Electrodynamics: from nuclei to neutron stars

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Chapter 1

General Introduction

Relativistic Astrophysics differs from the other branches of physics for the spectacularly large scales of the observables involved in physical processes. Following the well known case of supernova with energies $\lesssim 10^{53}$ ergs on time scales of months, gamma ray bursts (GRBs) have offered an extreme example of the most energetics ($E \lesssim 10^{55}$ ergs) and the fastest transient ($\Delta t \lesssim 10^{-3} - 10^4$ s) phenomena ever observed in the universe [RBB+07].

One of the fundamental issues in Physics and Astrophysics is the creation of an electron-positron plasma in overcritical electric fields larger than (see [RVX09] and references therein)

$$E_c = \frac{m_e^2 c^3}{\hbar},$$

which governs the dynamics of such astrophysical processes [RVX09].

Basic progress toward the understanding of the thermalization process of such a plasma have been achieved [ARV07]. The existence of such an electron-positron plasma has a central role in a variety of problems ranging from the acceleration process in GRBs [RVX09] to the sharp trigger process in supernova phenomena [PVCI09, PVCF09].

A theoretical model based on vacuum polarization process [DR75] occurring in a Kerr-Newman geometry can indeed explain such enormous energetics and the sharp time variability. The formation of such black holes is expected from a large variety of binary systems composed of neutron stars, white dwarfs and massive stars at the end point of their thermonuclear evolution [KJR08] in all possible combinations. In particular, in the merging of two neutron stars and in the final process of gravitational collapse to a black hole is expected the occurrence of electromagnetic fields with strength larger than the critical value $E_c$ [RBB+07].

This has motivated us to reconsider the standard treatment of neutron stars in order to find a theoretical explanation for the emergence of a wide variety of astrophysical situations involving such overcritical electric fields [VPX09, MRX09, ARX09b, ARX09a]. In order to approach this complex problem, we decided to go by steps, starting our study from nuclei. The aim of this work is to present a unified treatment from nuclei, superheavy nuclei, and neutron stars, based on the Thomas-Fermi model of the atom appropriately generalized and adapted to treat each one of these problems.

We start in Chapter 2 with the study the problem of uncompressed nuclear matter for systems with a small number of nucleons $A \lesssim 10^6$, namely nuclei, heavy nuclei and superheavy nuclei. We review there briefly the classical Thomas-Fermi model of the atom and the relativistic generalization by Ruffini et al. [VPX09], which we analyze with some detail. We discuss the charge to mass ratio relation obtained from this model, and compare it with the phenomenological one of Weizsacker. It is also discussed a simpler model based on an incompressible Fermi liquid approach, from which we derive explicit analytic formula for the proton fraction for a pure electrostatic model of the nucleus as well as for a model including not only the Coulomb interaction but also the strong interaction based on the so-called Relativistic Mean Field model of the nucleus.
In Chapter 3 we extend the treatment of Ruffini et al. [VPX09] to the case of compressed nuclear matter. This extension not only represent the generalization of the previous relativistic model of uncompressed nuclear matter by Ruffini et al. [VPX09] but it also represents the generalization of the non-relativistic treatment of Feynman, Metropolis and Teller [FMT49] of compressed atoms. In particular, we compare and contrast the results obtained from the non-relativistic and the relativistic one in the description of the behavior of the Fermi energy of the electron gas. Therefore, we discuss the range of validity of a non-relativistic approach and it is also discussed the range of validity of the uniform approximation for the electron gas as compared with the correct result given by the solution of the relativistic Thomas-Fermi equation.

We then in Chapter 4 extrapolate this relativistic treatment of compressed nuclear matter to the case of massive nuclear density cores, which are beta equilibrated configurations of neutrons, protons, and electrons, characterized by a number of baryons of the order of $\sim 10^{57}$ and constrained to satisfy global charge neutrality. This is accomplished by using the recently established scaling laws of the ultrarelativistic Thomas-Fermi equation obtained with the assumptions of having ultrarelativistic electrons and applying the planar (one-dimensional) approximation for the Poisson equation, which is already valid for cores with mass numbers $\gtrsim 10^4$. We discuss the ground-state configuration of these cores, and their stability against fission and Coulomb repulsion.

With the gained experience in treating compressed matter in nuclei, heavy and superheavy nuclei and in massive nuclear density cores, we formulate in Chapter 5 a novel approach to the construction of neutron star equilibrium configurations satisfying global charge neutrality instead of local neutrality. The corresponding equilibrium equations then follow from self-consistent solution of the general relativistic Thomas-Fermi equation, the Einstein-Maxwell equations and the beta equilibrium condition, properly expressed in general relativity. It is demonstrated analytically the inconsistency of the local neutrality condition with the Einstein-Maxwell field equations. We show as an example, the application of our approach for two different models of equation of state: the one of Baym, Bethe, and Pethick [BBP71] based on a compressible liquid drop model, and the one given by a degenerate non-interacting gas of neutrons, protons, and electrons.

Having addressed the existence of neutron star configurations globally neutral but possessing internal electric fields, which become overcritical in the core-crust transition surface, we study in Chapter 6 the polarization process around an already formed Kerr-Newman black hole. It may represent a physical state asymptotically reached in the process of gravitational collapse. What it is most important is that by performing this theoretical analysis we can gain a direct evaluation of the energetics of the spectra and dynamics of the $e^-e^+$ plasma created on the extremely short time scales due to the quantum phenomena of $\Delta t = \hbar/(m_e c^2) \approx 10^{-21}$ s. This study leads to the generalization of the concept of dyadosphere introduced for Reissner-Nordström geometries in [PRX98] to the concept of daydotorus for Kerr-Newman geometries.

In Chapters 2–5 we use units $\hbar = c = 1$, therefore we have mass = energy = length$^{-1}$. The gravitational constant in these units is given by $m_{\text{Planck}} = G^{-2}$. In Chapter 6 we use units $G = c = 1$ so mass = energy = length.
2.1 The classical Thomas-Fermi model

The Thomas-Fermi model assumes that the electrons of an atom constitute a fully degenerate gas of fermions confined in a spherical region by the Coulomb potential of a point-like nucleus of charge $+eN_p$ [Tho27, Fer28], which is located at the origin $r = 0$. We recall that the condition of equilibrium of the electrons in an atom, in the non-relativistic limit, is expressed by [Tho27, Fer28]

$$E_F^e = \left(\frac{P_F^e}{m_e}\right)^2 - eV = \text{constant}, \quad (2.1)$$

where $m_e$ is the electron mass, $V$ is the electrostatic potential and $E_F^e$ is the electron Fermi energy.

The electrostatic potential fulfills for $r > 0$ the Poisson equation

$$\nabla^2 V = 4\pi en_e, \quad (2.2)$$

where the electron number density $n_e$ is related to the Fermi momentum $P_F^e$ by

$$n_e = \left(\frac{P_F^e}{3\pi^2}\right)^3. \quad (2.3)$$

Introducing the dimensionless potential $\chi(r)$ by

$$eV(r) + E_F^e = \alpha N_p \frac{\phi(r)}{r}, \quad (2.4)$$

where $\alpha$ is the fine structure constant, we obtain the following expression for the electron number density

$$n_e(\eta) = \frac{1}{3\pi^2} \frac{N_p}{4\pi b^3} \left[ \frac{\phi(\eta)}{\eta} \right]^{3/2}, \quad (2.5)$$

where the new coordinate $\eta$ is defined by $r = b\eta$, with

$$b = \frac{(3\pi)^{2/3}}{2^{2/3}} \frac{\lambda_e^2}{\alpha} \frac{1}{N_p^{1/3}}, \quad (2.6)$$

where $\lambda_e = 1/m_e$ is the electron Compton wavelength. Then, from Eq. (2.2) we obtain

$$\frac{d^2\phi(\eta)}{d\eta^2} = \frac{\phi(\eta)}{\eta^{1/2}}^{3/2}, \quad (2.7)$$
which is the classic Thomas-Fermi equation \[\text{Fer28}\]. A first boundary condition for this equation follows from the point-like structure of the nucleus

\[\phi(0) = 1.\] (2.8)

A second boundary condition comes from the particle number conservation

\[1 - \frac{N_e}{N_p} = \phi(\eta_0) - \eta_0\phi'(\eta_0),\] (2.9)

where \(N_e = \int_0^{R_{WS}} 4\pi n_e(r)r^2dr\) is the total number of electrons inside the Wigner-Seitz cell radius \(R_{WS}\), which defines the boundary of the configuration. The dimensionless radius of the cell is given by \(\eta_0 = R_{WS}/b\).

The general behavior of the solutions of the classical Thomas-Fermi equation (2.7) is shown in Fig. 2.1. The solution corresponding to a free neutral atom, i.e. without any exerted pressure at its boundary, is obtained for \(\phi'(0) \approx -1.58807\), and its boundary is located at infinity, i.e. \(R_{WS} \to \infty\).

Correspondingly, as can be seen from the equilibrium condition (2.1), the electron Fermi energy of a free (uncompressed) neutral atom must be \(E_F^e = 0\), fixing the level of reference of the Coulomb potential \(V(R_{WS}) = 0\).

### 2.2 Relativistic Thomas-Fermi model for free neutral atoms

The main difference in the relativistic generalization of the Thomas-Fermi treatment is that the point-like approximation of the nucleus must be abandoned \[\text{FRS80, RS81}\], since the relativistic generalization of the equilibrium condition (2.1)

\[E_F^e = \sqrt{(P_F^e)^2 + m_e^2} - m_e - eV(r) = 0,\] (2.10)

leads to a divergent number of electrons near the origin. This can be seen as follows. In order to obtain a finite value of the number of electrons close to the origin, the function \(r^2 n_e(r)\), which is the integrand involved in the calculation, must be well-behaved as \(r \to 0\). In the non-relativistic case, we have when \(r \to 0\), \(n_e \propto (eV)^{3/2} \propto r^{-3/2}\) and correspondingly, \(r^2 n_e \propto \sqrt{r}\). On the contrary, in the relativistic case when \(r\) approaches the origin we have \(n_e \propto (eV)^{3} \propto r^{-3}\), which leads to \(r^2 n_e \propto r^{-1}\), which diverges for \(r \to 0\).

The study of neutral atoms with nuclei of mass number \(A \sim 10^2\text{–}10^6\) is a classic problem of theoretical physics \[\text{ZP72, RXV09}\]. Special attention has been given to a possible vacuum polarization process and the creation of \(e^+e^-\) pairs \[\text{PG69, ZP72, RXV09}\] as well as to the study of nuclear stability against Coulomb repulsion \[\text{GG82}\].

