show that the solution to the aforementioned system of equations is

$$B_2^\pm(R_0) = \frac{\nu^\tau R_0[h_2]^+ - 2[h_2]^+}{\partial \nu \partial R_0},$$

$$A_2 = 0,$$  \hspace{1cm} (6.25)

and

$$A_0 = -\left( h_0 e^{-\nu} + \frac{B_0 \partial \nu}{2 \partial R} \right) \tilde{\tau},$$

$$\tilde{\tau} = B_0(R_0)$$

with $B_0 = B_0(R_0)$ having zero-discontinuity and being an arbitrary function, related to our freedom in making transformations like $\tilde{\tau} \rightarrow f(\tilde{\tau})$ in Eq. (6.23). Notice that Eqs. (6.24)–(6.26) do not depend in general on $j(r, \theta)$ and are generic solutions to the static case. In this case the induced metric, Eq. (6.23), does not depend upon $\tilde{\tau}$, as one would expect. It can be checked this is the only case where such an aspect gives rise concomitantly with the self-consistency of the system of equations leading to the well-definiteness of the geometry of $\Sigma$. This justifies the choice of Eqs. (6.17)–(6.22).

It is worth emphasizing that $R(\tilde{\tau}) = \text{Const} = R_0$ does not automatically guarantee the stability of the system upon perturbations. This is just the case whenever $\Sigma$ is bounded. From Eq. (6.2), one sees that stability will be ensured
just when both $R(\tilde{\tau})$ and $B(\tilde{\tau}, \tilde{\phi})$ are bound functions. Actually, the fact that the latter function be bound is a requirement for our perturbative approach to be meaningful. Therefore, this should be the paramount aspect to be searched for. After having met this demand, the only left one is the boundedness of $R(\tilde{\tau})$. This can be analyzed in the scope of the thin shell formalism in the spherically symmetric case and we know that it is summarized by the search of minima of an effective potential [93]. As a result in the stable case, $\dot{R}$ is automatically bound and could be approximated by a harmonic function around $R_0$ with a very small amplitude. It is not difficult to see that $B_2(\tilde{\tau}, \tilde{\phi})$, as well as the other coordinate transformation functions [given by the solutions to Eqs. (6.17–6.22) for the aforesaid case], shall also be proportional to an oscillating function around $r = R_0$. We thereupon conclude that the stability of $\Sigma$ in the spherically symmetric case also implies its stability in the presence of small rotations. This is what one expects due to the fact generically rotation tends to stabilize gravitational systems.

6.4 Energy-momentum tensor for a slowly rotating thin shell

Now we proceed with the determination of the energy-momentum tensor on $\Sigma$ that guarantees the glue of two slowly rotating spacetimes. In order to do so, we must foremost find the normal vector to $\Sigma$. Generally, the normal-vector to a given hypersurface $\Psi$ is [6]

$$n_\mu \equiv \frac{\epsilon_n \partial_\mu \Psi}{|\delta^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi|^{\frac{1}{2}}} \quad (6.27)$$

where $\epsilon_n = \pm 1$, depending on $\Sigma$ being space-like or time-like, respectively. Besides, Eq. (6.27) ensures that $n_\mu$ is an unit vector in the same direction as the growth of $\Sigma$.

Let us calculate now the gradient to $\Sigma, \partial_\mu \Psi$. Again from Eq. (6.2),

$$\partial_\mu \Psi = \left(-\frac{1}{l} [\dot{R} + e^2 \dot{B}], 1, -\frac{\partial B}{\partial \tilde{\theta}}, 0 \right). \quad (6.28)$$
First of all, the inverse to the metric given by Eq. (6.1) is

\[
\begin{align*}
g^{00} &= e^{-\nu}[1 - 2e^2h], \\
g^{11} &= -e^{-\lambda}[1 - 2e^2j], \\
g^{22} &= -\frac{[1 - 2e^2k]}{r^2}, \\
g^{33} &= -\frac{[1 - 2e^2k]}{r^2 \sin^2 \theta} + e^{-\nu}e^2\omega^2, \\
g^{03} &= e^{-\nu}e\omega. 
\end{align*}
\] (6.29)

At the hypersurface \(\Sigma\), given by Eq. (6.2), the inverse of the metric, taking into account Eq. (6.29), reads

\[
\begin{align*}
g^{00} &= e^{-\nu(R)} \left[ 1 - 2e^2h - \frac{\partial \nu}{\partial R} e^2 B \right], \\
g^{11} &= -e^{-\lambda(R)} \left[ 1 - 2e^2j - \frac{\partial \lambda}{\partial R} e^2 B \right], \\
g^{22} &= -\frac{1}{R^2} \left[ 1 - 2e^2k - \frac{2e^2B}{R} \right], \\
g^{33} &= -\frac{1}{R^2 \sin^2 \theta} \left[ 1 - 2e^2k - \frac{2e^2B}{R} \right] + e^{-\nu(R)}e^2\omega^2. 
\end{align*}
\] (6.30)

