DOCTORAL THESIS

Kinetic approach
to pair production in strong electric fields
and to transparency of relativistic outflows

A thesis submitted in fulfilment of the requirements
for the degree of Doctor in Relativistic Astrophysics

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Preface

In my thesis I present the theoretical study of two physical phenomena in plasma physics, namely (a) electron-positron pairs production in strong electric fields and (b) transparency of relativistic outflows.

From the theoretical point of view, both these systems have been long under study. The pair production due to Schwinger effect is one of the key problems in modern Quantum Field Theory. Currently an international effort is underway aiming at generation of ultrahigh intensities of laser fields and thereby creating electric fields near the critical value for the pair production. There is an ongoing debate on whether the critical electric field can be potentially reached in these experiments. In the literature many theoretical studies of pair production can be found. However very few of them account for back reaction of created pairs on the initial electric field. Concerning transparency of relativistic outflows, there is currently a consensus in the literature that emission produced by the photosphere of relativistically expanding plasma should be present in all gamma-ray bursts (GRBs). The intensity of this photospheric component emission may be weak compared to the total intensity of the GRB. However when detected, this emission may provide unique information on physical properties of relativistic outflows which produce GRBs.

Even though very different in what concerns their macroscopic features, the microphysics of both phenomena (a) and (b) can be described in detail using a common method. This is what we aim to do in the study of pairs creation and photospheric emission using a theoretical approach based on Relativistic Kinetic Theory (RKT). One of the main motivations behind this strategy is related to the possibility, given by such theory, to study physical systems out of equilibrium from a very general point of view. To do that we solve the Relativistic Boltzmann Equation (RBE) with exact collision integrals corresponding to relevant microscopic processes, taking into account the anisotropy of the distribution function in the phase space. One one hand, with this technique we are able to describe the evolution in time of the initially out of equilibrium system of electric field and
electron-positron-photon plasma up to thermalization. On the other hand when transparency of relativistic outflow is considered, the opposite behavior of departure of electron-photon plasma from thermal equilibrium can be followed.

In Chapter 1, the principles and the framework of RKT are recalled. In Section 1.2 the concept of distribution function and its properties are presented first. Then we introduce the RBE as the master equation that describes deterministically the distribution function evolution with time taking into account particle collisions and external forces. We give the expression of the distribution function at thermal equilibrium and we explain its functional form in relation with classical and quantum statistics. Here we discuss the role of 2 and 3-particle interactions during the approach to the thermal state giving the motivations that allow us to consider only 2-particle interactions in the calculations we have performed in this thesis. In Section 1.3 we illustrate the technique we have used to solve the RBE. Here we delineate the structure of the transport terms and that of collision integrals in a general form. Then, after having explicated the appropriate coordinate systems for both momentum and physical space, we can specify the discretization procedure and the finite difference numerical method we adopted.

A historical introduction to the problem of electron-positron pairs creation in strong electric fields is firstly given in Chapter 2. We recall the concept of critical field in Section 2.1 which is mostly dedicated to the calculation of the electron-positron pair production rates in quantum electrodynamics (QED). Afterwards, we delineate several approaches concerning the dynamical evolution of the full system consisting of pairs and electric field in Section 2.2. Here we show how the phenomenon of pairs oscillations in strong electric fields can be described in different ways, starting from a semiclassical point of view till the numerical solution for a system of coupled Vlasov-Boltzmann equations. In Section 2.3 we work out one ordinary differential equation which gives the opportunity to describe the process of pair production, its back reaction onto the external field and the oscillating response. This procedure allows the computation of the emitted radiation spectrum that is the unique visible signature of the pairs oscillation phenomenon. In Section 2.4 we study the entire dynamics of energy conversion from initial strong electric field, ending up with thermalized electron-positron-photon plasma. To do that we use the kinetic approach delineated in Chapter 1 and solve numerically the system of coupled partial integro-differential Vlasov-Boltzmann equations for electron-photons pairs and photons respectively. The system is assumed to be homogeneous as well as anisotropic and 2-particle interactions between elec-
trons and photons are taken into account by collision terms. The results of this last Section can be regarded as the main outcome of this thesis concerning the process of electron-positron pairs creation in strong electric fields.

Chapter 3 starts with a brief introduction to the concept of transparency in different astrophysical systems. Being GRBs one of the possible applications, we summarize their key features in Section 3.1. In particular we focus our attention onto the component of their spectra that comes from the photosphere. So we briefly illustrate a few approaches that have been used to study the transparency of relativistic outflows. In Section 3.2 we firstly summarize the motivations that, in our opinion, justify the need of a kinetic treatment to throw more light on the physical processes occurring when a relativistic outflow becomes transparent. Then we report the theoretical framework one needs to adopt in order to compute the optical depth of the outflow and that will allow us to obtain the parameters for the relativistic wind. Making use of the theoretical and numerical apparatus introduced in Chapter 1 we solve the Boltzmann equations for a relativistically and spherically symmetric expanding plasma. Starting from an initially optically thick outflow, we follow the time evolution for electron and photon distribution functions during the transition from opaque to transparent plasma.

The main results about pairs production in strong electric fields and transparency of relativistic outflows are summarized in the Conclusions Chapter. The outcomes corresponding to these two physical systems are discussed and the relative conclusions are presented.
Chapter 1

Relativistic Boltzmann equation

1.1 Introduction

The relativistic Boltzmann equation (RBE) is the tool provided by relativistic kinetic theory (RKT) to describe the distribution function of particles moving with relativistic velocities. Kinetic theory (KT) plays a significant role in many areas of physics and can be applied to a wide range of physical phenomena. KT is closely related to the statistical mechanics, thermodynamics and hydrodynamics. Beyond that, the main ideas and principles of KT influenced the development of many other sciences, including mathematics (probability theory, ergodic theory), biology (evolutionary biology, population genetics) and economics (financial markets, econophysics).

By definition, KT is the microscopic theory of processes in systems not in statistical equilibrium \[1\]. The classical theory was developed by D. Bernoulli, Clausius, Maxwell and Boltzmann. The concept of distribution function was introduced at that time as the object from which one can extract all the informations about the physical system. It was found that at thermal equilibrium particles follow the so called Maxwell distribution function which depends on the temperature only. The Maxwell formula is correct when the quantum nature of particles can be neglected (classical system) and when particle velocities are much smaller than the speed of light (non relativistic). The main goal of KT is to determine the correct equation that describes deterministically how the distribution function evolves with time. This fundamental equation was devised by Boltzmann and it was named in his honor.

The relativistic domain of KT started to be explored soon after the publication of the Einstein paper on special relativity “Zur Elektrodynamik
bewegter Körper” [2]. Jüttner [3] generalized the Maxwell equilibrium distribution function to the relativistic case, even though not in the manifestly covariant form. The relativistic kinetic equation was derived by Walker for collisionless systems and by Licherowicz and Marrot including collision effects. The latter is the relativistic generalization of the Boltzmann equation.

In this Chapter we recall the formalism of RKT following [4] that will be used throughout this Thesis. The concept of distribution function will be introduced and the relativistic Boltzmann equation will be presented as well as some of its approximations.

1.2 Basic concepts

1.2.1 Distribution function

In classical (also relativistic) mechanics a complete description of a system composed of $N$ interacting particles is given by their $N$ equations of motion. In relativistic kinetic theory one deals with the phase space $\mathcal{M}$ of positions and momenta of all particles ($6N$-dimensional space). The most complete knowledge about a physical system is achieved once one has solved all the equations of motion, which is the same as knowing the trajectory in $\mathcal{M}$. However, even for a small number of particles such goal is very hard to achieve and also useless to some extent. In fact, usually one is interested in knowing the macroscopical properties of the ensemble of particles under study and not the particular status of one single particle.

A tremendous simplification occurs for some systems, where $N$ is very large. Under certain conditions they can be described by a function defined on the 6-dimensional phase space $\mathcal{M}^6$. Such function depends only on 7 variables: 3 space coordinates, 3 momentum components, and time. In such a case the DF is called the one particle distribution function (DF) $f(x^\mu, p^\mu)$. This is the basic object used in statistical (probabilistic) description of a system composed of large number of particles. For brevity in what follows denote the coordinates in momentum space as $x^\mu = (ct, x)$, $p^\mu = (p^0, p)$, where $c$ is the speed of light. Notice that $p^0$ is not an independent variable and it satisfies the relativistic energy equation $p^0 = \frac{\sqrt{\gamma m^2 c^4}}{c}$.

\footnote{In what follows Greek indices run from 0 to 3, while Latin ones run from 1 to 3. Einstein summation rule is adopted.}
\[ \sqrt{p^2 + m^2c^2}. \] The DF is defined such that the integral

\[
N \equiv \int_{M^6} f(p, x, t) \, d^3p \, d^3x
\]
gives the total number of particles. Notice that the integral is clearly Lorentz invariant. Then one observes that \( f(p, x, t) \, d^3p \, d^3x \) is an average number of particles having momenta in the range \((p, p + d^3p)\) and coordinates in the range \((x, x + d^3x)\) at the moment \(t\), and the integral (1.1) is taken in the whole phase space \(M^6\).

Notice that despite symmetrical form of \( f(p, x, t) \) there is a conceptual difference between \(x\) and \(p\). In particular, the following integral over the momentum space

\[
n(x, t) \equiv \int_{-\infty}^{+\infty} f(p, x, t) \, d^3p
\]
gives the number density in the physical space. On the other hand, if the previous integral would have been performed over the physical coordinates this would give a particle number density in the momentum space.

The one particle DF defined by Eq. (1.1) is not written in a Lorentz invariant way. Nevertheless, in \([5, 6]\) its invariance has been demonstrated, see also \([7]\).

### 1.2.2 Kinetic equations

Let us consider a mixture of \(N\) kinds of particles where each component, identified by the index \(k = 1, \ldots, N\), is described by a scalar distribution function \(f_k = f_k(x, p, t)\). Then such system has to satisfy the following system of \(N\) transport equations (see e.g. \([6]\))

\[
p^\mu \partial_\mu f_k + m_k F_k^{\mu} \frac{\partial f_k}{\partial p^\mu} = \text{St} f_k, \quad k = 1, \ldots, N.
\]

where \(\partial_\mu = \left( c^{-1} \frac{\partial}{\partial t}, \nabla \right)\), \(m\) is the particle mass, \(F_k^{\mu}\) represents an external four-force acting on the particle \(k\) and \(\text{St} f_k\) is the symbolic representation of the collision integral. Formula (1.3) is the so called relativistic transport equation and, in its full generality, a system of coupled partial integro-differential equations. As we will see later, the coupling is given by the collision integrals.

One of the main goals of KT is to establish the form of the collision integral. We now illustrate how \(\text{St} f_k\) appears in the simplest case of elastic
collision between identical particles. Let us describe the following interaction scheme

\[(1.4) \quad 1 + 2 \rightarrow 1' + 2',\]

where particles 1 and 2 have 4-momenta \(p^\mu\) and \(k^\mu\) which change after the collision to \(p'^\mu\) and \(k'^\mu\), respectively. Their 4-momenta are related by the energy-momentum conservation law

\[(1.5) \quad p^\mu + k^\mu = p'^\mu + k'^\mu.\]

The average number of such collisions is proportional to 1) the number of particles per unit volume with momenta \(p^\mu\) in the range \(d^3p\), 2) the number of particles per unit volume with momenta \(k^\mu\) in the range \(d^3k\) and 3) the intervals \(d^3p', d^3k'\) and \(d^4x\). The proportionality coefficients, depending only on four-momenta before and after the collision are represented as \(W(p, k | p', k') / (p^0 k^0 p'^0 k'^0)\). The quantity \(W(p, k | p', k')\) is called the transition rate and it is a scalar. By this process particles leave the phase volume \(d^3p\) around \(p^\mu\). Collisions also bring particles back into this volume by the inverse process with the corresponding rate \(W(p', k' | p, k)\).

In the absence of external forces, the RBE can be written in differential form as

\[(1.6) \quad p^\mu \partial_\mu f = \frac{1}{2} \int \frac{d^3p' d^3k d^3k'}{p'^0 k'^0} \cdot \left[ f(x, p') f(x, k') W(p', k' | p, k) - f(x, p) f(x, k) W(p, k | p', k') \right].\]

The same equation in vector notation becomes

\[(1.7) \quad \frac{\partial f}{\partial t} + v \cdot \nabla f = \frac{1}{2} \int d^3p' d^3k^3k' \left[ f(x, p') f(x, k') \omega_{p'k';pk} - f(x, p) f(x, k) \omega_{pk;p'k'} \right],\]

where \(v = c p / p^0\) and \(\omega_{pk;p'k'} = c W(p, k | p', k') / (p^0 k^0 p'^0 k'^0)\). If in this expression particle momenta are substituted by their velocities, this equation will coincide with the one derived first by Boltzmann [8]. Notice that the factor \(1/2\) in front of collision integral is due to indistinguishability of particles.

It is very important to notice that (1.6) refers to only one kind of particle and one type of interaction. If there are different particles and each of them
can interact in many ways, then one has to use (1.3) where the right hand side has to be replaced by a sum running over all the interaction types. Besides, above we have considered the case where only two particles are present both in the initial and final states. This situation can be easily generalized to one particle in the initial state (particle decay) or to three or more particles in the final state.

It can be easily verified that Eq. (1.6) guarantees particle number and energy-momentum conservation laws as well as eventual quantum numbers involved in the interaction processes. Particle number is not conserved in the case when three-particle interactions are considered.

When quantum nature of particles cannot be neglected, interactions can be describe by the so called Uehling-Uhlenbeck collision integrals [9, 10]. Their expression can be obtained from the classical one by means of the following phenomenological changes

\[
\text{St} f = \frac{1}{2} \int \frac{d^3 p'}{p'^0} \frac{d^3 k}{k^0} \frac{d^3 k'}{k'^0} \cdot \{ f(x, p') f(x, k') \left[ 1 + \theta \varphi(x, p) \right] \left[ 1 + \theta \varphi(x, k) \right] W(p', k' | p, k) - f(x, p) f(x, k) \left[ 1 + \theta \varphi(x, p') \right] \left[ 1 + \theta \varphi(x, k') \right] W(p, k | p', k') \},
\]

where \( f(x, p) = g \varphi(x, p) / (2\pi \hbar)^3 \), \( \hbar \) is the reduced Planck constant, \( g \) is the degeneracy factor, \( \theta = \pm 1, 0 \) for respectively Bose-Einstein, Fermi-Dirac and Boltzmann statistics. Comparing this expression to eq. (1.6) one finds additional multipliers \( 1 \pm \hbar^3 f(x, p) / g \), which guarantee that equilibrium distribution functions are indeed Bose-Einstein and Fermi-Dirac ones, respectively, see e.g. [11, 12].

Jüttner [3] was the first one who obtained the correct distribution function for a classical ensemble of relativistic particles in thermal equilibrium

\[
f^{\text{EQ}}(p) = \frac{1}{\hbar^3} \exp \left[ \frac{\varphi - p^\mu U_\mu}{k_B T} \right],
\]

where \( \varphi \) is the chemical potential, \( T \) is the temperature and \( U_\mu \) is the 4-velocity of the reference frame, \( \hbar \) and \( k_B \) are Planck’s and Boltzmann’s constants. However, when the quantum nature of particles cannot be ne-
1.2. BASIC CONCEPTS

glected the equilibrium distribution function has to be

\[ f^{EQ}(p) = \frac{1}{h^3} \left[ \exp \left( \frac{pU_\mu - \phi}{k_BT} \right) \pm 1 \right]^{-1}, \]

where the positive and negative signs are for fermions and bosons respectively in agreement with the well known Fermi-Dirac and Bose-Einstein quantum statistics.

1.2.3 The role of 2- and 3-particle interactions

Up to now in this Chapter we gave an explicit form for the collision integrals taking into account both classical and quantum statistics, but only when there are two particles in the initial state and two particles in the final one. Such collision integrals were first considered by Boltzmann in his theory of gases [8].

However it has been demonstrated in [13, 14] that an initially out of equilibrium optically thick relativistic plasma cannot reach the thermal equilibrium configuration if 3-particle interactions are not considered. This is due to the fact that 3-particle processes can change the chemical potentials for each component of the plasma while 2-particle ones cannot. By definition a plasma is in thermal equilibrium if all particles have common temperature and chemical potential. Even though 3-particle interactions have such fundamental role in plasma physics, we now give the reasons why for the physical processes examined in this Thesis only 2-particle interactions are essential.

Since interactions among photons, electrons, positrons and protons are described perturbatively by QED Feynman diagrams, we know that 3-particle interactions have a cross section that is about \( \alpha^{-1} = 137 \) times smaller than those for 2-particle collisions. This difference is very much related the the time-scale at which a specific process manifest itself appreciably. In this sense 2-particle collisions are the first ones to give a main contribution in collisional relaxation of the distribution function toward the thermal state. Only on a much longer time-scale triple interactions intervene changing the number of particles and, as a consequence, the chemical potentials.

As we will see in Chapter 2 electron, positron and photon distribution functions are strongly anisotropic in the phase space at the very beginning. That means they are initially very far from thermal equilibrium and the effects of 2-particle interactions have to be taken into account first. This first
1.3. NUMERICAL METHOD

stage is what we aim to study because the electric field is still extremely active in producing and accelerating electrons and positrons but, at the same time, also pair annihilation into photons becomes crucial. Only at later times, when the electric field is small, collisions dominate the future evolution of the system toward a fully thermalized plasma.

Studying the transparency of a relativistic outflow in Chapter 3 we start with electrons and photons in thermal equilibrium. At not very large optical depth the rate of 3-particle collisions (which is much smaller than the 2-particle one) is small compared with the expansion time-scale. The temperature of the plasma is such that electron-positron annihilation and creation can be neglected. When the concentration of photons greatly exceeds the one of electrons Coulomb scatterings can be neglected as well. As a matter of fact only Compton scattering between electrons and photons can give rise to appreciable effects during the relativistic expansion.

1.3 Numerical method

We dedicate this Section to the numerical scheme adopted to solve a system of relativistic Boltzmann equations. This technique has been applied in a number of different physical and astrophysical problems [15, 13, 14, 16]. Now we apply it to both the topics we have studied in this Thesis, namely a) electro-positron pair creation in strong electric fields in Chapter 2 and b) transparency of relativistic outflows in Chapter 3.

In both processes we consider the system to be composed by electrons, positrons and photons. The circumstance of having three different particles in our system, implies that the system of equations to be solved is given by Eqs. (1.3) which can be rewritten as

\begin{align}
    p^\mu \partial_\mu f_\pm &\pm m_e F^\mu \frac{\partial f_\pm}{\partial p^\mu} = S^\pm f_\pm, \\
    p^\mu \partial_\mu f_\gamma & = S^\gamma f_\gamma, \tag{1.11}
\end{align}

where the symbols $-, +, \gamma$ are used to indicate electron, positron and photon respectively. In the case of pair production, the external force in Eq. (1.11) is due to the electric field which is the source from which electrons and positrons are produced and then accelerated. Eqs. (1.11) can be simplified in such a way that only one equation has to be solved instead of two, see also [16, 17]. In fact, since we consider systems with no net electric charge, the number density of electrons has to be equal to the number density of positrons $n_+ = n_-$. Besides, under the effect of an external elec-
tric field they move to opposite direction in space as well as with opposite 3-momentum. Therefore, once we know the electron distribution function we also know the one of the positron because they are symmetric with respect to a plane in the phase space which is perpendicular to their velocity. This symmetry can be written as follows

\[(1.13) \quad f_-(p) = f_+(p).\]

When we study the approach to transparency of relativistic outflows there is no external force acting on particles.

As we have already mentioned, Eqs. (1.11) and (1.12) are coupled because of the collision integrals on the right hand side. We are now going to give explicit expressions for them. To do that we first assume particles interact electromagnetically only via two-particle collisions. Given such assumptions, the best theory at our disposal from which cross sections can be computed is QED. According to this theory, electrons, positrons and photons can interact via electron-positron creation or annihilation, Compton, Bhabha and Möller scatterings.

### 1.3.1 The transport equation

The relativistic Boltzmann Eqs. (1.3) can be cast in the form of transport equations used in radiative transfer theory [18, 19]. In Chapter 3 we are going to consider the expansion of a spherically symmetric outflow and the distribution function will depend on one spatial coordinate. In the literature the independent variables on which the DF depends are usually taken as the particle energy \( \epsilon \) and the angle \( \alpha \) between the particle 3-velocity and the radial direction of motion. As a result Eq. (1.3), can be rewritten as [18, 19, 20]

\[(1.14) \quad \frac{\partial f_k}{\partial t} + c \beta_k \left( \cos \alpha \frac{\partial f_k}{\partial r} + \frac{1 - \cos^2 \alpha}{r} \frac{\partial f_k}{\partial \cos \alpha} \right) = \frac{c^2}{\epsilon_k} \text{St} f_k,\]

where \( \beta_k = \sqrt{\epsilon_k^2 - (m_k/m_e)^2}/\epsilon_k. \)

However, there is a better way to write down the left hand side (LHS) in Eqs. (1.14) when one has to solve it numerically. In particular the equiv-
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alent so called conservative form is given by

\[
\frac{\partial f_k}{\partial t} + \frac{\cos \alpha}{r^2} \frac{\partial}{\partial r} \left( r^2 c \beta_k f_k \right) + \\
\frac{1}{r} \frac{\partial}{\partial \cos \alpha} \left( (1 - \cos^2 \alpha) c \beta_k f_k \right) = \frac{c^2}{\epsilon_k} \text{St}_k f_k.
\]

Eq. (1.15) is also said to be “conservative” because the “transport” terms can be expressed as the derivative of the flux divided by the volume [21].

1.3.2 Structure of collision integrals

In the radiative transfer theory instead of usual collision integrals of the KT the so called emission and absorption coefficients \( \eta \) and \( \chi \) are introduced. According with [18] the time derivative of the distribution function due to collisions only can be generally written as

\[
\left. \frac{\partial f_k}{\partial t} \right|_{\text{coll}} = \frac{c^2}{\epsilon_k} \text{St}_k f_k = \sum_q \left( \eta^q_k - \chi^q_k f_k \right),
\]

where the index \( k \) identifies the kind of particle and \( q \) labels a specific interaction. From Eq. (1.16) it is clear that \( \eta \), giving a positive contribution, corresponds to emission of certain number of particles \( k \) at a given point in space and with a given momentum. On the contrary, \( \chi f \) shows how many particles are absorbed at the point with same space and momentum coordinates.

To go furthermore, we represent schematically a general 2- particle interaction as

\[
\begin{array}{cccc}
1 & 2 & \rightarrow & 3 & 4 \\
p_1^\mu & p_2^\mu & & p_3^\mu & p_4^\mu
\end{array}
\]

which means that particles 1 and 2 having respectively 4-momenta \( p_1^\mu \) and \( p_2^\mu \) are absorbed; while in the same process particles 3 and 4 with 4-momenta \( p_3^\mu \) and \( p_4^\mu \) are emitted, see Figure [1.1]. The interactions considered are shown in Table [1.1]. Then, it turns out that the absorption and
Figure 1.1: Interaction scheme for a general 2-particle interaction. Incoming particles 1 and 2 are described by the absorption coefficient $\chi$ while outgoing particles 3 and 4 contribute to the emission coefficient $\eta$.

Table 1.1: Exemplification of the schematic interaction for each of the considered QED 2-particles interactions.

<table>
<thead>
<tr>
<th>Interaction (q)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pair Annihilation (pa)</td>
<td>$e^-$</td>
<td>$e^+$</td>
<td>$\gamma$</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>Pair Creation (pc)</td>
<td>$\gamma$</td>
<td>$\gamma$</td>
<td>$e^-$</td>
<td>$e^+$</td>
</tr>
<tr>
<td>Compton Scattering (cs)</td>
<td>$e^\pm$</td>
<td>$\gamma$</td>
<td>$e^\pm$</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>Bhabha Scattering (bs)</td>
<td>$e^\pm$</td>
<td>$e^+$</td>
<td>$e^\pm$</td>
<td>$e^+$</td>
</tr>
<tr>
<td>Möller Scattering (ms)</td>
<td>$e^\pm$</td>
<td>$e^\pm$</td>
<td>$e^\pm$</td>
<td>$e^\pm$</td>
</tr>
</tbody>
</table>
emission coefficients for the specified process \( q \) are given by

\[
\chi^q_1(p_1) f_1(p_1) = \int d^3 p_2 \int d^3 p_3 \int d^3 p_4 \ w^q_{1,2,3,4} \ f_1(p_1) \ f_2(p_2),
\]

(1.17)

\[
\chi^q_2(p_2) f_2(p_2) = \int d^3 p_1 \int d^3 p_3 \int d^3 p_4 \ w^q_{1,2,3,4} \ f_1(p_1) \ f_2(p_2),
\]

(1.18)

\[
\eta^q_3(p_3) = \int d^3 p_1 \int d^3 p_2 \int d^3 p_4 \ w^q_{1,2,3,4} \ f_1(p_1) \ f_2(p_2),
\]

(1.19)

\[
\eta^q_4(p_4) = \int d^3 p_1 \int d^3 p_2 \int d^3 p_3 \ w^q_{1,2,3,4} \ f_1(p_1) \ f_2(p_2),
\]

(1.20)

where the integrals must be calculated all over the phase space. In Eqs. (1.18) to (1.20) the distribution function dependence on \( x \) has been dropped because all the quantities have to be evaluated in the same point in space. The “transition rate” \( w^q_{1,2,3,4} \) is given by [1]

\[
w^q_{1,2,3,4} = \frac{1}{(2\pi)^2} \frac{|M^q_{fi}|^2}{16\epsilon_1\epsilon_2\epsilon_3\epsilon_4} \cdot \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \delta^{(3)}(p_1 + p_2 - p_3 - p_4)
\]

(1.21)

where \( M^q_{fi} \) is the matrix element which describe the transition between the initial \( i \) and final \( f \) states of the particular process \( q \) and Dirac Delta’s are needed to satisfy the energy and momentum conservation law in the given process.