It has been known since the classic works of Fermi \[\text{E.50}\], that the phenomenological drop model of the nucleus gives excellent results for a variety of properties. In fact, the well-known Weissacker semi-empirical mass formula can be used to predict the stability of nuclei against particle emission, as well as the energy release and stability of nuclei for fission \[\text{E.50}\], in addition to do precise prediction of the masses of stable nuclei. Written in terms of the mass number \(A\) and the number of protons \(N_p\), the mass formula of the nucleus given by Weissacker reads

\[M(A, N_p) = m_n(A - N_p) + m_pN_p - a_v A + a_{sym} \frac{(A - 2N_p)^2}{A} + a_C \frac{N_p^2}{A^{1/3}} + a_{surf} A^{2/3} + \delta,\] (2.11)

where \(a_v = 15.8\ \text{MeV}, a_{surf} = 18.3\ \text{MeV}, a_{sym} = 23.3\ \text{MeV}, a_C = 0.714\ \text{MeV},\) and \(\delta \approx 12\ \text{MeV}\) are the volume, surface, symmetry, Coulomb, and pairing coefficients, which are obtained from fitting of the experimental data.
From the extremization of the mass formula
\[
\frac{\partial M(A, N_p)}{\partial N_p} = 0,
\]
the following relation between the mass number \( A \) and the number of protons \( N_p \) is obtained
\[
N_p \frac{A}{2} + \left( \frac{a_C}{2a_{\text{sym}}} \right) A^{2/3} \lesssim \frac{A}{2 + 0.015A^{2/3}},
\]
which in the limit of small \( A \) gives
\[
N_p \approx \frac{A}{2}.
\]

The analysis of the stability of the nucleus against finite deformation leads to a stability condition against fission relating the surface energy and the Coulomb energy. This leads to the condition
\[
\frac{N_p^2}{A} \lesssim 45.
\]
A novel situation occurs when super-heavy nuclei \((A > 10^4)\) are examined \([FRS80, RRX07]\). The distribution of electrons penetrates inside the nucleus: a much smaller effective net charge of the nucleus occurs due to the screening of relativistic electrons \([MVP76, FRS80]\). In \([RS81]\) a definition of an effective nuclear charge due to the penetration of the electrons was presented. A treatment based on the relativistic Thomas-Fermi model was developed in order to describe the penetration of the electrons and their effective screening of the positive nuclear charge. In particular, by assuming \(N_p \simeq A/2\), Greiner et al. \([PG69, MPRG72, GG82, Pop71, ZP72, MVP76]\) in a series of papers were able to solve the non-linear Thomas-Fermi equation.

It was demonstrated in \([MVP76]\) that the effective positive nuclear charge is confined to a small layer of thickness \(\sim \frac{1}{\alpha} \frac{m_\pi}{\pi}\) where \(\lambda_\pi = \frac{1}{m_\pi}\) is the pion Compton wavelength. Correspondingly electric fields of strength much larger than the critical value \(E_c\) for vacuum polarization at the surface of the core are created. However, the creation of electron-positron pairs due to the vacuum polarization process does not occur because of the Pauli blocking by the degenerate electrons \([RVX09]\).

However, in all these works, the charge to mass ratio \(N_p/A\) has been assumed to be either \(N_p \simeq A/2\) as adopted by Greiner and Popov \([PG69, MPRG72, GG82, Pop71, ZP72, MVP76]\), or \(N_p \simeq A/(2 + 0.015A^{2/3})\) as adopted by Ferreirinho, Ruffini and Stella \([FRS80]\).

In this line, an improvement is represented by the model of Ruffini et al. \([VPX09]\), which we will discuss now with some detail. The major difference with preceding treatments is that, the charge to mass ratio, which traditionally had been imposed using phenomenological expressions, is obtained self-consistently by the explicit imposition of beta equilibrium between neutrons, protons, and electrons.

The density of protons \(n_p\) is taken to be constant inside the nucleus radius

\[
R_c = \frac{\Delta}{m_\pi N_p^{1/3}},
\]

and zero outside the nucleus. The parameter \(\Delta\) is obtained from the condition that the baryon density must be the nuclear density, i.e \(n = n_0 \simeq m_\pi^3/2\). Then, the parameter \(\Delta\) can be written as

\[
\Delta \simeq \left(\frac{3}{2\pi}\right)^{1/3} \left(\frac{A}{N_p}\right)^{1/3},
\]

so \(\Delta \sim 1\) when applied to ordinary nuclei with \(A/N_p \simeq 2\).

The overall Coulomb potential satisfies the Poisson equation

\[
\nabla^2 V(r) = -4\pi e [n_p(r) - n_e(r)],
\]

with the boundary conditions \(V'(\infty) = 0\) (due to the global charge neutrality of the system) and finiteness of \(V(0)\). The density \(n_e(r)\) of the electrons is determined by

\[
n_e(r) = \frac{p_F^3}{3\pi^2} = \frac{1}{3\pi^2} \left[ e^2 V^2(r) + 2m_e eV(r) \right]^{3/2},
\]

where we have used Eq. (2.10). By introducing the dimensionless quantities \(x = m_\pi r, x_c = m_\pi R_c\) and \(\chi/r = eV(r)\), the relativistic Thomas-Fermi equation takes the form

\[
\frac{1}{3x} \frac{d^2 \chi(x)}{dx^2} = -\frac{\alpha}{\Delta^3} \theta(x_c - x) + \frac{4\alpha}{9\pi} \left[ \chi^2(x) + 2\frac{m_e}{m_\pi} \frac{\chi(x)}{x} \right]^{3/2},
\]

where \(\chi(0) = 0, \chi(\infty) = 0\).
The neutron density \( n_n(r) \) is determined by the Fermi energy condition on their Fermi momentum \( P_n^F \) imposed by beta decay equilibrium

\[
E_n^F = \sqrt{(P_n^F)^2 + m_n^2} - m_n = \sqrt{(P_p^F)^2 + m_p^2} - m_p + eV(r),
\]

which in turn is related to the proton and electron densities by Eqs. (2.18), (2.19) and (2.20).

In Fig. 2.2 we show the \( N_p-A \) relation obtained from the Ruffini et al. treatment of uncompressed neutral nuclear matter [VPX09], which we have just reviewed above. The integration has been extended to large mass numbers.

Figure 2.2: The \( N_p-A \) relation of Ruffini et al. [VPX09] obtained from first principles (solid line) compared with the phenomenological expressions given by \( N_p \approx A/2 \) (dashed line) and Eq. (2.13) (dotted line). The asymptotic value for large mass numbers \( A \), is approximately \( N_p/A \approx 0.0046 \).

In the limit of small \( A \), we can see that the behavior of the charge-mass relation obtained from the Ruffini et al. treatment [VPX09], is very similar to the one obtained from the phenomenological relations (2.13) and (2.14). It appears that the explicit evaluation of the beta equilibrium, leads to an effect comparable in magnitude and qualitatively similar to the asymmetry energy in the phenomenological liquid drop model.

However, the \( N_p-A \) relation of Weizsacker deviates from the one obtained from the full integration of the relativistic Thomas-Fermi equation for large numbers \( A \gtrsim 10^3 \). The essential difference is that the semi-empirical formula of Weizsacker (2.11) considers a bare nucleus, i.e. it does not take
into account the effect of the penetration of the surrounding electrons, which becomes important for superheavy nuclei, as can be seen from Fig. 2.2. The electrons, which initially are located outside the nucleus, start to penetrate into the nucleus, screening its positive charge. As a result of this penetration, the Coulomb energy decreases with increasing $A$. On the contrary, the Coulomb energy of the nucleus given by the semi-empirical formula of Weizsacker

$$E_C = 0.71 \frac{N_p^2}{A^{1/3}} \text{ MeV} \simeq 0.71 \frac{A^{5/3}}{(2 + 0.015A^{2/3})^2} \text{ MeV}, \quad (2.22)$$

is an increasing function of $A$. Consequently, the Weizsacker stability curve predicts erroneously nuclei with very large proton-neutron asymmetry, i.e. nuclei characterized with a small proton fraction $N_p/A$. This behavior can be clearly individuated in Fig. 2.2 where $N_p$ grows much less with $A$ for the Weizsacker model respect to the $N_p$ given by the Ruffini et al. model [VPX09], which takes into account properly the screening of the nucleus charge due to the penetration of the electrons.

![Figure 2.3: Weizsacker Coulomb energy (2.22) as a function of the number of protons $N_p$ inside the nucleus given by Eq. (2.13).](image)

After having discussed the effect of the penetration of the electrons on screening the nucleus charge and hence on the determination of the proton fraction, we turn now to the lower part of the $N_p$-$A$ relation of Fig. 2.2. In Fig. 2.3, we show the $N_p$-$A$ relation for the Ruffini et al. model [VPX09] as well as for the Weizsacker model given by Eq. (2.13) at a scale of $A \sim 10^2$. On these scales, some quantitative differences between the two curves can be appreciated. Actually, the difference should be due to the additional contributions to the beta equilibrium that has not been taken into account within the Ruffini et al. treatment [VPX09]. What is indeed remarkable is that from first principles it
is possible to obtain a qualitatively similar stability curve for nuclei without doing phenomenological assumptions.

![Graph](image)

Figure 2.4: The $N_p$-$A$ relation of Ruffini et al. [VPX09] obtained from first principles (dashed line), compared with the phenomenological expression of Weizsacker given by Eq. (2.13) (solid line).

The similarities between the $N_p$-$A$ relation given by the Ruffini et al. treatment based on the solution of the relativistic Thomas-Fermi equation [VPX09], and the one given by the phenomenological model expressed by the Weizsacker formula (2.11) should be investigated in a deeper way. We will show through a simpler model the basic assumptions one needs to do in order to obtain a Weizsacker-like $N_p$-$A$ relation without imposing any phenomenological assumption.

### 2.3 The nucleus as an incompressible Fermi gas

Let us assume the nucleus as composed by an incompressible degenerate gas of non-relativistic baryons (neutrons and protons) surrounded by a relativistic degenerate gas of electrons that warranty the overall neutrality of the system. The equilibrium equation governing the system is represented by the condition of beta equilibrium, which using the equilibrium condition for the electron gas (2.10) can be expressed as

$$
\mu_n = \mu_p + U_{pn},
$$

where

$$
\mu_i = \frac{(p_i^F)^2}{2m_i}, \quad i = p, n,
$$

(2.24)
is the non-relativistic chemical potential of protons and neutrons. The potential $U_{pn}$ denotes the difference between the proton potential energy $U_p$ and the neutron potential energy $U_n$, i.e.

$$U_{pn} = U_p - U_n.$$ (2.25)

Using the relation between Fermi momentum and particle density

$$P^F_i = \left( \frac{3\pi^2 n_i}{2} \right)^{1/3},$$ (2.26)

from Eq. (2.23) we obtain

$$\left( \frac{n_p}{n_n} \right)^{2/3} = 1 - \frac{U_{pn}}{\mu_n},$$ (2.27)

where we have assumed for simplicity $m_n = m_p$. Using the fact that $U_{pn}/\mu_n \ll 1$ at small $A$, and integrating over the nucleus volume we obtain the generalized $N_p$-$A$ relation

$$Y(1 - Y)^{2/3} = \frac{(1 - Y)^{2/3}}{2} - \frac{3}{4} \frac{m_n}{(3\pi^2 n)^{2/3}} \frac{n}{A} \int_0^{R_c} U_{pn} d^3r,$$ (2.28)

where we have defined the proton fraction

$$Y = \frac{N_p}{A},$$ (2.29)

and $n = n_p + n_n$ denotes the baryon number density. As usual, the nuclear radius is $R_c$. We can see how the charge to mass ratio depends explicitly on the potential $U_{pn}$. Due to the assumption of the equal mass of the proton and the neutron, the potential $U_{pn}$ is the only responsible for the difference between the kinetic energy of the proton and the neutron, being in this way the responsible of the asymmetry inside the nucleus. In fact, the $N_p$-$A$ relation given by Eq. (2.28) possesses the good property that when $U_{pn} = 0$ we obtain $Y = 1/2$ as should be expected. We explore now the resulting form of the $N_p$-$A$ relation (2.28) for different assumptions of the potential $U_{pn}$. 