From the above equation and Eqs. (6.3), (6.8) and (6.28), we have

\[
\begin{align*}
\left| \partial \mu \Psi \partial \nu \Psi \right|^{-\frac{1}{2}} &= e^{\nu + \mu} T \left[ 1 + e^2 \left( \frac{\dot{A}}{T} + e^2 \dot{R}^2 \left[ \frac{B}{R} - \frac{B}{2} \frac{\partial \nu}{\partial R} - h \right] \right) - e^2 \dot{T}^2 \left[ \frac{\dot{A}}{T} - \frac{B}{2} \frac{\partial \lambda}{\partial R} - j \right] \right]. 
\end{align*}
\] (6.31)

Equation (6.28), with the help of Eq. (6.3), can be reduced to

\[
\partial \mu \Psi = \frac{1}{T} \left[ -\dot{R} - e^2 \left( B - \frac{\dot{A}}{T} \right), \dot{T}, -e^2 \frac{\partial B}{\partial \theta} \dot{T}, 0 \right] 
\] (6.32)

Therefore, by gathering Eqs. (6.31) and (6.32), one obtains \(n_\mu\), as given by Eq. (6.27), namely

\[
\begin{align*}
n_0 &= e^{\nu + \lambda} R \left[ 1 - e^2 \left( \frac{\dot{B}}{R} + e^2 \dot{R}^2 \left[ \frac{B}{R} - \frac{B}{2} \frac{\partial \nu}{\partial R} - h \right] - e^2 \dot{T}^2 \left[ \frac{\dot{A}}{T} - \frac{B}{2} \frac{\partial \lambda}{\partial R} - j \right] \right) \right], \\
n_1 &= -e^{\nu + \mu} T \left[ 1 + e^2 \left( \frac{\dot{A}}{T} + e^2 \dot{R}^2 \left[ \frac{B}{R} - \frac{B}{2} \frac{\partial \nu}{\partial R} - h \right] - e^2 \dot{T}^2 \left[ \frac{\dot{A}}{T} - \frac{B}{2} \frac{\partial \lambda}{\partial R} - j \right] \right) \right], \\
n_2 &= e^{\nu + \mu} \dot{T} e^2 \frac{\partial B}{\partial \theta} 
\end{align*}
\] (6.33)
and \( n_3 = 0 \). In the above normal components, we assumed that \( \Sigma \) is time-like, which accounts for \( \epsilon_n = -1 \) to the normal vector. The contravariant components of the normal vector can be worked out by means of \( n^\mu = g^{\mu\nu} n_\nu \). To our ends, such a calculation is not needed.

Now we proceed with the calculations of the extrinsic curvature, defined as

\[
K_{ab} = n_\mu \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} = -n_\mu \left( \frac{\partial^2 x^\mu}{\partial y^a \partial y^b} + \Gamma^\mu_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b} \right). 
\] (6.36)

For slowly rotating spacetimes, off-diagonal terms in \( K_{ab} \) rise. Nevertheless, one does not need to compute all of its components, as we shall show generally in the follow-up. First off, we recall that in the spherically symmetric case

\[
K^0_0 = \frac{\nu'(e^{-\lambda} + \ddot{R}^2) + 2\ddot{R} + \lambda' \dot{R}^2}{2\sqrt{e^{-\lambda} + \dot{R}^2}},
\] (6.37)

and

\[
K^1_1 = K^2_2 = \frac{\sqrt{e^{-\lambda} + \dot{R}^2}}{R}.
\] (6.38)

For the first order corrections on “\( \epsilon \)”, we additionally have

\[
K^0_2 = \frac{1}{2} R^2 \sin^2 \theta \omega' e^{-\frac{\nu + \lambda}{2}} = -R^2 \sin^2 \theta K^2_0. 
\] (6.39)

According to Lanczos’ equation [91, 148], the surface energy-momentum leading to a distributional solution to Einstein’s equations is

\[
8\pi S^a_b = [K^q_b]^+ - \delta^q_b [K]^+. 
\] (6.40)

where \( K \doteq h^{ab} K_{ab} \) is the trace of the extrinsic curvature and we assumed the hypersurface splitting the spacetimes is time-like.