\( |M^q_{fi}|^2 \) can be conveniently expressed in any reference frame in terms of dimensionless Mandelstam kinematic invariants [22]. They are defined by the following expressions

\[
s = \frac{(p_1^\mu + p_2^\mu)^2}{m^2 c^2} = \frac{(p_3^\mu + p_4^\mu)^2}{m^2 c^2},
\]

(1.22)

\[
t = \frac{(p_1^\mu - p_3^\mu)^2}{m^2 c^2} = \frac{(p_2^\mu - p_4^\mu)^2}{m^2 c^2},
\]

(1.23)

\[
u = \frac{(p_1^\mu - p_4^\mu)^2}{m^2 c^2} = \frac{(p_2^\mu - p_3^\mu)^2}{m^2 c^2}.
\]

(1.24)

Each of them corresponds to a particular interaction channel in the lan-
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Table 1.2: Values of emission and absorption coefficients for each of the considered QED 2-particles interactions. For each interaction $q$, $\eta (\chi)$ equals 0 ($\chi = 0$) means that no particle can be produced (absorbed). Otherwise the coefficient is non zero (n.z.).

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\eta^q$</th>
<th>$\chi_-$</th>
<th>$\eta^q$</th>
<th>$\chi_+$</th>
<th>$\eta^q$</th>
<th>$\chi^q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>pa</td>
<td>0</td>
<td>n.z.</td>
<td>0</td>
<td>n.z.</td>
<td>0</td>
<td>n.z.</td>
</tr>
<tr>
<td>pc</td>
<td>n.z.</td>
<td>0</td>
<td>n.z.</td>
<td>0</td>
<td>n.z.</td>
<td>n.z.</td>
</tr>
<tr>
<td>cs</td>
<td>n.z.</td>
<td>n.z.</td>
<td>n.z.</td>
<td>n.z.</td>
<td>n.z.</td>
<td>n.z.</td>
</tr>
<tr>
<td>bs</td>
<td>n.z.</td>
<td>n.z.</td>
<td>n.z.</td>
<td>0</td>
<td>0</td>
<td>n.z.</td>
</tr>
<tr>
<td>ms</td>
<td>n.z.</td>
<td>n.z.</td>
<td>n.z.</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

language of Feynman diagrams. $s$ and $t$ are the square of the center-of-mass (CM) energy (invariant mass) and the square of the momentum transfer respectively. Below we list matrix elements relative to the five processes considered in this Thesis using the Mandelstam invariants defined above with proper modifications, see e.g. [23]. We stress here that polarization of particles has been already averaged out giving the so called unpolarized matrix elements.

For the Compton scattering the matrix element squared is

$$\left| M_{f_i}^{cs} \right|^2 = 2^6 \pi^2 \alpha^2 \left[ \frac{1}{s-1} + \frac{1}{u-1} + \left( \frac{1}{s-1} + \frac{1}{u-1} \right)^2 - \frac{1}{4} \left( \frac{s-1}{u-1} + \frac{u-1}{s-1} \right) \right].$$

From Eq. (1.25) we can obtain the matrix elements squared $\left| M_{f_i}^{pc/pa} \right|^2$ relative to pair creation and annihilation processes by means of the following substitution

$$s \rightarrow s' = \frac{(p_1^\mu - p_3^\mu)^2}{m_e^2 c^2},$$

$$u \rightarrow u' = \frac{(p_1^\mu - p_4^\mu)^2}{m_e^2 c^2}.$$
For Möller scattering we have

\[
|M_{ms}^{mf_i}|^2 = 2^6 \pi^2 \alpha^2 \left\{ \frac{1}{t^2} \left[ \frac{s^2 + u^2}{2} + 4 (t - 1) \right] + \right.
\]
\[
+ \frac{1}{u^2} \left[ \frac{s^2 + t^2}{2} + 4 (u - 1) \right] + \frac{4}{tu} \left( \frac{s}{2} - 1 \right) \left( \frac{s}{2} - 3 \right) \right\}.
\]

The matrix elements squared \(|M_{fi}^{bs}|^2\) due to Bhabha scattering are recovered from Eq. (1.28) through the substitution

\[
s \rightarrow s' = \frac{(p_1^\mu - p_4^\mu)^2}{m_e^2 c^2},
\]
\[
t \rightarrow u' = \frac{(p_2^\mu - p_4^\mu)^2}{m_e^2 c^2},
\]
\[
u \rightarrow u' = \frac{(p_1^\mu + p_2^\mu)^2}{m_e^2 c^2}.
\]

### 1.3.3 Cylindrically symmetric momentum space

For both the systems under study, we make assumptions about the symmetries of the physical and momentum spaces. This is because calculations would be otherwise too complicated and it would be probably harder to focus our study on specific aspects we are interested in. In particular, assuming an homogeneous physical space in the case of electron-positron pair production, we do not consider any spatial dependence into our equations. That also means we do not need to solve Maxwell equations in their full complexity but the energy conservation law is enough to calculate the electric field. On the other hand, to study the approach to transparency of relativistic outflows we assume the physical space is spherically symmetric. Despite this difference, we assume that distribution functions have cylindrical symmetry in the momentum space for both systems. The axis of symmetry in the case of pair production is given by the direction of the electric field. Whereas in the case of spherically symmetric outflows this symmetry axis is given by the radial direction of motion.

Because of the assumed symmetry, we can describe the momentum of a particle using its parallel \((p_\parallel)\) and orthogonal \((p_\perp)\) components with
Figure 1.2: Geometrical relations between cartesian and cylindrical coordinates adopted in this Thesis. In Chapter 2 in which we study the problem of pairs creation in strong electric fields, the parallel momentum is considered to be aligned with the external electric field. Then in Chapter 3 where we consider the problem of transparency of relativistic outflows, \( p || \) will be parallel to the radial direction of motion.

respect to the direction of the symmetry axis plus the azimuthal angle \((\phi)\) which describes the rotation of \( p \perp \) around \( p || \). These cylindrical coordinates in the momentum space are defined in the following intervals \( p || \in (-\infty, +\infty), p \perp \in [0, +\infty), \phi \in [0, 2\pi] \). The correspondence between the new coordinates and the cartesian ones \((p_x, p_y, p_z)\) is depicted in Figure 1.2. DF cylindrical symmetry means that \( f_k \) does not depend on the azimuthal angle \( \phi \), namely

\[
 f_k = f_k(p ||, p \perp, t).
\]

This symmetry reduces the number of integrals in \( St f_k \) we should calculate every time step decreasing the computation time with respect to the case with no symmetries at all. Within the chosen phase space configuration, the prescription for the integral over the entire momentum space is

\[
 \int d^3 p \rightarrow \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dp || \int_{0}^{+\infty} dp \perp p \perp
\]
and the relativistic energy is given by the following equation

\begin{equation}
\epsilon_k = \sqrt{p_\parallel^2 + p_\perp^2 + m_k^2},
\end{equation}

where \( m_k \) is the mass of the considered particle \( k \). In the previous equation we set \( c = 1 \). The number density is given by

\begin{equation}
n_k = \int d^3 p \, f_k = 2\pi \int_{-\infty}^{+\infty} dp_\parallel \int_0^{+\infty} dp_\perp f_k,
\end{equation}

where the integral over \( \phi \) has been performed giving the \( 2\pi \) pre-factor is the result.

### 1.3.4 Phase space discretization and averaging

The discretization of the phase space is done defining a finite number of elementary volumes which are uniquely identified by triplets of integer numbers \( (i,k,l) \). Their values run over the ranges \( \{1, \ldots, I\} \), \( \{1, \ldots, K\} \) and \( \{1, \ldots, L\} \) respectively. Since we are dealing with an axially symmetric phase space, the parallel momentum is aligned with our preferential direction while the orthogonal component lays on the plane orthogonal to this preferential axis. Each elementary volume encloses only one momentum vector which can be written explicitly in cylindrical coordinates as \( (p_\parallel i, p_\perp k, \phi_l) \). The corresponding boundaries are marked by semi-integer indices \( [p_\parallel i-1/2, p_\parallel i+1/2], [p_\perp k-1/2, p_\perp k+1/2], [\phi_l-1/2, \phi_l+1/2] \). Due to axial symmetry, the DFs do not depend on the azimuthal angle \( \phi \) and the index \( l \) will be used only to identify angles explicitly. We use the greek letter \( \nu \) which identifies the kind of particle under consideration, \( \{\gamma, -, +\} \) for photons, electrons and positrons respectively. From these definitions, the energy \( 2.72 \) of a particle with mass \( m_\nu \) corresponding to the grid point \( (i,k) \) is

\begin{equation}
\epsilon_{\nu ik} = \sqrt{m_\nu^2 + p_\parallel^2 + p_\perp^2}, \quad (m_\gamma = 0, m_{\pm} = m_e).
\end{equation}

In this finite difference representation the distribution function has a Klimontovich form \[24\] and can be seen as a sum of Dirac deltas centered on the grid points \( (i,k) \) and multiplied by the energy density of particles on the same grid point \( n_{\nu ik} \)

\begin{equation}
f_\nu(p_\parallel, p_\perp) = \frac{1}{2\pi p_\perp} \sum_{ik} \delta(p_\parallel - p_\parallel i) \delta(p_\perp - p_\perp k) f_{\nu ik},
\end{equation}

---

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where \( f_{\nu ik} = (m_\nu c)^3 f_\nu (p_\parallel_{ik}, p_\perp_{ik}) \), \( \Sigma_{ik} = \sum_{i=1}^{I} \sum_{k=1}^{K} \) and time dependence has been dropped for simplicity. As a consequence of Eq. (1.37) energy and number densities of particle \( \nu \) are given by

\[
\rho_\nu = \sum_{ik} f_{\nu ik} \epsilon_{\nu ik}, \tag{1.38}
\]

\[
n_\nu = \sum_{ik} f_{\nu ik}, \tag{1.39}
\]

from which the total energy and number of particles in the system are

\[
\rho = \sum_\nu \rho_\nu, \tag{1.40}
\]

\[
n = \sum_\nu n_\nu. \tag{1.41}
\]

Then the mean parallel momentum, its mean squared value and the mean squared orthogonal momentum are

\[
\langle p_\parallel \rangle_\nu = \frac{1}{n_\nu} \sum_{ik} f_{\nu ik} p_{\parallel i}, \tag{1.42}
\]

\[
\langle p_\parallel^2 \rangle_\nu = \frac{1}{n_\nu} \sum_{ik} f_{\nu ik} (p_{\parallel i} - \langle p_\parallel \rangle_\nu)^2, \tag{1.43}
\]

\[
\langle p_\perp^2 \rangle_\nu = \frac{1}{n_\nu} \sum_{ik} f_{\nu ik} p_{\perp k}^2. \tag{1.44}
\]

Due to axial symmetry the mean orthogonal momentum must be null identically \( \langle p_\perp \rangle_\nu = 0 \).

### 1.3.5 Transport terms and radial grid discretization

Now we take Eq. (1.14) replacing the spherical coordinates \( \epsilon, \alpha \) with cylindrical ones \( p_\parallel, p_\perp \). At this point we also have to change variables with respect to which derivatives are calculated. Changing variables by means of the Jacobian of the coordinate transformation we can rewrite \( \partial / \partial \cos \alpha \) obtaining the following system of coupled integro-differential equations

\[
\frac{\partial f_\nu}{\partial t} + \frac{p_\parallel}{p} \frac{\partial}{\partial r} \left( r^2 c \beta_\nu f_\nu \right) + \frac{\partial}{\partial p_\parallel} \left( \frac{p_\parallel^2}{p} c \beta_\nu f_\nu \right) - \frac{\partial}{\partial p_\perp} \left( \frac{p_\parallel p_\perp}{p} c \beta_\nu f_\nu \right) = \frac{c^2}{\epsilon_\nu} \text{St} f_\nu, \tag{1.45}
\]
where $\beta_\nu = p c / \epsilon_\nu$. These are the basic equations to be solved numerically in this Thesis.

The discretization procedure for Eqs. (1.45) can be done following the method of finite difference presented in [15]. In the numerical scheme we define a radial grid which is divided into $j_{\text{max}}$ spherical shells whose boundaries, as the momentum ones, are designated with half-integer indices. Therefore the $j$-th shell ($1 < j < j_{\text{max}}$) is between $r_{j-1/2}$ and $r_{j+1/2}$ and its width is $\Delta r_j = r_{j+1/2} - r_{j-1/2}$. The inner and outer radii are $r_{\text{in}} = r_{1/2}$ and $r_{\text{ext}} = r_{j_{\text{max}}+1/2}$.

The quantity we compute are the average of the distribution function over phase-space cells

$$\langle f_{vikj}(t) \rangle = \frac{2\pi}{\Delta(r_j^3)} \int_{\Delta r_j} dr^2 \int_{\Delta ||} dp \int_{\Delta \perp} dp \, p_\perp f_\nu(r, p_\parallel, p_\perp, t)$$

where $\Delta(r_j^3) = (r_{j+1/2}^3 - r_{j-1/2}^3)/3$. Replacing the derivatives in Eqs. (1.45) with finite differences, we have the following set of ordinary differential equations for $f_{vikj}$ specified on the computational grid:

$$\frac{df_{vikj}}{dt} + c \beta_{vik} \Delta(r^2 p_\parallel f_{vik} / p_{ik})_j +$$

$$+ c \left\langle \frac{1}{r} \right\rangle_j \left[ \frac{\Delta(\beta_{vik} p_\perp^2 f_{vik} / p_{ik})_i}{\Delta p_\perp} - \frac{\Delta(\beta_{vik} p_\parallel f_{vik} / p_i)_k}{\Delta p_\parallel} \right] =$$

$$= \sum_q \left( \eta_{vikj}^q - \chi_{vikj}^q \right),$$

where $p_i, p_k, \beta_{vik}, \ldots$ are obtained substituting in their definition $p_\parallel, p_\perp$ if the index is specified and $p_\parallel, p_\perp$ otherwise. Besides, the following notation have been used

$$\left\langle \frac{1}{r} \right\rangle_j = \frac{(r_{j+1/2}^2 - r_{j-1/2}^2)}{2\Delta(r_j^3)},$$

$$\Delta(r^2 p_\parallel f_{vik} / p_{ik})_j =$$

$$= r_{j+1/2}^2 \left( p_\parallel f_{vik} / p_{ik} \right)_{r_{j+1/2}} - r_{j-1/2}^2 \left( p_\parallel f_{vik} / p_{ik} \right)_{r_{j-1/2}}.$$
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\begin{equation}
(1.50) \quad \left( p_{||} f_{vik} / p_{ik} \right)_{r_{j+1/2}} = (1 - \tilde{\chi}_{vikj+1/2}) \cdot \left( \frac{p_{||} + |p||}{2 p_{ik}} f_{vikj} + \frac{p_{||} - |p||}{2 p_{ik}} f_{vikj+1} \right) + \tilde{\chi}_{vikj+1/2} f_{vikj} + f_{vikj+1} / 2
\end{equation}

\begin{equation}
(1.51) \quad \tilde{\chi}_{vikj+1/2} = 1 + \frac{1}{\chi_{vikj}\Delta r_{j}} + \frac{1}{\chi_{vikj}\Delta r_{j+1}}
\end{equation}

\begin{equation}
(1.52) \quad \Delta(\beta_{vk} p_{\perp k}^2 f_{vk} / p_{k})_i = \Delta(\beta_{vi+1/2 k} f_{vi+1/2 k})_i - \Delta(\beta_{vi-1/2 k} f_{vi-1/2 k})_i = p_{\perp k}^2 \left( \beta_{vi+1/2 k} p_{i+1/2} / f_{vi+1/2 k} - \beta_{vi-1/2 k} p_{i-1/2} / f_{vi-1/2 k} \right)
\end{equation}

\begin{equation}
(1.53) \quad f_{vi+1/2 k} = \frac{f_{vi+1/2 k} \Delta p_{||} + f_{vi+1/2 k} \Delta p_{||+1}}{\Delta p_{||} + \Delta p_{||+1}}
\end{equation}

\begin{equation}
(1.54) \quad \Delta(\beta_{vi} p_{||} p_{\perp f_{vi}} / p_{i})_k = \Delta(\beta_{vi+1/2} p_{\perp k+1/2} / f_{vi+1/2} - \beta_{vi-1/2} p_{\perp k-1/2} / f_{vi-1/2})_k = p_{||} \left( \beta_{vi+1/2} p_{\perp k+1/2} / f_{vi+1/2} - \beta_{vi-1/2} p_{\perp k-1/2} / f_{vi-1/2} \right)
\end{equation}

\begin{equation}
(1.55) \quad f_{vikj+1/2} = \frac{f_{vikj+1/2} \Delta p_{\perp k} + f_{vikj+1/2} \Delta p_{\perp k+1}}{\Delta p_{\perp k} + \Delta p_{\perp k+1}}
\end{equation}

\begin{equation}
(1.56) \quad \eta_{vikf} = 2\pi \int_{\Delta_{||}} dp_{||} \int_{\Delta_{\perp k}} dp_{\perp} \eta_{q}(r_{j}, p_{||}, p_{\perp}, t)
\end{equation}

\begin{equation}
(1.57) \quad \chi_{vikj} f_{vikj} = 2\pi \int_{\Delta_{||}} dp_{||} \int_{\Delta_{\perp k}} dp_{\perp} \chi_{q}(r_{j}, p_{||}, p_{\perp}, t) f_{q}(r_{j}, p_{||}, p_{\perp}, t)
\end{equation}

In Eq. (1.50) the dimensionless coefficient \( \tilde{\chi} \) defined by Eq. (1.51) is introduced to describe correctly both the optically thick and optically thin computational cells by means of compromise between the high-order method.
and the monotonic transport scheme, see [25, 20, 26].

The particle number conservation can be written as

\[
\sum_{\nu i k} \frac{df_{\nu ik}}{dt} \Delta(r_j^3) + F_{j+1/2} - F_{j-1/2} = 0 , 
\]

where

\[
F_{j+1/2} = c r^2_{j+1/2} \sum_{\nu i k} \beta_{\nu ik} \left( p_{i f_{\nu ik}} / p_{i k} \right) \]

is the energy flux through the sphere of radius \( r_{j+1/2} \). In such scheme this is satisfied exactly and energy conservation is automatically fulfilled. The set of Eqs. (1.47) is solved by the implicit Gear method [27].

### 1.3.6 Emission and absorption coefficients

We now briefly delineate the technique we have used to calculate \( \eta \) and \( \chi \) according to the adopted discretization procedure for the momentum space. The dependence on the physical space coordinates does not need to be specified, therefore Eqs. (1.18)-(1.20) can be discretized using the integral prescriptions given below

\[
\chi_{1i, k_1}^q f_{1, i, k_1} = \int_{V_{i k_1}} d^3 p_1 \chi_1(p_1) f_1(p_1) ,
\]

\[
\chi_{2i, k_2}^q f_{2, i, k_2} = \int_{V_{i k_2}} d^3 p_2 \chi_2(p_2) f_2(p_2) ,
\]

\[
\eta_{3i, k_3}^q = \int_{V_{i k_3}} d^3 p_3 \eta_3(p_3) ,
\]

\[
\eta_{4i, k_4}^q = \int_{V_{i k_4}} d^3 p_4 \eta_4(p_4) ,
\]

where \( V_{i k_0} \) is the volume in the phase space which contains only the grid point with \( p_{i_{||}} \) and \( p_{k_{\perp}} \). It is actually a ring with inner radius \( p_{k_{-1/2}} \), outer radius \( p_{k_{+1/2}} \) and thickness \( p_{i_{||+1/2}} - p_{i_{||-1/2}} \). An explicit expression for the collision integrals can be obtained inserting Eqs. (1.18)-(1.20) into the corresponding Eqs. (1.60)-(1.63). Then we replace all the integrals over the entire momentum space with a sum of integrals over the elemen-
tary volumes $V_{ik}$

\[
\int d^3 p \to \sum_{ik} \int_{V_{ik}} d^3 p .
\]

Therefore, using Eq. (1.64) and the arguments above we get

\[
\chi_{1i_1k_1}^q f_{1i_1k_1} = \sum_{i_2k_2} \sum_{i_3k_3} \sum_{i_4k_4} R_{i_1i_2i_3i_4k_4}^q ,
\]

\[
\chi_{2i_2k_2}^q f_{2i_2k_2} = \sum_{i_1k_1} \sum_{i_3k_3} \sum_{i_4k_4} R_{i_1i_2i_3i_4k_4}^q ,
\]

\[
\eta_{3i_3k_3}^q = \sum_{i_1k_1} \sum_{i_2k_2} \sum_{i_4k_4} R_{i_1i_2i_3i_4k_4}^q ,
\]

\[
\eta_{4i_4k_4}^q = \sum_{i_1k_1} \sum_{i_2k_2} \sum_{i_3k_3} \sum_{i_4k_4} R_{i_1i_2i_3i_4k_4}^q .
\]

Even though the sums are different in this set of equations, there is a common factor which is given by the following series integrals on specific momentum space volumes

\[
R_{i_1i_2i_3i_4k_4}^q = \int_{V_{i_1k_1}} d^3 p_1 \int_{V_{i_2k_2}} d^3 p_2 \int_{V_{i_3k_3}} d^3 p_3 \int_{V_{i_4k_4}} d^3 p_4 = w_{1,2,3,4}^q f_1(p_1) f_2(p_2) .
\]

Now we have to calculate $R$ explicitly and eventually simplify its expression. Using Eq. (1.37) for $f_1(p_1)$ and $f_2(p_2)$, we see that their integrals over $V_{i_1k_1}$ and $V_{i_2k_2}$ simply give the values $f_{1i_1k_1}$ and $f_{2i_2k_2}$ respectively, see also [14]. As a result we have that

\[
R_{i_1i_2i_3i_4k_4}^q = \frac{f_{1i_1k_1} f_{2i_2k_2}}{(2\pi)^2} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int_{V_{i_3k_3}} d^3 p_3 \int_{V_{i_4k_4}} d^3 p_4 w_{1,2,3,4}^q ,
\]

which implies that the momentum component of initial particles 1 and 2 are given by the discrete grid values $(p_{||i_1}, p_{\perp i_1})$ and $(p_{||i_2}, p_{\perp i_2})$ respectively. In other words the kinematic condition for the incoming particles has been fixed for this particular interaction. As we can see, the time dependence contained in $f_{1i_1k_1}$ and $f_{2i_2k_2}$ is shifted out of the integrals. Therefore, the integrations we are left with in Eq. (1.70) can be performed only once at the beginning of the computation because they only depend on the kinematics through the presence of $w_{1,2,3,4}^q$, see also [15, 14]. To go further, we
know that momentum conservation forces two-particle scatterings to occur in a plane which means that the azimuthal angle of one particle can be fixed arbitrarily. In this respect we can set \( \phi_1 = 0 \), but doing that we must add a factor \( 2\pi \) because this is a variable on which we have to perform an integral. Therefore we can now rewrite Eq. (1.70) as follows

\[
R_{1i1j2j3j4k4}^q = \frac{f_{1i1j} f_{2j2k}}{2\pi} \int_0^{2\pi} d\phi_2 \int_{V_{i3j3}} d^3 p_3 \int_{V_{i4k4}} d^3 p_4 \omega_{1234}^q .
\]

Dirac Deltas in the transition rate defined by Eq. (1.21) can now be used to eliminate four one-dimensional integrals. This means that we have chosen four variables (angles or momentum components) which can be given as functions of the remaining eight. These variables, being dependent on others variable quantities, can fall out of the discrete points chosen at the beginning. Nevertheless, we want to calculate the emission coefficients of particles 3 and 4 on the grid points with corresponding coordinates given by \((p_{i3j3}, p_{i4k4})\) respectively. However, they do not satisfy conservation laws. The solution to this problem turns out to be a distribution of the coefficient \( R \) over different discrete grid points in such a way that energy and momentum are conserved. In [14] the same procedure was adopted for the energy grid. Finally, emission and absorption coefficients can be rewritten in the following form

\[
\chi_{1i1j1}^q f_{1i1j} = \sum_{i2k2} f_{1i1j} f_{2i2k} \sum_{i3j3i4k4} \hat{R}_{1i1j2j3j4k4}^q ,
\]

(1.72)

\[
\chi_{2i2k2}^q f_{2i2k2} = \sum_{i1k1} f_{1i1k} f_{2i2k} \sum_{i3j3i4k4} \hat{R}_{1i1j2j3j4k4}^q ,
\]

(1.73)

\[
\eta_{3i3j3r}^q = \sum_{i1k1} f_{1i1k} f_{2i2k} \sum_{i3j3i4k4} \hat{R}_{1i1j2j3j4k4}^q x_r , \quad r = 1, \ldots, n_3 ,
\]

(1.74)

\[
\eta_{4i4k1v}^q = \sum_{i1k1} f_{1i1k} f_{2i2k} \sum_{i3j3} \hat{R}_{1i1j2j3j4k4}^q x_v , \quad v = 1, \ldots, n_4 ,
\]

(1.75)
where the indices $r$ and $v$ indicate the grid points on which redistribution is performed, $x_r$ and $x_v$ are the redistribution coefficients and $\hat{R}$ is given by

$$
\hat{R}^q_{i_1 k_1 i_2 k_2 i_3 k_3 i_4 k_4} = \frac{1}{2\pi} \int_0^{2\pi} d\phi_2 \int_{V_{i_3 k_3}} d^3 p_3 \int_{V_{i_4 k_4}} d^3 p_4 \hat{w}^q_{i_1 i_2 i_3 i_4}.
$$

For sake of generality we left Eq. (1.76) without giving any explicit choice of the variables we could get rid of, but $x_r$ and $x_v$, as well as $n_3$ and $n_4$, depend on this choice and have to satisfy constraints about particle, energy and momentum conservation laws.