### 2.3.1 Proton fraction in a pure Coulomb model

Let us assume that the asymmetry potential $U_{pn}$ is given only by the Coulomb potential due both to protons and electrons. For free nuclear matter ($E_F^e = 0$) the screening of the Coulomb potential at small $A$ due to electron penetration inside the nucleus is very poor (see [RRX07, VPX09] for details), and the potential can be approximated to the one produced by a uniform distribution of protons

$$eV = \frac{\alpha N_p}{R_c} \left[ \frac{3}{2} - \frac{1}{2} \left( \frac{r}{R_c} \right)^2 \right].$$ (2.30)

We then have

$$\int_0^{R_c} U_{pn} d^3r = \int_0^{R_c} eVd^3r = 6 \frac{\alpha N_p}{5 \frac{R_c}{R_c}} = \frac{6}{5} \left( \frac{2\pi}{3} \right)^{1/3} a m_n \frac{N_p}{A^{1/3}},$$ (2.31)

where we have used Eqs. (2.16) and (2.17). Feeding back into Eq. (2.28) we obtain

$$Y(1 - Y)^{2/3} = \frac{(1 - Y)^{2/3}}{2} - k_1 Y A^{2/3}, \quad k_1 = \frac{3\alpha}{5\pi} m_n.$$(2.32)

In the case of small proton-neutron asymmetry, i.e. for $2Y = 1 - \epsilon$, with $\epsilon \ll 1$, Eq. (2.32) becomes

$$N_p = \frac{A}{2 + 2^{5/3}k_1 A^{2/3}} \simeq \frac{A}{2 + 0.0307 A^{2/3}}.$$ (2.33)
2.3. THE NUCLEUS AS AN INCOMPRESSIBLE FERMI GAS

Figure 2.5: The $N_p$-$A$ relation for the different models. The dots correspond to the full numerical integration of the relativistic Thomas-Fermi equation performed by Ruffini et al. [VPX09], while the dashed curve corresponds to the analytic expression given by Eq. (2.33). The Weizsacker stability curve given by Eq. (2.13) is represented by the solid curve.

The above Eq. (2.33) has the same form as the stability curve (2.13) obtained from the phenomenological Weizsacker formula (2.11). In Fig. 2.5 we have plotted the charge to mass ratio of the Ruffini et al. treatment [VPX09] as well as the one given by the analytical formula (2.33), together with the $N_p$-$A$ relation of the Weizsacker (2.13).

From Fig. 2.5 we can see how the full numerical integration of the equations of the relativistic Thomas-Fermi treatment of Ruffini et al. [VPX09], follow the analytic curve given by Eq. (2.33) at small $A$. It demonstrates the accuracy of the approximations we applied in the derivation of the analytic formula (2.33). It is evident the importance of beta equilibrium in determining the correct particle fraction and so the correct charge to mass ratio at equilibrium. Indeed, already for the simple case of a pure electrostatic potential it is possible to understand the nature of the Weizsacker stability curve (2.13), without phenomenological assumptions.

Nevertheless, from the quantitative point of view, both the $N_p$-$A$ relation given by the pure electrostatic model and the one given by the Weizsacker are different. It is clear from our approach the necessity of additional contributions to the potential $U_{pn}$, in order to modify the beta equilibrium in such a way that it can explain appropriately the observed proton fraction $N_p/A$ of ordinary nuclei.

Based on that, we will modify the beta equilibrium based on the Relativistic Mean Field (RMF) model of the nucleus (see [Wal74, Rin96] and Appendix A for details and references on the subject). In this way, we find a new potential $U_{pn}$, which is integrated following the same procedure as in the pure electrostatic model, obtaining an analytic expression for the proton fraction at small $A$. 
CHAPTER 2. UNCOMPRESSED NUCLEAR MATTER

2.3.2 Proton fraction in the Relativistic Mean Field model

The beta equilibrium condition in the Relativistic Mean Field (RMF) model is given by (see Eq. (A.59) in Appendix A)

\[
\sqrt{(P_n^F)^2 + (m^*)^2 + C\rho (n_n - n_p)} = \sqrt{(P_p^F)^2 + (m^*)^2 - C\rho (n_n - n_p) + eV}, \tag{2.34}
\]

where \( m^* \) is the Dirac effective nucleon mass, related to the \( \sigma \)-meson by \( m^* = m + g_\sigma \sigma \), with \( g_\sigma \) the coupling constant between the scalar \( \sigma \)-meson and the nucleons, whereas \( C\rho = (g_\rho / m_\rho)^2 \), being \( g_\rho \) the coupling constant between the \( \rho \)-meson and nucleons, and \( m_\rho \) is the \( \rho \)-meson mass.

Correspondingly, the potential \( U_{pn} \) is given by

\[
U_{pn} = eV - 2C\rho (n_n - n_p) = eV - 2nC\rho (1 - 2Y). \tag{2.35}
\]

Integrating Eq. (2.28) with the new potential \( U_{pn} \) given by Eq. (2.35), we obtain the \( N_p-A \) relation

\[
N_p = \frac{A}{2 + \left( \frac{2^{5/3}k_2}{1 + 2^{5/3}k_2} \right) A^{2/3}}, \quad k_2 = \frac{(3\pi^2/2)^{1/3}}{2\pi^2} m^* m_\pi C\rho, \tag{2.36}
\]

where we have assumed the density \( n \approx n_0 = m_\pi^3 / 2 \). Once again, we obtain a stability curve similar to Eq. (2.13) obtained from the Weizsacker semi-empirical formula (2.11).

We can fix the combination of parameters \( m^* C\rho \) in order to fit the Weizsacker proton fraction given by Eq. (2.13). Therefore, the value of \( m^* C\rho \) that fits the \( N_p-A \) relation (2.13) is

\[
m^* C\rho \simeq 0.0196 \text{ MeV}^{-1}. \tag{2.37}
\]

As should be expected, we cannot constraint independently the parameters \( m^* \) and \( C\rho \) of the RMF model, which can be understood from the fact that we are fitting only one property of ordinary nuclei. In fact, the complete set of parameters within a RMF model are obtained from the fitting of several observed properties of the nuclear matter at the so-called saturation density \( n_0 \approx m_\pi^3 / 2 \).

Among the observed properties of nuclei that should be explained for any nuclear model we have the binding energy of the nucleus \( \sim -15.8 \text{ MeV} \), the nuclear asymmetry energy \( \sim 23.3 \text{ MeV} \), and the effective nucleon mass \( m^* \sim 0.7 - 0.8 m_n \), which does not coincide with the Dirac effective nucleon mass \( m^* \) but instead they are related in some way \( m^* = m^*(m, n_0) \). Another properties that can help to constrain the parameters of a determined model are, although less constrained by experiments, the surface energy and the nucleus compressibility modulus.

For instance, in the NL3-model by Lalazissis et al. [LKR97], which is one of the most prolific models to describe several properties of nuclei, the best fit of the Dirac effective nucleon mass is found to be \( m^* \approx 0.6 m_n \approx 563.4 \text{ MeV} \), whereas for the \( \rho \)-meson mass and coupling constant they found \( m_\rho \approx 763.0 \text{ MeV} \) and \( g_\rho \approx 4.474 \). In this case we obtain

\[
(m^* C\rho)_{NL3} \simeq 0.0194 \text{ MeV}^{-1}, \tag{2.38}
\]

which is in perfect agreement with the result obtained from our analytic expression (2.36).

2.4 Conclusions

We have first generalized the treatment of heavy nuclei by enforcing the condition of beta equilibrium in the relativistic Thomas-Fermi equation, avoiding the imposition of \( N_p \approx A/2 \) or any other relation based on phenomenological assumptions between \( N_p \) and \( A \) traditionally assumed in the
2.5. PERSPECTIVES

literature. In doing so we have obtained (see Fig. 2.2) an $N_p/A$ relation which extends the ones adopted in the literature.

Although quantitatively different, the charge to mass ratio of Weizsacker and the one of Ruffini et al. [VPX09] shown in Fig. 2.2 are qualitatively similar, which would suggest that the stability curve of nuclear matter is mainly determined by the beta equilibrium of its constituents. Furthermore, it was seen that through the self-consistent treatment of the electron gas it is possible to take into account the effect of the screening of the positive charged nucleus, due to penetration of the electrons. This effect appears to be a major contribution in the determination of the charge to mass ratio for increasing $A$ (see Fig. 2.2).

By constructing a very simple model based on an incompressible Fermi liquid model of the nucleus, we confirmed the above statements, and even more interesting, it was possible through such a simple model to obtain an analytic Weizsacker-like $N_p/A$ relation for small $A$ of the order of ordinary nuclei (see Eqs. (2.33) and (2.36)), which depends only on the potential $U_{pn} = U_p - U_n$, where $U_{pn}$ is the potential energy of protons, neutrons. In particular, we showed that, when applied to a pure electrostatic model $U_{pm} = eV$, the analytic formula obtained from this simplistic incompressible model of the nucleus fits quite well the charge to mass ratio obtained from full numerical integration of the relativistic Thomas-Fermi equation done by Ruffini et al. in [VPX09] (see Fig. 2.3). Subsequently, we extended the application of this simple model by including the effects due to the strong interaction given by the RMF model of the nuclear matter (see [Wal74] [Rin96] and Appendix A for details). In such case the potential $U_{pn}$ has the contribution of the Coulomb potential as well as the contribution of the $\rho$-meson. As we said before, we obtained once again a Weizsacker-like $N_p/A$ relation, but now with adjustable parameters related to the Dirac nucleon effective mass and to the $\rho$-meson coupling constant (see Eq. (2.36)). We then have found the values of these parameters in order to fit the Weizsacker stability curve. The welcome result was that the values we found for these parameters agree quite good with those found in literature. We had success in obtaining from first principles a stability curve in complete agreement with what we know from experimental data of ordinary nuclei, avoiding the necessity of invoking any phenomenological assumption.

2.5 Perspectives

It is clear that the correct proton fraction $N_p/A$ in nuclei at equilibrium is the result of a combined effect of the penetration of the electrons and the strong interaction, duly taken into account in the beta equilibrium condition for nuclear matter. We have explored these effects through simplistic models from which we successfully extracted crucial information about their contribution at different regimes, and for systems characterized with different mass numbers.

It would be very interesting to construct the stability curve from first principles for a more realistic model, taking into account both the effect of screening of the nucleus charge and the strong interaction contribution without additional assumptions, in such a way that we can construct a full charge to mass relation starting from light nuclei, passing for heavy and superheavy nuclei, to finish with the extrapolation of the model to high mass number systems. Such a calculation could be carried out using the RMF model summarized in Appendix A by solving together with the Klein-Gordon equations for the meson fields the beta equilibrium condition for neutrons, protons, and electrons, and imposing the relativistic Thomas-Fermi equilibrium condition for the electron gas (2.10), under the constraint of global neutrality.
Chapter 3
Compressed nuclear matter

3.1 The Feynman-Metropolis-Teller treatment

Feynman, Metropolis, and Teller showed that the Thomas-Fermi model can be used to derive the equation of state of matter at high pressures by considering neutral atoms confined in a Wigner-Seitz cell \[ \text{Wigner-Seitz} \]. We will denote the radius of such a cell as \( R_{WS} \).

In Chapter 2, we have shown that the electron Fermi energy \( E_F^e \) of a free neutral atom is equal to zero, which can be deduced from Eq. (2.1). In that case the Fermi momentum of the electron gas goes smoothly to zero at the boundary \( R_{WS} \rightarrow \infty \). Following the same analysis, we can see that for a compressed neutral atom, the Fermi energy satisfies necessarily \( E_F^e > 0 \), i.e. it is positive. Consequently, the electron Fermi momentum does not vanish at the boundary \( R_{WS} \) (see curve 1 in Fig. 2.1 of Chapter 2).

In the case of compressed neutral atoms, from Eqs. (2.1) and (2.3), we find that the Fermi energy of electrons can be expressed by

\[
E_F^e = \frac{N_p e^2}{b} \frac{\phi(\eta_0)}{\eta_0} .
\]

Therefore in the classic treatment \( \eta_0 (R_{WS}/b) \) can approach zero and, consequently, the range of the possible values of the Fermi energy extends from zero to infinity. In Fig. 3.1 we show the behavior of the Fermi energy of the electrons for iron \( (N_p = 26) \) subjected to different levels of compression \( \eta_0 \).

However, at high compressions, the electron kinetic energy reaches values of the order of the electron rest-mass, and even larger. In this case, we expect the Fermi energy obtained from the non-relativistic treatment to be larger that the Fermi energy obtained from a correct relativistic formulation. We then proceed to formulate the Thomas-Fermi relativistic treatment of compressed atoms \[ \text{Ruffini et al.} \]. This treatment represents the generalization of the Ruffini et al. \[ \text{VPX09} \] approach of free neutral atoms as well as the generalization of the non-relativistic treatment of Feynman-Metropolis-Teller to relativistic regimes.