In the spherically symmetric (ss) case, \( S^a_b \) is a diagonal tensor concerning the \((\tau, \theta, \phi)\) coordinates. In other words

\[
8\pi S^a_b = diag(-2[K^1]^+, -[K^0_0 + K^1]^+, -[K^0_0 + K^1]^+) = diag(\sigma_{ss}, -\mathcal{P}_{ss}, -\mathcal{P}_{ss}). 
\] (6.41)
6. Matching arbitrary slowly rotating spacetimes

This is the form of the energy-momentum tensor for a comoving frame. Therefore, the fluid on \( \Sigma \) is a perfect-like one,

\[
8\pi S^{ab} = \sigma_{ss}u^a u^b + \mathcal{P}_{ss}(u^a u^b - h^{ab}).
\] (6.42)

Whenever one works up to first order of approximation on \( \epsilon \), similar results as the above ones also hold in the coordinate system \((\tilde{\tau}, \tilde{\theta}, \tilde{\phi})\), as \( \tilde{\phi} \) defined by Eq. (6.3). Nevertheless, for this case, local observers on \( \Sigma \) (who utilize the aforementioned coordinates) experience the effect of frame dragging, whose velocity of dragging is proportional to \( K^2_0 \).

Now we turn to the case where up to second order corrections on \( \epsilon \) are worked out. From what we have pointed out previously, the form of the surface energy-momentum in this case in the coordinate system \((\tilde{\tau}, \tilde{\theta}, \tilde{\phi})\) should generically resemble as

\[
S^a_b = \begin{bmatrix}
S^0_0 + \epsilon^2 \bar{S}^0_0 & \epsilon^2 \bar{S}^0_1 & \epsilon \bar{S}^0_2 \\
\epsilon^2 \bar{S}^1_0 & S^1_1 + \epsilon^2 \bar{S}^1_1 & 0 \\
\epsilon \bar{S}^2_0 & 0 & S^1_1 + \epsilon^2 \bar{S}^1_1
\end{bmatrix}.
\] (6.43)

From a continuity argument, we still expect that the fluid on \( \Sigma \) be a perfect one. In order to diagonalize Eq. (6.43), we should just express \( S^a_b \) in the frame of observers who make use of the coordinates \((\tilde{\tau}, \tilde{\theta}, \tilde{\phi})\), according to Eq. (6.3), by taking into account the velocity they infer to \( \Sigma \) (which is not related to a comoving frame). Therefore, our task is simply to find this velocity. From Eq. (6.42), we see that in general for a perfect-like fluid

\[
S^a_b u^b = \sigma u^a,
\] (6.44)

which states that the velocity of \( \Sigma \) as seen by the observers with coordinates \((\tilde{\tau}, \tilde{\theta}, \tilde{\phi})\) is the eigenvector of \( S^a_b \) and its eigenvalue is the energy density.

The results coming from inferior orders of approximation tell us that we should just accept solutions of the form

\[
u^a = [u_0 + \epsilon^2 \bar{u}_0, \epsilon^2 \bar{u}_1, \epsilon \bar{u}_2], \quad \text{and} \quad \sigma = \sigma_{ss} + \epsilon^2 \bar{\sigma}.
\] (6.45)

It can be shown that the only solution coming from Eq. (6.44) that satisfies such
prerequisites is
\[ u^a = 1 + \epsilon^2 \left\{ \frac{1}{2} \left( R \sin \tilde{\theta} S^2_0 \right)^2 - \{a_2\}_+ p_2(\cos \tilde{\theta}) \right\} \left( \frac{\epsilon S^1_0}{S^0_0 - S^1_1}, \frac{\epsilon S^2_0}{S^0_0 - S^1_1} \right), \] (6.46)

where we fixed the arbitrary quantities \( u_0 \) and \( \bar{u}_0 \) coming from the eigenvalue approach by the normalization condition \( u^a u_a = 1 \), and

\[ \sigma = S^0_0 + \epsilon^2 \left( S^0_0 + \frac{S^0_2 S^0_0}{S^0_0 - S^1_1} \right). \] (6.47)

For the pressure, we have generically

\[ P = \frac{1}{2} S^0_0 (u^b u_a - \delta_a^b). \] (6.48)

For the particular case analyzed, we have

\[ P = -S^1_1 + \epsilon^2 \left[ -\tilde{S}^1_1 + \frac{S^0_2 S^0_0}{2(S^0_0 - S^1_1)} \right]. \] (6.49)

These general results tell us which components of the extrinsic curvature have a physical meaning. We stress that the component \( S_{12} \) is not present in the calculations due to the assumed axial symmetry to both spacetimes.