At this point, since the coefficient matrix in Eq. (1.76) can be calculated only once at the beginning, the time evolution of each particle distribution function is reduced to summations of $\hat{R}^q_{i_1 k_1 i_2 k_2 i_3 k_3 i_4 k_4}$ multiplied by initial distribution functions.
Chapter 2

Electron-positron pairs production in strong electric fields

2.1 Introduction

The concept of critical electric field was introduced by Sauter [28], Heisenberg and Euler [29], and Schwinger [30] by means of their pioneer works concerning the possibility of producing electron-positron pairs from vacuum. Nowadays this is one of the most popular topics in relativistic field theory [31]. The value of the critical electric field at which copious pair creation occurs can be estimated from a semiclassical viewpoint equating the work done by the electric field in a Compton wavelength $\lambda_C$ to the rest mass energy of an electron-positron pair

$$eE_c\lambda_C = m_ec^2$$

from which we get the critical value to be

$$E_c \equiv \frac{m_e^2c^3}{e\hbar} \simeq 1.3 \cdot 10^{16} \text{ V/cm}$$

where $m_e$, $e$, $c$ and $\hbar$ are respectively the electron mass and charge, the speed of light and the reduced Planck’s constant.

At present such electron-positron pair production is understood within QED as non-perturbative effect. Nonlinear effects in high intensity fields can be observed already in undercritical electric field, see e.g. [32, 33]. Strong electric fields possibly up to several percents of the critical value will be reached by advanced laser technologies in laboratory experiments [34, 35], X-ray free electron laser facilities [36], optical high-intensity laser
facilities such as Vulcan or the Extreme Light Infrastructure [37], for a recent review see [38]. Electron beam-laser interactions seem also promising in reaching high Lorentz transformed electromagnetic fields capable for multiple pair production [39, 40].

Considerable effort has been made over last two decades in increasing the intensity of high power lasers in order to explore these high field regimes. Yet, the Schwinger field $E_c$ is far from being reached, see e.g. for recent review [40]. It has been claimed recently [41] that critical Schwinger field could never be reached in high power lasers due to avalanche-like QED cascade operating mainly through nonlinear Compton scattering combined with nonlinear Breit-Wheeler process [42, 43], see also [31], and via the trident process [44, 45]. In particular, as soon as one single pair is generated by the Schwinger process such electromagnetic cascade of secondary electron-positron pairs is expected to deplete the electromagnetic energy thus preventing further pair production from vacuum. The requirements for the avalanche to occur are twofold: a) the probability to emit photon should not be suppressed and b) the photon must be energetic enough to produce pair by interaction with another photon. It is shown that for a specific electromagnetic field configuration considered in [42, 43] as well as in [41], namely circularly polarized standing electromagnetic wave both these conditions may fulfill for undercritical electric field $E < E_c$. However, as it was shown in [46] for linearly polarized standing wave such electromagnetic cascade is not expected to dominate over the Schwinger process. It is easy to understand these results looking at the energy loss rate of charged particle in classical electrodynamics

\[
\frac{dW}{dt} = \frac{2}{3} \frac{a^2}{m_e^2} \gamma^2 \left[ (E + v \times H)^2 - (E \cdot v)^2 \right].
\]

When magnetic field is absent (the case considered in [42, 43] and [41]) if directions of particle velocities and electric field are collinear the radiation loss turns out independent on particle energy. In such case, as we have shown previously [47] for overcritical electric field the radiation loss is smaller than the rate of energy conversion from electromagnetic field to electron-positron pairs via Schwinger process. Notice that in the case of plasma oscillations considered in this Thesis the velocity and acceleration vectors are indeed collinear, so curvature radiation considered in [42, 43] does not occur. When electric field changes with time not only its amplitude but also direction, as for instance in circularly polarized electromagnetic wave, the acceleration and velocity vectors become misaligned and curvature radiation becomes much more efficient due to quadratic depen-
dence on particle energy in (2.3). We also notice that the backreaction of electron-positron pairs on the initial electric field, which is the topic of the present Section, is not taken into account in [42, 43] and [41], see however [48].

While dynamical mechanisms involving increase of small initial electric field toward its critical value appear problematic because of avalanches, existence of overcritical electric field in which pair production is blocked [49] is widely discussed in astrophysical context in compact stars, e.g. hypothetical quark stars [50, 49], neutron stars [51], see also [52] for review. Pair production in such overcritical field may occur due to several reasons, e.g. heating [53] or gravitational collapse of the compact object.

### 2.1.1 Pair production in QED

Now following [52] we briefly discuss some aspects of QED which are relevant for this Thesis. QED, the relativistic quantum theory of electrons, positrons, and photons, was established by Tomonaga [54], Feynman [55, 56, 57], Schwinger [58, 59, 60], Dyson [61, 62] and others in the 1940’s and 1950’s [63]. QED is the best theory we can use to describe interactions between electrically charged particles such that an astonishing agreement exists between theory and experiment.

Even though there are no particles in the vacuum state, it does harbor zero-point oscillations of the electron, positron and photon fields. In the worldline description, the vacuum is represented by a grand-canonical ensemble of interacting closed world lines. These are called virtual particles. In the Fourier decomposition of the fluctuating fields, virtual particles correspond to Fourier components, or modes, in which the 4-vectors of energy and momentum \( k^\mu \equiv (k^0, k) \equiv (\mathcal{E}, k) \) do not satisfy the mass-shell relation

\[
k^2 \equiv (k^0)^2 - c^2 |k|^2 = \mathcal{E}^2 - c^2 |k|^2 = m^2 c^4,
\]

valid for real particles.

The only way to evaluate physical consequences from QED is based on the smallness of the electromagnetic interaction which is characterized by the dimensionless fine structure constant \( \alpha \). All theoretical results derived from QED are found in the form of series expansions in powers of \( \alpha \), which are expansions around the non-interacting system. The number of terms contributing to the same order of \( \alpha \) grows factorially fast, i.e., faster than any exponential, leading to a zero radius of convergence. Fortunately, however, the coupling \( \alpha \) is so small that the series possess an apparent convergence up to order \( 1/\alpha \approx 137 \).
In perturbation expansions, all physical processes are expressible in terms of *Feynman diagrams*. These are graphic representations of the interacting world lines of all particles. Among these lines, there are some which run to infinity. They satisfy the mass shell relation \( (2.4) \) and describe real particles observable in the laboratory. Those which remain inside a finite space-time region are virtual.

An interesting aspect of virtual particles both theoretically and experimentally is the possibility that they can become real by the effect of external fields. In this case, real particles are excited out of the vacuum. This possibility was first pointed out in the framework of quantum mechanics by Klein, Sauter, Euler and Heisenberg \([64, 65, 28, 66]\) who studied the behavior of the Dirac vacuum in a strong external electric field. Afterward, Schwinger studied this process and derived the probability (*Schwinger formula*) in the field theory of Quantum Electro-Dynamics, which will be described in this Chapter.

The total Lagrangian describing the interacting system of photons, electrons, and positrons reads, see e.g. \([23]\)

\[
\mathcal{L} = \mathcal{L}_0^\gamma + \mathcal{L}_0^{e^+e^-} + \mathcal{L}_{\text{int}},
\]

where the free Lagrangians \( \mathcal{L}_0^{e^+e^-} \) and \( \mathcal{L}_0^\gamma \) for electrons and photons are expressed in terms of quantized Dirac field \( \psi(x) \) and quantized electromagnetic field \( A_\mu(x) \) as follows:

\[
\mathcal{L}_0^{e^+e^-} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m_e)\psi(x),
\]

\[
\mathcal{L}_0^\gamma = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \text{gauge-fixing term}.
\]

Here \( \gamma^\mu \) are the \( 4 \times 4 \) Dirac matrices, \( \bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0 \), and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) denotes the electromagnetic field tensor. Minimal coupling gives rise to the interaction Lagrangian

\[
\mathcal{L}_{\text{int}} = -e j^\mu(x) A_\mu(x), \quad j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x).
\]

After quantization, the photon field is expanded into plane waves as

\[
A_\mu(x) = \int \frac{d^3k}{2k_0(2\pi)^3} \cdot \sum_{\lambda=1}^3 \left[ a^{(\lambda)}(k) e^{(\lambda)}_\mu(k) e^{-ikx} + a^{(\lambda)}_\dagger(k) e^{(\lambda)*}_\mu(k) e^{ikx} \right],
\]
where $\epsilon^{(\lambda)}_\mu$ are polarization vectors, and $a^{(\lambda)}$, $a^{(\lambda)\dagger}$ are annihilation and creation operators of photons. The quantized fermion field $\psi(x)$ has the expansion

$$
\psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{k^0} \cdot \sum_{\alpha=1,2} \left[ b_{\alpha}(k,s_3) u^{(\alpha)}(k,s_3) e^{-ikx} + d_{\alpha}^\dagger(k,s_3) v^{(\alpha)}(k,s_3) e^{ikx} \right],
$$

where the four-component spinors $u^{(\alpha)}(k,s_3)$, $v^{(\alpha)}(k,s_3)$ are positive and negative energy solutions of the Dirac equation with momentum $k$ and spin component $s_3$. The operators $b(k,s_3)$, $b^\dagger(k,s_3)$ annihilate and create electrons, the operators $d(k,s_3)$ and $d^\dagger(k,s_3)$ do the same for positrons [23].

In the framework of QED the transition probability, also called scattering amplitude, from an initial to a final state for a given process is represented by the imaginary part of the unitary S-matrix squared

$$
P_{f\leftarrow i} = |\langle f, \text{out} | \text{Im} \mathcal{S} | i, \text{in} \rangle|^2,
$$

where

$$\text{Im} \mathcal{S} = (2\pi)^4 \delta^4(P_f - P_i) \left| M_{fi} \right|,
$$

$M_{fi}$ is called matrix element and $\delta$-function stays for energy-momentum conservation in the process.

The S-matrix is computed through the interaction operator as

$$\mathcal{S} = \mathcal{T} \exp \left( i \int \mathcal{L}_{\text{int}} d^4x \right),
$$

where $\mathcal{T}$ is the chronological operator.

We turn now to a Sauter-Heisenberg-Euler process in QED. An external electromagnetic field is incorporated by adding to the quantum field $A_\mu$ in (2.8) an unquantized external vector potential $A^e_\mu$, so that the total interaction becomes

$$\mathcal{L}_{\text{int}} + \mathcal{L}^e_{\text{int}} = -e \bar{\psi}(x) \gamma^\mu \psi(x) \left[ A_\mu(x) + A^e_\mu(x) \right].
$$

Instead of an operator formalism, one can derive the quantum field theory from a functional integral formulation, see e.g. [67], in which the quantum
mechanical partition function is described by

\[ Z[A^e] = \int [D\psi D\bar{\psi} DA_{\mu}] \exp \left[ i \int d^4x (\mathcal{L} + \mathcal{L}_{\text{int}}) \right], \]

to be integrated over all fluctuating electromagnetic and Grassmannian electron and positron fields. The normalized quantity \( Z[A^e] \) gives the amplitude for the vacuum to vacuum transition in the presence of the external classical electromagnetic field:

\[ \langle \text{out}, 0|0, \text{in} \rangle = \frac{Z[A^e]}{Z[0]}, \]

where \(|0, \text{in}\rangle\) is the initial vacuum state at the time \( t = t_- \rightarrow -\infty \), and \(|\text{out}, 0\rangle\) is the final vacuum state at the time \( t = t_+ \rightarrow +\infty \). By selecting only the one-particle irreducible Feynman diagrams in the perturbation expansion of \( Z[A^e] \) one obtains the effective action as a functional of \( A^e \):

\[ \Delta A_{\text{eff}}[A^e] \equiv -i \ln \langle \text{out}, 0|0, \text{in} \rangle. \]

In general, there exists no local effective Lagrangian density \( \Delta \mathcal{L}_{\text{eff}} \) whose space-time integral is \( \Delta A_{\text{eff}}[A^e] \). Under the assumption that the external field \( A^e(x) \) varies smoothly over a finite space-time region, one may define an approximately local effective Lagrangian \( \Delta \mathcal{L}_{\text{eff}}[A^e(x)] \),

\[ \Delta A_{\text{eff}}[A^e] \approx \int d^4x \Delta \mathcal{L}_{\text{eff}}[A^e(x)] \approx V \Delta t \Delta \mathcal{L}_{\text{eff}}[A^e], \]

where \( V \) is the spatial volume and time interval \( \Delta t = t_+ - t_- \).

For a large time interval \( \Delta t = t_+ - t_- \rightarrow \infty \), the amplitude of the vacuum to vacuum transition (2.16) has the form,

\[ \langle \text{out}, 0|0, \text{in} \rangle = e^{-i(\Delta E_0 - i\Gamma/2)\Delta t}, \]

where \( \Delta E_0 = E_0(A^e) - E_0(0) \) is the difference between the vacuum energies in the presence and the absence of the external field, \( \Gamma \) is the vacuum decay rate, and \( \Delta t \) the time over which the field is nonzero. The probability that the vacuum remains as it is in the presence of the external classical electromagnetic field is

\[ |\langle \text{out}, 0|0, \text{in} \rangle|^2 = e^{-2\text{Im}\Delta A_{\text{eff}}[A^e]}. \]

This determines the decay rate of the vacuum in an external electromag-
netic field:

\[
\frac{\Gamma}{V} = \frac{2 \text{Im}\Delta A_{\text{eff}}[A^e]}{V\Delta t} \approx 2 \text{Im}\Delta L_{\text{eff}}[A^e].
\]

The vacuum decay is caused by the production of electron and positron pairs.

The Dirac fields appears quadratically in the partition functional (2.15) and can be integrated out, leading to

\[
Z[A^e] = \int DA_\mu \text{Det} \{i\not\partial - e [\not\partial A(x) + A^e(x)] - m_e + i\eta\},
\]

where \(\text{Det}\) denotes the functional determinant of the Dirac operator. Ignoring the fluctuations of the electromagnetic field, the result is a functional of the external vector potential \(A^e(x)\):

\[
Z[A^e] \approx \text{const} \times \text{Det} \{i\not\partial - e A^e(x) - m_e + i\eta\}.
\]

The infinitesimal constant \(i\eta\) with \(\eta > 0\) specifies the treatment of singularities in energy integrals. From Eqs. (2.16)–(2.23), the effective action (2.20) is given by

\[
\Delta A_{\text{eff}}[A^e] = -i \text{Tr} \ln \left\{ \left[i\not\partial - e A^e(x) - m_e + i\eta\right] - \frac{1}{i\not\partial - m_e + i\eta} \right\},
\]

where \(\text{Tr}\) denotes the functional and Dirac trace. In physical units, this is of order \(\hbar\). The result may be expressed as a one-loop Feynman diagram, so that one speaks of one-loop approximation.

If only a constant electric field \(E\) is present, it may be assumed to point along the \(\hat{z}\)-axis, and one can choose a gauge such that \(A^e_z = -Et\) is the only nonzero component of \(A^e_\mu\). The result is (see [52] for details)

\[
\frac{\Gamma}{V} = \frac{\alpha E^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left(-\frac{n\pi E_c}{E}\right).
\]

This result, i.e. the Schwinger formula [30, 68, 69] is valid to lowest order in \(\hbar\) for arbitrary constant electric field strength.

The Schwinger formula above is derived for uniform constant in time electric field. However, it still can be used for slowly time-varying electric field. In particular for oscillating field in e.g. laser wave the constant field approximation is justified provided the inverse adiabaticity parameter [70].
is much larger than one,

\[ \eta = \frac{m_e}{\omega} \frac{E_{\text{peak}}}{E_c} = \tilde{T} \tilde{E}_{\text{peak}} \gg 1 , \]

where \( \omega \) is the frequency of oscillations, \( \tilde{T} = m_e/\omega \) is dimensionless period of oscillations. Eq. \( (2.26) \) implies that time variation of the electric field is much slower than the rate of pair production. As it will be shown in Section \( 2.3 \) the range of electric fields considered in this Thesis is such that the condition \( (2.26) \) Schwinger formula can be used.

### 2.2 Dynamical evolution

In the previous Section we have recalled the Schwinger formula for rate of electron-positron pairs production in the presence of an external strong electric field. Now we review the main results concerning the dynamical evolution of the system considering the back reaction of pairs on the external electric field once they have been produced. In particular we are going to focus our attention on the screening effect of pairs on the external electric field strengths and the motion of pairs and their interactions.

When these dynamical effects are considered, the pair production in an external electric field is no longer only a process of quantum tunneling in a constant static electric field. In fact, it turns out to be a much more complex process during which all the three above mentioned effects play an important role. More precisely, a phenomenon of electron–positron oscillation, plasma oscillation, takes place.

The most general framework for considering the problem of back reaction of created matter fields on initial strong electric field is QED. Up to now the pair production in strong electric fields has been studied in QED in 1+1 dimension case for both scalar \[72\] and fermion \[73\] fields. It was shown there that pair creation is indeed followed by plasma oscillations due to back reaction of pairs on initial electric field. The results were compared with the solutions of the relativistic Vlasov-Boltzmann equations and shown to agree very well. That means the classical description can be used to describe such system, even though intrinsically quantum, with a reasonable degree of accuracy.

Along the same line, the Vlasov type kinetic equation for description of \( e^+e^- \) plasma creation under the action of a strong electric field was used in the following works \[74\ \[75\ \[76\] and the back reaction problem in the same framework was considered by \[77\ \[78\]. The KT was applied to the
description of the vacuum quark creation under action of a supercritical chromo-electromagnetic field by [79, 80]. Kinetic equations for electron-positron-photon plasma in strong electric field were obtained in [81] from the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy.

A simpler model was developed later, starting from Vlasov-Boltzmann equations [82] and assuming that all particles are in the same momentum state at a given time, that allowed to consider pair-photon interactions. In this way the system of partial integro-differential equations was reduced to the system of ordinary differential equations which was integrated numerically. This model was studied in details in [47, 83], where existence of plasma oscillations was confirmed and extended to undercritical electric fields. It was also shown that photons are generated and reach equipartition with pairs on a time scale much longer than the oscillation period.

2.2.1 Semiclassical theory of plasma oscillations

In the semi-classical QED [84, 85], one quantizes only the Dirac field \( \psi(x) \), while an external electromagnetic field \( A^\mu(x) \) is treated classically as a mean field. The electron-positron field is described by a Dirac equation in an external classical electromagnetic potential \( A^\mu(x) \)

\[
[\gamma_\mu (i\partial^\mu - eA^\mu) - m] \psi(x) = 0
\]

and the semi-classical Maxwell equations

\[
\partial_\mu F^{\mu\nu} = \langle j^\nu(x) \rangle, \quad j^\nu(x) = \frac{i}{2} [\bar{\psi}(x), \gamma^\nu \psi(x)],
\]

where \( j^\nu(x) \) is the electron and positron current and the expectation value is with respect to the quantum states of the electron and positron fields. The dynamics that these equations describe is not only the motion of electron and positron pairs, but also their back reaction on the external electromagnetic field.

The problem of the back reaction was first solved in scalar QED in Refs. [84, 85] and a numerical analysis was made in (1+1)-dimensional case [72]. Eqs. (2.27), (2.28) are replaced by the scalar QED coupled equations for a charged scalar field \( \Phi(x) \)

\[
[(i\partial^\mu - eA^\mu)^2 - m^2_e] \Phi(x) = 0.
\]

The current \( j^\nu(x) \) of the charged scalar field in the semi-classical Maxwell
2.2. DYNAMICAL EVOLUTION

equations (2.28) is

\[ j^\nu(x) = \frac{i e}{2} [\Phi^*(x) \partial^\nu \Phi(x) - \Phi(x) \partial^\nu \Phi^*(x)]. \]

If there is a spatially homogeneous electric field \( E = E_z(t) \hat{z} \) in the \( \hat{z} \)-direction, then the corresponding gauge potential is \( A = A_z(t) \hat{z}, A_0 = 0 \). Defining \( E \equiv E_z, A \equiv A_z \) and \( j \equiv j_z \), the Maxwell equations (2.28) reduce to the single equation

\[ \frac{d^2 A}{dt^2} = \langle j(x) \rangle, \]

for the potential and \( E = -dA/dt \).

The quantized scalar field \( \Phi(x) \) in Eq. (2.29) can be expanded in terms of plane waves with operator-valued amplitudes \( f_k(t) a_k \) and \( f^*_k(t) b_k^\dagger \)

\[ \Phi(x) = \frac{1}{V^{1/2}} \sum_k [f_k(t) a_k + f^*_k(t) b_k^\dagger] e^{-ikx}, \]

where \( V \) is the volume of the system and the time-independent creation and annihilation operators obey the commutation relations

\[ [a_k, a_k^\dagger] = [b_k, b_k^\dagger] = \delta_{k,k'}, \]

and each \( k \)-mode function \( f_k \) obeys the Wronskian condition,

\[ f_k f^*_k - f^*_k f_k = i. \]

The time dependency in this basis \( (a_k, b_k^\dagger) \) \( (2.32, 2.33) \) is carried by the complex mode functions \( f_k(t) \) that satisfy the following equation of motion, as demanded from the QED coupled Eq. (2.29) of Klein–Gordon type

\[ \left( \frac{d^2}{dt^2} + \omega^2_k(t) \right) f_k(t) = 0, \]

where the time-dependent frequency \( \omega^2_k(t) \) is given by

\[ \omega^2_k(t) \equiv [k - eA]^2 + m^2 = [k - eA(t)]^2 + k^2_\perp + m^2_e. \]

Here \( k \) is the constant canonical momentum in the \( \hat{z} \)-direction which should be distinguished from the gauge invariant, but time-dependent kinetic mo-
mentum

\begin{equation}
\label{eq:2.37}
p(t) = k - eA(t), \quad \frac{dp}{dt} = eE,
\end{equation}

which reflects the acceleration of the charged particles due to the electric field, while in the directions transverse to the electric field the kinetic and canonical momenta are the same \( k_\perp = p_\perp \).

The mean value of electromagnetic current (2.30) in the \( \hat{z} \)-direction is then

\begin{equation}
\langle j(t) \rangle = 2e \int \frac{d^3k}{(2\pi)^3} \left| k - eA(t) \right| f_k(t) \left| f_k(t) \right|^2 \left[ 1 + N_+(k) + N_-(k) \right],
\end{equation}

where \( N_+(k) = \langle a_k^\dagger a_k \rangle \) and \( N_-(k) = \langle b_k^\dagger b_k \rangle \) are the mean numbers of particles and antiparticles in the time-independent basis (2.32,2.33). The mean charge density must vanish

\begin{equation}
\langle j^0(t) \rangle = e \int \frac{d^3k}{(2\pi)^3} \left[ N_+(k) - N_-(k) \right] = 0,
\end{equation}

by the Gauss law for a spatially homogeneous electric field (i.e., \( \nabla \cdot E = 0 \)). As a result, \( N_+(k) = N_-(k) \equiv N_k \). For the vacuum state, \( N_k = 0 \).

The Maxwell equation (2.31) for the evolution of electric field becomes

\begin{equation}
\frac{d^2A}{dt^2} = 2e \int \frac{d^3k}{(2\pi)^3} \left| k - eA(t) \right| f_k(t)^2 \sigma_k, \quad \sigma_k = (1 + 2N_k).
\end{equation}

These two scalar QED coupled Eqs. (2.35) and (2.39) in (1+1)-dimensional case were numerically integrated in Ref. [72]. Starting with a critical electric field, one clearly finds the phenomenon of oscillating electric field \( E(t) \) and current \( j(t) \), i.e., plasma oscillation.

We can understand this phenomenon as follows. In a classical kinetic picture, we have the electric current \( j = 2en\langle v \rangle \) where \( n \) is the density of electrons (or positrons) and \( \langle v \rangle \) is their mean velocity. Driven by the external electric field, the velocity \( \langle v \rangle \) of electrons (or positrons) continuously increases, until the electric field of electron and positron pairs screens the external electric field down to zero, and the kinetic energy of electrons (or positrons) reaches its maximum. The electric current \( j \) saturates as the velocity \( \langle v \rangle \) is close to the speed of light. Afterward, these electrons and positrons continuously move apart from each other further, their electric field, whose direction is opposite to the direction of the external electric
field, increases and decelerates electrons and positrons themselves. Thus
the velocity $\langle v \rangle$ of electrons and positrons decreases, until the electric field
reaches negative maximum and the velocity vanishes. Then the velocity
$\langle v \rangle$ of electrons and positrons starts to increase in backward direction and
the electric field starts to decrease for another oscillation cycle.

## 2.2.2 Vlasov-Maxwell equations

The same problem of pair creation in homogeneous electric field described
in the previous Section can be studied using an alternative method with
respect to the semiclassical one, namely the model based on the relativis-
tic Boltzmann-Vlasov equation [72]. Due to the possibility of creation of
electron-positron pairs from vacuum, the source term has to be added to
Eq. (1.3). In particular, the probability density rate $S(E, p)$ for the cre-
ation of a pair with 3–momentum $p$ in the electric field $E$ is given by the
Schwinger formula (see also Refs. [72, 73]):

\[
S(E, p) = -\frac{|eE|}{m^2} \log \left[ 1 - \exp \left( -\frac{\pi (m^2 + p^2)}{|eE|} \right) \right] \delta(p_{||}),
\]

where $p_{||}$ and $p_{\perp}$ are the components of the 3-momentum $p$ parallel and
orthogonal to $E$, respectively. Then the kinetic equations (1.11) for electron
and positron classical distribution function become

\[
\frac{\partial f_\pm}{\partial t} \pm eE \frac{\partial f_\pm}{\partial p_{||}} = S(E, p),
\]

where $E = |E|$. Let us note that Eq. (2.41) does not include the collision
integrals, therefore such model cannot describe interaction between parti-
cles.