3.2 Relativistic Thomas-Fermi model for compressed atoms

As we have seen, the main difference between the relativistic and non-relativistic treatments is that the point-like approximation of the nucleus must be abandoned \[ \text{FRS80 RS81} \]. The first difference between the relativistic treatment for free neutral atoms of Ruffini et al. \[ \text{VPX09} \] and the one for compressed neutral atoms is that the relativistic Fermi equilibrium condition \( 2.10 \) must be changed by

\[
E_F^e = \sqrt{(p_F^e)^2 + m_e^2} - m_e - eV(r) > 0 .
\]
Figure 3.1: The electron Fermi energy for iron, in units of the electron mass, are plotted as a function of the dimensionless compression parameter $\eta_0$. Points refer to the numerical integrations of the Thomas-Fermi equation (2.7) performed originally by Feynman, Metropolis and Teller in [FMT49].

In this case, the electron density is given by

$$n_e(r) = \left(\frac{P_{Fe}}{3\pi^2}\right)^3 = \frac{1}{3\pi^2} \left[ e^2 \hat{V}^2(r) + 2m_e e \hat{V}(r) \right]^{3/2},$$

(3.3)

where $e \hat{V} = eV + E_{Fe}^f$. Then, by introducing the same dimensionless quantities as before $x = m_\pi r$, $x_c = m_\pi R_c$, and $\chi/r = e \hat{V}(r)$, we obtain from the Poisson equation the relativistic Thomas-Fermi equation

$$\frac{1}{3x} \frac{d^2 \chi(x)}{dx^2} = -\frac{\alpha}{\Delta^3} \theta(x_c - x) + \frac{4\pi e}{9\pi} \frac{\chi^2(x)}{x^2} + 2\frac{m_e}{m_\pi} \frac{\chi(x)}{x}^{3/2}.$$  

(3.4)

In the case of compressed neutral atoms, the boundary conditions are given by $\chi(0) = 0$, $\chi(x_{WS}) \geq 0$, $x_{WS} = m_\pi R_{WS}$.

The neutron density $n_n(r)$, related to the neutron Fermi momentum $P_{Fn} = (3\pi^2 n_n)^{1/3}$, is determined, as usual from the condition of beta equilibrium

$$E_{Fn}^f = \sqrt{(P_{Fn})^2 + m_n^2} - m_n$$

$$= \sqrt{(P_{Fe}^f)^2 + m_p^2} - m_p + eV(r) + E_{Fe}^f.$$  

(3.5)
3.3 Comparison and contrast between the different treatments

In order to compare and contrast non-relativistic and the relativistic treatments of compressed atoms, we first express the non-relativistic equations in terms of the dimensionless variables used for the relativistic treatment $x = \frac{m_{n}r}{\pi}$, and $eV = \frac{\chi}{r}$. In these variables, the classical Thomas-Fermi equation (2.7) becomes

$$\frac{1}{3x} \frac{d^{2}\chi(x)}{dx^{2}} = \frac{4\alpha}{9\pi} \left[ \frac{m_{e}}{m_{\pi}} \frac{\chi(x)}{x} \right]^{3/2},$$

with the boundary conditions

$$\chi(0) = \alpha N_{p}, \quad x_{WS} \chi'(x_{WS}) = \chi(x_{WS}),$$

and dimensionless Wigner-Seitz cell radius $x_{WS} = m_{\pi}R_{WS}$.

In these new variables the electron Fermi energy is given by

$$E_{F}^{e} = \frac{\chi(x_{WS})}{x_{WS}} m_{\pi}.$$ (3.8)

The two treatments, the relativistic and the non-relativistic one can be now directly compared by using the same units (see Fig. 3.2 for details).

We find here two major differences:

1. By compression, the Fermi energy in the non-relativistic treatment increases much more than the one obtained in the relativistic treatment.

2. While in the non-relativistic treatment, by compression, the Fermi energy can reach infinite values as $R_{WS} \rightarrow 0$, in the relativistic treatment it reaches a perfectly finite value given by

$$E_{F}^{e} \approx \left[ \frac{m_{e}}{m_{\pi}} + \sqrt{\left( \frac{m_{e}}{m_{\pi}} \right)^{2} + \left( \frac{3\pi^{2}n}{2} \right)^{2/3} \left( \frac{N_{p}}{A} \right)^{2/3}} \right] m_{\pi},$$

when $R_{WS}$ coincides with the nuclear radius $R_{c}$. At nuclear density $n \simeq \frac{m_{\pi}^{3}}{2}$ the above formula becomes

$$E_{F}^{e} \simeq \left[ \frac{m_{e}}{m_{\pi}} + \sqrt{\left( \frac{m_{e}}{m_{\pi}} \right)^{2} + \left( \frac{3\pi^{2}}{2} \right)^{2/3} \left( \frac{N_{p}}{A} \right)^{2/3}} \right] m_{\pi}.$$ (3.10)

3.4 The uniform approximation for the electron gas

There exist in the literature a large variety of semi-qualitative approximations adopted in order to describe the electron component of a compressed atom. In particular, we will discuss the uniform approximation for the electron gas.

Consider the electron gas as uniformly distributed inside the Wigner-Seitz radius $R_{WS}$ (see e.g. [BMG07] for instance), which is determined by the overall neutrality condition at the boundary $R_{WS}$

$$N_{p} = \frac{4\pi}{3} R_{WS}^{3} n_{e},$$

where $n_{e} = \left( P_{F}^{e} \right)^{3} / (3\pi^{2})$. 

Correspondingly, the electron Fermi energy can be written as

\[
E_F^e \simeq \left[ -\frac{m_e}{m_\pi} + \sqrt{\left( \frac{m_e}{m_\pi} \right)^2 + \left( \frac{9\pi}{4} \right)^{2/3} \frac{N_e^{2/3}}{x_{WS}} \right]} \right] m_\pi. \tag{3.12}
\]

In Fig. 3.3 we show the behavior of the Fermi energy \(E_F^e\) for the relativistic generalization of the Feynman-Metropolis-Teller treatment as well as \(E_F^e\) in the uniform approximation for the electron gas given by Eq. (3.12).

Any analysis of nuclear composition, determined in function of the electron Fermi energy \(E_F^e\), will be definitely very sensitive to the approximation adopted. Any approximation which does not follow the results obtained from the relativistic Thomas-Fermi equation presented above, leads necessarily to incorrect results. The difference represented in Fig. 3.3 has been obtained for a specific model of the nucleus. We expect that in the case of a different nuclear model the dependence of the Fermi energy from compression may be different (see Section 3.5). For any fixed nuclear model, however, the approximation given by Eq. (3.12) and the correct one obtained using the relativistic Thomas-Fermi equation, will remain.
3.5 Dependence of the Fermi energy on the nuclear model

In order to analyze the effect of the nuclear model adopted on the behavior of the electron Fermi energy, we will compare the non-interacting model with the RMFT model, assuming an incompressible degenerate gas of electrons, protons, and neutrons. The Fermi energy for both the non-interacting case and the RMFT case can be written as

\[ E_F^e \simeq \left[ -\frac{m_e}{m_\pi} + \sqrt{\left(\frac{m_e}{m_\pi}\right)^2 + \left(\frac{3\pi^2}{2}\right)^{2/3}} \frac{Y^{2/3}}{\tilde{R}_{WS}} \right] m_\pi, \]  

(3.13)

where as before \( Y = N_p/A \) denotes the proton fraction, and we have used the global neutrality condition (3.11). The parameter \( \tilde{R}_{WS} = R_{WS}/R_c \) is the Wigner-Seitz cell radius normalized to the radius of the nucleus.
The beta equilibrium condition in the non-interaction model reads

\[
\sqrt{\left(\frac{3\pi^2}{2}\right)^{2/3} m_n^2 (1 - Y)^{2/3} + m_n^2} = \sqrt{\left(\frac{3\pi^2}{2}\right)^{2/3} m_p^2 Y^{2/3} + m_p^2} + \sqrt{\left(\frac{3\pi^2}{2}\right)^{2/3} m_\pi^2 \frac{Y^{2/3}}{\tilde{R}_{WS}} + m_\pi^2}.
\] (3.14)

For the RMFT model, the beta equilibrium condition is modified (see Appendix A). In particular, it is modified explicitly by the \(\sigma\)-meson through the effective nucleon mass and by the \(\rho\)-meson as can be seen from the modified beta equilibrium condition

\[
\sqrt{\left(\frac{3\pi^2}{2}\right)^{2/3} m_n^2 (1 - Y)^{2/3} + (m^*)^2} = \sqrt{\left(\frac{3\pi^2}{2}\right)^{2/3} m_p^2 Y^{2/3} + (m^*)^2 + m_\pi^2 C_\rho (1 - 2Y)} + \sqrt{\left(\frac{3\pi^2}{2}\right)^{2/3} m_\pi^2 \frac{Y^{2/3}}{\tilde{R}_{WS}} + m_\pi^2}.
\] (3.15)

For each given \(\tilde{R}_{WS}\), we solve numerically the above algebraic equation for \(Y\), in such a way that we can determine the electron Fermi energy (3.13) as a function of \(\tilde{R}_{WS}\). In Fig. 3.4 we have plotted the behavior of the Fermi energy of the two models as a function of the normalized Wigner-Seitz cell radius \(\tilde{R}_{WS}\).

Figure 3.4: The electron Fermi energy \(E^F_e\) for different compressions of a degenerate incompressible gas of electrons, protons, and neutrons. The solid line corresponds to the non-interacting model, while the dashed line corresponds to the RMFT model.
3.6. CONCLUSIONS

As can be seen from Eq. (3.13) the Fermi energy depends only on the proton fraction, which at the same time depends on the nuclear model adopted, which modifies the beta equilibrium of the configuration. However, from Fig. 3.4 we see that the difference becomes very small at moderate compressions, for instance for $E_F^e << 0.1m_{\pi}$. Typical Fermi energies of the electrons in the outer crust of a neutron star are smaller than $\sim 0.2m_{\pi}$ (see [JARX09b] and Chapter 5, Section 5.5 for details). Therefore, in the bottom layers of a neutron star the specification of a particular nuclear model should give only a very small correction to the nuclear composition. On the contrary, with the increasing of compression the influence of the nuclear model adopted on the nuclear composition and on the Fermi energy of the electron gas increases. In Fig. 3.5 we have plotted the charge to mass ratio $N_p/A$ (proton fraction) for the two studied models. In general, the proton fraction given by the RMFT model is larger than the one of the non-interacting model. We confirm that the difference between the two approaches is appreciable overall at very high compressions.

![Graph showing proton fraction $N_p/A$ vs. $R_{WS}/R_c$](image)

Figure 3.5: Proton fraction $N_p/A$ of an incompressible degenerate gas of electrons, protons, and neutrons. The solid line corresponds to the non-interacting model, while the dashed line corresponds to the RMFT model.

3.6 Conclusions

We have first considered the problem of a compressed atom described by a relativistic Thomas-Fermi equation. As in the previous works [FRS80, RS81, RRX07] the protons in the nuclei have been assumed to be at constant density, the electron distribution has been derived by the Thomas-Fermi relativistic equation and the neutron component has been derived by the beta equilibrium between neutrons, protons and electrons. The effect of compression has been described by constraining the system in a Wigner-Seitz cell. In doing so we have generalized the classic results obtained in the
non-relativistic treatment by Feynman, Metropolis and Teller. In the non-relativistic treatment the Fermi energy of electrons can vary from zero to infinity in view of the point-like structure of the nucleus. In the relativistic Thomas-Fermi equation, a perfectly finite maximum value of the Fermi energy is reached. These results, generalize the Feynman-Metropolis-Teller treatment and will be certainly verifiable in forthcoming experiments of confined high temperature plasma \cite{RVX09}. The relativistic generalization introduce corrections with two major results:

1. The softening of the dependence of the electron Fermi energy on the compression factor.
2. The reaching of a limiting value of the electron Fermi energy.