### 6.5 Dragging of inertial frames

In this section we briefly discuss some kinematical effects apropos of the match of two slowly rotating axially symmetric spacetimes [see Eq. (6.1)]. We see from the coordinate transformation given by Eq. (6.3) that an static observer in a region of a spacetime is actually moving w.r.t. the fixed stars (observers very far away from \( \Sigma \)).

Let us analyze first an unmoving observer inside the rotating thin shell. This observer can be described by \( d\phi^- = 0 \). From Eq. (6.3), we have that this case is equivalent to \( d\phi = -(\epsilon C)^{-1} d\tau \). Therefore, in terms of the fixed stars, such an
observer is rotating with the velocity

\[
\frac{d\varphi}{dt^+} = (\omega^+ - \omega^-) \frac{d\bar{\tau}}{dt^+},
\] (6.50)

where we have used also Eq. (6.7). We recall that the \( T^\pm \) can easily be read off from Eq. (6.8). Equation (6.50) states that far away external observers see internal ones rotating, even in the case where \( \epsilon^- = 0 \) [the inner spacetime is spherically symmetric]. Such an effect is the well-known dragging of inertial frames, or Lense-Thirring effect [36]. The case where \( \epsilon^- = 0 \) is beguiling because it shows the rotation of the shell intrinsically induces a rotation of observers inside it. Whenever the inner spacetime is also endowed with a rotational parameter \( \epsilon^- \), naturally it also contributes to the final angular velocity fixed stars ascribe to internal observers at rest, as evidenced by Eq. (6.50). Even more remarkable in this case is the possibility of the disappearance of the dragging of inertial frames [see again Eq. (6.50)]. This is so even in the case where \( \epsilon^\pm \) and the shell parameters are given, by convenient choices of the inner black hole parameters.

Also pertaining to distant observers, the rotation of the thin shell can be obtained. We know that w.r.t. the frame with coordinates \((\bar{\tau}, \bar{\theta}, \bar{\varphi})\), the shell rotates with velocity

\[
\frac{d\varphi}{d\bar{\tau}} = \frac{\epsilon S_0^2}{S_0^1 - S_1^1},
\] (6.51)

where we just remember that the “\( \epsilon \)” dependence on Eq. (6.46) is merely an indicator that the associated surface energy-momentum tensor is of a given order on the rotational parameters \( \epsilon^\pm \). Both \( \epsilon^\pm \) contribute to \( u^a \) since \( S_0^a \) is itself obtained by means of jumps. Taking into account Eqs. (6.51) and (6.3), we have that the velocity of the shell w.r.t. the fixed stars is

\[
\frac{d\varphi}{dt^+} = \frac{d\varphi}{d\bar{\tau}} + \epsilon^+ C^+ \frac{d\bar{\tau}}{dt^+} = \frac{\epsilon S_0^2}{(S_0^0 - S_1^1) T^+} + \epsilon^+ \omega^+, \] (6.52)

where also Eq. (6.7) has been taken into account and we recall that \( T \) is given by Eq. (6.8). For completeness, we just remark that the relative velocity of observers inside the shell with the shell itself could be obtained out of Eqs. (6.50) and (6.52) and always vanishes at the associated “gravitational radius” of the exter-
nal spacetime \((e^\nu = 0)\), showing that a Machian behavior always does take place in this case, as we already know [139]. Nevertheless, whenever an inner nontrivial spacetime is considered, in principle there always exists configurations that would lead the observers inside the shell to corotate with the thin shell.

The interest in the effect of frame dragging naturally lies on the fact that its measurement could give direct information about the stratification of the spacetime. This could be probed through the use of gyroscopes, as for instance in experiments similar to gravity probe B [147].

### 6.6 Matching Kerr-Newman spacetimes

Our approach of gluing two slowly rotating spacetimes is general. Therefore, one could conceive any case and pore over its properties. A case of interest would be the one where the regions above and below the surface of discontinuity be analogous to the Kerr-Newman solution. This could be for instance a model for an atom in an excited state. A natural advantage of this match is the geometric simplicity of both regions. We know generally that the Kerr-Newman metric in the Boyer-Lindquist coordinates \(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi}\) is [36]

\[
\begin{align*}
   ds^2 &= \left(1 - \frac{2Mr - Q^2}{\rho^2}\right) d\bar{t}^2 - \frac{2(2Mr - Q^2)a \sin^2 \bar{\theta}}{\rho^2} d\bar{t}d\bar{\phi} \\
   &\quad - \frac{\bar{\rho}^2}{\Delta} d\bar{r}^2 - \rho^2 d\bar{\theta}^2 - \frac{Y}{\rho^2} \sin^2 \bar{\theta} d\bar{\phi}^2,  
\end{align*}
\]