Because pair creation back reacts on the electric field, Vlasov equations
(2.41) are coupled with the homogeneous Maxwell equations, which read

\[
\partial_t E = -j_p(E) - j_c(t),
\]

where

\[
j_p(E) = 2 \frac{E}{E^2} \int d^3 p \epsilon_p S(E, p),
\]
is the polarization current and

\[ j_c(t) = 2ne \int d^3p \frac{p}{\epsilon_p} f_c(p), \]

is the conduction current (see Ref. [77]).

The relativistic Boltzmann-Vlasov equation (2.41) and field equation (2.31) with the conduction current (2.44) and the polarization current (2.43) were numerically integrated [72] in (1+1)-dimensional case. The numerical integration shows that the system undergoes plasma oscillations. Besides, comparing the semi-classical analysis and the numerical integration of the Boltzmann equation one sees that they are in good quantitative agreement. The discrepancies are because in addition to spontaneous pair production, the quantum theory takes into account pair production via bremsstrahlung (“induced” pair production), which are neglected in Eq. (2.41).

In Refs. [73, 86], the study of plasma oscillations was extended to the fermionic case. On the basis of semi-classical theory of spinor QED, expressing the solution of the Dirac equation (2.27) as

\[ \psi(x) = [\gamma_\mu (i\partial^\mu - eA^\mu) + m_e] \phi(x), \]

where \( \phi(x) \) is a four-component spinor, one finds that \( \phi(x) \) satisfies the quadratic Dirac equation,

\[ \left[ (i\partial^\mu - eA^\mu)^2 - \frac{e^2}{2} \sigma^{\mu\nu} F_{\mu\nu} - m_e^2 \right] \phi(x) = 0. \]

The electric current of spinor field \( \phi(x) \) couples to the external electric field that obeys the field equation (2.31).

Analogously to the scalar QED case, both semi-classical theory of spinor QED and kinetic Boltzmann-Vlasov equation have been analyzed and numerical integration was made in the (1+1)-dimensional case [73, 86]. Again, the numerical results show that plasma oscillations of electric field, electron and positron currents are similar to those computed for the bosonic case.

### 2.2.3 The role of photons

In this Section, we review the studies of electron-positron pairs production in strong electric fields based on the electro-fluidodynamic equations for electrons, positrons and photons [82]. Collision terms originated from
annihilation of electron-positron pair into two photons and \textit{vice versa} are included. As we will see, these collision terms lead to the damping of plasma oscillations and to energy equipartition between different types of particles. Since the initial state is far from equilibrium the collisions cannot be modeled by an effective relaxation time term in the transport equations.

Furthermore in \cite{82} the authors assume that the electric field varies on macroscopic length scale and therefore one can approximate an electric field as a homogeneous one. Also, transport equations are used for electrons, positrons and photons, with collision terms, coupled to Maxwell equations, as introduced in the previous Sections. Besides the collision terms can be exactly computed, because the QED cross-sections are known.

As we already mentioned in Chapter 1, see (1.13), particle-antiparticle symmetry allows us to write

\begin{equation}
  f_{e^+} (t, p) = f_{e^-} (t, -p) \equiv f_e (t, p),
\end{equation}

and therefore only electrons can be described. Following the notation we have introduced in Section 1.3.2 for the collision integrals and using Eq. (2.41) to take into account the pairs production rate, the transport Boltzmann–Vlasov equations for electron and photon DFs can now be written as

\begin{align}
  \partial_t f_e + eE \cdot \nabla_p f_e &= S (E, p) + \eta_e - \chi_e f_e, \\
  \partial_t f_\gamma &= \eta_\gamma - \chi_\gamma f_\gamma,
\end{align}

where only pairs annihilation into photons $e^- e^+ \rightarrow \gamma \gamma$ and the inverse process $\gamma \gamma \rightarrow e^- e^+$ are taken into account. Note that the collisional terms $\eta_k, \chi_k f_k$ can be safely neglected when created pairs do not produce a dense plasma and hence Eq. (2.47) is reduced to Eq. (2.41). The importance of collision can be easily evaluated by means of the optical depth $\tau$ inside the plasma \cite{83}. As a matter of fact such parameter gives an idea about the time-scale at which a given interaction cannot be neglected anymore. To have an idea, if a particle moves at speed $v$ and interacts with particles with number density $n$ with cross-section $\sigma$, then the time-scale for this process to occur is roughly given by

\begin{equation}
  t_{\text{coll}} = \frac{1}{\sigma n v}.
\end{equation}

The equation above helps to estimate the time at which every particle on average underwent the interaction under consideration one time at least. Then the optical depth, interpreted as the average number of collisions in
2.2. DYNAMICAL EVOLUTION

the interval $\Delta t$ is given by

(2.50) \[ \tau_{\text{coll}} = \sigma n v \Delta t. \]

The definition in Eq. (2.50) will be used in the next Sections in order to estimate if the collisionless approximation can be considered valid or not.

Because pair creation back reacts on the electric field, Vlasov equations (2.47) and (2.48) are coupled with the homogeneous Maxwell equation (2.42) with conduction and polarization currents given by Eqs. (2.44) and (2.43). This system of equations describes the dynamical evolution of the electron-positron pairs, the photons and the strong homogeneous electric field due to the Schwinger process of pair creation, the pair annihilation into photons and the two photons annihilation into pairs. In [82] the authors did not integrate the Vlasov-Boltzmann equations but integrated Eqs. (2.47) and (2.48) over the phase spaces of positrons (electrons) and photons to get differential equations for mean values. These equations can be regarded as rate equations for number and energy densities of photons and electron-positron pairs as well as Maxwell equation for the electric field:

\[
\frac{dn_e}{dt} = S(E) - n_e^2 \langle \sigma_1 v' \rangle_e + n_e^2 \langle \sigma_2 v'' \rangle_e, \\
\frac{dn_\gamma}{dt} = 2n_e^2 \langle \sigma_1 v' \rangle_e - 2n_\gamma^2 \langle \sigma_2 v'' \rangle_\gamma, \\
\frac{d}{dt} \left( n_e \langle \epsilon_p \rangle_e \right) = en_e E \cdot \langle v \rangle_e + \frac{1}{2} E \cdot j_p - n_e^2 \langle \epsilon_p \sigma_1 v'' \rangle_e + n_\gamma^2 \langle \epsilon_k \sigma_2 v'' \rangle_\gamma, \\
\frac{d}{dt} \left( n_\gamma \langle \epsilon_k \rangle_\gamma \right) = 2n_e^2 \langle \epsilon_p \sigma_1 v'' \rangle_e - 2n_\gamma^2 \langle \epsilon_k \sigma_2 v'' \rangle_\gamma, \\
\frac{d}{dt} \left( n_e \langle p \rangle_e \right) = en_e E - n_e^2 \langle p \sigma_1 v' \rangle_e, \\
\frac{dE}{dt} = -2en_e \langle v \rangle_e - j_p(E),
\]

(2.51)

where

(2.52) \[ S(E) = \frac{m_e^4}{4\pi^3} \left( \frac{E}{E_c} \right)^2 \exp \left( -\pi \frac{E_c}{E} \right), \]

is the total probability rate for pair production. In Eqs. (2.51), $v'' = c$ the velocity of light and $v' = 2|p|/E_{p\text{CoM}}$ is the relative velocity between electrons and positrons in the reference frame of the center of mass, where
\( p = |p_{e^+}|, p_{e^-} = -p_{e^+} \) are 3–momenta of electron and positron and \( \epsilon_{p_{e^\pm}} = \epsilon_p^\text{CoM} \) are their energies. \( \sigma_1 = \sigma_1 \left( \epsilon_p^\text{CoM} \right) \) is the total cross-section for the process \( e^+ e^- \rightarrow \gamma \gamma \), and \( \sigma_2 = \sigma_2 \left( \epsilon_k^\text{CoM} \right) \) is the total cross-section for the process \( \gamma \gamma \rightarrow e^+ e^- \), here \( \epsilon_p^\text{CoM} \) is the energy of a particle in the reference frame of the center of mass.

In order to evaluate the mean values in system (2.51) some further hypotheses on the distribution functions are used. One defines \( \bar{p}_{\parallel}, \bar{\epsilon}_p \) and \( \bar{p}_{\perp}^2 \) such that

\[
\langle p_{\parallel} \rangle_e \equiv \bar{p}_{\parallel},
\]

\[
\langle \epsilon_p \rangle_e \equiv \bar{\epsilon}_p \equiv \sqrt{\bar{p}_{\parallel}^2 + \bar{p}_{\perp}^2 + m_e^2}.
\]

In [82] the following strong ansatz is made about the electron distribution function

\[
f_e (t, p) \propto n_e (t) \delta \left( p_{\parallel} - \bar{p}_{\parallel} \right) \delta \left( p_{\perp}^2 - \bar{p}_{\perp}^2 \right).
\]

Since in the scattering \( e^+ e^- \rightarrow \gamma \gamma \) the coincidence of the scattering direction with the incidence direction is statistically favored, it is also assumed

\[
f_\gamma (t, k) \propto n_\gamma (t) \delta \left( k_{\parallel} - \bar{k}_{\parallel} \right) \left[ \delta \left( k_{\perp}^2 - \bar{k}_{\perp}^2 \right) + \delta \left( k_{\parallel} + \bar{k}_{\parallel} \right) \right],
\]

where \( k_{\parallel} \) and \( k_{\perp} \) have analogous meaning as \( p_{\parallel} \) and \( p_{\perp} \) and the terms \( \delta \left( k_{\parallel} - \bar{k}_{\parallel} \right) \) and \( \delta \left( k_{\parallel} + \bar{k}_{\parallel} \right) \) account for the probability of producing, respectively, forwardly scattered and backwardly scattered photons. Since the Schwinger source term (2.40) implies that the positrons (electrons) have initially fixed \( p_{\parallel} \), namely \( p_{\parallel} = 0 \), assumption (2.55) (2.56) means that the distribution of \( p_{\parallel} (k_{\parallel}) \) does not spread too much with time and, analogously, that the distribution of energies is sufficiently peaked to be describable by a \( \delta \)-function. As we show in Section 2.4 this assumption is oversimplified. As we showed in [87] that pairs are produced and accelerated to relativistic speeds on the same time-scale. Then a more complete description in terms of particle distribution functions is needed. The actual dependence on the momentum of the distribution functions has been discussed in Ref. [73, 78]. If Eqs. (2.55) and (2.56) are substituted into the system (2.51) one gets a new system of ordinary differential equations.
One can introduce the inertial reference frame which on average coincides with the center of mass frame for the processes $e^+e^- \leftrightarrow \gamma\gamma$, and has $e_{\text{CoM}} \simeq \bar{\epsilon}$ for each species, and therefore substituting Eqs. (2.55) and (2.56) into Eqs. (2.51) one finds

\[
\frac{dn_e}{dt} = S(E) - 2n_e^2 \sigma_1 \rho_e^{-1} \left| \pi_e \right| + 2n_\gamma^2 \sigma_2 ,
\]

\[
\frac{dn_\gamma}{dt} = 4n_e^2 \sigma_1 \rho_e^{-1} \left| \pi_e \right| - 4n_\gamma^2 \sigma_2 ,
\]

\[
\frac{d\rho_e}{dt} = en_eE \rho_e^{-1} \left| \pi_e \right| + \frac{1}{2} E j_p - 2n_e \rho_e \sigma_1 \rho_e^{-1} \left| \pi_e \right| + 2n_\gamma \rho_\gamma \sigma_2 ,
\]

\[
\frac{d\rho_\gamma}{dt} = 4n_e \rho_e \sigma_1 \rho_e^{-1} \left| \pi_e \right| - 4n_\gamma \rho_\gamma \sigma_2 ,
\]

\[
\frac{d\pi_e}{dt} = en_eE - 2n_e \pi_e \sigma_1 \rho_e^{-1} \left| \pi_e \right| ,
\]

\[
\frac{dE}{dt} = -2en_e \rho_e^{-1} \left| \pi_e \right| - j_p(E) ,
\]

(2.57)

where

\[
\rho_e = n_e \bar{\epsilon}_p ,
\]

\[
\rho_\gamma = n_\gamma \bar{\epsilon}_k ,
\]

\[
\pi_e = n_e \bar{\rho}_p ,
\]

(2.58)

(2.59)

(2.60)

are the energy density of positrons (electrons), the energy density of photons and the density of “parallel momentum” of positrons (electrons), $E$ is the electric field strength and $j_p$ the only component of $j_p$ parallel to $E$. $\sigma_1$ and $\sigma_2$ are evaluated at $e_{\text{CoM}} = \bar{\epsilon}$ for each species. Finally Eqs. (2.57) are duly consistent with energy density conservation:

\[
\frac{d}{dt} \left( \rho_e + \rho_\gamma + \frac{1}{2} E^2 \right) = 0 .
\]

(2.61)

Eqs. (2.57) have been integrated in [82] with the following initial conditions

\[
n_e(t_0) = n_\gamma(t_0) = \rho_e(t_0) = \rho_\gamma(t_0) = \pi_e(t_0) = 0 , \quad E(t_0) = E_0 .
\]

(2.62)

The conditions above say that only the electric field is present at the beginning without any particle.

The numerical integration confirms previous results obtained in [72, 73] that the system undergoes plasma oscillations. The electric field does
not abruptly reach the equilibrium value but rather oscillates with decreasing amplitude, electrons and positrons oscillate along the electric field direction, reaching ultrarelativistic velocities and the role of the $e^+e^- \rightleftharpoons \gamma\gamma$ scatterings is marginal in the early time of the evolution. This last point can be easily explained as follows: since the electrons are extremely relativistic, the annihilation probability is very low and consequently the density of photons builds up very slowly.

It is found in [82] that the electric field is screened to about the critical value; the initial electromagnetic energy density is distributed over electron–positron pairs and photons, indicating energy equipartition; photons and electron-positron pairs number densities are comparable asymptotically, indicating number equipartition.

2.3 The frequency of pairs oscillations

Now following [83] we focus our attention on the frequency of the plasma oscillations we have been discussing in the previous Sections. The interest about oscillation frequency is simply due to the fact that such event could be possibly observed in laboratory experiments. In particular the oscillating pairs produce oscillating currents and therefore radiation emission from such system is possible.

In this Section we reproduce the arguments adopted in Section 2.3. We do that for the electric field strength $E$ in the range $0.2 E_c < E < 10 E_c$. Following the approach adopted in [47] we work out one second order ordinary differential equation for a variable related to the velocity from which we can recover the classical plasma oscillation equation when $E \to 0$. Thereby, we follow the system evolution in time studying how this oscillation frequency approaches the plasma frequency. The time-scale needed to approach to the plasma frequency is computed for different initial fields. Hence we study the radiation spectrum as the key signature of the presence of a critical field by means of pairs oscillations. We show that the characteristic frequency of the power spectrum is determined uniquely from the initial value of the electric field strength.

In [47] it was pointed out that the pairs oscillation phenomenon occurs also when $E \leq E_c$ giving emphasis on the fact that, for overcritical (undercritical) field, a large (small) fraction of the initial electromagnetic energy is converted into the rest mass of pairs, whereas a small (large) fraction is converted into kinetic energy. In [88] the case of spatially inhomogeneous electric field has been considered, the emitted radiation spectrum far from the oscillation region was obtained, presenting a narrow feature.
2.3. THE FREQUENCY OF PAIRS OSCILLATIONS

2.3.1 Electro-fluidodynamic approximation

Following [47] the conservation laws and Maxwell equations written for electrons, positrons and electromagnetic field are

\[
\frac{\partial (\tilde{n} U^\mu)}{\partial x^\mu} = S, \tag{2.63}
\]

\[
\frac{\partial T^{\mu\nu}}{\partial x^\nu} = -F^{\mu\nu} J_\nu, \tag{2.64}
\]

\[
\frac{\partial F^{\mu\nu}}{\partial x^\nu} = -4\pi J^\mu, \tag{2.65}
\]

where \(S\) is the rate of pairs production, given by Eq. (2.90), \(\tilde{n}\) is the comoving number density of electrons, \(J^\mu = J^\mu_{\text{cond}} + J^\mu_{\text{pol}}\) is the total 4-current and \(T^{\mu\nu}\) is energy-momentum tensor of electrons and positrons defined as

\[
T^{\mu\nu} = m_e \tilde{n} \left(U^\mu_{(+)} U^\nu_{(+)} + U^\mu_{(-)} U^\nu_{(-)}\right), \tag{2.66}
\]

where \(U^\mu\) is four velocity respectively of positrons and electrons. Electrons and positrons move along the electric field lines in opposite directions therefore both giving the same contribution to the current.

It has been shown in [47] that in a uniform electric field, from the system (2.63)-(2.65), the following system of four coupled ordinary differential equations may be obtained

\[
\frac{d\tilde{n}}{dt} = \tilde{S}, \tag{2.67}
\]

\[
\frac{d\tilde{\rho}}{dt} = \tilde{n} \tilde{E} \tilde{\vartheta} + \tilde{\gamma} \tilde{S}, \tag{2.68}
\]

\[
\frac{d\tilde{p}}{dt} = \tilde{n} \tilde{E} + \tilde{\gamma} \tilde{\vartheta} \tilde{S}, \tag{2.69}
\]

\[
\frac{d\tilde{E}}{dt} = -8\pi\alpha \left(\tilde{n} \tilde{\vartheta} + \tilde{\gamma} \tilde{S}/\tilde{E}\right), \tag{2.70}
\]

where \(n = m_e^2 \tilde{n}\) is dimensionless number density normalized by the Compton length \(\lambda_c = 1/m_e\), \(\rho = m_e^4 \tilde{\rho}\) is energy density of positrons\(^1\), \(p = m_e^4 \tilde{p}\) is momentum density of positrons, \(E = E_c \tilde{E}\) is electric field strength, and \(t = m_e^{-1} \tilde{t}\) is time, normalized by the Compton time \(t_c = 1/m_e\). The rate of pair production is \(\tilde{S} = \tilde{E}^2 \exp(-\pi/\tilde{E})/(4\pi^3)\), velocity is \(\tilde{\vartheta} = \tilde{p}/\tilde{\rho}\) and

\(^1\)Total energy density of electrons and positrons is twice this value.
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Lorentz factor is \( \tilde{\gamma} = \left(1 - \tilde{\sigma}^2\right)^{-1/2} \).

For the system (2.67)-(2.70) there exist two integrals (conservation laws)

\[
\begin{align*}
\tilde{\rho}^2 &= \tilde{\rho}^2 + \tilde{n}^2, \\
16\pi\alpha\tilde{\rho} &= \tilde{E}_0^2 - \tilde{E}^2,
\end{align*}
\]

so the particle energy density vanishes for initial value of the electric field \( \tilde{E}_0 \). Combining together the previous two equations, we get for the maximum number density of pairs that can be created

\[
\tilde{n}_{\text{max}} = \frac{\tilde{E}_0^2}{16\pi\alpha}.
\]

In [47] the system (2.67)-(2.70) was reduced to two equations using (2.71)-(2.72) and analyzed on the phase plane \((\tilde{E}, \tilde{\sigma})\).

Notice that the equation of motion of single particle in our approximation is just

\[
\dot{\tilde{u}} = \tilde{E},
\]

where we have defined \( \dot{\tilde{u}} = d\tilde{u}/dt \) and introduced a new variable constructed from hydrodynamic velocity as \( \tilde{u} = \tilde{\gamma}\tilde{\sigma} = \gamma v/c \). Then this equation can be combined with (2.70) to obtain a single master equation

\[
\ddot{\tilde{u}} + \frac{\tilde{E}_0^2 - \tilde{u}^2}{2(1 + \tilde{u}^2)}\ddot{\tilde{u}} + \frac{2\alpha}{\pi^2}\sqrt{1 + \tilde{u}^2}\exp\left(-\frac{\pi}{|\tilde{u}|}\right)\dot{\tilde{u}} = 0.
\]

The key point of our treatment is the physical interpretation of Eq. (2.75) which can be rewritten symbolically as

\[
\ddot{\tilde{u}} + \tilde{\omega}_p^2\tilde{u} + k\dot{\tilde{u}} = 0.
\]

With constant coefficients Eq. (2.76) would describe damped harmonic oscillations with frequency \( \tilde{\omega}_p \) and friction \( k \). In our case \( \tilde{\omega}_p \) and \( k \) are time dependent, but Eq. (2.75) still possesses an oscillating behavior with damping. With our definitions the number density of pairs is

\[
\tilde{n} = \frac{\tilde{E}_0^2 - \tilde{u}^2}{16\pi\alpha\sqrt{1 + \tilde{u}^2}}.
\]
Using Eqs. (2.67)-(2.70) we then identify $\tilde{\omega}_p$ in (2.76) as

$$\tilde{\omega}_p = \sqrt{8\pi \alpha \tilde{n}} \sqrt{1 + \tilde{u}^2},$$

i.e. the relativistic plasma frequency$^2$.

The function

$$k = \frac{2\alpha}{\pi^2} \sqrt{1 + \tilde{u}^2} \exp \left( -\frac{\pi}{|\tilde{u}|} \right),$$

in Eq. (2.76) accounts for the rate of pair production (2.90). It describes the increase of inertia of electron-positron pairs due to increase of their number and causes decrease of the amplitude of oscillations.

For small electric fields Eq. (2.75) is reduced to classical plasma oscillations equation describing Langmuir waves, since in that case $k$ is exponentially suppressed. For this reason we expect that as the amplitude of oscillations of electric field gets smaller the frequency of oscillations $\omega$ tends to the plasma frequency $\omega_p$.

In [47] the authors solved numerically the system of four coupled ordinary differential equations (2.67)-(2.70). Now it is possible to solve just one second order differential equation (2.75) which allows us to study its asymptotic behavior as well.

### 2.3.2 Numerical results

We solve numerically Eq. (2.75) with the initial conditions $\tilde{u}(0) = \tilde{E}_0$ and $\tilde{u}(0) = 0$, corresponding to no pairs in the initial moment, taking for initial electric field strength $\tilde{E}_0 = \{0.2, 0.5, 1, 2, 5, 10\}$. Once this equation has been solved, we have the solution for the number density from (2.77) and for the plasma frequency by means of (2.78). In Figure 2.1 we show the evolution in time of $\tilde{u}$ where the amplitude of the oscillations decreases, while its frequency increases with time.

We present the number density of pairs in Figure 2.2 for the case $E_0 = 2 E_c$ as a fraction of the maximum achievable value $\tilde{n}_{\text{max}}$. In all the cases under interest, this number is asymptotically achieved indicating that the

$^2$The factor $8\pi$ is in this formula due to the presence of two charge carriers with the same mass - electrons and positrons. This is different from the classical electron-ion plasma where only electron component oscillates and the corresponding factor is twice smaller.
2.3. THE FREQUENCY OF PAIRS OSCILLATIONS

Figure 2.1: In this picture we show how the function \( \tilde{u} = \gamma v/c \) changes with time when Eq. (2.75) is solved numerically for \( E_0 = 2E_c \). The oscillating behavior is reproduced and two features are evident. Firstly the amplitude decreases with time very rapidly indicating that the energy is going to be converted into rest mass of particles instead of their kinetic energy. Secondly the frequency increases with time.
Figure 2.2: Ratio between number density and maximum achievable number density $n_{\text{max}} = 1/(4\pi a) \lambda c^{-3}$ for $E_0 = 2E_c$ computed from Eq. (2.75); it becomes close to unity after $10^5 t_c$. From Figure 2.1 and Eq. (2.74) we see that the maxima of $\tilde{u}$ correspond to $E = 0$ and quenching of pair creation giving rise to flattening of $n$. This happens for each oscillation, but it is more evident at the beginning due to the double logarithmic scale on this figure.
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Figure 2.3: In this figure are shown, for all the cases under interest, the half period of the first oscillation $t_1$ (triangles), the characteristic time-scales $t_a$ (squares) needed to the pairs oscillation frequency to reach the plasma frequency and the time $t_\gamma$ (circles) which satisfies the condition $\tau(t_\gamma) \simeq 1$ for the optical depth defined in (2.85), see below. The value of $t_a$ for the case $E_0 = 2E_c$ is shown in Figure 2.5 by the vertical line.
final result of the process will be the complete conversion of the electromagnetic energy density into the rest mass of the pairs. Moreover, looking at Figure 2.2 we recover the result obtained in [47]; in fact we can see that after the first oscillation, higher is $E_0$ larger is $\tilde{n}$. This means that the first oscillation gives the leading contribution to the process in which the electromagnetic energy of the field is converted in the rest mass of pairs, with a moderate contribution to their kinetic energy for $E_0 > E_c$. The values of the half periods of the first oscillation for each considered case are represented in Figure 2.3 by triangles.

We computed the frequency of the $i$-th oscillation as $\tilde{\omega}_i = \pi / \tilde{T}_{1/2}^i$, where $\tilde{T}_{1/2}^i$ is the corresponding half period, calculated considering the time interval between the $i$-th and $(i+1)$-th subsequent roots of $\tilde{u}$, see Figure 2.1.