It is also appropriate to remark that the correct treatment via a relativistic Thomas-Fermi equation, essential in determining the electron distribution in a compressed atom, is not equivalent to current treatments which have been often adopted in the literature using a variety of approximations (see e.g. \cite{BMG07}).

In addition, we analyze the effect of the nuclear model adopted to describe strong nucleon interactions on the behavior of the proton fraction and of the Fermi energy of the electrons with compression. For sake of simplicity, we perform the calculation for a simple incompressible Fermi liquid model adopting a RMF model for the nucleon strong interaction. We concluded that the effect of the interaction between nucleons due to strong force can be neglected at small electron Fermi energies $\ll 0.1 m_\pi$ while at larger energies, it becomes important allowing the electrons to possess larger Fermi energies.

We consider the understanding of these compressed atoms a necessary step in order to approach for instance, the problem of describing the nuclei in the crust of a neutron star.

### 3.7 Perspectives

As for the case of uncompressed atoms in Chapter 2, we have analyzed the problem of compressed atoms using simple models, which has helped us to approach the problem of nuclear matter under extreme conditions of compression from first principles. However, we consider a well defined and interesting problem the analysis of compressed nuclear matter by using more realistic models. Again, it can be accomplished by integrating the equations of the RMF model of the nucleus (see \cite{Wal74, Rin96} and Appendix A for details), together with the now familiar equations of equilibrium given by the beta equilibrium and the relativistic Thomas-Fermi equilibrium conditions, following the same approach that we have introduced in this Chapter.
Chapter 4

Compressed massive nuclear density cores

4.1 Analytic solution to the ultrarelativistic Thomas-Fermi equation

One of the most active field of research has been to formulate a unified approach both to superheavy nuclei, up to atomic numbers of the order of \(10^5 - 10^6\), and to what we have called “Massive Nuclear Density Cores” (see [VPX09] and references therein for some historical remarks on the subject). A massive nuclear density core is a system composed by a degenerate gas of \(N_n\) neutrons, \(N_p\) protons and \(N_e\) electrons in beta equilibrium. These massive cores are constrained to satisfy overall neutrality \(N_e = N_p\), in contrast to the very stringent traditional condition of local neutrality \(n_e = n_p\). The cores are held at nuclear density \(n \approx m_\pi^3/2\), and have mass numbers of the order of \(A \sim (m_{\text{Planck}}/m_\pi)^3 \sim 10^{57}\) [RRX07, VPX09].

With the increasing of the mass number, the distribution of electrons starts to penetrate inside the nucleus. Already for the case of superheavy nuclei with \(A \gtrsim 10^4\), a much smaller effective net charge of the nucleus occurs due to the screening of relativistic electrons [MVP76, FRS80]. Correspondingly, the effective positive nuclear charge is confined to a small layer of thickness \(\sim 1/(m_\pi a)\) around the surface of the nucleus (see [MVP76] for details).

Under such conditions, the electrodynamic problem involves only the surface of the configuration, and the use of the so-called planar or one-dimensional approximation to solve the Poisson equations is justified. In this approximation, the curvature term of the laplacian of the Coulomb potential \(r^{-1}dV/dr\) is neglected.

As we will show, the combination of the mentioned planar approximation and the ultrarelativistic approximation of the electron gas, leads to the existence of scaling laws of the resulting ultrarelativistic Thomas-Fermi equation, from which it is indeed possible to extrapolate all the treatment amply used in the study of superheavy nuclei to macroscopic systems of \(M \sim M_\odot\) and \(R_c \sim 10\) km.

For positive values of the Fermi energy \(E_F\), we introduce the new function

\[
\phi = \left(\frac{4}{9\pi}\right)^{1/3} \Delta \frac{\chi}{\hat{x}},
\]

and the new coordinate \(\hat{x} = kx\) where \(k = (12/\pi)^{1/6} \sqrt{\Delta} a^{-1}\), as well as the shifted coordinate \(\xi = \hat{x} - \hat{x}_c\) in order to describe better the region around the core radius, which is defined as usual by

\[
R_c = \frac{\Delta}{m_\pi} N_p^{1/3}.
\]
CHAPTER 4. COMPRESSED MASSIVE NUCLEAR DENSITY CORES

Using the planar approximation as well as the ultra-relativistic approximation of the electron gas

$$\sqrt{(P_F^e)^2 + m_e^2} - m_e \simeq P_F^e,$$

the Eq. (3.4) in the new variables and coordinates becomes

$$\frac{d^2\hat{\varphi}(\xi)}{d\xi^2} = -\theta(-\xi) + \hat{\varphi}(\xi)^3,$$

where $\hat{\varphi}(\xi) = \varphi(\xi + \hat{x}_c)$.

We then have the Coulomb potential energy

$$eV(\xi) = \left(\frac{9\pi}{4}\right)^{1/3} \frac{m_e}{\Delta} \hat{\varphi}(\xi) - E_F^e,$$

and correspondingly the electric field

$$E(\xi) = -\left(\frac{3^5\pi}{4}\right)^{1/6} \left(\frac{m_e}{\Delta}\right)^2 \hat{\varphi}'(\xi),$$

and the ultra-relativistic electron number density

$$n_e(\xi) = \frac{1}{3^2} \left(\frac{9\pi}{4}\right)^{3/4} \left(\frac{m_e}{\Delta}\right)^3 \hat{\varphi}^3(\xi).$$

One of the boundary conditions for Eq. (4.4) is given by the condition of local neutrality at the center of the core $n_e(r = 0) = n_p(r = 0)$, from which we obtain $\hat{\varphi}(-\infty) = 1$. In order to consider a compressed massive nuclear density core, we then introduce a Wigner-Seitz cell determining the outer boundary of the electron distribution which, in the new radial coordinate $\xi$ is characterized by $\xi^{WS}$. In view of the overall charge neutrality of the system the electric field goes to zero at $\xi = \xi^{WS}$. This implies the boundary condition $\hat{\varphi}'(\xi^{WS}) = 0$.

This boundary-value problem can be solved analytically and indeed Eq. (4.4) has the first integral,

$$2 \left[ \frac{d\hat{\varphi}(\xi)}{d\xi} \right]^2 = \left\{ \begin{array}{ll} \hat{\varphi}^4(\xi) - 4\hat{\varphi}(\xi) + 3, & \xi < 0, \\ \hat{\varphi}^4(\xi) - \hat{\varphi}^4(\xi^{WS}), & \xi > 0, \end{array} \right.$$  \hspace{1cm} (4.8)

with the continuity conditions at $\xi = 0$:

$$\hat{\varphi} \bigg|_{\xi=0} = \frac{\hat{\varphi}^4(\xi^{WS}) + 3}{4},$$

$$\frac{d\hat{\varphi}}{d\xi} \bigg|_{\xi=0} = -\sqrt{\frac{\hat{\varphi}^4(0) - \hat{\varphi}^4(\xi^{WS})}{2}}. \hspace{1cm} (4.9)$$

Then, the analytical solution of Eq. (4.8) satisfying all the boundary and continuity conditions is

$$\hat{\varphi}(\xi) = \left\{ \begin{array}{ll} 1 - 3 \left[ 1 + 2^{-1/2} \sinh(a - \sqrt{3}\xi) \right]^{-1}, & \xi \leq 0, \\ \hat{\varphi}(\xi^{WS}) \left\{ \cos \left[ \text{am}\left[\hat{\varphi}(\xi^{WS})(\xi - \xi^{WS}), \frac{1}{\sqrt{2}}\right]\right] \right\}^{-1}, & \xi > 0, \end{array} \right.$$  \hspace{1cm} (4.10)

where the integration constant $a$ has the value

$$\sinh(a) = \sqrt{2} \left(\frac{11 + \hat{\varphi}^4(\xi^{WS})}{1 - \hat{\varphi}^4(\xi^{WS})}\right) \hspace{1cm} (4.11)$$
4.1. ANALYTIC SOLUTION TO THE ULTRARELATIVISTIC THOMAS-FERMI EQUATION

Figure 4.1: Solutions of the ultrarelativistic Thomas-Fermi equation (4.4) for different values of the Wigner-Seitz cell radius $R_{WS}$ and correspondingly of the electron Fermi energy in units of the pion rest mass as in Fig. 4.2 near the core surface. The solid line corresponds to the case of null electron Fermi energy.

and $am(u,k) = \varphi = F^{-1}(u,k)$ is the Jacobi Amplitude, i.e. the inverse function of the elliptic function of the first kind $F(\varphi,k)$.

In the Figs. 4.1, 4.2, 4.3 we have plotted the function $\hat{\varphi}(\xi)$, the electron potential energy $-eV$ and the electric field $E$ for selected values of the electron Fermi energy for a given number of baryons $A$ and core radius $R_c$. For the case $E_{Fe} = 0$, the analytical solution (4.10) reduces to the one obtained by Ruffini et al. [VPX09] for the case of free massive nuclear density cores.

In the present case of $E_{Fe} > 0$, the ultra-relativistic approximation is indeed always valid up to $\xi = \xi_{WS}$ for high compression factors, i.e. when $R_{WS}$ approaches $R_c$. In the uncompressed case $E_{Fe} = 0$, for which $\xi_{WS} \to \infty$, there is a breakdown of the ultra-relativistic approximation when $\xi \to \xi_{WS}$.

It is interesting that similar treatments to the one exposed here for massive nuclear density cores have been already used in the literature, not only for the study of superheavy nuclei (see [Pop71, ZP72, MVP76] for instance), but also for the study of the electrodynamical properties of macroscopic objects as the hypothetical strange (quark) stars (see [Wit84, AFO86, Itu70, KWWG95] and references therein). In particular, Alcock et al. [AFO86] solved the ultra-relativistic Thomas-Fermi equation to describe the electrodynamical properties of the surface of bare strange stars as well as the core-crust interface of strange stars with a crust of white-dwarf-like material as the one
Figure 4.2: The electron Coulomb potential energies in a massive nuclear density core with $A \simeq 10^{57}$ and $R_c \simeq 10$ km, are plotted as a function of the dimensionless variable $\xi$, for selected values of $E_F^e$. The solid line corresponds to the case $E_F^e = 0$ and presents the maximum binding energy. By increasing the value of $E_F^e$ the electron Coulomb potential energy depth is reduced.

found in the bottom layers of neutron stars (see Chapter 5 for details). However, no explicit analytical expression for the solution, analogous to one reported here can be found there.

It is worth to mention that, although the results of the numerical integration of the ultra-relativistic Thomas-Fermi equation given by Alcock et al. in [AFO86] are correct, the Fig. 6 of that paper could lead to some confusion about the behavior of the Coulomb potential in these configurations. In Fig. 6 of [AFO86], it was plotted what they called the Coulomb potential energy, which we will denote as $V_{\text{Alcock}}$. The potential $V_{\text{Alcock}}$ was plotted for different values of the electron Fermi momentum at the edge of the crust. Actually, such a potential $V_{\text{Alcock}}$ is not the Coulomb potential but it coincides with our function $e\hat{V} = eV + E_F^e$. Namely, the potential $V_{\text{Alcock}}$ corresponds to the Coulomb potential shifted by the Fermi energy of the electrons. We then have from Eq. (4.5)

$$e\hat{V}(\xi) = \left(\frac{9\pi}{4}\right)^{1/3} \frac{m_\pi}{\Lambda} \hat{\phi}(\xi) = V_{\text{Alcock}}. \quad (4.12)$$

This explains why in [AFO86], for different values of $V_c$, the depth of the potential $V_{\text{Alcock}}$ remains unchanged (see Fig. 6 of Alcock et al. [AFO86] and Fig. 4.1 for details). Instead, the correct behavior of the Coulomb potential is quite different and, indeed, its depth decreases with the increasing of compression as can be seen from Fig. 4.2.
4.1. ANALYTIC SOLUTION TO THE ULTRARELATIVISTIC THOMAS-FERMI EQUATION

Figure 4.3: The electric field in units of the critical field $E_c$ is plotted as a function of the coordinate $\xi$, for selected values of $E_F^e$. The solid line corresponds to the case $E_F^e = 0$. To an increase of the value of the electron Fermi energy it is found a reduction of the peak of the electric field.