(6.53)

where

\[
\Delta \doteq \bar{r}^2 - 2Mr + a^2 + Q^2,  
\]

(6.54)

\[
Y \doteq (\bar{r}^2 + a^2)^2 - \Delta a^2 \sin^2 \bar{\theta}  
\]

(6.55)

and finally

\[
\rho^2 \doteq \bar{r}^2 + a^2 \cos^2 \bar{\theta}.  
\]

(6.56)
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We recall that this solution has three arbitrary constants, namely the total mass $M$ of the system, its total charge $Q$ and its total angular momentum per unit mass $a$. The above metric has horizons at $\Delta = 0$. We shall not elaborate anymore on the properties of this solution.

Since charges are present in the system, electromagnetic fields appear. Naturally they are axially symmetric. Their components are\footnote{We are not talking about local fields here, but simply components of the electromagnetic field tensor $F_{\mu \nu}$. Local fields are related to the components of the tetrad decomposition of $F_{\mu \nu}$ (see Ref. [2] for further details) and we shall not be interested in such aspects in this chapter.} [36]

\[
E_{\bar{r}}(\bar{r}, \bar{\theta}) = \frac{Q(\bar{r}^2 - a^2 \cos^2 \bar{\theta})}{\rho^4}, \quad (6.57)
\]

\[
E_{\bar{\theta}}(\bar{r}, \bar{\theta}) = -\frac{2Qa^2 \bar{r} \cos \bar{\theta} \sin \bar{\theta}}{\rho^4} = -\frac{aB_{\bar{r}}(\bar{r}, \bar{\theta})}{\bar{r}^2 + a^2}, \quad (6.58)
\]

\[
B_{\bar{r}}(\bar{r}, \bar{\theta}) = \frac{2Qa^2 \bar{r} \cos \bar{\theta} \sin \bar{\theta}(\bar{r}^2 + a^2)}{\rho^4} \quad (6.59)
\]

and

\[
B_{\bar{\theta}}(\bar{r}, \bar{\theta}) = \frac{Qa \sin^2 \bar{\theta}(\bar{r}^2 - a^2 \cos^2 \bar{\theta})}{\rho^4} = a \sin^2 \bar{\theta} E_{\bar{r}}(\bar{r}, \bar{\theta}). \quad (6.60)
\]

It is simple to see (e.g. Ref. [36]) that the Eqs. (6.57)–(6.60) are derived from the four-potential

\[
A_{\mu} = -\frac{QF}{\rho^2} (1, 0, 0, -a \sin^2 \bar{\theta}). \quad (6.61)
\]

We emphasize that the Hartle-Thorne coordinates $(t, r, \theta, \phi)$ are different from the Boyer-Lindquist ones. By imposing specific induced forms to the metric components due to a coordinate change, it is simple, though tedious to show that the coordinate transformations linking the Hartle-Thorne to the Kerr-Newman
metrics up to second order on “\(\epsilon\)” (here \(\epsilon = a/r\)) are

\[
\tilde{\theta} = \theta - \frac{a^2}{2r^2} \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \cos \theta \sin \theta
\]  

(6.62)

and

\[
\tilde{r} = r - \frac{a^2}{2r} \left[ \left( 1 + \frac{2M}{r} \right) \left( 1 - \frac{M}{r} \right) - \frac{2Q^2}{3r^2} \left( 1 + \frac{Q^2}{r^2} - \frac{7M}{2r} \right) \right.
\]

\[
- \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \left( 1 + \frac{3M}{r} - \frac{2Q^2}{r^2} \right) \cos^2 \theta \left] \right.
\]

(6.63)

The coordinate transformations given by Eqs. (6.62) and (6.63) are necessary in order to apply the formalism we developed previously for matching two slowly rotating spacetimes. The Kerr-Newman metric components up to second order on “\(a/r\)” in the Hartle-Thorne coordinate system are

\[
h_{0}^{KN}(r) = -\frac{2M^2 r^2 - Q^2 (3M + r) r + Q^4}{6r^4} e^{-\nu}
\]  

(6.64)

\[
h_{2}^{KN}(r) = -\frac{e^{-\nu}}{3r^6} \left[ 2Q^4 - Q^2 (9M - r) r + Q^2 (13M^2 - 2Mr - 2r^2) r^2 
\right.
\]

\[
- M (6M^2 - Mr - 3r^2) \right] 
\]

(6.65)

\[
j_{0}^{KN}(r) = -\frac{6M^2 r^2 + Q^2 (2r - 7M) r + 2Q^4}{6r^4} e^{-\nu},
\]  