Notice that $\tilde{\omega}_p$ is an oscillating function of time, due to the presence of $\tilde{u}$ in (2.78), besides the frequency of these oscillations increases in time. For this reason, in order to get a smooth function we calculated the average of $\tilde{\omega}_p$. We use this new function $\tilde{\omega}_p^{av}$ for the plasma frequency to make a comparison with the frequency of oscillations of pairs. In Figure 2.4, for the case $E_0 = 2E_c$, the blue area represents the plasma frequency as defined by (2.78), the yellow curve is its average $\tilde{\omega}_p^{av}$, while the pairs oscillation frequency $\tilde{\omega}$ is represented by the red curve. For the same initial electric field, in Figure 2.5 the trend of the ratio $\tilde{\omega} / \tilde{\omega}_p^{av}$ is shown, which indicates that the averaged plasma frequency is achieved asymptotically as expected from (2.75). Notice that the oscillation frequency $\tilde{\omega}$ is always smaller than $\tilde{\omega}_p^{av}$ since the number density of pairs is constantly increasing with time during each oscillation cycle.

We computed the power spectrum of radiation in the far zone, assuming dipole radiation following [88]. The power spectrum, namely the energy radiated per unit solid angle per frequency interval and per unit volume, is given by

$$
\tilde{P}(\tilde{\omega}) = \frac{d\tilde{I}}{d\tilde{\omega} d\Omega} = 2\alpha |\tilde{D}(\tilde{\omega})|^2,
$$

where the amplitude $\tilde{D}(\tilde{\omega})$ is proportional to the Fourier transform of the electric current time derivative [88]

$$
\tilde{D}(\tilde{\omega}) \propto \int_T d\tilde{t} e^{i\tilde{\omega}\tilde{t}} \left[ \frac{\partial \tilde{J}(\tilde{t})}{\partial \tilde{t}} \right].
$$

The electric current is simply related to the new variables by the following
expression

\[ \tilde{J} = 2 \sqrt{\tilde{a} \tilde{n}} \frac{\tilde{u}}{\sqrt{1 + \tilde{u}^2}}. \]

From the Figure 2.6 it is clear that the main contribution is given by the final and fastest oscillations which last for a longer time. Therefore, the power spectrum shows a peak close to the plasma frequency being reached asymptotically. We can easily estimate the frequency corresponding to this peak combining Eq. (2.73) with Eq. (2.78) as

\[ \tilde{\omega}_{\text{peak}} \simeq \frac{\tilde{E}_0}{\sqrt{2}}, \]

with the corresponding energy \( \hbar \tilde{\omega}_{\text{peak}} \simeq 0.72 \frac{\tilde{E}_0}{\tilde{E}_c} \text{ MeV} \). The energy loss due to the dipole radiation for \( \tilde{E} = 2\tilde{E}_c \) considered in Figure 2.6 for \( t = 10^5 \tilde{t}_c \) is less than one percent.

Once we know the frequency of the pairs oscillations and the plasma frequency, we obtain their ratio as it is shown in Figure 2.5. Besides, we use \( \tilde{\omega}/\tilde{\omega}_{\text{av}} \) in order to compute the characteristic time scale \( \tilde{t}_{a} \) needed for the pairs oscillation frequency to reach the plasma frequency. This has been done considering the ratio between \( \tilde{\omega}/\tilde{\omega}_{\text{av}} \) and its time derivative; the result of this procedure gives us a numerical function from which we have taken the average. For all the considered cases, this quantity is shown in Figure 2.3, from which we understand that the general trend is that larger is the initial electric field, larger will be the starting oscillation frequency.

It is worth noting the effect of degeneracy on the pair production. One may think that concentration of pairs should reach quickly the maximum allowed value by the Pauli principle and the pair production would be immediately blocked. This would imply that Pauli blocking factors should be necessarily taken into account in Vlasov-Boltzmann equations. For particles at rest this would happen when two pairs with opposite spins occupy a Compton volume. Considering asymptotic number of pairs given by (2.73) one finds that it would happen for \( \tilde{E} > 4\sqrt{\tilde{a} \pi \tilde{E}_c} \simeq 0.6\tilde{E}_c \). However, one has to keep in mind that particles produced at rest are accelerated by external electric field and thus leave the quantum state with zero momentum which can be subsequently filled by a new pair. These effects are independent since they operate in ortogonal directions of the phase space, so one can estimate the value of external electric field at which phase space blocking occurs by comparing their rates. Such analysis gives us the fol-
Figure 2.4: In this plot the blue area represents the plasma frequency as defined in Eq. (2.78); it appears like a continuum because of the fast oscillations. The yellow curve is its average in time $\bar{\omega}_p^{av}$ which can be compared with the pairs oscillation frequency $\bar{\omega}$ given by the red curve. This plot corresponds to the case $E_0 = 2E_c$. 
Figure 2.5: Behavior of the ratio $\tilde{\omega}/\tilde{\omega}_{p}^{av}$ in time for $E_0 = 2E_c$. The plasma frequency is attained asymptotically because of the limit $\tilde{S}(\tilde{t} \to \infty) \to 0$. The vertical line corresponds to the time-scale $t_a$ that is needed to attain the plasma frequency in this specific case. This corresponds to the red square in Figure 2.3 for the same value of the initial field.
lowing inequality

\( \frac{1}{4\pi^3} \tilde{E} \exp (-\frac{\pi}{\tilde{E}}) \geq 1 , \)

having the solution \( E \gtrsim 127E_c \), which is much higher than electric fields considered here.

Another effect, relevant for large enough electric field, is interaction of pairs with photons discussed in some details in \[47, 82\] and in previous Sections. Following the definition for the optical depth given in Eq. (2.50) one can estimate the optical depth for electron-positron annihilation in our specific case as

\[
\tau(t) \simeq \int_0^t \frac{\sigma_T}{\gamma^2} n \nu d\tilde{t} = \int_0^t \frac{8\pi a^2}{3} \frac{|\tilde{u}|}{(1 + \tilde{u}^2)^{3/2}} \tilde{n} d\tilde{t} ,
\]

where \( \sigma_T \) is the Thomson’s cross-section, and we approximated \( \sigma \simeq \sigma_T / \gamma^2 \). Equating (2.85) to unity we find the timescale \( t_\gamma \) at which the probability of electron to interact with positron and create a pair of photons reaches unity. From that time moment interaction of pairs with photons can no longer be neglected. This timescale is represented in Figure 2.3 by circles.

Summarizing the informations presented in Figure 2.3 we conclude that independent on the initial value of the electric field, there is a hierarchy between the following time scales \( t_1 < t_\gamma < t_a \). It means that many oscillations occur before electron-positron collisions turn out to be important, thus justifying our collisionless approximation. The condition \( t_\gamma < t_a \) means that the estimation of the maximal frequency of oscillations (2.83) is an approximate one: photons produced by interaction of pairs will also distort the spectrum shown in Figure 2.6.

As long as electric field does not reach critical values for creation of muons and pions their production from electron-positron collisions \[89]-\[90\] is suppressed because of two different mechanisms. Both these processes have a kinematic threshold given by the rest mass of the produced particles. For this reason the Lorentz factor of the relative motion of colliding electron and positron should exceed \( \sim 10^2 \), restricting initial electric fields to be undercritical, \( E_0 < E_c \), see Figure 3 in Ref. \[47\]. On the other hand, the number density of pairs produced is exponentially suppressed for undercritical fields. Besides the cross-section for all these processes decreases as \( \sigma \propto \gamma^{-2} \) which further decreases the rate of electron-positron collisions.
Figure 2.6: The power spectrum in arbitrary units has been obtained using Eq. (2.80). The peak is almost reached at the maximum plasma frequency (2.78) corresponding to the maximum achievable number density (2.73), indicated here by the vertical line. The main contribution is given by the oscillations which last for a long time, namely when the asymptotic limit of the plasma frequency is attained.
2.4 Phase space evolution toward thermalization

In this last Section of this Chapter we study the process of energy conversion from overcritical electric field into electron, positron, photon plasma \[87\]. To do that we solve numerically Vlasov-Boltzmann equations (2.47) and (2.48) for pairs and photons respectively assuming the system to be homogeneous and anisotropic. All the 2-particle interactions between pairs and photons are described by collision terms, see Table 1.1. We are going to evidence several epochs of this energy conversion, where each of them is associated to a specific physical process. Firstly pair creation occurs, secondly back reaction results in plasma oscillations. Thirdly photons are produced by electron-positron annihilation. Finally particle interactions lead to completely equilibrated thermal electron-positron-photon plasma.

Here for the first time we study the entire dynamics of energy conversion from initial strong electric field, ending up with thermalized electron-positron-photon plasma which is assumed to be optically thick. With this goal we generalize previous treatments \[72, 83\] that have been presented in the previous Sections of this Chapter. In particular, we relax the delta-function approximation of particle momenta expressed by Eqs. (2.55) and (2.56) adopted in \[82\]. In contrast, we obtain the system of partial integro-differential equations which is solved numerically on large timescales, exceeding many orders of magnitude several characteristic timescales of the problem under consideration.

2.4.1 Kinetic approach

Assuming invariance under rotations around the direction of the electric field we can use the set of cylindrical coordinates for the phase space we have introduced in Section 1.3.3. Now \(p_\parallel\) and \(p_\perp\) are parallel and orthogonal with respect to the direction of the initial electric field. For computational convenience we use a new DF defined as

\[
F_\nu = 2\pi p_\perp \epsilon f_\nu ,
\]

which allows to write down Boltzmann-Vlasov equations (2.47) and (2.48) in conservative form \[15, 14\], essential for numerical computations. Using Eq. (2.86) the energy density for each type of particle is given by its integral over the parallel and orthogonal component of the momentum

\[
\rho_\nu = \int_{-\infty}^{+\infty} dp_\parallel \int_{0}^{+\infty} dp_\perp F_\nu .
\]
In isotropic momentum space this DF is reduced to the spectral energy density $d\rho_\nu / d\epsilon$.

The time evolution of the electron DF is described by the relativistic Boltzmann-Vlasov equation (2.47)

$$\frac{\partial F_e(p_\parallel, p_\perp)}{\partial t} + eE \frac{\partial F_e(p_\parallel, p_\perp)}{\partial p_\parallel} = \sum_q \left( \eta^{\nu^q}_e(p_\parallel, p_\perp) - \chi^{\nu^q}_e(p_\parallel, p_\perp) F_e(p_\parallel, p_\perp) \right) + S(p_\parallel, p_\perp, E),$$

where $\eta^{\nu^q}_e = 2\pi \epsilon_e p_\perp \eta^{\nu^q}_e$, $\chi^{\nu^q}_e$ are the emission and absorption coefficients due to the interaction denoted by $q$, and the source term $S$ is the rate of pair production. The sum over $q$ covers all the 2-particle QED interactions considered in this work. In particular the electron-positron DFs in Eq. (2.88), varies due to the acceleration by the electric field, the creation of pairs due to vacuum breakdown and the interactions. Indeed, the Vlasov term describes the mean field produced by all particles, plus the external field. In our approach particle collisions, including Coulomb ones, are taken into account by collision terms. Particle motion between collisions is assumed to be subject to external field only, and the mean field is neglected. This is an assumption, but in dense collision dominated plasma such as the one considered in this paper this assumption is justified, see e.g. [6]. The rate of pair production already distributes particles in the momentum space according to Eq. (2.40) we report below

$$S(p_\parallel, p_\perp, E) = \frac{|eE|}{m_e^3 (2\pi)^2} \epsilon p_\perp \cdot \log \left[ 1 - \exp \left( -\frac{\pi (m_e^2 + p_\perp^2)}{|eE|} \right) \right] \delta(p_\parallel).$$

For $E < E_c$ this rate is exponentially suppressed. Besides, Eq. (2.90) already indicates that pairs are produced with orthogonal momentum, up to about $m_e (E/E_c)$ but at rest along the direction of the electric field.

The Boltzmann equation for photons is taken from Eq. (2.48) substituting $f_\gamma$ with the corresponding one given by Eq. (2.86), hence giving

$$\frac{\partial F_\gamma(p_\parallel, p_\perp)}{\partial t} = \sum_q \left( \eta^{\nu^q}_\gamma(p_\parallel, p_\perp) - \chi^{\nu^q}_\gamma(p_\parallel, p_\perp) F_\gamma(p_\parallel, p_\perp) \right),$$
and their DF changes due to the collisions only. In more detail, photons must be produced first by annihilating pairs, then they affect the electron-positron DF through Compton scattering. Besides, also photons annihilation into electron-positron pairs becomes significant at later times. Eqs. (2.90) and (2.88) are coupled by means of the collision integrals, therefore they are a system of partial integro-differential equations that must be solved numerically. Efficient methods for solving such equations in optically thick case was developed in [15] and later generalized in [13, 14].

It is well known [82, 47] that both acceleration and pair creation terms in Eq. (2.88) operate on a much shorter time-scale than interactions with photons described by collision terms in Eqs. (2.88) and (2.90). For this reasons we run two different classes of simulations, one neglecting collision integrals which is referred to as “collisionless” and another one including them called “interacting”. Before doing that we now give the initial conditions from which we start solving Eqs. (2.88) and (2.90) as well as some quantities that will turn to be useful in interpreting our results.

The boundary condition is set when the initial electric field \( E_0 \) and the initial DF \( F_{\nu 0}(p_\|, p_\perp) \) are specified. For simplicity we performed several runs with different initial electric fields, but always with no particles at the beginning. When in addition to external electric field also particles are present from the beginning, oscillations still occur, but with higher frequency, as given by the plasma frequency Section 2.3. So we start computations with DFs null identically in the whole momentum space and our initial conditions can be written as

\[
\begin{cases}
E_0 = \xi E_c, \\
F_{\nu 0}(p_\|, p_\perp) = 0, \\
p_\perp \in [0, +\infty), \\
p_\| \in (-\infty, +\infty).
\end{cases}
\]

Consequently, at the very beginning electrons and positrons are produced exclusively by the Schwinger process.

In order to interpret meaningfully our results, we introduce first some useful quantities. Initially the energy is stored in the electric field and it fixes the energy budget available as given by

\[
\rho_0 = \frac{E_0^2}{8\pi}.
\]

We expect therefore the final state of the equilibrated thermal electron-
positron-photon plasma to be characterized by the temperature

\[ T_{eq} = \sqrt[4]{\frac{\rho_0}{4\sigma}} \simeq 1.7 \sqrt{\frac{E_0}{E_c}} \text{ MeV}, \]

where \( \sigma \) is the Stefan-Boltzmann constant. The total energy density of pairs \( \rho_\pm \) and photons \( \rho_\gamma \) are related to the actual and initial electric fields, \( E \) and \( E_0 \), by the energy conservation law

\[ \rho_\pm = \rho_+ + \rho_- = \frac{E_0^2 - E^2}{8\pi} - \rho_\gamma. \]

Following [83], we define the maximum achievable pairs number density

\[ n_{max} = \frac{E_0^2}{8\pi m_e}, \]

which corresponds to the case of conversion of the whole initial energy density into electron-positron rest energy density

\[ \rho_{\pm \text{rest}} = (n_- + n_+) m_e, \]

where \( n_- \) and \( n_+ \) are the electrons and positrons number densities respectively. From the electrons and positrons DFs we can extrapolate their bulk parallel momentum \( \langle p_\parallel \rangle \) and the symmetry of our problem implies that \( \langle p_{\parallel -} \rangle = -\langle p_{\parallel +} \rangle \). We make use of this identity to define the kinetic energy density of pairs

\[ \rho_{\pm \text{kin}} = \rho_{\pm \text{rest}} \left( \sqrt{\left( \frac{\langle p_\parallel \rangle}{m_e} \right)^2 + 1} - 1 \right). \]

Therefore \( \rho_{\pm \text{kin}} \) is the energy density as if all particles are put together in the momentum state with \( p_\parallel = \langle p_\parallel \rangle \) and \( p_\perp = 0 \) while their rest energy density is \( \rho_{\pm \text{rest}} \). The difference between the total energy density and all the others defined above is denoted as internal energy density

\[ \rho_{\pm \text{in}} = \rho_\pm - \rho_{\pm \text{rest}} - \rho_{\pm \text{kin}}. \]

The term "internal" refers here to the dispersion of the DF around a given point with coordinates \( (\langle p_\parallel \rangle, \langle p_\perp \rangle) \) in the momentum space.
2.4. Collisionless evolution

Since interactions with photons operate on much larger time scale than the pair creation by vacuum breakdown we first present the results obtained solving the relativistic Boltzmann equation (2.88) for electrons and positrons with $\chi^q_{\pm} = \eta^q_{\pm} = 0$. With these assumptions we expect the results to be closely related to those reported in Section 2.3.

For all the explored initial conditions, there are important analogies between the approach adopted in Section 2.3 and the one presented in this work. For each initial field the first half period of the oscillation $t_1$ is nearly equal to the corresponding one obtained in Section 2.3. Also the evolution with time of $\langle p_{\parallel} \rangle_{\pm}$ during this time lapse is very similar to the result given by their analytic method. The time evolution of electric field $E$ and $\langle p_{\parallel} \rangle_{\pm}$ in Compton units with $t_c = 1/m_e$ are shown in Figure 2.7 for $E_0 = 30 E_c$.

In addition to these similarities new important features emerge from the current study. The manifestation of these new aspects is represented in Figure 2.8 where we show how the various forms of $e_{\pm}$ energy defined in the previous Section evolve with time. These energy densities are normalized to the total initial energy density $\rho_0$ defined by Eq. (2.91). One of the most important evidences of this figure is that the rest energy density of pairs $\rho_{\pm \text{kin}}$ saturates to a small fraction of the maximum achievable one. This is in contrast with the result presented in Section 2.3 where the value given in Eq. (2.94) was reached asymptotically.

As a consequence the energy is mainly converted into other forms: the kinetic $\rho_{\pm \text{kin}}$ and internal $\rho_{\pm \text{int}}$ ones. Both these quantities oscillate with the same frequency but with shifted phase. Relative maxima and minima of $\rho_{\pm \text{kin}}$ correspond to the peaks of the bulk parallel momentum shown in Figure 2.7 as can be grasped from its definition in Eq. (2.96). Looking at Figure 2.8 we see their relative importance changing progressively with time. Even if they oscillate, the internal component dominates over the kinetic one as time advances. This trend points out that all the initial energy will be converted mostly into internal energy, while the contribution of the kinetic one will eventually be small.

In this respect, from the electron and positron DFs we obtain the mean squared values of the parallel and orthogonal momentum using Eqs. (1.42)-(1.44). These quantities give us some insight about the spreading of the DF along the parallel and orthogonal components of the momentum. In Table 2.1 we report their values at the end of runs with different initial fields. It is clear that the larger the initial electric field the larger is $\langle p^2_{\perp} \rangle_{\pm}$. This is a direct consequence of the rate of pair production given by Eq. (2.90) that already distributes particle along the orthogonal direction in the momen-
2.4. PHASE SPACE EVOLUTION TOWARD THERMALIZATION

Figure 2.7: Evolution of electric field $E$ and pairs bulk parallel momentum $\langle p_\parallel \rangle_{\pm}$ obtained from the numerical solution of Eq. (2.88) setting $E_0 = 30 E_c$.

The mean squared value of the parallel momentum $\langle p_\parallel^2 \rangle_{\pm}$ reaches a minimum value between 3 and 10 critical electric fields. This minimum was first found in [47], see Figure 3 in that paper. In Table 2.1 we report also $\langle p_\parallel \rangle_{\pm1}$ which is the peak value of the bulk parallel momentum at the moment when electric field vanishes for the first time. We see from the table that also this quantity has a minimum in the same range of initial fields as $\langle p_\parallel^2 \rangle_{\pm}$. Both these minima are linked to the combined effects of pairs creation and acceleration processes.

However, it is important to compare $\langle p_\parallel^2 \rangle_{\pm}$ and $\langle p_\perp^2 \rangle_{\pm}$ for different initial fields. Indeed, this juxtaposition gives us quantitative informations about the anisotropy of the DFs in the phase space. Looking at the numerical values we observe how this anisotropy decreases with the increase of the initial electric field, which points out how an eventual approach toward isotropy, and therefore thermalization, would be much more difficult for lower initial fields.
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Figure 2.8: Evolution with time of the pairs energies as defined by Eqs. (2.93), (2.95), (2.96) and (2.97) for the collisionless case $E_0 = 30E_c$. All of them are normalized by the total initial energy $\rho_0$ given by Eq. (2.91).

In Table 2.1 we compare also two different number densities $n_1$ and $n_s$ normalized to the maximum achievable one. The first is the number density of pairs at the first zero of the electric field $t_1$. The second is the saturation number density of pairs at the end of the run. We found that the values of $n_1$ are very close to the same densities computed in Section 2.3 with a significant amount of pairs produced already in a very small time lapse. Let us note that there are maxima of both $n_1$ and $n_s$ in the range between 1 and 10$E_c$ in correspondence with minima of $\langle p_\parallel^2 \rangle_{\pm}$ and $\langle p_\perp^2 \rangle_{\pm}$.

2.4.3 Collisional evolution

Now we turn to the dynamics of our system on much larger time scales. As discussed above, in long run interactions between created pairs become important. We consider the following 2-particle interactions: pair
Table 2.1: Square root of the mean squared value of orthogonal $\langle p^2_\perp \rangle_{\pm}$ and parallel $\langle p^2_\parallel \rangle_{\pm}$ momentum, parallel momentum $p^1_\parallel$, in units of $m_e$, and number density $n_1$ of pairs at the first zero of the electric field, saturation number density $n_s$ normalized by the maximum achievable one given by Eq. (2.94) for different initial electric fields.

<table>
<thead>
<tr>
<th>$E/E_c$</th>
<th>$\sqrt{\langle p^2_\perp \rangle_{\pm}}$</th>
<th>$\sqrt{\langle p^2_\parallel \rangle_{\pm}}$</th>
<th>$\langle p^1_\parallel \rangle_1$</th>
<th>$\frac{n_1}{n_{max}}$</th>
<th>$\frac{n_s}{n_{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
<td>75</td>
<td>160</td>
<td>0.006</td>
<td>0.018</td>
</tr>
<tr>
<td>3</td>
<td>0.8</td>
<td>37</td>
<td>82</td>
<td>0.018</td>
<td>0.037</td>
</tr>
<tr>
<td>10</td>
<td>1.3</td>
<td>35</td>
<td>77</td>
<td>0.013</td>
<td>0.041</td>
</tr>
<tr>
<td>30</td>
<td>2.0</td>
<td>87</td>
<td>192</td>
<td>0.005</td>
<td>0.016</td>
</tr>
<tr>
<td>100</td>
<td>3.5</td>
<td>127</td>
<td>284</td>
<td>0.003</td>
<td>0.011</td>
</tr>
</tbody>
</table>

creation and annihilation, Compton, Bhabha and Möller scatterings. As a consequence, we describe them by the collision integrals in Eqs. (2.88) and (2.90) using the same range of initial fields used for the collisionless systems. More sparse computational grid is used as calculation of collision terms imply performing multidimensional integrals in the phase space.

The larger the electric field the higher the rate of pairs production and consequently their number density. Since the interaction rate is proportional to particle number densities, we expect them to be important sooner for higher initial field. In this respect, it is worth mentioning that in Section 2.3 the time $t_\gamma$ was estimated at which the optical depth for electron-positron annihilation equals unity $\tau(t_\gamma) = 1$. There, it was found that $t_\gamma$ decreases when the initial electric field increases. Besides, the order of magnitude of their estimations is in agreement with the time at which the photons number density is around a few percent of the pairs number density.

In the previous Subsection dedicated to collisionless systems, we described the anisotropy of the pairs DF by means of mean squared values of parallel and orthogonal momentum reported in Table 2.1. In that case, we knew approximately the range of orthogonal momentum in which the most part of electrons and positron were located and the orthogonal grid was chosen and kept fixed from the beginning. This choice was possible because the dispersion along the orthogonal direction was determined uniquely by the rate $S$. The extension of the grid was chosen in such a way
that the value of each DF at the grid boundaries was small compared to the maximum value. For the reason that interactions redistribute particles in the phase space and tend to isotropize their distributions, the orthogonal grid must be extended to values comparable to the kinetic equilibrium temperature. To do that, we use initially an orthogonal grid with the same extension as in the collisionless system. We extend it later when particles are scattered toward higher orthogonal momenta and therefore the tails of the DF at the boundaries is not negligible. The extension of the parallel grid remains essentially the same as the collisionless case.

In order to correctly describe the pairs acceleration process, the time step of the computation must be a small fraction of their oscillation period. This constraint prevents us to study the evolution up to the kinetic equilibrium within a reasonable time. After hundreds of oscillations, the energy density carried by the electric field is a small fraction of the pairs and photons energy densities. In other words, most energy has already been converted into electron-positron plasma. Due to this fact, the acceleration of electrons and positrons does not affect their DFs appreciably. This
allows us to neglect the presence of the electric field hereafter. To do that we use the distribution function at this instant as initial condition for a new computation in which the condition $E = 0$ is imposed. By neglecting oscillations induced by the electric field the constraint on the time step of the numerical calculation is released, and it is now determined by the rate of the interactions.

In Figure 2.9 we show the time evolution of the pairs and photons energy densities for $E_0 = 100 E_c$. From this plot we can understand the hierarchy of time scales associated to the distinct physical phenomena we are dealing with. In presence of an overcritical electric field, electron-positron pairs start to be produced in a shortest time according to Eq. (2.90). As soon as they are created, electrons and positrons are accelerated toward opposite directions as the back reaction effect on the external field. The characteristic duration of this back reaction corresponds approximately to the first half oscillation period. At early times, even after many oscillations, the energy density of photons is negligible compared to that of pairs, meaning that interactions do not play any role. Such a starting period, during which the real system can be considered truly collisionless, exists independently on the initial electric field even if its duration depends on it. From Figure 2.9 it is clear that the photons energy density increases with time as a power law approaching the pairs energy density.