We can now estimate two crucial quantities of the solutions: the Coulomb potential at the center of the configuration and the electric field at the surface of the core

$$eV(0) \simeq \left( \frac{9\pi}{4} \right)^{1/3} \frac{m_\pi}{\Delta} - E_F^e,$$

(4.13)

$$E_{\text{max}} \simeq -2.4 \frac{\sqrt{\pi}}{\Delta^2} \left( \frac{m_\pi}{m_e} \right)^2 E_c \left( \frac{d\hat{\phi}}{d\xi} \bigg|_{\xi=0} \right).$$

(4.14)

These functions depend on the value $\hat{\phi}(S_{WS})$ through Eq. (4.19), which in turn can be related to the Fermi energy $E_F^e$ from Eq. (4.5), which evaluated at the boundary $R_{WS}$, where the Coulomb potential satisfies $V(R_{WS}) = 0$ gives

$$E_F^e = \left( \frac{9\pi}{4} \right)^{1/3} \frac{m_\pi}{\Delta} \hat{\phi}(S_{WS}).$$

(4.15)

From the above Eq. (4.15), one can see that there exists a solution, characterized by the value of Fermi energy $E_F^e$

$$\left( E_F^e \right)_{\text{max}} = \left( \frac{9\pi}{4} \right)^{1/3} \frac{m_\pi}{\Delta},$$

(4.16)
such that $\bar{\phi}(\xi^{WS}) = 1$. This solution corresponds to have $\bar{\phi}(0) = \bar{\phi}(-\infty) = 1$ and $\xi^{WS} = 0$. In this case the configuration does not possess any electrodynamic structure, and indeed it satisfies both overall neutrality $N_e = N_p$ and local neutrality $n_e = n_p$.

In this special case, starting from the beta equilibrium condition given by Eq. (3.5) and $A = N_p + N_n$, we obtain

$$\left(E_F^e\right)_{max} = \frac{9\pi A}{4 R_c^3} - (E_F^e)_{max}^{3/2} \left[ \left(\frac{9\pi A}{4 R_c^3} - (E_F^e)_{max}^{3/2} + m_n^2 \right) \right]^{3/4}. \quad (4.17)$$

In the ultra-relativistic approximation $(E_F^e)_{max}^{3/2} \frac{9\pi A}{4 R_c^3} \ll 1$ so Eq. (4.17) can be approximated to

$$(E_F^e)_{max} = 2^{1/3} m_n \gamma \left(-1 + \sqrt{1 + \frac{\beta}{2\gamma^3}}\right) m_n, \quad (4.18)$$

where

$$\beta = \frac{9\pi}{4} \frac{1}{m_n^2 R_c^3}, \quad \gamma = \sqrt{1 + \beta^{2/3}}. \quad (4.19)$$

Correspondingly, the limiting value of the proton fraction $N_p / A$ is given by

$$\frac{N_p}{A} = 2\gamma^3 \left(-1 + \sqrt{1 + \frac{\beta}{2\gamma^3}}\right)^2. \quad (4.20)$$

As expected, the Fermi energy obtained from Eq. (4.18), using Eqs. (4.19) and (4.20), coincides with the ultra-relativistic limit of Eq. (3.9), and obviously with (4.16).

In Fig. 4.4 we have plotted the Fermi energy of electrons as a function of the dimensionless parameter $\xi^{WS}$. We can see that as $\xi^{WS} \rightarrow 0$, the limiting value given by Eq. (4.18) is clearly displayed.

### 4.2 Ground-state of massive nuclear density cores

We turn now to study the energetics of these family of compressed nuclear density cores each characterized by a different Fermi energy of the electrons. The kinematic energy-spectra of complete degenerate electrons, protons and neutrons are

$$e^i(p) = \sqrt{p^2 + m_i^2}, \quad p \leq p_F^i, \quad i = e, p, n. \quad (4.21)$$

So the total energy of the system is given by

$$E_{tot} = E_B + E_e + E_{em}, \quad E_B = E_p + E_n, \quad (4.22)$$

$$E_i = 2 \int \frac{d^3r d^3p}{(2\pi)^3} e^i(p), \quad i = e, p, n, \quad E_{em} = \int \frac{E^2}{8\pi} d^3r. \quad (4.23)$$

Using the analytic solution (4.10) we calculate the energy difference between two systems, $I$ and $II$,

$$\Delta E_{tot} = E_{tot}(E_F^e(II)) - E_{tot}(E_F^e(I)), \quad (4.24)$$

with $E_F^e(II) > E_F^e(I) \geq 0$, at fixed $A$ and $R_c$. 

4.2. GROUND-STATE OF MASSIVE NUCLEAR DENSITY CORES

Figure 4.4: The Fermi energy of electrons in units of the pion rest mass is plotted as a function of the compression parameter $\xi_{WS}$ in the ultra-relativistic approximation. In the limit $\xi_{WS} \to 0$ the electron Fermi energy approaches asymptotically the value $(E^F_e)_{\text{max}}$ given by Eq. (4.18).

We first consider the infinitesimal variation of the total energy $\delta E_{\text{tot}}$ with respect to the infinitesimal variation of the electron Fermi energy $\delta E^F_e$

$$\delta E_{\text{tot}} = \left[ \frac{\partial E_{\text{tot}}}{\partial N_p} \right]_{VWS} \left[ \frac{\partial N_p}{\partial E^F_e} \right] \delta E^F_e + \left[ \frac{\partial E_{\text{tot}}}{\partial V_{\text{WS}}} \right]_{N_p} \left[ \frac{\partial V_{\text{WS}}}{\partial E^F_e} \right] \delta E^F_e. \quad (4.25)$$

For the first term of this relation we have

$$\left[ \frac{\partial E_{\text{tot}}}{\partial N_p} \right]_{VWS} \left[ \frac{\partial N_p}{\partial E^F_e} \right]_{VWS} \approx \left[ \frac{\partial E^F_p}{\partial E^F_e} \right]_{VWS}, \quad (4.26)$$

where the general definition of chemical potential $\partial E_i / \partial N_i = \partial E_i / \partial N_i$ is used ($i = e, p, n)$ neglecting the mass defect $m_n - m_p - m_e$. Further using the condition of the beta-equilibrium (2.21) we have

$$\left[ \frac{\partial E_{\text{tot}}}{\partial N_p} \right]_{VWS} = \left[ \frac{\partial E_{\text{em}}}{\partial N_p} \right]_{VWS}. \quad (4.27)$$

For the second term of the Eq. (4.25) we have

$$\left[ \frac{\partial E_{\text{tot}}}{\partial V_{\text{WS}}} \right]_{N_p} = \left[ \frac{\partial E^F_p}{\partial V_{\text{WS}}} \right]_{N_p} + \left[ \frac{\partial E^F_e}{\partial V_{\text{WS}}} \right]_{N_p} + \left[ \frac{\partial E_{\text{em}}}{\partial V_{\text{WS}}} \right]_{N_p}, \quad (4.28)$$
since in the process of increasing the electron Fermi energy namely, by decreasing the radius of the Wigner-Seitz cell, the system by definition maintains the same number of baryons $A$ and the same core radius $R_c$.

Now $\delta E_{\text{tot}}$ reads

$$\delta E_{\text{tot}} = \left\{ \left[ \frac{\delta E_e}{\delta V^{\text{WS}}_{N_p}} \right]_{N_p} - \frac{\delta V^{\text{WS}}_{V^{\text{WS}}} \partial N}{\delta N_{N_p}} + \left[ \frac{\delta E_{\text{em}}}{\delta V^{\text{WS}}_{N_p}} \right]_{N_p} \frac{\partial N_{N_p}}{\delta E_e} \right\} \delta E_{e_1},$$

so only the electromagnetic energy and the electron energy give non-null contributions.

From this equation it follows that

$$\Delta E_{\text{tot}} = \Delta E_{\text{em}} + \Delta E_e.$$  (4.30)

In the particular case in which $E_{\text{em}}(I) = (E_{\text{em}})^{\text{max}}$ and $E_{e_1}(I) = 0$ we have $\Delta E_{\text{em}} = -E_{e_1}$ because the limiting case $(E_{\text{em}})^{\text{max}}$ corresponds to a locally neutral system. Using Eq. (4.6) the electromagnetic energy difference can be written as

$$E_{\text{em}} \simeq 0.15 \frac{3^{5/3}}{2} \left( \frac{\pi}{4} \right)^{1/3} \frac{(\pi/12)^{1/6}}{\Delta \sqrt{\alpha}} N_{N_p}^{2/3} m_{\pi},$$

while the electron energy difference is given by

$$\Delta E_e \simeq 0.20 \left( \frac{9\pi}{4} \right)^{4/3} \frac{(\pi/12)^{1/6}}{\Delta \sqrt{\alpha}} N_{N_p}^{2/3} m_{\pi}.$$  (4.32)

Finally the total energy difference can be written as

$$\Delta E_{\text{tot}} \simeq 0.75 \frac{3^{5/3}}{2} \left( \frac{\pi}{4} \right)^{1/3} \frac{(\pi/12)^{1/6}}{\Delta \sqrt{\alpha}} N_{N_p}^{2/3} m_{\pi},$$

which is positive. For instance, for a system with $A \sim 10^{57}$ and $R_c \sim 10$ km, the above energy difference is of the order of $10^{34}$ erg. Therefore, the total energy of a massive nuclear density core increases with its electron Fermi energy, and consequently, the ground state of a massive nuclear density core corresponds to the configuration with $E_{e_1} = 0$.

We consider the study of massive nuclear density cores to be necessary to clarify basic conceptual issues prior to a correct description of a neutron star. Neutron stars are composed of two sharply different components: the liquid core at nuclear and/or supra-nuclear density consisting of neutrons, protons and electrons and a crust of white-dwarf-like material, namely of degenerate electrons in a lattice of nuclei [BBP71, HTWW65]. The pressure and the density of the core are mainly due to the baryons while the pressure of the crust is mainly due to the electrons. The density of the crust is due to the nuclei and to the free neutrons due to neutron drip when this process occurs (see e.g. [BBP71]). Consequently, the Fermi energy of the electrons at the surface of the neutron star core will be generally positive in order to take into account the compressional effects of the neutron star crust on the core (see [JARX09a, JARX09b] and Chapter 5 for details). The case of zero electron Fermi energy corresponds to the limiting case of absence of the crust.

### 4.3 Massive cores with smooth proton distribution

We have proved the existence of electric fields close to the critical value $E_c = m_e^2/\alpha$ for massive cores with $\sim 10^{57}$ nucleons using a proton distribution of constant density and a sharp step function at
4.3. MASSIVE CORES WITH SMOOTH PROTON DISTRIBUTION

Let us to introduce the proton distribution function $f_p(z)$ by mean of $n_p(z) = n_c^p f_p(z)$, where $n_c^p$ is the proton density at the center. We also introduce the dimensionless radial coordinate \( z = (r - \bar{R}_c)/a \), with \( a^{-1} = \sqrt{4\pi a n_c^p/m_e} \), and \( \bar{R}_c = R_c + \delta R_c \), where \( \delta R_c << R_c \), so \( \bar{R}_c \simeq R_c \).