(6.66)

\[
j_{2}^{KN}(r) = \frac{1}{3r^4} \left[ 8Q^4 + 3M (5M - r) r^2 - Q^2 (23M - 4r) r \right],
\]  

(6.67)

\[
k_{2}^{KN}(r) = \frac{Q^2 (7M + r) r - 3Mr^2 (2M + r) - 2Q^4}{3r^4}
\]  

(6.68)

\[
e^{\nu} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}
\]  

(6.69)
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and

$$\omega = \frac{2Mr - Q^2}{r^2}. \quad (6.70)$$

We shall now study the nature of equilibrium points for collapsing shells of matter and charge. This is noteworthy since it allows us to scrutinize which entities could “linger on” in the spacetimes they themselves contribute to. In this regard, it is interesting to notice that Lopez [149] has proposed a time ago a scenario where rotating thin shells could be conceptually relevant. His analyses led him to glue the Minkowski spacetime with the Kerr-Newman one. The key point there was perceiving that besides the infinity, there is another special radial coordinate where the Kerr-Newman metric is Minkowskian, namely at $r = Q^2/(2M)$. This is half of the radius one would attribute to a classical massive and charged particle. It can be specially applied to the electron case. Bearing the aforementioned analogies in mind, Lopez built up a model of an electron as a spinning thin shell at $r = Q^2/(2M)$, whose external region is associated with the Kerr-Newman spacetime and the inner region with the Minkowski one. Several aspects were analyzed concerning this proposal. Due to the axial symmetry, no gravitational radiation can be emitted. Nevertheless, in Lopez’s model, the electron does have an electric quadrupole moment, quite differently from what one would expect from quantum mechanics. Besides, it was not of interest of his work to analyze the stability of the equilibrium point the shell is placed. Naively speaking one could state the aforesaid equilibrium is stable due to the fact we are talking about a model for a fundamental particle. In the sequel we shall analyze the stability of slowly rotating shells at $R = Q^2/(2M)$, as well as in other positions, just as a special application of the formalism we developed in this chapter.

As we showed previously, our matching approach leads automatically to stable axially symmetric configurations whenever their nonperturbed counterparts are so. Therefore, it suffices just to analyze the spherically symmetric configurations regarding its stability. Nevertheless, for details about the charge and mass distribution on the slowly rotating thin shell, obviously one has to make use of the full set of equations appropriate for this case.

A point of interest are the properties of the induced energy density in a slowly rotating thin shell. Incontestably the total energy density shall be constrained solely by the nature of the spherically symmetric case when it is nonva-
nishing. Whenever the latter vanishes, our approach should be seen with wariness and would just suggest possible trends. The aforementioned query is of relevance because it would point towards systems whose induced surface energy could decrease due to the rotation in the nonperturbative case, even possibly being negative in some regions of the shell.

Whenever one analyzes the case \( R = Q^2_+/ (2M_+) \), by assuming that \( Q_- = 0 \) and \( M_- = 0 \), our formalism breaks down since there the induced energy density vanishes. In the case where \( \sigma_{ss} \) is positive (though small), one has actually to consider \( R = Q^2_+ / (2M_+) + \delta \), with \( \delta > 0 \). For this case, it can be checked by numerical analyses that the thin shell formalism gives stable solutions for any equation of state with \( R / M_+ = Q^2_+ / (2M^2_+) \approx 1.0 \) and unstable solutions for any \( R / M_+ \lesssim 0.33 \). Whenever the interior spacetime is a Schwarzschild black hole, it is impossible to have positive surface energies for \( R = Q^2_+ / (2M_+) \). For having \( \sigma_{ss} > 0 \), one should consider larger radii and outer masses bigger than the inner ones. If the interior spacetime is a charged black hole and has approximately the same mass as the external one, then positive spherically symmetric surface energy densities can just be attained if \( Q_- > Q^2_+ \) [for \( R = Q^2_+ / (2M_+) \)]. Nevertheless, it implies that only interior naked singularities would be physically relevant (because just in this case the radius of the shell is bigger than the associated outer Reissner-Nordstrom horizon of the inner solution). Figure 6.1 depicts the stability analyses for some of the scenarios commented above. Notice that there are cases where certain equations of state (represented by \( \eta \), the square of the speed of the sound) could render unstable solutions to the thin shell (the equilibrium points of the effective potential [93], that describes the dynamics of the surface of discontinuity, are maxima \( \tilde{V}''(R) < 0 \), where \( \tilde{V}'' \) is given by Eq. (4.18)).