Only when hundreds oscillations have taken place, interactions start to affect the evolution of the system appreciably and can not be neglected any further. The slope of the photons curve in Figure 2.9 reduces indicating that pairs annihilation has become less efficient than the photons annihilation process. Now the evolution of the system is mostly governed by interactions. Möller, Bhabha and Compton scatterings give rise to momentum and energy exchange between electron, positron and photon populations. Besides the same collisions have the tendency to distribute particles more isotropically in the momentum space. After some time, the photons energy density becomes equal and then overcomes the pairs energy density. This growth continues until the equilibrium between pairs annihilation and creation processes is established $e^- e^+ \leftrightarrow \gamma \gamma$. For this reason both pairs and photons curve are flat on the right of Figure 2.9, see also [13]. However, at this point the DF is not yet isotropic in the momentum space indicating that the kinetic equilibrium condition is not yet satisfied. In fact, kinetic equilibrium is achieved only at later times when also Möller, Bhabha and Compton scatterings are in detailed balance. At that time the electron-positron-photon plasma can be identified by a common temperature and nonzero chemical potential. This is also the last evolution stage attainable by our study because only 2-particle interactions are taken into
Figure 2.10: Phase space distributions of electrons (left column) and photons (right column) for the initial condition $E_0 = 100E_c$. Top: $2.3t_c$, middle: $2.3 \cdot 10^2t_c$, bottom: $4.6 \cdot 10^6t_c$. 
2.4. PHASE SPACE EVOLUTION TOWARD THERMALIZATION

account while thermalization is expected to occur soon after kinetic equi-
librium if also 3-particle interactions would be included [14].

As an example, in Figure 2.10 density plots of $f_-$ and $f_\gamma$ are shown on
the left and right columns respectively, for the initial condition $E_0 = 100$. Their time evolution starts from the top line to the bottom one correspond-
ing to three different times. After $2.3 t_c$ both DFs are highly anisotropic as it is well established by the ratio
\[ R = \sqrt{\langle p^2_\parallel \rangle_\pm / \langle p^2_\parallel \rangle_\pm} = 0.06. \]
At this stage, the electric field is highly overcritical and a very small fraction of initial energy has been converted into rest mass energy of electrons and positrons. For this reason $e^-e^+$ are easily accelerated up to relativistic velocities explaining why the electron DF is shifted on the right side of the phase space plane characterized by $p_\parallel > 0$. At this instant electrons are characterized by a relativistic bulk velocity corresponding to a Lorentz factor 170. On the second line the time is $2.3 \cdot 10^2 t_c$ and the DFs are still anisotropic if we look at the parameter $R$ introduced above. However the situation is different with respect to the previous stage because the electric field is only slightly overcritical and many more pairs and photons have been generated. As a consequence interaction rates are much larger than before and an efficient momentum exchange between electron and positron populations occurs. Both small electric field and collisions pre-
vent particles to reach ultra-relativistic velocities and for this reason the electron DF is now perfectly symmetric with respect to the plane $p_\parallel = 0$. Only later on, at $4.6 \cdot 10^6 t_c$ for the bottom line, collisions dominate the evo-
lution of the system whereas the presence of the electric field can be safely neglected. The pictures show a prominent DFs widening toward higher orthogonal momenta which is confirmed by the value $R \simeq 0.23$. This re-
markable evidence allows us to predict the forthcoming fate of the system to be an electron-positron-photon plasma in thermal equilibrium. The DFs isotropy in the momentum space not only indicates that the kinetic equi-
librium condition is approached but also that the system is going to lose information about the initial preferential direction of the electric field. In the case of isotropic DF, the timescale on which thermal equilibrium is achieved can be estimated as $\tau_{th} \simeq 1/(n\sigma_T)$ [14]. For our anisotropic DF the thermalization timescale is remarkably longer.
Chapter 3

Transparency of relativistic outflows

Emission from optically thick stationary plasma is an important topic in astrophysics. Such plasma confined by the gravitational field constitutes stars, accretion disks and other objects. The light from these systems is coming from the so called photosphere defined as a region where the optical depth computed from the interior of the optically thick plasma outwards reaches unity.

There are also dynamical sources where bulk velocities of plasma reach ultrarelativistic values such as microquasars [91, 92], active galactic nuclei (AGNs) [93] and gamma-ray bursts (GRBs) [94]. While in the former two objects there is clear evidence for jets which contain optically thin plasma, in the latter objects the issue of jets is controversial, and the source is required to be optically thick. This observational fact poses a new problem: the emission from (spherically) expanding plasma which initially is optically thick, for a review see [95]. Such plasma eventually becomes optically thin during its expansion, and initially trapped photons should be released.

In this Chapter we briefly introduce the GRBs phenomenon from the observational point of view summarizing some established facts. Then we describe the problematics related to photospheric emission and the current theoretical approaches devoted to its study. We also list the motivations that stimulated our interests of studying the transparency of a spherically symmetric relativistic outflow by means of a kinetic approach. Finally we describe the method and the results we have obtained.
3.1 Gamma-ray bursts

3.1.1 Main features

GRBs are catastrophic astrophysical events occurring at cosmological distances, every two or three days on average, and with isotropic distribution in the sky. They are detected by satellites orbiting around the Earth as short and bright flashes of gamma-rays. Their observed duration $T_{90}$ goes from few milliseconds to tens of minutes and the associated photon energy flux varies in the range $10^{-7}$-10$^{-4}$ ergs/cm$^2$ [24]. GRBs parameters make them the most luminous objects in the gamma-ray sky. Knowing redshift and detected fluence one can calculate the emitted isotropic energy to be $10^{51}$-10$^{55}$ ergs making GRBs the most energetic events in the Universe as well. In most cases an afterglow is detected after a prompt emission and several days after some of them a supernova is also observed at the same position in the sky.

There are other crucial and well established facts about GRBs that can be grasped from observational data:

- **Ultra-relativistic motion.** GRBs spectra peak in the MeV range and, in many cases, their high energy tail reaches several GeV. If photons are emitted by a region which is stationary with respect to the observer, most of the photons would have an energy above their annihilation threshold into electron-positron pairs. As a consequence optical depth would increase so much that photons could not reach the observer. The way out to this contradiction is the ultra-relativistic motion of the outflow in which photons scatter for the last time.

- **Compactness.** Time variability $\delta t$ of about 10 milliseconds implies the source size to be smaller than $\delta t / c \approx 3000$ Km.

- **Large initial optical depth.** From variability, distance and fluence one can calculate the initial average photon optical depth to be $\tau_i \approx 10^{13}$ [96] meaning that no radiation can escape from this region of space.

Looking at their “zoo” of light curves in Figure 3.1 GRBs do not really seem to have a common origin. Spectra are never purely thermal, though time-resolved, and vary a lot from burst to burst. The so called Band

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1In the GRB community the $T_{90}$ is defined as the time interval over which 90% of the total background-subtracted counts are observed, with the interval starting when 5% of the total counts have been observed.
Figure 3.1: Diversity of "zoo" GRBs light curves detected by BATSE [97].
model [98] is a phenomenological analytical expression which is very successful in fitting most of the observed time-integrated spectra, but, unfortunately, there is no physical explanation for its own parameters. A detailed time-resolved spectra analysis is not always possible due to the low number of detected photons and statistical significance puts strong limits on the minimum time interval of data from which a spectrum can be extracted.

The first GRB was detected by the Vela satellite in 1967 and, since the first publication in 1973 [99], a lot of studies have been carried out to understand the full sequence of involved physical processes. However, it is not yet clear whether their progenitor is due to collapsar, magnetar, BH-BH, BH-NS, NS-NS mergers or BH formation. There is also uncertainty about the outflow composition which can be leptonic, hadronic or poynting-flux dominated and it can have a spherical or jet-like geometry. Possible sub-photospheric dissipation mechanisms, such as internal shocks, that should transform fraction of the outflow kinetic energy into radiation are not well understood and their efficiency seems to be limited. Comptonization, magnetic reconnection, pairs production are the radiative processes occurring at the photosphere that would be able to broaden the Planckian spectrum, but their role and importance are poorly understood.

### 3.1.2 Photospheric emission

The importance of photospheric component in GRBs spectra has been revived in the last decade. Instruments onboard of satellites orbiting around the Earth have been improved allowing a better time resolution as well as an unprecedented spectral energy coverage. In fact the Large Area Telescope (LAT) [100] and the Gamma-ray Burst Monitor (GBM) [101] onboard of the Fermi Gamma-ray Space Telescope is particularly suitable for such purposes. The key result of such detections is that the photospheric component is present in prompt emission of many GRBs [102]. For several bursts the radiation coming from the photosphere is only a small fraction of the total detection. However, in a very few events the photospheric component dominates over the non-thermal one. In that respect, GRB090902B represents a unique case of study because observations were interpreted as nearly 100% of the total energy is emitted by the photosphere [103]. A very important aspect is that detection of photospheric components in GRBs would provide unique information on physical characteristics of the outflow, and consequently on the properties of the source. In particular,
constraints on Lorentz factor \([104]\), radius at the base of the outflow \([105]\) and comoving temperature \([106]\) may be obtained.

From the theoretical viewpoint the problem of photospheric emission from GRBs has been considered already by Goodman \([107]\) and Paczynski \([108]\) in 1986, albeit in completely different frameworks. These papers laid the basis for two main models of GRBs: coined as \textit{shell} and \textit{wind} respectively. Goodman considered instantaneous creation of plasma in a given volume, while Paczynski proposed a steady generation of energy at a given boundary. In both models created plasma expands driven by the radiative pressure. The dynamics of plasma expansion, being subject to relativistic hydrodynamics, becomes similar as soon as ultrarelativistic bulk velocities are reached. The key difference in these frameworks is however, that in the shell model of Goodman expanding plasma occupies only finite region of space, and hence the model is intrinsically dynamic, while in the wind model of Paczynski expanding plasma occupies all the space, and the problem is intrinsically steady.

This difference resulted in the different treatment of the transition between optically thick and optically thin regimes in plasma expansion. In the wind model of Paczynski such transition occurs at some position which does not depend on time, this position was referred to as the ”photosphere”, in analogy with stellar photospheres. Actually, it was realized later by Abramowitz et al. \([109]\) that the shape of this surface, as it appears to a distant observer for which the source of plasma is at rest, has nothing to do with the sphere: because of relativistic effects it has a concave shape. Now the entire literature on ”photospheric emission” from GRBs refers to this wind model.

In contrast, in the shell model of Goodman the ”photosphere” is a dynamic surface. Its dynamics is, in principle, different from the dynamics of the expanding shell itself. For this reason the term ”transparency” introduced by Ruffini et al. \([110]\), which denotes transition from optically thick to optically thin regime in plasma expansion, appears more suitable and less confusing. Goodman computed the spectrum of photons assuming the photon distribution is thermal at the ”photosphere”. He realized that the relativistic radiative transfer problem is much more difficult than the approach he adopted.

Depending on the GRB origin, the hydrodynamic expansion can be very different from case to case. As an example the expanding plasma can have spherical symmetry or a jet-like structure according to the nature of the central engine. The outflow can be purely leptonic, contaminated by baryons or dominated by magnetic fields and, since the radiation is trapped in it, all this aspects can affect photospheric emission through
many different physical processes.

For example, in the fireshell model of GRBs (see e.g. the review presented in [111] and references therein), assumes that the GRBs originate from almost instantaneous creation of an electron-positron plasma in the process of vacuum polarization occurring in the formation of a black hole [112, 113]. In this model radiation pressure accelerates an optically thick electron-positron-photon plasma up to relativistic velocities. At this point it can become transparent before or after its interaction with a baryonic environment [114]. In both these cases the first radiation emitted is named proper GRB [115].

In the fireball model, see review [116], the nature of the central source is not specified, but the expanding plasma is baryon contaminated since the beginning. Also in this case there is a transition from an optically thick regime to an optically thin one when most of the radiation initially trapped in the plasma is emitted and such process is called photospheric emission.

All the models have to deal with the observational evidence of photon spectra which are significantly broader than the Planck one even though some of the detected radiation comes from the photosphere. To solve this problem optically thin models were proposed at the beginning [117, 118], nevertheless mainly the ones based on synchrotron emission were found to contradict observations [119]. There are essentially two effects which lead to broadening of the observed spectrum of photospheric emission, with respect to the black body shape, termed as “geometric” and “physical”. “Geometric” broadening of the spectrum occurs due to several reasons. Firstly, the photosphere is not a sharp surface, but is a region in space and time characterized by a probability of photons to be scattered with electrons for the last time [120, 121]: this is an analog of the last scattering surface in cosmology. Secondly, the emission arriving to an observer with given arrival time originates from different parts of the outflow, with different radial coordinates and angles. Hence the observed spectrum represents a superposition of spectra produced at these different parts of the outflow. Broadening results if dissipation of some part of kinetic energy of the outflow occurs before the plasma becomes transparent. Such dissipation may happen due to several reasons: magnetic reconnection [122], inelastic nuclear collisions [123] and shocks [124, 125]. All dissipative photospheric models are based on simple idea: if temperature of electrons is not equal to the temperature of photons Compton scattering produce distortions of the photon spectrum. Photospheric models with dissipation are mostly concerned with the part of the spectrum above its peak. However, also low energy part of the spectrum may be modified due to Compton scattering, see [126].
3.2 Kinetic approach

Up to now, several different methods have been used to compute spectra and light curves of photospheric emission. In particular integration over photospheric equi-temporal surface (PhE) \cite{127, 128}; integration over volume with attenuation factors \cite{120, 129}; approximations to the radiative transfer \cite{121, 127}; Monte Carlo simulations of photon scattering \cite{123, 130, 131, 132}; Kompaneets equation with anisotropic photon field \cite{126}. Besides, a technique based on rate equations is described in \cite{133}.

Basically all the methods mentioned above agree on one key point, namely that the photospheric spectrum has to be broader with respect to the Planck one showing an increase of the low energy slope of the spectrum. We now briefly review some of them listing their main strengths and disadvantages.

- Though very simplistic, the easiest estimation of the observed spectrum can be obtained superposing thermal spectra coming from different emitting regions at different view angles and with temperatures obtained from hydrodynamic equations. A step forward can be done using approximate solutions of the radiative transfer equations and also such methods show that the spectrum coming from a given region is no longer thermal, although the source function in the comoving frame is still assumed to be isotropic and thermal.

- Monte Carlo simulation is completely independent method and it has been used widely in the literature. Here each photon is followed in the frame where the plasma is initially at rest while it experience numerous collisions with the cross sections of Compton and isotropic scattering models, until it ceases scattering. Photons are injected in the expanding plasma well before it becomes transparent. Resulting photons constitute the final spectrum. The drawbacks of this approach are: 1) a prescribed distribution of electron component and 2) impossibility to account for stimulated emission of photons. The first limitation originates from the fact that Monte Carlo simulations need a prescribed background of electrons, and any backreaction of photons on electron distribution can be only accounted for by iterative scheme. The second limitation constrains the spectrum of photons in optically thick region to have a Wien shape, instead of the Planck one, if the model of Compton scattering is used. In addition, good statistics is required to resolve both low and high energy parts of photon spectrum implying the need of large number of photons which in turn demands long computational times.
• The Fokker-Planck approximation to the Boltzmann equation allows to take into account stimulated emission of photons. However it does not allow to account of variations in electron component. In this approach which solves partial differential equations a rather good resolution in spectrum can be achieved.

There are several reasons due to which one can understand why a kinetic treatment would be more appropriate with respect to the approaches listed above.

• It is known that interactions between particles play a significant role during the approach to transparency of a relativistic outflow. In particular, Compton scattering dominates over the other processes and it is the most efficient in changing the photon spectrum. To describe this kind of interactions, kinetic theory is the most general approach one can adopt and it allows to compute interaction rates. In particular this is true in all circumstances, included the case in which particles are not in thermal equilibrium and collision rates has to be calculated from first principles. This is of crucial importance in describing the photospheric emission in GRBs since near the photosphere photons are no longer in thermal equilibrium with the expanding plasma.

• Then the Boltzmann equation determines the time evolution for each particle DF that is the central mathematical object from which all the physical properties of the system can be extracted. This is done by means of proper integrations of the DF over the phase space. Therefore temperature, mean momenta and energy spectra of particles can be easily determined following simple averaging procedures.

• Another important aspect is that RKT allows us to study a given physical event in an arbitrary reference frame. This feature becomes necessary when a “comoving”reference frame cannot be properly defined and this is exactly the case when a relativistic outflow becomes transparent to photons. By definition, a reference frame is comoving with respect to an ensemble of particles if the mean momentum of the system is zero. This is possible when an outflow expands but photons are trapped in it making the bulk velocity of photons equal to the bulk velocity of the plasma. However, during the transition from opaque to transparent plasma, photons decouple from electrons. Photons carry energy and momentum and electrons recoil at the last scattering of photons. This effect can be taken into account only within kinetic treatment in the laboratory reference frame.
3.2. KINETIC APPROACH

• Old previously discussed approaches assume electrons having thermal distribution at a temperature given by the adiabatic cooling condition. In the kinetic approach both photon and electron DFs are followed self-consistently through solution of coupled Boltzmann equations.

So Compton scattering of photons can be followed from high optical depth regions to low optical depth ones. The main problematic here is that the system of coupled Boltzmann equations for electrons and photons has to be solved numerically. As finite difference methods are involved when calculating particle interactions, the only limitation in this approach is, depending on the available computational power, the size of the grid in the phase space.

3.2.1 Transparency of a relativistic outflow

In this Section we introduce the theory of photospheric emission from relativistic outflows following the approach present in [127]. The same methodology will be used in the reminder of this Chapter to set up proper initial conditions for our computations. We define the 4-velocity of the outflow in the laboratory frame as

$$U^\mu = c \Gamma (1, \beta) = c \Gamma (1, \beta, 0, 0),$$

where $\beta c$ is the radial velocity of the outflow with respect to the laboratory frame and the Lorentz factor is related to this velocity by the relation $\Gamma = 1/\sqrt{1 - \beta^2}$.

Let us consider a spherically symmetric wind of finite duration $\Delta t$ which emerges from the source with radius $R_0$, luminosity $L(t) = dE/dt = \text{const}$ and mass ejection rate $\dot{M} = dM/dt = \text{const}$. It is already known that under the condition that the bulk Lorentz factor $\Gamma \gg 1$ there is an approximate analytic solution of relativistic hydrodynamic equations [108, 134]. This solution can be divided in two phases: accelerating and coasting ones. At accelerating phase

$$\Gamma \approx r/R_0, \quad R_0 < r < \eta R_0,$$

while at coasting phase

$$\Gamma \approx \eta = \text{const}, \quad r > \eta R_0,$$

where $r$ is the radius and $\eta = L/(\dot{M}c^2)$ is dimensionless entropy [135].
Continuity equation implies for laboratory density of baryons
\begin{equation}
(3.4) \quad n = n_c \Gamma \simeq n_0 \left( \frac{r}{R_0} \right)^{-2}, \quad R(t) \leq r \leq R(t) + l,
\end{equation}
where $n_c$ is the comoving number density of baryons, $n_0$ is its value at the base of the wind, $l \simeq c \Delta t$ and $R(t) = \beta ct + l$. The solutions of relativistic hydrodynamic equations (3.2), (3.3), and (3.4) are also valid in thin shell model of fireball \cite{136, 116} and in the asymptotic regimes for the fireshell model provided that $\Gamma \gg 1$.

Since we know the number density of particles with which photons interact, we can now compute the optical depth within the outflow in order to obtain the shape and position of the photosphere. The optical depth along the light-like world line $\mathcal{L}$ can be defined in \cite{12} as
\begin{equation}
(3.5) \quad \tau = \int_{\mathcal{L}} \sigma j_\mu dx^\mu,
\end{equation}
where $\sigma$ is cross section, $j_\mu$ is the 4-current of particles, and $dx^\mu$ is the element of the world line. Therefore the optical depth (3.5) can be defined in any reference frame.

Consider the light-like world line starting at time $t$ at the interior boundary $r = R$ of the outflow and directed outwards. The optical depth given by equation (3.5) is then (see e.g. \cite{109, 137})
\begin{equation}
(3.6) \quad \tau = \int_{R}^{R + \Delta R} \sigma n \left( 1 - \beta \cos \theta \right) \frac{dr}{\cos \theta},
\end{equation}
where $\beta \simeq 1 - 1/2 \Gamma^2$, $r$ is used as a parameter along the world line, $R + \Delta R$ is the radial coordinate at which the world line crosses the outer boundary of the outflow, and $\theta$ is the angle between the world line and the velocity vector of the outflow, $n$ is the laboratory number density of electrons\footnote{If positrons are also present in the outflow, their contribution has also to be added.}. The quantity $\Delta R$ is found from the equation of motion of the outflow. In particular, for outflow expanding with constant velocity (coasting phase) one has from (3.3) for its outer boundary $r(t) = \beta ct + l$. For radially directed light-like world line $r(t) = ct$. Equating these expressions, we find $ct = \Delta R = l/(1 - \beta) \simeq 2 \Gamma^2 l$.

When contribution of positrons to (3.6) can be neglected, the density of electrons coincides with density of baryons given by equation (3.4). Hence using (3.2) and (3.3) and assuming $\sigma = \text{const}$, three asymptotic expres-
sions for the optical depth along the line of sight (LOS) are recovered from (3.6): respectively, one for accelerating phase and two for coasting phase, namely

\[
\tau = \begin{cases}
\frac{1}{6} \tau_0 \left( \frac{R_0}{R} \right)^3, & R_0 \ll R \ll \eta R_0, \\
\frac{1}{2\eta^2} \tau_0 \left( \frac{R_0}{R} \right), & \eta R_0 \ll R \ll 2\eta^2 l, \\
\frac{\tau_0 R_0 l}{R^2}, & R \gg 2\eta^2 l,
\end{cases}
\]

where the initial optical depth is

\[
\tau_0 = \sigma n_0 R_0 = \frac{\sigma E_0}{4\pi m_p c^2 R_0 \eta} = \frac{\sigma L}{4\pi m_p c^3 R_0 \eta},
\]

where \( E_0 \simeq L \Delta t \) is the total initial radiative energy of the outflow and \( m_p \) is the proton mass. It turns out that this definition of \( \tau_0 \) implies that at \( \eta R_0 \) the energy density of photons computed in the laboratory frame exactly coincides with the kinetic energy of particles. This clearly indicates that the conversion of radiation energy starts to be converted into kinetic energy of particles by means of their relativistic motion becomes substantial at this radius.

By definition, the photospheric radius \( R_{ph} \) is obtained by equating the optical depth to unity

\[
R_{ph} = \begin{cases}
R_0 \left( \frac{\tau_0}{6} \right)^{1/3}, & \tau_0 \ll \eta^3, \\
R_0 \frac{\tau_0}{2\eta^2}, & \eta^3 \ll \tau_0 \ll \eta^4 l/R_0, \\
(\tau_0 R_0 l)^{1/2}, & \tau_0 \gg \eta^4 l/R_0.
\end{cases}
\]

In [127] the authors give a very interesting physical interpretation of the different asymptotic expressions in Eqs. (3.7) for the optical depth, renaming them in relation with the time spent by the photon within the outflow. On one hand, the first two equations in (3.7) imply that the radial distance \( \Delta R \) is sufficiently large so that the number density substantially decreases between \( R \) and \( R + \Delta R \). In this respect the outflow is a “thick wind”, even if the laboratory thickness of the outflow may be small, \( l \ll R \). They refer to this case as a photon thick outflow. On the other hand, the last equation in (3.7) implies that the number density of the outflow does
not change substantially along the radial distance $\Delta R$. In this respect the outflow is a “thin shell” even if the duration is long $\Delta t \gg R_0/c$. They refer to this latter case as a \textit{photon thin outflow}.

In the next Sections we are going to consider only a photon thick outflow during its coasting phase of expansion. We focus our attention on this specific asymptotic case because it is one in which the spectrum of the photospheric emission is found to deviate at the largest extent from the Planck spectrum.

### 3.3 Assumptions

Here we define our main assumptions in order to follow each particle DF evolution with time when a relativistic outflow becomes transparent.

A key point of our method is that we aim to describe the whole system in the laboratory reference frame. As we have already mentioned at the beginning of Section 3.2, this is due to the impossibility of defining a proper comoving reference frame during the phase of decoupling between electrons and photons.

In order to simplify our treatment, we take the physical space to be spherically symmetric. This means that physical parameters of the outflow such as number and energy densities, temperature and Lorentz factor can be functions of the radial coordinate only. Still, there is much freedom in choosing the radial profiles for the physical quantities one is interested in. Our initial radial profiles will be the ones corresponding to a finite relativistic wind. In particular, considering a photon thick outflow during its coasting phase we take the Lorentz factor to be constant within the width of the outflow and much larger than unity.

Analogously to the study of pair production in strong electric fields, we assume the momentum space to be axially symmetric. In this case the parallel momentum will be aligned along the radial direction of motion while the perpendicular component will be orthogonal to it. It is then clear that the dependence of the DF on the parallel momentum will be very much related to the Lorentz factor. On the other hand the dependence on the orthogonal component will be linked with the comoving temperature. The reason for using momentum component coordinates in the phase space is very much related to the problem we are going to study. In the literature spherical coordinates are usually adopted to model the phase space, however this becomes a problem if the numerical grid is kept constant and $\Gamma$ becomes large. In such case the main part of the distribution function tends to occupy a very small region in the momentum space making its
3.3. ASSUMPTIONS

Evolution with time very hard to follow. Such problematic would manifest also in the case in which the distribution function is strongly anisotropic. Our choice of coordinates allows for better representation of the DFs in momentum space, as the DF of photons occupies large part of the grid and it is more uniform, see Figure 3.4 below.