In this case, the relativistic Thomas-Fermi equation (2.20) can be rewritten as

\[
\frac{d^2 \xi_e}{dz^2} + \left( \frac{2}{z + \bar{R}_c/a} \right) \frac{d \xi_e}{dz} - \frac{\left( \xi_e^2(z) - 1 \right)^{3/2}}{\mu} + f_p(z) = 0, \tag{4.34}
\]

where \( \mu = 3\pi^2 n_c^p/m_e^3 \) and we have introduced the normalized free electron chemical potential \( \xi_e = \sqrt{(P_{Fe})^2 + m_e^2}/m_e \).

For a given distribution function $f_p(z)$ and a central proton density $n_c^p$, the above Eq. (4.34) can be integrated numerically with the boundary conditions

\[
\xi_e(0) = \sqrt{1 + \left[ \mu \delta f_p(0) \right]^{2/3}}, \quad \xi_e'(0) < 0, \tag{4.35}
\]

where \( \delta \equiv n_e(0)/n_p(0) \).

We use a monotonically decreasing proton distribution fulfilling a Woods-Saxon dependence

\[
f_p(z) = \frac{\gamma}{\gamma + e\beta z}, \tag{4.36}
\]

where \( \gamma > 0 \) and \( \beta > 0 \). Fig. 4.5 shows the proton distribution for a particular values of \( \gamma \) and \( \beta \).

![Proton distribution function](image)

Figure 4.5: Proton distribution function for $\gamma = 1.5$, $\beta \simeq 0.0586$.

We have integrated numerically Eq. (4.34) for different parameters and initial conditions. As an example, we show here the results for the set of parameters and initial conditions of Table 4.1.

In Figs. 4.6 we show the electron and the proton density. In Fig. 4.8 we have plotted the electric field, and Fig. 4.7 shows the charge separation function given by

\[
\Delta(z) = \frac{n_p(z) - n_e(z)}{n_c^p}, \tag{4.37}
\]

which let us to appreciate clearly the proton-electron separation around the core surface.
CHAPTER 4. COMPRESSED MASSIVE NUCLEAR DENSITY CORES

Figure 4.6: Electron and proton density for the initial conditions of Table 4.1

Figure 4.7: Charge separation for the initial conditions of Table 4.1

From Fig. 4.7 we see how the system reaches indeed global charge neutrality in a very small scale of the order of $1/(\alpha m_\pi)$ as noted by Migdal et al. [MVP76] in their classical paper, and confirmed by others (see for instance [FRS80] [RRX07] [VPX09] [MRX09] and references therein). There are two well defined zones with oppositely charged. In the first zone we have $n_p > n_e$ so we have a positive charged shell, while in the second one, we have $n_p < n_e$ and a negative charged shell develops. At the point $n_e = n_p$ we find the maximum of the electric field, which is screened by the negative charged shell until reach global charge neutrality (see fig. 4.8).

The electric field is overcritical but smaller respect to the case of a sharp step proton distribution. However, it is yet well above the critical field $E_c$. The maximum of the electric field occur at the point where the transition from the positive charged shell to the negative charged shell takes place, i.e., where $n_e = n_p$ (see fig. 4.7).

We confirm the existence of overcritical fields on massive nuclear density cores with smooth proton distributions at the core surface. The intensity of the electric field depends on the proton density mainly due to the value of $n_p^c$ and to the diffuseness (sharpness) of the distribution around the core radius.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\Delta \xi'(0)$</th>
<th>$n_p^c$ (cm$^{-3}$)</th>
<th>$N_e = N_p$</th>
<th>$A$</th>
<th>$E_{peak}/E_c$</th>
<th>$R_c$ (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9662053</td>
<td>-0.868051</td>
<td>$1.38 \times 10^{36}$</td>
<td>$10^{54}$</td>
<td>$1.61 \times 10^{55}$</td>
<td>95</td>
<td>5.56</td>
</tr>
<tr>
<td>0.9782929</td>
<td>-0.899201</td>
<td>$2.76 \times 10^{36}$</td>
<td>$2 \times 10^{54}$</td>
<td>$2.35 \times 10^{56}$</td>
<td>125</td>
<td>5.56</td>
</tr>
</tbody>
</table>

Table 4.1: Example of initial conditions for the numerical integration of Eq. (4.34). We also show the corresponding physical quantities obtained from the integration.
4.4 Stability of massive nuclear density cores

The question about the stability of these massive nuclear density cores against nuclear fission and Coulomb repulsion is a very important issue that worths to be discussed. We recall that the analytic formulas that we have derived in this chapter based on the ultrarelativistic Thomas-Fermi equation are also applicable to the case of superheavy nuclei, where the effect of the penetration of the electrons is already relevant, and in fact, the electric field is located only in a very narrow shell on the surface of the superheavy nucleus.

4.4.1 Stability against nuclear fission

The charge-to-mass ratio of the effective charge \( Q \) at the core surface to the core mass \( M \) is given by

\[
\frac{Q}{\sqrt{GM}} \sim \frac{E_{\text{max}} R_c^2}{\sqrt{G m_n A}} \simeq \frac{m_{\text{Planck}}}{m_n} \left( \frac{1}{N_p} \right)^{1/3} \frac{N_p}{A}.
\]

For superheavy nuclei with \( N_p \simeq 10^3 \), the above charge-to-mass ratio for the nucleus gives

\[
\frac{Q}{\sqrt{GM}} > \frac{1}{20} \frac{m_{\text{Planck}}}{m_n} \sim 10^{18}.
\]
Gravitation obviously plays no role in the stabilization of these nuclei. For a massive core at nuclear density the criterion of stability against fission

\[ E_{\text{em}} < 2E_{\text{surf}}, \]  

(4.41)

where \( E_{\text{em}} \) and \( E_{\text{surf}} \) are the Coulomb and surface energy is satisfied. In order to see this we use Eqs. (2.11) and (4.31)

\[ \frac{E_{\text{em}}}{2E_{\text{surf}}} \simeq 0.15 \frac{3}{8} \sqrt{\frac{3\pi}{\alpha}} \frac{N_p}{A} \frac{2/3}{m_{\pi} a_{\text{surf}} / \Delta} \sim 0.1 < 1, \]  

(4.42)

where we have used the estimation of the Coulomb energy given by Eq. (4.31) and the surface energy given by the semi-empirical formula of Wèizsacker (2.11).

### 4.4.2 Estimates of gravitational effects in a Newtonian approximation

First of all, let us to evaluate the charge to mass ratio given by Eq. (4.39) for massive nuclear density cores. In this case we have \( N_p \simeq (m_{\text{planck}} / m_n)^3 \), then Eq. (4.39) is simply

\[ \frac{Q}{\sqrt{GM}} \simeq \frac{N_p}{A} << 1, \]  

(4.43)

as can be seen from Fig. 2.2. It is well-known that the charge-to-mass-ratio (4.43) smaller than 1 evidences the equilibrium of self-gravitating mass-charge system both in Newtonian gravity and general relativity.

In order to investigate the possible effects of gravitation on these massive neutron density cores we proceed to some qualitative and quantitative estimates based on the Newtonian approximation.

1. The maximum Coulomb energy per proton is given by Eq. (4.13) where the potential is evaluated at the center of the core. The Newtonian gravitational potential energy per proton (of mass \( m_p \)) in the field of a massive nuclear density core with \( A \simeq (m_{\text{planck}} / m_n)^3 \) is given by

\[ E_g = -G \frac{m_p M}{R_c} = - \frac{1}{\Delta} \frac{m_{\text{planck}}}{m_n} \frac{m_{\pi} A^{1/3}}{N_p^{1/3}} \simeq - \left( \frac{A}{N_p} \right)^{1/3} \frac{m_{\pi}}{\Delta}. \]  

(4.44)

Since \( A/N_p \simeq 200 \) (see Fig. 2.2) the gravitational energy is larger in magnitude than and opposite in sign to the Coulomb potential energy per proton given by Eq. (4.13)

\[ eV \simeq \left( \frac{9\pi}{4} \right)^{1/3} \frac{m_{\pi}}{\Delta}, \]  

(4.45)

so the system should be gravitationally stable.

2. There is yet a more accurate derivation of the gravitational stability based on the analytic solution of the Thomas-Fermi equation Eq. (4.4). The Coulomb energy \( E_{\text{em}} \) given by Eq. (4.31) is mainly distributed within a thin shell of width

\[ \delta R_c \simeq \frac{\Delta}{\sqrt{\hat{a}m_{\pi}}}, \]  

(4.46)

and proton number

\[ \delta N_p = n_p 4\pi R_c^2 \delta R_c, \]  

(4.47)
4.4. **STABILITY OF MASSIVE NUCLEAR DENSITY CORES**

At the surface, to ensure the stability of the system, the attractive gravitational energy of the thin proton shell

\[ E_{gr} \simeq -3 \frac{A^{4/3}}{\sqrt{\alpha \Delta}} \left( \frac{N_p}{A} \right)^{1/3} \left( \frac{m_n}{m_{\text{Planck}}} \right)^2 m_n, \]  

(4.48)

must be larger than the repulsive Coulomb energy \( E_{\text{nuclear}} \). For small \( A \), the gravitational energy is always negligible. However, since the gravitational energy increases proportionally to \( A^{4/3} \) while the Coulomb energy only increases proportionally to \( A^{2/3} \), the two must eventually cross, which occurs at

\[ A_R = 0.039 \left( \frac{N_p}{A} \right)^{1/2} \left( \frac{m_{\text{Planck}}}{m_n} \right)^{3/2} \]  

(4.49)

This establishes a lower limit for the mass number \( A_R \) necessary for the existence of an island of stability for massive nuclear density cores. The upper limit of the island of stability will be determined by general relativistic effects.

3. Having established the role of gravity in stabilizing the Coulomb interaction of the massive nuclear density core, we outline the importance of the strong interactions in determining its surface. We find for the neutron pressure at the surface:

\[ P_n = \frac{9}{40} \left( \frac{3}{2\pi} \right)^{1/3} \left( \frac{m_n}{m_{\pi}} \right) \left( \frac{A}{N_p} \right)^{5/3} m_{\pi}^4, \]  

(4.50)

and for the surface tension, as extrapolated from nuclear scattering experiments,

\[ P_{\text{surf}} = -\left( \frac{0.13}{4\pi} \right) \left( \frac{A}{N_p} \right)^{2/3} m_{\pi}^4 \]  

(4.51)

We then obtain

\[ \frac{|P_s|}{P_n} = 0.39 \Delta^3 \left( \frac{N_p}{A} \right) = 0.24 \rho_{\text{nuc}} \rho_{\text{surf}}' \]  

(4.52)

where

\[ \rho_{\text{nuc}} = \frac{3m_nA}{4\pi R_c^2}. \]  

(4.53)

The relative importance of the nuclear pressure and nuclear tension is a very sensitive function of the density \( \rho_{\text{surf}} \) at the surface.

4. It is important to emphasize a major difference between nuclei and the massive nuclear density cores treated in this chapter: the gravitational binding energy per nucleon in these massive nuclear density cores is instead

\[ E_{gr} \simeq \frac{GM_c m_n}{R_c} \simeq 0.1m_n \simeq 94 \text{ MeV}, \]  

(4.54)

is much bigger than the binding energy in ordinary nuclei

\[ E_{\text{nuclear}} \simeq 8 \text{ MeV}. \]  

(4.55)
4.5 Conclusions

We extrapolated the results obtained for compressed atoms to the case of massive neutron density cores for $A \simeq (m_{\text{Planck}}/m_n)^3 \sim 10^{57}$. In both systems of the compressed atoms and of the massive nuclear density cores a maximum value of the Fermi energy has been reached corresponding to the case of Wigner-Seitz cell radius $R_{WS}$ coincident with the core radius $R_c$. The results generalize the considerations presented in the article corresponding to a massive nuclear density core with null Fermi energy of the electrons [VPX09].