We stress that when one analyzes the stability of thin shells apropos of \( \eta \), one is in reality building up a catalogue to these systems, scrutinizing all possible situations. The physical equation of state of a system should be unique and it is hence a representative of the aforementioned catalogue and its knowledge would tell us with certainty its underlying stability.

We turn now to the induced surface energy-density \([\Delta \sigma \doteq 8 \pi (\sigma - \sigma_{ss})]\) analyses for slowly rotating hypersurfaces. As it is clear from Eq. (6.2), \( \sigma \) should be polar angle dependent. We sift now the sign of the induced surface energy density. We shall be concerned here only with a representative of the ellipticities a hypersurface may have. As a first example, let us consider the case where the
internal spacetime is flat, while the external one is of Kerr-Newman. We are only interested in the case \( R/M_+ = 1/2(Q_+^2/M_+^2 + \delta) \), \( \delta > 0 \). Figure 6.2 shows the result for \( Q_+/M_+ = 3 \). As it can be checked for the matching of Minkowski with Kerr-Newman spacetimes, for any \( Q_+/M_+ > 0 \) the induced energy density is negative for all the polar angles. Naturally this does not mean that the mass of the system is negative, since the surface pressure also contributes to the final mass [139], whose answer we already know to be \( M_+ \).

Another interesting case to study is the one where we glue a Kerr spacetime with a Kerr-Newman one. Here we already know that it is impossible to have spherically symmetric solutions that present positive surface energy density for \( R = Q_+^2/(2M_+) \). Figure 6.3 depicts a case in which the aforementioned problem is circumvented, where the inner and outer regions have the same rotational parameter. From this figure one can clearly see that there are regions where the induced surface energy is negative.

A final case that we investigate is the gluing of two Kerr-Newman spacetimes
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Figure 6.2: Induced surface energy density ($\Delta\sigma$) due to rotation for the case $Q_+/M_+ = 3$ and $R/M_+ = 1/2(Q_+/M_+)^2 + \delta$, $\delta > 0$ as a function of the polar angle $\theta$ when the internal spacetime is flat. It can be checked that for $Q_+/M_+ > 0$, $\Delta\sigma$ is always negative. The mass of outer spacetime is always positive ($M_+$), being a contribution also from the surface tension.

\[
\Delta\sigma M_+/(a^2/M_+^2)
\]

Figure 6.3: Induced surface energy density ($\Delta\sigma$) due to rotation for the case $Q_+/M_+ = 1.2$, $M_+/M_- = 2$ and $R/M_+ = 2$ as a function of the polar angle $\theta$ when the internal solution is of Kerr and the outer one of Kerr-Newman and the difference in their rotational parameters is negligible.

\[
\Delta\sigma M_+/(a^2/M_+^2)
\]

with the same rotational parameter, negligible differences in their masses and rotational parameters and with $R/M_+ = 1/2(Q_+/M_+)^2$. For this case, solutions with $\sigma_{ss} > 0$ raise just when one analyzes naked-like singularities. Figure 6.4 shows the induced energy density as a function of the polar angle for the case $Q_-/M_+ = 1.45$ and $Q_+/M_+ = 1.4$. 

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![Figure 6.4](image)

Figure 6.4: Induced surface energy-density ($\Delta\sigma$) due to rotation for the case $Q_+/M_+ = 1.4$, $Q_-/M_+ = 1.45$ and $R/M_+ = 1/2(Q_+/M_+)^2$ as a function of the polar angle $\theta$ when the internal and external spacetimes are of Kerr-Newman with negligible changes in their masses and rotational parameters. For this case, a region where $\Delta\sigma$ is negative raises around the equatorial plane.

Just as another exemplification of our analyses, Fig. 6.5 shows the behavior of the induced surface tension for the same configuration as in Fig. 6.4, where also regions with $\Delta\mathcal{P} < 0$ [$\Delta\mathcal{P} \doteq 8\pi(\mathcal{P} - \mathcal{P}_{ss})$] rise. We shall not elaborate on induced surface tensions anymore. It is worth stressing that the equations of state in the spherically symmetric case automatically guarantee the well-behaved notion of the speed of the sound on the surfaces under interest.

Finally, just for completeness, in order to calculate the induced surface charge quantities, it suffices to recall that from the non-homogeneous part of the Maxwell equations, one learns easily that the induced currents on the surface of discontinuity are given by $4\pi j^a = [F_{\mu\nu} n_\mu e_\nu^a]_+$ (see Ref. [150] for further details), $e_\mu^a \doteq \partial y^a / \partial x^\mu$, $y^a$ the coordinate system on $\Sigma$. The continuity of the tangential decomposition of the electromagnetic field tensor is always physically satisfied, given that the shells are actually thin and slowly rotating.