Figure 3.2: In this picture the radial direction of motion is aligned with the $x$-axis, therefore the 3-velocity of the outflow $\beta$ is parallel to this direction. The angle between the 3-momentum $p$ and $\beta$ has been called $\alpha$.

In GRBs the relativistic expanding plasma is composed by photons, electrons, positrons and protons in such proportions that the system has no net electric charge. Since we are going to start our computations with non-relativistic comoving temperatures, electron-positron number density is exponentially suppressed and their presence can be safely neglected. Then the main contribution to the outflow optical depth is given by the electrons associated to baryons. Moreover we know that in coasting expansion all protons move with nearly the same momentum and their DF is peaked around the corresponding Lorentz factor. Being much more massive that the electron, the proton DF cannot be affected by interactions with photons or electrons and therefore we can avoid to solve the corresponding Boltzmann equation. Therefore we show in the next Sections that the
approach to transparency of a relativistic outflow can be investigated considering only electrons and photons. In this picture, electrons will interact with photons via Compton scattering, therefore responsible for the optical depth, and they will carry the kinetic energy of the outflow as well.

We choose to start our calculations at relatively large optical depths. This allows to follow decoupling of photons starting with thermal DFs with the temperature given by the adiabatic cooling condition.

The geometrical relations between cartesian coordinates, particle momentum components $p$ and the radial velocity $\beta c$ is depicted in Figure 3.2. In this picture the $x$-axis is aligned along the radial coordinate of our spherically symmetric physical space. In the same figure we also call $\alpha$ the angle between the radial direction of motion of the outflow and the 3-momentum of the particle. With the chosen coordinates in the phase space the 4-momentum of the considered particle becomes

\begin{equation}
    p^\mu_k = \left( \frac{E_k}{c}, p, \frac{E_k}{c} \beta, p_\perp \cos \phi, p_\perp \sin \phi \right),
\end{equation}

where the energy is $E_k = c \sqrt{p^2 + m_k^2 c^2}$ and $m_k$ is the mass of the particle of kind $k$.

Particles are initially assumed to be in thermal equilibrium. This has been proven to be true if the initial optical depth of the plasma is large. If so the DF in thermal equilibrium for a particle of kind $k$ is given by Eq. (1.10) where the sign depends on the quantum statistics this particle obey. In the laboratory frame we have that 4-dimensional scalar product to be

\begin{equation}
    p^\mu_k U_\mu = \Gamma \left( E_k - \beta p_\parallel c \right),
\end{equation}

Eq. (1.10) can be rewritten specifying the dependence of the DF on the radial coordinate and the two momentum components as follows

\begin{equation}
    f_k(p_\parallel, p_\perp, r) = \frac{1}{h^3} \left[ \exp \left( \frac{\Gamma \left( E_k - \beta p_\parallel c \right)}{k_B T} - \phi_k \right) \pm 1 \right]^{-1},
\end{equation}

where the sign (+) corresponds to electrons, while the sign (−) corresponds to photons. In principle also Lorentz factor, comoving temperature $T$ and chemical potential can depend on the radial coordinate. Eq. (3.12)
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satisfies the following integral normalization relation

\[ n_k(r) = 2\pi \int_{-\infty}^{\infty} dp_\parallel \int_{0}^{\infty} dp_\perp f_k(p_\parallel, p_\perp, r) = \Gamma n'_k(r) , \]

where \( n_k(r) \) is the number density measured in the laboratory rest frame while \( n'_k(r) \) is the comoving number density. As a consequence the initial DFs would be

\[ f_{\Gamma\theta}^\parallel (\mu_\parallel, \mu_\perp, r) = \frac{1}{\hbar^3} \left[ \exp \left( \frac{\Gamma \left( \sqrt{\mu_\parallel^2 + \mu_\perp^2} - 1 \right) - \psi_e}{\theta} \right) + 1 \right]^{-1} , \]

\[ f_{\Gamma\theta}^\parallel (\mu_\parallel, \mu_\perp, r) = \frac{1}{\hbar^3} \left[ \exp \left( \frac{\left( \epsilon_\gamma - \beta \mu_\parallel \right) - \psi_\gamma}{\theta} \right) - 1 \right]^{-1} , \]

where we have used dimensionless units \( \mu_\parallel = p_\parallel / (m_e c) \), \( \mu_\perp = p_\perp / (m_e c) \), \( \theta = k_B T / (m_e c^2) \), \( \epsilon_k = E_k / (m_e c^2) \), \( \psi_k = \phi_k / (m_e c^2) \). For sake of completeness we report below the value of \( n'_k(r) \) for electrons, protons and photons

\[ n'_e(r) = 4\pi \theta K_2(1/\theta) , \]
\[ n'_p(r) = 8\pi \zeta(3) \theta^3 , \]

where \( K_2 \) and \( \zeta \) are the Bessel function of the second kind and the Zeta function respectively.

In analogy with the kinetic treatment we used to follow the electron-positron pair production, now we introduce a new distribution function \( F_k \) as

\[ F_k(\mu_\parallel, \mu_\perp, r) = \epsilon_k f_{\Gamma\theta}^\parallel (\mu_\parallel, \mu_\perp, r) , \]

from which the energy density of particle \( k \) in the laboratory frame turns out to be

\[ \rho_k(r) = 2\pi \int_{-\infty}^{\infty} d\mu_\parallel \int_{0}^{\infty} d\mu_\perp \mu_\perp F_k(\mu_\parallel, \mu_\perp, r) . \]
Figure 3.3: Contour plots for $F_{\Gamma \theta}(\mu_\parallel, \mu_\perp)$. Two different Lorentz factors $\Gamma = \{10, 10^3\}$ and dimensionless temperatures $\theta = \{10^{-2}, 1\}$ have been chosen. For each pair $(\Gamma, \theta)$ we have 5 contour lines corresponding to $F_{\Gamma \theta} = \zeta \max(F_{\Gamma \theta})$ for $\zeta = \{5 \cdot 10^{-1}, 10^{-1}, 10^{-3}, 10^{-6}, 10^{-9}\}$. The maximum is given by $\max(F_{\Gamma \theta}) = F_{\Gamma \theta}(\mu_\parallel^*, 0)$ where $\mu_\parallel^* \simeq \Gamma \beta$. Lines with same thickness correspond to equal Lorentz factors, the thicker the line the larger the $\Gamma$. Curves with same patterns correspond to equal dimensionless temperatures; solid curves have smaller temperatures than the dashed ones.
Figure 3.4: Contour plots for $\mathcal{F}_{\gamma}^{\Gamma \theta}(\mu_{\parallel}, \mu_{\perp})$. Two different Lorentz factors $\Gamma = \{10, 10^3\}$ and dimensionless temperatures $\theta = \{10^{-2}, 1\}$ have been chosen. For each pair $(\Gamma, \theta)$ we have 5 contour lines corresponding to $\mathcal{F}_{\gamma}^{\Gamma \theta} = \xi \max(\mathcal{F}_{\gamma})$ for $\xi = \{5 \cdot 10^{-1}, 10^{-1}, 10^{-3}, 10^{-6}, 10^{-9}\}$, where $\max(\mathcal{F}_{\gamma}^{\Gamma \theta}) = \mathcal{F}_{\gamma}^{\Gamma \theta}(0, 0)$. Lines with same thickness correspond to equal Lorentz factors, the thicker the line the larger the $\Gamma$. Curves with same patterns correspond to equal dimensionless temperatures; solid curves have smaller temperatures than the dashed ones.
We work with $F_k$ also because its maximum value is bounded in the phase space for both fermions and bosons. Indeed, we know that $f_\gamma$, being the occupation number for a gas of particles obeying to the Bose-Einstein quantum statistics, diverges for small momenta indicating the tendency to form the so called Bose-Einstein condensate. It is very easy to compute the maxima for $F_k$ in the phase space. In particular the $F_\gamma$ has a finite maximum when both parallel and orthogonal momentum components are zero. Instead, for massive particles their maximum is located in the point in the phase space with coordinates $\mu_\parallel \simeq \Gamma \beta$ and $\mu_\perp = 0$.

We present $DF\,F_k$ for electrons and photons for different temperatures and Lorentz factors in Figures 3.3 and 3.4. The contour plots show that an increase in $\Gamma$ corresponds to a shift along the $\mu_\parallel$ axis with small changes in shape. On the contrary, if the temperature increases then $F_k$ becomes wider covering a larger area in the momentum space.

### 3.3.1 Distribution function in spherical coordinates

In this Section we give two analytical formulas for electron and photon energy spectra in the laboratory reference frame with respect to which they move with bulk velocity $\beta c$. We also show that in the non relativistic limit such spectra reduce to the well known Maxwell and Planck distributions.

In alternative to momentum components, we now express the normalized DFs for electrons and photons using spherical dimensionless coordinates energy $\epsilon$ and angle $\alpha$. Then for non-degenerate electrons and photons in thermal equilibrium we have that

\begin{align}
(3.20) \quad f_e^{\Gamma \theta}(\epsilon, \alpha, r) &= \frac{n'_e(r)}{2\pi \theta K_2(1/\theta)} \exp \left[ \frac{-\Gamma \left( \epsilon - \beta \sqrt{\epsilon^2 - 1 \cos \alpha} \right)}{\theta} \right],
\end{align}

\begin{align}
(3.21) \quad f_\gamma^{\Gamma \theta}(\epsilon, \alpha, r) &= \frac{n'_\gamma(r)}{8\pi \theta^3 \zeta(3)} \left[ \exp \left( \frac{\Gamma \epsilon (1 - \beta \cos \alpha)}{\theta} \right) - 1 \right]^{-1},
\end{align}

where $\epsilon = E/mc^2$, $\alpha = \arccos(\mathbf{p} \cdot \beta/(||\mathbf{p}|\mathbf{\beta}||))$. They all fulfill the usual relativist relation between number density of particles in comoving and laboratory frame

\begin{align}
(3.22) \quad n_\nu(r) &= 2\pi \int_{m_\nu}^\infty d\epsilon \, \sqrt{\epsilon^2 - \hat{m}_\nu^2} \int_0^\pi d\alpha \, \sin \alpha \, f_\nu^{\Gamma \theta}(\epsilon, \alpha, r) = \Gamma^\nu \cdot n'_\nu(r),
\end{align}
where $\hat{m}_\nu = 0, 1$ for photons and electrons respectively.

The spectrum in energy, which gives the differential number of particles at a specified energy, is defined by the following integral

\begin{equation}
(3.23) \quad n_\nu(r) = \int_{\hat{m}_\nu}^{\infty} d\epsilon \frac{d\nu_\nu(\epsilon, r)}{d\epsilon}.
\end{equation}

As a result, it can be obtained performing the integral over $\alpha$ in Eq. (3.22) taking into account for the factors out of such an integral. This can be done analytically and the results for electrons is

\begin{equation}
(3.24) \quad \frac{d\nu_e(\epsilon, r)}{d\epsilon} = 2\pi \epsilon \sqrt{\epsilon^2 - 1} \int_0^\pi d\alpha \sin \alpha f_e^{\ell \theta} (\epsilon, \alpha, r) =
\end{equation}

\begin{equation}
(3.25) \quad = \frac{n'_e(r)}{2K_2(1/\theta) \Gamma \beta} \exp \left( -\frac{\Gamma \epsilon}{\theta} \right) \cdot
\end{equation}

\begin{equation}
\cdot \left[ \exp \left( \frac{\Gamma \beta \sqrt{\epsilon^2 - 1}}{\theta} \right) - \exp \left( -\frac{\Gamma \beta \sqrt{\epsilon^2 - 1}}{\theta} \right) \right],
\end{equation}

while for photons we obtain

\begin{equation}
(3.26) \quad \frac{d\nu_\gamma(\epsilon, r)}{d\epsilon} = 2\pi \epsilon^2 \int_0^\pi d\alpha \sin \alpha f_\gamma^{\ell \theta} (\epsilon, \alpha, r) =
\end{equation}

\begin{equation}
(3.27) \quad = \frac{n'_\gamma(r)}{4\theta^2 \zeta(3) \Gamma \beta} \left[ \ln \left( \frac{1 - \exp \left( \frac{\Gamma \epsilon (1 + \beta)}{\theta} \right)}{1 - \exp \left( \frac{\Gamma \epsilon (1 - \beta)}{\theta} \right)} \right) - \frac{2\Gamma \beta \epsilon}{\theta} \right].
\end{equation}

One can check very straightforwardly that the limit $\beta \to 0$ of Eq. (3.25) gives the correct expression for the electron energy spectrum. Indeed, in the considered limit we have that

\begin{equation}
(3.28) \quad \exp \left( \pm \frac{\Gamma \beta \sqrt{\epsilon^2 - 1}}{\theta} \right) \simeq 1 \pm \frac{\Gamma \beta \sqrt{\epsilon^2 - 1}}{\theta},
\end{equation}

which inserted in Eq. (3.25) and carrying out the algebra gives

\begin{equation}
(3.29) \quad \frac{d\nu_e(\epsilon, r)}{d\epsilon} = \frac{n'_e(r)}{\theta K_2(1/\theta)} \epsilon \sqrt{\epsilon^2 - 1} \exp \left( -\epsilon/\theta \right),
\end{equation}

which is the Maxwell energy distribution of a system of non-degenerate
3.3. ASSUMPTIONS

Electrons in thermal equilibrium. The same can be done for photons starting from Eq. (3.27). For $\beta \to 0$ we have that

$$
\frac{1 - \exp \left( \frac{\Gamma \epsilon (1 + \beta)}{\theta} \right)}{1 - \exp \left( \frac{\Gamma \epsilon (1 - \beta)}{\theta} \right)} \simeq 1 ,
$$

as a consequence the logarithm can be expanded around 1 and the result inserted into Eq. (3.27) and remembering that for the considered limit we have

$$
\exp \left( \pm \frac{\Gamma \beta \epsilon}{\theta} \right) \simeq 1 \pm \frac{\Gamma \beta \epsilon}{\theta} .
$$

Simplifying and performing the limit $\beta \to 0$ we get the following finite expression for the comoving energy spectrum of photons

$$
\frac{dn_{\gamma}(\epsilon, r)}{d\epsilon} = \frac{n'_{\gamma}(r)}{2 \theta^3 \zeta(3)} \frac{\epsilon^2}{\exp(\epsilon/\theta) - 1} ,
$$

which is the well known Planck spectrum.

3.3.2 $\mathcal{F}_\nu$ - energy spectrum

Here we give explicit integral equations for $\mathcal{F}_\nu$ in order to compute the spectra we have defined in the previous Section starting from $\mathcal{F}_\nu$ in cylindrical coordinates.

Let us define a finite energy grid identified by integer indices $m = \{1, \ldots, M\}$, energy intervals $\Delta \epsilon_m = \epsilon_{m+1/2} - \epsilon_{m-1/2}$ and cell centered value $\epsilon_m \in [\epsilon_{m-1/2}, \epsilon_{m+1/2}]$. Then, from the definition for the energy spectrum given in Eq. (3.23) we can write the number density as

$$
n_{\nu}(r) = \sum_{m=1}^{M} \int_{\Delta \epsilon_m} d\epsilon \frac{dn_{\nu}(\epsilon, r)}{d\epsilon} = \sum_{m=1}^{M} \Delta \epsilon_m \frac{dn_{\nu}(r)}{d\epsilon} m ,
$$

where the discrete value for the energy spectrum is given by

$$
\frac{dn_{\nu}(r)}{d\epsilon} \bigg|_m = \frac{2\pi}{\Delta \epsilon_m} \int_{\nu_m^b} d\mu_\parallel \int_{k_{\nu m}^a(\mu_\parallel)}^{k_{\nu m}^b(\mu_\parallel)} d\mu_\perp \frac{\mu_\perp}{\epsilon_v} \mathcal{F}_\nu(\mu_\parallel, \mu_\perp, r) ,
$$
and where the following notation has been used

\begin{align}
I^a_{vm} & = -\sqrt{\epsilon^2_{m+1/2} - \hat{m}_v^2}, \\
I^b_{vm} & = \sqrt{\epsilon^2_{m+1/2} - \hat{m}_v^2}, \\
K^a_{vm}(\mu_\|) & = \max\left(0, \sqrt{\epsilon^2_{m-1/2} - \mu_\| - \hat{m}_v^2}\right), \\
K^b_{vm}(\mu_\|) & = \sqrt{\epsilon^2_{m+1/2} - \mu_\|^2 - \hat{m}_v^2},
\end{align}

(3.35) \hspace{1cm} (3.36) \hspace{1cm} (3.37) \hspace{1cm} (3.38)

to define a ring in the phase space such that the inner radius is identified by the energy \(\epsilon_{m-1/2}\) and the outer one is identified by the energy \(\epsilon_{m+1/2}\). In discrete form, it can be rewritten as follows

\begin{align}
\left. \frac{dn_v(r)}{de} \right|_m & = \frac{2\pi}{\Delta\epsilon_m} \sum_{\mu'_l} \Delta\mu'_l \Delta\mu_{\perp k'_l} \frac{\mu_{\perp k'_l}}{\epsilon_{v_l k'_l}} \mathcal{F}_v(\mu_\|, \mu_{\perp k'_l}, r),
\end{align}

(3.39)

with the constraint

\begin{align}
\epsilon_{m-1/2} < \sqrt{\mu_\|^2 + \mu_{\perp k'_l}^2 + \hat{m}_v^2} < \epsilon_{m+1/2}.
\end{align}

(3.40)

3.3.3 \textbf{\(\mathcal{F}_v\) - angular energy spectrum}

Now we work out the general formula for the energy spectrum of particles when different viewing angles are considered.

Let us start writing the number density as

\begin{align}
n_v(r) = \int_{\hat{m}_v}^{\infty} \int_0^\pi \frac{dn_v(\epsilon, \alpha, r)}{de} \, d\alpha.
\end{align}

(3.41)

Then the discrete grid for the viewing angles is identified by integer numbers \(l = \{1, \ldots, L\}\), angle intervals \(\Delta\alpha_l = \alpha_{l+1/2} - \alpha_{l-1/2}\) with cell centered value \(\alpha_l \in [\alpha_{l-1/2}, \alpha_{l+1/2}]\). In analogy to what has been done in the previous Section we have that

\begin{align}
n_v(r) = \sum_{m=1}^M \sum_{l=1}^L \int_{\Delta\epsilon_m} \int_{\Delta\alpha_l} \frac{dn_v(\epsilon, \alpha, r)}{de} \, d\alpha = \sum_{m=1}^M \sum_{l=1}^L \Delta\epsilon_m \Delta\alpha_l \left. \frac{dn_v(r)}{de} \right|_{ml},
\end{align}

(3.42)
where the new defined angle-energy spectrum is given by

\[ \frac{d n_v(r)}{d \epsilon d \alpha} \bigg|_{ml} = \frac{2\pi}{\Delta \epsilon_m \Delta \alpha_l} \int_{\epsilon_{m-1/2}}^{\epsilon_{m+1/2}} d\epsilon \int_{\alpha_{l-1/2}}^{\alpha_{l+1/2}} d\alpha \int_{K_{vlml}(\mu_\parallel)}^{K_{vlml}(\mu_\parallel)} d\mu_\parallel \frac{\mu_\perp}{\epsilon_v} \mathcal{F}_v(\mu_\parallel, \mu_\perp, r). \]  \tag{3.43}

In the previous equation we used definitions given by Eqs. (3.35) and (3.36) plus the following ones

\[ K_{vlml}(\mu_\parallel) = \max \left( \mu_\parallel K_{vlml}^0, \sqrt{\epsilon_{m-1/2}^2 - \mu_\parallel^2 - \hat{m}_v^2} \right), \]  \tag{3.44}

\[ K_{vlml}^b(\mu_\parallel) = \max \left( \mu_\parallel \tan \alpha_{l-1/2}, \mu_\parallel \tan \alpha_{l+1/2}, \sqrt{\epsilon_{m+1/2}^2 - \mu_\parallel^2 - \hat{m}_v^2} \right), \]  \tag{3.45}

where

\[ K_{vlml}^0 = \min \left( \tan \alpha_{l-1/2}, \tan \alpha_{l+1/2} \right), \]  \tag{3.46}

if \( \alpha_{l-1/2} \) and \( \alpha_{l+1/2} \) are both smaller or both larger than \( \pi/2 \), otherwise

\[ K_{vlml}^0 = \begin{cases} \tan \alpha_{l-1/2} & \mu_\parallel \geq 0, \\ \tan \alpha_{l+1/2} & \mu_\parallel < 0. \end{cases} \]  \tag{3.47}

In discrete form Eq. (3.43) can be rewritten as follows

\[ \frac{d n_v(r)}{d \epsilon d \alpha} \bigg|_{ml} = \frac{2\pi}{\Delta \epsilon_m \Delta \alpha_l} \sum_{l'} \Delta \mu_\parallel^{l'} \Delta \mu_\perp^{l'} \frac{\mu_\perp^{l'}}{\epsilon_v^{l'}} \mathcal{F}_v(\mu_\parallel^{l'}, \mu_\perp^{l'}, r), \]  \tag{3.48}

with the constraints

\[ \epsilon_{m-1/2} < \sqrt{\mu_{\parallel^{l'}}^2 + \mu_{\perp^{l'}}^2 + \hat{m}_v^2} < \epsilon_{m+1/2}, \]  \tag{3.49}

\[ \alpha_{l-1/2} < \arccos \left( \frac{\mu_\parallel^{l'}}{\sqrt{\mu_{\parallel^{l'}}^2 + \mu_{\perp^{l'}}^2}} \right) < \alpha_{l+1/2}. \]  \tag{3.50}
3.4 Equations and initial conditions

In this Section we specify the system of equations we must solve as well as the starting conditions from which our numerical computation begins. To begin with, we take the Boltzmann Eqs. (1.45) for spherically symmetric physical space and cylindrically symmetric momentum space. Then with collision integrals defined by Eq. (1.16) and replace \( f_k \) with the new DF defined by Eq. (3.18). As a result we obtain the following system of coupled integro-differential equations for electrons and photons

\[
\frac{\partial F_k}{\partial t} + \frac{\mu_\parallel}{\mu} \frac{r}{r^2} \frac{\partial}{\partial r} \left( r^2 c \beta_k F_k \right) + \frac{\partial}{\partial \mu_\parallel} \left( \frac{\mu^2 c \beta_k F_k}{\mu} r \right) - \frac{\partial}{\partial \mu_\perp} \left( \frac{\mu \mu_\perp c \beta_k F_k}{\mu} r \right) = \sum_q \left( \eta_k^{q*} - \chi_k^{q} F_k \right), \quad k = e, \gamma,
\]

where the new emission coefficient is defined as \( \eta_k^{q*} = \epsilon_k \eta_k^{q} \). Emission and absorption coefficient are computed for Compton scattering only with Eq. (1.25). The discretization scheme that allows us to solve numerically Eqs. (3.51) is the same as the one given by Eqs. (1.47) where \( f_{\nu ij} \) has to be replaced by \( F_{\nu ij} \).

The initial electron and photon DFs must be taken from Eqs. (3.14) and (3.15) and then inserted into Eq. (3.18). Because the plasma is assumed to be in thermal equilibrium, we know that the photon chemical potential has to be zero \( \psi_\gamma = 0 \). Then the temperature of the plasma is such that the Fermi-Dirac distribution function can be well approximated with the classical Maxwell distribution for a thermal and non degenerate electron gas. As a consequence the exponential which contains the electron chemical potential can be factorized out as \( A_e = \exp(\psi_e/\theta) \). Then the initial electron and photon DFs are given by

\[
F_e(\mu_\parallel, \mu_\perp, r) = \frac{A_e \epsilon_e}{\hbar^3} \exp \left( -\frac{\Gamma \left( \epsilon_e - \beta \mu_\parallel \right)}{\theta} \right),
\]

\[
F_\gamma(\mu_\parallel, \mu_\perp, r) = \frac{\epsilon_\gamma}{\hbar^3} \left[ \exp \left( \frac{\Gamma \left( \epsilon_\gamma - \beta \mu_\parallel \right)}{\theta} \right) - 1 \right]^{-1}.
\]
Here it is important to note that the normalization $A_e$ for the electrons is chosen to satisfy the electron number conservation law given by Eq. (3.4).

Now we have to explain in detail how the outflow parameters can be determined in order to set up the initial DFs by means of Eqs. (3.52) and (3.53). This can be done using the theory of transparency of relativistic outflows we have briefly delineated in Section 3.2.1.

First of all, having assumed the outflow to be already in the coasting phase of relativistic expansion, the Lorentz factor is equal to a constant $\Gamma = \eta \gg 1$ all throughout the outflow, see Eq. (3.3). Then the initial conditions are completely determined by the following parameters:

- $R_i$: inner radius of the outflow;
- $R_e = R_i + l$: external radius of the outflow;
- $\Gamma = \eta$: constant Lorenz gamma factor;
- $\tau_i$: optical depth at $R_i$;
- $\theta_i$: comoving temperature at $R_i$.

The variables introduced above can be related to their value at the base of the wind $R_0$. This can be done quite easily because we know how optical depth and comoving temperature evolved since the outflow was launched at $R_0$. In particular the optical depth is described by the first formula in Eq. (3.7) during the first acceleration phase up to the saturation radius $R_s = \eta R_0$. From $R_s$ up to $R_i$ the optical depth evolves according with the second expression in Eq. (3.7). Matching the two profiles at $R_s$ we can estimate the optical depth at $R_i$ as

$$\tau_i = \frac{\sigma_T E_0}{24\pi m_e c^2 \eta^3 (R_e - R_i) R_i},$$

where the definition of $\tau_0$ has been taken from Eq. (3.8) in which the electron mass appears instead of the proton mass.

Then Eq. (3.54) could be inverted in order to have an expression for the total initial energy $E_0$. However we know that at the coasting phase $E_0$ has been converted, almost entirely, into kinetic energy of electrons, therefore their energy in the laboratory frame is larger than the photon energy in the laboratory frame. As a consequence we can approximate the energy of the
electrons in the laboratory frame as

\begin{equation}
E_{\text{lab}}^{e\gamma} \simeq \frac{24\pi m_e c^2}{\sigma_T} \eta^3 R_i^2 (\xi_e - 1) \tau_i = 9.3 \cdot 10^{45} \eta_2^3 R_{i10}^2 (\xi_e - 1) \tau_i \ \text{erg}
\end{equation}

where the parametrization \( R_e = \xi_e R_i \) has been used. Moreover we used the notation \( Q = 10^n Q_n \) that will be adopted also in the reminder of this Chapter. Eq. (3.55), giving the total energy of electrons, represents the value to which the distribution function in Eq. (3.52) has to be normalized, namely it determines \( A_e \).

The temperature profile in the outflow can be found following the same logic we have adopted to calculate the optical depth. The final result is that the temperature has the radial profile that is given by the following power law

\begin{equation}
\theta(r) = \frac{\theta_0}{\eta^{1/3}} \left( \frac{R_0}{r} \right)^{2/3}.
\end{equation}

From the temperature profile and the definition of \( \theta_0 \) given in Section 3.2.1 one can easily obtain the temperature at \( R_i \) as

\begin{equation}
\theta_i = 1.4 \cdot 10^{-4} \tau_i^{1/4} \eta_2^{5/12} R_{06}^{1/6} R_{i10}^{-5/12}.
\end{equation}

The total photon energy is automatically given by the integral of Eq. (3.53). Nevertheless it is convenient to give its expression as a function of the five parameters we introduced at the beginning of this Section. In particular the total energy of the electrons is given by the integral of their energy

\begin{equation}
E_{\text{lab}}^{e\gamma} = 4\pi \int_{R_e}^{R_i} dr \ r^2 \rho_{\gamma}^{\text{lab}}(r),
\end{equation}

where the energy density of photons in the laboratory reference frame is given by

\begin{equation}
\rho_{\gamma}^{\text{lab}}(r) = \eta^2 \frac{4\sigma_{SB}}{c} \left( \frac{m_e c^2}{k_B} \right)^4 \theta(r)^4,
\end{equation}

\( \theta(r) \) is given by Eq. (3.56) and the factor \( \eta^2 \) comes from the Lorentz transformation for the energy density from the comoving to the laboratory ref-
ence frame [139]. Making use of our variables we can rewrite Eq. (3.58) as

\begin{equation}
E_{\gamma}^{lab} = 0.14 \eta^{2/3} R_{06}^{2/3} R_{i10}^{-2/3} \frac{\xi_e^{1/3}}{\xi_e} - 1 \quad E_{\gamma}^{lab}.
\end{equation}

At this point only the parameters \( R_0, \eta, \tau_i, \xi_e, \) are to be chosen before performing the calculations. However one has to keep in mind a few constraints that are needed in order to avoid inconsistencies with our initial assumptions. For example, to be sure that we are in the coasting phase, the condition \( R_i > R_0 \) has to be satisfied. Then one has to pay attention to the initial temperature \( \theta_i \). In fact it is shown in Figures 3.3 and 3.4 that the DF shape in the phase space strongly depends on temperature and Lorentz factor. In particular, if the temperature is too small, the main parts of photon and electron DFs do occupy very separate regions in the phase space and this fact requires a more detailed computational grid in order to follow the time evolution of both photon and electron components. Finally, one has to verify that the outflow is photon thick, namely \( 2\eta^2 l \gg R_{ph} \).

### 3.5 Evolution of electron and photon DFs

We present the results of our method when it is applied to the study of transparency of a relativistic outflow. Due to large difference in reaction rate on the one hand and expansion rate on the other hand we cannot consider realistic total energy and size of plasma typical for GRBs. However we can study relativistic outflows with similar comoving temperatures and Lorentz factors to those inferred from GRBs. Then we perform simulations of macroscopic initially optically thick plasma with parameters similar to those considered in [140] but with the transparency occurring in the coasting phase of expansion.

Let us consider an optically thick and spherically symmetric outflow in the form of finite relativistic wind in the coasting phase with electron number density given by Eq. (3.4), Lorentz factor \( \Gamma \) given by Eq. (3.3) and comoving temperature given by Eq. (3.56) and composed by electrons and photons. The parameters are

\begin{equation}
\eta = 10^2, \quad n_0 = 10^{29} \text{ cm}^{-3}, \quad R_0 = 5 \text{ cm}, \quad \theta_0 = 5.36, \quad l = 4.5 \cdot 10^3 \text{ cm}.
\end{equation}

From Eq. (3.8) we find \( \tau_0 = 10^7 \). Then from Eq. (3.9) we find \( R_{ph} = 5 \cdot 10^3 \text{ cm} \). Due to numerical limitations we start with initial optical depth...
Figure 3.5: Comparison between time evolution of photon DFs during expansion neglecting (left) and taking into account (right) Compton scattering. Top: $t_1 = 0$ s, middle: $t_2 = 1.2 \cdot 10^{-8}$ s, bottom: $t_3 = 6.1 \cdot 10^{-7}$ s.
\[ \tau_i = 10 \text{ which gives us } R_i = 5 \cdot 10^2 \text{ cm}. \]

Inserting these numbers into Eqs. \((3.57), (3.55)\) and \((3.60)\) one finds

\[ n_i^{\text{lab}} = 10^{25}, \theta_i = 5.36 \cdot 10^{-2}, \]
\[ E_e^{\text{lab}} = 8.37 \cdot 10^{31} \text{ erg}, \quad E_\gamma^{\text{lab}} = 3.24 \cdot 10^{31} \text{ erg}. \]

Initial electron and photon DFs are given by Eqs. \((3.52)\) and \((3.53)\), where \(\eta = \text{const}, \theta = \theta(r)\) is given by Eq. \((3.56)\) for both electrons and photons while the factor \(A_e\) is determined from Eq. \((3.4)\).

Since photon and electron DFs extension in the phase space covers three or four orders of magnitude in both parallel and orthogonal directions, we must set up a logarithmic computational grid. Besides, due to the limitations on the maximum number of grid points and because the electron DF peaks sharply around \(\mu_|| \simeq \eta\), a few cells have to be placed around this value of the parallel momentum. Below we list the cell centered values of the grid in momentum space we have chosen

\[ \mu_|| = \{0.208, 1.74, 6.58, 20.8, 62.5, 10^2, 1.33 \cdot 10^2, 3.30 \cdot 10^2\}, \]
\[ \mu_{\perp k} = \{10^{-4}, 4.02 \cdot 10^{-3}, 3.74 \cdot 10^{-2}, 1.87 \cdot 10^{-1}, 6.54 \cdot 10^{-1}\}. \]

The radial grid is linear, there are 60 equivalent intervals, and it goes from \(R_i = 5 \cdot 10^2 \text{ cm up to } R_f\). The choice of \(R_f\) is crucial because the spectrum changes substantially even if the optical depth is slightly smaller that one. Considering the values for \(R_i\) and \(\tau_i\) given in \((3.61)\) and taking into account that \(\tau \propto r^{-1}\) we can take the external radius to be \(R_f = 6 \cdot 10^4 \text{ cm}\).

The results of the numerical computation are summarized in Figures 3.5 and 3.6, where we show the time evolution of \(F_\gamma, e\), and in Figures 3.7 and 3.8 where we plot for both photons (thin curves) and electrons (thick curves) the physical quantities that are helpful in interpretation of results. Final energy spectra of photons and electrons are shown in Figures 3.9 and 3.10.

In Figures 3.5 and 3.6 we show snapshots of DFs at three different times: top \(t_1 = 0 s\), middle: \(t_2 = 1.2 \cdot 10^{-8} s\), bottom: \(t_3 = 6.1 \cdot 10^{-7} s\).

Instead of computing the observed quantities by integrating over space and time as done in \([127]\), we are interested only in physical quantities computed at the radius corresponding to the inner boundary of the outflow given approximately by the following equation

\[ R_n = R_i + c t_n. \]

Inserting the numerical values of the chosen times into Eq. \((3.63)\) one ob-
Figure 3.6: The same as in Figure 3.5 for electron DFs.
contains $R_1 = R_i$, $R_2 = 8.6 \cdot 10^2$ cm, $R_3 = 1.8 \cdot 10^4$ cm. Other choices for $R_n$ are possible but the one we made above, being $R_i$ the point in the relativistic wind with largest optical depth, would make the effects we are looking for much more visible. In Figures 3.5 and 3.6 one has to compare the left and the right panels. In fact, with the aim of evaluating the net effect of Compton scattering, we studied the evolution of the system by means of Eqs. (3.51) putting $\eta^{\gamma \ast} = \eta^{e \ast} = \chi^\gamma = \chi^e = 0$. This way, electron and photon DFs evolve independently and no change in their DFs is expected to occur, hence we refer to this case as “free expanding”. On the contrary, the right panel corresponds to the time evolution for photon and electron DFs where Compton scattering is considered and we call this set of plots as “interacting”.

To start our analysis, we first consider the top rows in both Figures 3.5 and 3.6 which correspond to the initial conditions of thermal equilibrium for $F_{\gamma , e}$ given by Eqs. (3.53) and (3.52) respectively with parameters (3.61). To get a more quantitative picture of the outflow onset, we have shown the initial radial profiles for energy density $\rho$ and bulk parallel momentum $\langle \mu \parallel \rangle$ on the left and right panels of Figure 3.7 respectively. Looking at the energy density profiles one can easily realize that $\rho_e > \rho_\gamma$ at each radii which is necessary condition to guarantee coasting expansion. The sudden drop in the same plot identifies the position of the wind outer boundary. In course of expansion there is a radial spreading of the outflow, as seen from bottom line of Figure 3.7 resulting in Gaussian shape of the energy density profile. This spreading is not significant, keeping the width of the outflow nearly constant. The average parallel momentum plotted on the top right of Figure 3.7 has different meanings for electrons and photons. For the first ones in relativistic motion it is almost the same as their Lorentz factor and it is constant throughout the whole outflow, namely $\langle \mu \parallel \rangle_e \simeq \eta$. For the second ones such parameter measures their mean energy in the laboratory frame. Finally, in Figure 3.8 we give the radial profiles for parallel and orthogonal momenta dispersions $\sqrt{\langle \mu^2 \parallel \rangle}$ and $\sqrt{\langle \mu^2 \perp \rangle}$ from which one gets quantitative informations about the spreading of the distribution function along the corresponding directions in the phase space. For this reason, these two parameters are both tightly related to the comoving temperature of a given distribution of particles. It is worth noting that the top row in Figures 3.7 and 3.8 represent initial profiles for $\rho$, $\langle \mu \parallel \rangle$, $\sqrt{\langle \mu^2 \perp \rangle}$, $\sqrt{\langle \mu^2 \parallel \rangle}$ which corresponds to adiabatic profiles of temperature and number density of electrons given by Eqs. (3.56) and (3.4) respectively.

At $t_2$ the plasma has expanded outwards by a distance $\Delta R_2 = c t_2 =$
Figure 3.7: Comparison between time evolution of energy density in the laboratory frame (left panel) and bulk parallel momentum (right panel) for electrons (thick) and photons (thin) during expansion neglecting (dashed) and taking into account (solid) Compton scattering. Top: $t_1 = 0 \text{ s}$, middle: $t_2 = 1.2 \cdot 10^{-8} \text{ s}$, bottom: $t_3 = 6.1 \cdot 10^{-7} \text{ s}$. 
Figure 3.8: The same as in Figure 3.7 for dispersion of both orthogonal (left panel) and parallel (right panel) momenta.
3.6 \cdot 10^2\ cm which is well below the photospheric radius. One has to keep in mind that the optical depth in $R_2$ is still large and already some changes are visible indeed. The reason for that can be understood calculating the collisions time-scale given by Eq. (2.49) where the comoving density is $n_i^{lab}/\eta$. Doing so, one sees that every photon on average underwent Compton scattering at least once already after $t_{\text{coll}} \simeq 5 \cdot 10^{-10}\ s$. Then the average number of collisions between one photon and the electrons turns out to be 24 validating the efficiency of Compton scattering at so early times. In particular $F_\gamma$ has become much broader in the phase space with respect to his freely expanding counterpart on the left. In more detail, the orthogonal spread is more pronounced with respect to the parallel one. Also the $F_e$ has changed, but the effects are very small. Despite the short distance travelled by particles, the energy density within the outflow boundaries has decreased by a significant amount. The mean parallel momentum of electrons remains constant all the way from $R_i$ up to $R_f$ meaning that our initial parameters set up is consistent with the assumption of an actual coasting phase. On the contrary $\langle \mu_\parallel \rangle_\gamma$ shows a small deviation with respect to the initial profile in the figure above. The same effect is present in Figure 3.8 for both $\sqrt{\langle \mu_\perp^2 \rangle}$ and $\sqrt{\langle \mu_\parallel^2 \rangle}$. However this bump is due to the sparse grid in orthogonal momentum. At this stage one has to notice the perfect agreement between the freely expanding and interacting that cannot be distinguished indeed.

The most interesting effects are expected to occur when the outflow has already expanded up to several photospheric radii. This is the case after a time $t_3 = 6.1 \cdot 10^{-7}\ s$ since the beginning and the inner boundary of the outflow is at $R_3 = 3.6\ R_{\text{ph}}$, where the optical depth of the outflow is about $\tau_3 \simeq 0.27$. The plots of the photon and electron energy densities show that the whole wind is now far from $R_{\text{ph}}$. Looking carefully at the contour lines at the bottom line in Figure 3.5 one can recognize that both photon and electron DFs shrank in the phase space. This effect can be explained as follows. The average energy density of particles keeps decreasing up to the photosphere following the temperature law of adiabatic expansion. Instead beyond the photosphere this quantity approaches a constant, see [121][131]. Nevertheless, $F$ that evolved including Compton scattering is much broader in the phase space with respect to the one depicted on its left. For what concerns the electron DFs, it seems that the shape around their peak remained unchanged. There is a slight increase in the low parallel momentum region, which is probably a numerical effect. To confirm this fact we can see that electrons always move with the initial Lorentz
factor and the dispersions $\sqrt{\langle \mu_{||}^2 \rangle_e}$ and $\sqrt{\langle \mu_{\perp}^2 \rangle_e}$ did not change since the beginning. A more quantitative description about what we have now argued looking at the shapes of $F_{\gamma}$ can be obtained by considering the value of $\langle \mu_{||} \rangle_\gamma$, $\sqrt{\langle \mu_{\perp}^2 \rangle_\gamma}$ and $\sqrt{\langle \mu_{\parallel}^2 \rangle_\gamma}$ at $R_2$. In fact looking at the bottom line in Figure 3.7 one sees that the average photon energy is a few times larger than the one in the Figure above at the same radius and it becomes constant according to [121, 131]. Instead looking at left panel at the bottom line of Figure 3.8 it becomes clear that the $\sqrt{\langle \mu_{\perp}^2 \rangle_\gamma}$ is about one order of magnitude larger than the one predicted by the profile shown above at the same position. The dispersion value for $F_{\gamma}$ along the orthogonal momentum is now much closer to the one of electrons. Besides significant difference appear between the adiabatic and interacting cases. Also along the parallel momentum the dispersion of the photon DF increases indicating that $F_{\gamma}$ is stretched along the radial direction of motion. This is a clear indication of departure from thermal equilibrium and anisotropies development during expansion and transparency emission in particular.

![Figure 3.9: Comparison between the final photon spectrum obtained from our numerical calculation (dots) and the one given by Eq. (3.26) (solid line) with the parameters one would expect from adiabatic expansion $\Gamma_{ad} = \eta$ and $\theta_{ad} \simeq 1.7 \cdot 10^{-2}$.](image)
Figure 3.10: Comparison between the final electron spectrum obtained from our numerical calculation (dots) and the one given by Eq. (3.25) (solid line).

The apparently anomalous behavior at instant $t_3$ of $\langle \mu_\parallel \rangle_{\gamma,e}$, $\sqrt{\langle \mu_\perp^2 \rangle_{\gamma,e}}$ and $\sqrt{\langle \mu_\perp^2 \rangle_{\gamma,e}}$ for small radii is mainly due to the following reasons. First of all only particles with minimum parallel momentum are still in that position therefore their bulk parallel momentum has to be small as well. The second one is that at smaller radii the temperature is larger and, because $\sqrt{\langle \mu_\perp^2 \rangle_{\gamma,e}}$ is proportional to the temperature, these particles will keep the same dispersion in the orthogonal momentum as it was at the beginning.

The final energy spectra for photons and electrons have been extracted from the corresponding DFs following the definitions and the integration procedures delineated in Section 3.3.2 and are plotted in Figures 3.9 and 3.10 respectively. The photon spectrum is broader than the Planck one, as it was found in [131, 126] by means of Monte Carlo simulations and the Fokker-Planck approximation to the Boltzmann equation respectively. In our case we see that the low energy part of the spectrum is softer, in addition the high energy part of the spectrum is also affected by Compton scattering and shows significant departure from the Wien part of thermal spectrum. This occurs also for the part of the spectrum above the thermal peak.
3.6 Perspectives

The results we have presented in the previous Section refer to a particular situation in which the hydrodynamical behavior of the system is given. In our situation the initial temperature, density and Lorentz factor radial profiles are given by the hydrodynamic solution for a relativistic wind which is expanding adiabatically.

Nevertheless, our method can be used to explore arbitrary initial conditions. In fact one can change initial outflow parameters as well as temperature, density and Lorentz factor radial profiles. Doing so one could check how the final result depends on the choice of the initial conditions. In particular one should check the stability and the accuracy of the numerical calculation.

Also the composition of the outflow can be varied and baryons as well as electron-positron pairs can be added to the photon and electron system we have considered in this Thesis. In particular an outflow which is loaded with baryons would allow to consider initial outflow parameters that are typical for GRBs or other sources in which an initially optically thick relativistic outflow is launched. Correspondingly, other reactions, in particular Coulomb scatterings, have to be added in the collision integrals.

Then also the width of the wind could be changed in order to explore the photon thin case as well as intermediate conditions. Beside the assumption of steady wind could also be released and a time varying outflows can also be studied.

In our treatment we described particle interactions only via Compton scattering. This collision process is crucial during the transition from opaque to transparent outflow because it controls the energy exchange between electron and photon populations. If the initial optical depth is very large other 2- and 3-particle interactions have to be considered.

A further extension can be done if one considers more complex symmetries in the physical space. For example nearly axially symmetric structures like jets could be described.
Conclusions

Pair production in strong electric fields

Frequency of pairs oscillations

We showed that the study of plasma oscillations due to the vacuum polarization in uniform electric field can be reduced to the analysis of a single second order ordinary differential equation for the variable constructed from hydrodynamic velocity. From its solution one can easily obtain all the other physical quantities of interest such as electric field, electron-positron number and energy densities and their oscillation frequency as well. This simplification in the description of the problem allows to study the evolution of the system for very long time, much longer with respect to other approaches adopted in the literature.

We mainly focused our attention on the relation between the frequency of oscillation and plasma frequency. As expected, the plasma frequency is reached asymptotically for all the considered initial conditions $0.2 E_c \leq E_0 \leq 10 E_c$ and the difference between them rely on the time scale the system needs to approach the plasma frequency. This fact is very well summarized in Fig. 2.3 from which one can understand that the larger the initial electric field the shorter the time scale to attain the plasma frequency. Surprisingly, in all considered cases we find that averaged frequency of oscillations is very close to the plasma frequency even at the beginning of the computation when the rate of electro-positron pairs production is still important.

Also the power spectrum of pairs oscillations has been computed assuming the dipole radiation approximation to be valid. The characteristic peak of such power spectrum originated by plasma oscillations is shown to be very close to the plasma frequency, but always below such limit. The left tail in Fig. 2.6 is due to the first oscillations with frequencies smaller than averaged plasma frequency, while the main contribution is due to the final stage of the evolution when the pairs oscillate almost with the same
frequency close to the plasma one.

In this approximation, all the energy of the system which is initially stored into the electric field is finally converted into rest mass energy of electrons and positrons. The upper limit to the optical depth to pair annihilation into photons is obtained, showing that it never exceeds unity for $E < 45E_c$ for the time-scales we have considered. By comparing pair production rate and acceleration we also estimated the effect of degeneracy on the pair production process showing that Pauli blocking can be safely neglected if the initial field is much smaller than $127E_c$.

**Phase space evolution toward thermalization**

For the first time we studied the entire dynamics of energy conversion from initial overcritical electric field, ending up with thermalized electron-positron-photon plasma. Such conversion occurs in a complicated sequence of processes starting with Schwinger pair production which is followed by oscillations of created pairs due to back-reaction on initial electric field, then production of photons due to annihilation of pairs and finally isotropization of created electron-positron-photon plasma. We solved numerically the relativistic Vlasov-Boltzmann equations for electrons, positrons and photons, with collision integrals for 2-particle interactions computed from exact QED matrix elements.

In order to appreciate the consequences of the kinetic treatment and the relevance of interactions separately, two different computations have been performed for every single initial condition. We called them collisionless and interacting systems in view of the fact that collision terms have been discarded and accounted for, respectively. The collisionless runs allowed us to compare our results with those obtained earlier, and to resolve better the momentum space of pairs. In this way we found that number density of pairs always saturates without exceeding 5 per cent of the maximum achievable number density $2.94$, in contrast to earlier works. Surprisingly this number is not far from the thermal number density of pairs obtained from the temperature $2.92$. In particular, for the largest field $E_0 = 100E_c$ we obtained almost 30 per cent of the thermal number density of pairs when the interactions are not yet important. It is interesting that the energy stored in initial electric field is mainly converted into internal and kinetic energies of pairs, but the former becomes predominant as time advances. Even if the distribution in momentum space reminds Maxwellian, also at the very beginning it is highly anisotropic, with the dispersion along the direction of electric field exceeding orders of magni-
tude that in orthogonal direction. We conclude that simultaneous production of pairs and their acceleration in the same electric field is responsible for such peculiar form of particles DF.

We found that interactions become important at later times with respect to the average oscillation period, in agreement with estimates performed in Chapter 2. For higher initial fields interactions become significant earlier and for each initial condition there is a characteristic time scale after which they can not be neglected. We find that photons initially follow the distribution of pairs with nonzero parallel bulk momentum.

Collisions provide the mechanism of relaxation towards an equilibrium configuration. The first equilibrium manifests itself when the perfect symmetry between pair annihilation and creation rates is established. Only later on, when scatterings have distributed particles almost isotropically in the momentum space, the kinetic equilibrium is reached. In such state the electron-positron-photon plasma is generally described by a common temperature and nonzero chemical potentials for all particles. The subsequent evolution of the pair plasma towards thermal equilibrium is well understood.

Transparency of relativistic outflows

We presented the study of an important topic in modern astrophysics, namely the emission from optically thick relativistic wind. The photospheric emission from GRBs can be regarded as the main physical process to which our studies can be applied. In this respect, we first described GRBs phenomenon from an observational point of view focusing on their key features. Then we described their photospheric emission that is generated when the initially optically thick relativistically expanding plasma becomes transparent to radiation. The peculiarity of such event is represented by the observed photon energy spectrum that is never thermal but always broader with respect to a simple Planck spectrum.

For the computation of observed spectra of photospheric emission different methodologies have been adopted. Among them Monte Carlo simulations, radiative transfer approximations, Kompaneets equations with anisotropic photon field and finally rate equation formalism. Each of the approaches mentioned above present several limitations. Therefore we decided to study this problem within the most general framework that is given by relativistic kinetic theory.

In order to deal with relativistic particles in the laboratory frame, we introduced a system of cylindrical coordinates in the phase space. This
choice allowed us to calculate average quantities from the distribution function that are related to the comoving temperature of particles as well as their bulk motion properties such as the electron Lorentz factor. The physical system is assumed to be spherically symmetric. We introduced a new DF that is well defined in the phase space even when Bose-Einstein statistics for photons is considered. We have written the general system of relativistic Boltzmann equations for the newly defined distribution function as well as its finite difference representation in order to solve numerically such system. Moreover we specified the procedure that is needed to extract the total energy spectrum and the angular resolved spectrum of particles from this new DF.

We chose to study a photon thick relativistic wind during its coasting phase of expansion, assuming the outflow to be composed by electrons and photons. Given the assumptions above the initial conditions for the outflow parameters have been worked out from the theory of transparency of relativistic outflow summarized in Section 3.2.1. This specific case has been chosen since because the effects of transparency emission are expected to be more evident with respect to accelerating and photon thin outflows.

We solved numerically the system of partial integro-differential relativistic Boltzmann equations for photon and electron DFs with initial conditions given by adiabatic expansion profiles for temperature and electron number density. Interactions between electrons and photons have been described via Compton scattering.

Therefore we have been able to follow the time evolution of the system of photons and electrons starting from large optical depth up to small optical depths through transparency. In order to understand the role of interactions at transparency, we computed two different runs with same initial conditions: the first one called free expanding where we do not include collision integrals for Compton scattering; instead for the second one called interacting we do include Compton interaction processes.

We find evidence for photons becoming increasingly anisotropic during expansion through the photosphere. This resulted in strong deviations from Planck spectrum even without integrating the spectrum over space and time. This result is in agreement with previous results obtained by Fokker-Planck approximation. For what concerns electrons, we find that their DF is not affected by decoupling of photons for the considered initial conditions.
Bibliography