6.7 Conclusions

In this chapter we showed some of the subtleties present in the match of slowly rotating spacetimes. From our approach for a given hypersurface connecting two of such spacetimes, we obtained generically that its equilibrium points are auto-
6.8 Perspectives

The analysis done in this chapter was general. Therefore, many other situations still in the slowly rotating case should be examined, such as the conditions that would lead to counter-rotating spacetimes, solutions whose external spacetimes were neutral, like Kerr or Hartle-Thorne (which would introduce more parameters in the analyses), etc. The ultimate interest would be matching axially symmetric spacetimes nonperturbatively, since it could have a possible relevance in the analyses of rapidly rotating stratified stars.

Figure 6.5: Induced surface tension ($\Delta P$) due to rotation for the same case as in Fig. 6.4. As there for the induced surface energy density, here regions where $\Delta P < 0$ also rise.

Mathematically stable if their spherically symmetric counterparts are so. As we found in some particular examples, the surface energy-density and the surface tension could decrease concerning their spherical counterparts for some regions of the shell. This would suggest, for instance, that in non-perturbative calculations the assumption of having nowhere non-negative energy-densities could eventually be broken. Another likely and reasonable interpretation would be that these associated solutions are unphysical.
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General conclusions

In each chapter precise conclusions and perspectives can be found. Here we constrain ourselves to recapitulate some of them in a more succinct framework.

In Chapter 1 we found a simple and elegant way to decompose the energy of a spherically symmetric and nonlinear charged black hole based upon conservation laws (charge and energy). This was possible in part due to the high symmetry this system displays. This expression is important per se because it could substitute the first law of black hole mechanics for nonlinear electromagnetism. It would be in a sense its generic “solution”. Besides, it is the constraint equation that the parameters of any nonlinear charged black hole must satisfy. This restricts severely their arbitrariness as its occurs only in the realm of Einstein’s equations. Practically speaking, it could have a germaneness to all approaches that depend upon the energy extraction from charged black holes.

In Chapter 2 we saw that the constraint equation obtained in Chapter 1 simplifies tremendously the description of nonlinear charged black holes. This is so since one with it is studying only the cases that are in congruence with conservation laws. Hence, this should always be the case for the analyses of black hole systems. It even allows us to appraise certain situations that would be really cumbersome exactly, e.g. black hole interactions. We showed that the radiation coming from these interactions could shed a precious light on the discrimination of the black holes enmeshed, be either with gravitational waves or electromagnetic ones.

In Chapter 3 we concluded that neutrinos could be precious for scrutinizing spacetime aspects of charged black holes described by nonlinear Lagrangians to the electromagnetism. This is so due to their unique quantum aspects, which we properly explored there. Besides, kinematical effects, such as the dragging of inertial frames could be of relevance in probing the aforesaid axially symmetric spacetimes. We showed, as it is very intuitive, that the most significant departures from the classical analyses take place near the outer horizons of the
associated black holes. Therefore, QPO’s could play a role in characterizing such spacetimes. Since neutrinos are produced copiously during the process of gravitational collapse, nonlinear aspects of the black holes could influence their dynamics, that in turn could have an effect on astrophysical events, such as supernovas. The two aforesaid issues are yet ideas to be further explored.

Chapter 4 taught us that surface degrees of freedom could have a relevance to astrophysical systems endowed with different phases. The reason for that is because first off they are necessary to guarantee distributional solutions to Einstein’s equations and secondly their nontrivial dynamics could only per se already influence the stability of a stratified star. We showed that certain subtleties may be present for matching cores and crusts in stars. This suggests also that such degrees of freedom could be in shallow regions of these systems, which could also be linked with their outside dynamics near the surface, such as QPOs.

In Chapter 5 we showed that the distributional approach can be used in order to describe the stability (against radial perturbations) of stratified systems. It automatically gave us further boundary conditions that should be taken into account for their stability analyses. Besides, it makes evident the subtleties behind matching solutions to Einstein’s equations, as we demonstrated there. In general, additional boundary conditions change the set of eigenfrequencies a system may have and this could thus be used as a smoking gun to astrophysical analyses. It would be of interest to connect the aforementioned analyses with variational and numerical approaches, QPOs and glitches observed in stars.

Lastly, Chapter 6 dealt with aspects of surfaces degrees of freedom in the presence of slow rotations. It suggests that certain energy conditions could possibly be modified in nonperturbative analyses, which could leave us with even richer scenarios to be scrutinized in the future. It could also address issues related to the non-singularity of black holes.
Bibliography