An entire family of configurations exist with values of the Fermi energy of the electrons ranging from $E_F^e = 0$ to the maximum value $(E_F^e)_{\text{max}}$. The configuration with $E_F^e = (E_F^e)_{\text{max}}$ fulfills both the global and the local charge neutrality and correspondingly no electrodynamical structure is present in the core. The configuration with $E_F^e = 0$ has the maximum value of the electric field at the core surface, well above the critical value $E_c$ (see Fig. 4.3). All these cores with overcritical electric fields are stable against the vacuum polarization process due to the Pauli blocking by the degenerate electrons [RVX09].

We have compared and contrasted our treatment of Thomas-Fermi ultrarelativistic solutions to the corresponding one addressed in the framework of strange stars [AFO86] pointing out in that treatment some inconsistency in the definition of the Coulomb potential. In that article, by increasing the compression on the electrons of the core due to the crust, the depth of the Coulomb potential energy remains unchanged, which is clearly incorrect as can be seen from Fig. 4.2.

We have discussed the problem about the ground-state configuration of a massive nuclear density core by calculating their energetics for selected values of the electron Fermi energy. We found that the configuration with null Fermi energy of the electrons represent the ground-state of the system.

We have also discussed the effect of the proton density profile on the electromagnetic structure of the cores. We have then integrated the relativistic Thomas-Fermi equation for massive nuclear density cores assuming a proton distribution fulfilling a Woods-Saxon dependence. We have confirmed the existence of overcritical fields on the core surface also in this case. However, the field is less intense. We also analyzed the effect of the bulk value of the proton density and the effect of the diffuseness parameter of the proton distribution on the intensity of the electric field.

Exploiting the analytic solution to the ultrarelativistic Thomas-Fermi equation, we discuss the stability of massive nuclear density cores against nuclear fission and against Coulomb repulsion. We found that these systems are indeed stable against both fission and repulsion. The criterion of stability against Coulomb repulsion determines the existence of a lower limit for the mass number $A$ below which the gravitational interaction cannot balance the Coulomb repulsion on the protons at the surface of the core. Finally we compare the pressure due to surface tension and the neutron pressure on the core surface in order to establish if such an abrupt density decrease at nuclear density can indeed exist for these kind of objects. We found that assuming the surface tension per nucleon to be of the same order of the one acting on ordinary nuclei, such sharp density profile can occur.

The problem of compressed massive nuclear density cores, have been treated by the solution of the relativistic Thomas-Fermi equation and by enforcing the condition of beta equilibrium. This is a theoretically well defined problem and, in our opinion, a necessary step in order to approach a more complex problem of a neutron star core and its interface with the neutron star crust.

4.6 Perspectives

We have proved the existence of overcritical fields on the surface of massive nuclear density cores. Because the Pauli blocking, the electron-positron vacuum polarization is forbidden [RVX09]. An interesting problem that deserves attention is the analysis of the vacuum polarization instability
against small radial oscillations of these cores. The long time scale associated with the beta equilibrium process \( \sim \text{min.} \) respect to the one of vacuum polarization \( \sim \frac{\hbar}{m_e c^2} \sim 10^{-21} \text{ s.} \) could avoid the Pauli blocking and opening the phase space for electron-positron pair creation. However, further analysis must be done in order to determine if indeed this process can occur and, if this was the case what implications should it have for the structure of the configuration.
CHAPTER 4. COMPRESSED MASSIVE NUCLEAR DENSITY CORES
Chapter 5

Neutron star equilibrium configurations

Recently, assuming a simplified but rigorous model based on analytic solutions of the relativistic Thomas–Fermi equation, Ruffini et al. \[\text{VPX09}\] constructed equilibrium configurations of electrons, protons, and neutrons in beta equilibrium satisfying global charge neutrality instead of local neutrality \(n_e = n_p\). In their approach they do not consider any explicit dependence on the gravitational interaction. The sole effect of gravitation was simulated by imposing the core at nuclear density to be confined in a box. The density of protons was considered to be constant in the box and the relativistic Thomas-Fermi equation with the condition of beta equilibrium was solved analytically. The details of their approach can be found in Chapter 2, Section 2.2. They did not consider cores subjected to external pressure, which corresponds in a neutron star to neglect the crust component. Consequently, the Fermi energy of the electrons was assumed there to be equal to zero. They found that on the surface of such configurations an electric field develops and extends over a surface-shell with a thickness of the order of the electron Compton wavelength \(\lambda_e = 1/m_e\). The intensity of such a field was found to be larger than the critical field for vacuum polarization

\[
E_c = \frac{m_e^2}{\sqrt{\alpha}}, \quad (5.1)
\]

However, the \(e^+e^-\) pair creation does not occur because all possible electron states are occupied, so blocked by Pauli principle \[\text{VPX09}\].

The Chapter 4 was devoted to the detailed description of the recent article of Ruffini et al. \[\text{MRX09}\], which generalize the above treatment of uncompressed nuclear matter to the case of positive electron Fermi energies. There, it is still assumed a globally neutral distribution of baryons in a core of radius \(R_c\) and \(N_e\) electrons in beta equilibrium confined in a Wigner-Seitz cell of radius \(R_{WS} \geq R_c\) which is determined by the condition of overall neutrality \(\int_{0}^{R_{WS}} n_e d^3r = \int_{0}^{R_c} n_p d^3r\). Again the effect of gravity was not considered explicitly and simulated by a boxed nuclear density distribution with a constant proton density. One of the important aspects of this simplified but rigorous analytic treatment is that it is possible to analyze some of the features which characterize the presence of a crust in a realistic neutron star configuration. Namely, the effect of the crust on compressing the electrons of the core. They found a new family of equilibrium configurations of these massive nuclear density cores for positive values of the Fermi energy of the electrons. This new family of cores with specific electrodynamical properties, range from a value of \(E_F^e = 0\) to a maximum value of \(E_F^e\) obtained when \(R_{WS} = R_c\) given by

\[
E_F^e = \left[-\frac{m_e}{m_\pi} + \sqrt{\left(\frac{m_e}{m_\pi}\right)^2 + \frac{(3\pi^2 n)^{2/3}}{m_\pi^2} \left(\frac{N_p}{A}\right)^{2/3}}\right] m_\pi, \quad (5.2)
\]
For each value of the Fermi energy there exists a critical field at the surface which decreases by increasing the value of the electron Fermi energy of the configuration. The corresponding configuration with the maximum value of $E_F$ where $R_{WS} = R_c$ is also the one satisfying the local charge neutrality condition. What is most remarkable is that these configurations which have global charge neutrality are always energetically favorable with respect to the ones with local charge neutrality.

Nevertheless, in the approaches described in Chapters 3 and 4 on compressed nuclear matter, and in particular the one of compressed massive nuclear density cores as we said above, do not take explicitly into account the gravitational and the strong interactions. The aim of this chapter is to generalize all the above results extending the treatment to general relativity, in the non-rotating case, taking into account explicitly and self-consistently the effect of gravity. We focus on relaxing the traditional condition of local charge neutrality $n_e = n_p$, which appears to have been assumed only for mathematical convenience without any physical justification. Instead, we adopt the more general condition of global charge neutrality $N_e = N_p$. The corresponding equilibrium equations then follow from self-consistent solution of the relativistic Thomas-Fermi equation, the Einstein-Maxwell equations and the beta equilibrium condition, properly expressed in general relativity.

5.1 Matter phases in neutron star interiors

Neutron stars are composed of two sharply different components: the liquid core at nuclear and/or supra-nuclear density and a crust below nuclear density [BBP71, HTWW65, ST83]. The pressure and the density of the core are mainly due to the baryons while the pressure of the crust is mainly due to the electrons. The density of the crust is due to the nuclei and to the free neutrons due to neutron drip when this process occurs (see e.g. [BBP71]).

The core is composed mainly by neutrons, and a smaller presence of protons and electrons in beta equilibrium, which determines the equilibrium particle fraction. This uniform phase survives approximately until the ordinary nuclear density

$$\rho_0 \simeq m_n \frac{m_n^3}{2} \simeq 2.7 \times 10^{14} \text{ g cm}^{-3},$$

(5.3)

where the uniform phase is expected to be unstable against proton clustering. Explicit calculations on this unstability can be found in [BBP71].

The crust is generally divided in an outer and an inner crust. The outer crust is composed by white-dwarf-like material, a nuclei lattice (also known as Coulomb lattice) immersed in a sea of free electrons. It extends from the bottom layers of the neutron star until the so-called neutron drip density

$$\rho_{drip} \simeq 4.33 \times 10^{11} \text{ g cm}^{-3},$$

(5.4)

which triggers the dripping out of the less bounded neutrons from nuclei. Consequently, at higher densities, an inner crust is formed composed by a nuclei lattice in a background of electrons and neutrons. The extension and properties of the inner crust is currently matter of discussion (see e.g. [CH08]). We will turn back to this in Section .

5.2 Einstein-Maxwell system of equations

We are interested in constructing general relativistic neutron star equilibrium configurations that satisfy global neutrality instead of local neutrality. In order to do that, we need to formulate our structure equations taking into account the electromagnetic interaction. We start with the formulation of the Einstein-Maxwell system of equations for the case of a non-rotating neutron star.
5.2. EINSTEIN-MAXWELL SYSTEM OF EQUATIONS

The Einstein-Maxwell equations are given by

\[ G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -8\pi G T_{\alpha\beta}, \]

\[ F_{\alpha\beta} = 4\pi j^\alpha, \]

where \( G_{\alpha\beta} \) is the Einstein tensor, \( R_{\alpha\beta} \) is the Ricci tensor, \( R = R^a_a \) is the Ricci scalar, \( g_{\alpha\beta} \) denotes the metric tensor, \( T_{\alpha\beta} \) is the combined energy-momentum tensor of matter and fields, \( j^\alpha \) is the electromagnetic four-current, and the electromagnetic tensor \( F_{\alpha\beta} \) is given by

\[ F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \]

where \( A_\alpha \) denotes the electromagnetic four-potential.

The interior metric of a non-rotating neutron star can be written in the Schwarzschild-like form

\[ ds^2 = e^\nu dt^2 - e^{\nu/2} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \]

where \( \nu \) and \( \lambda \) are functions only of \( r \).

The general anisotropic energy-momentum tensor in spherical symmetry of a multi-component fluid can be written as (see Appendix B for details on the derivation)

\[ T_{\alpha\beta} = \sum_i T^{(i)}_{\alpha\beta} = \sum_i (\mathcal{E}^{(i)} + p^{(i)}_{\perp}) u_\alpha u_\beta - \sum_i p^{(i)}_{\perp} g_{\alpha\beta} + \sum_i (p^{(i)}_{\perp} + p^{(i)}_r) \lambda^\alpha \chi_\beta, \]

where \( \mathcal{E}^{(i)}, p^{(i)}_r, \) and \( p^{(i)}_{\perp} \) are the energy-density, the radial pressure, and the tangential pressure of the \( i \)-fluid. Here \( u_\alpha = e^{\nu/2} \delta_\alpha^0 \) is the four-velocity of an observer in the Schwarzschild frame and \( \chi_\alpha = e^{\lambda/2} \delta_\alpha^1 \).

The above energy-momentum tensor can be expressed as

\[ T_{\alpha\beta} = \text{diag} \left( \sum_i \mathcal{E}^{(i)}, -\sum_i p^{(i)}_{\perp}, -\sum_i p^{(i)}_r, -\sum_i p^{(i)}_{\perp} \right), \]

which in the case of a perfect fluid endowed with electric field gives

\[ T^a_\beta = \text{diag}(\mathcal{E} + \mathcal{E}^{\text{em}}, -P - P^{\text{em}}, -P + P^{\text{em}}, -P + P^{\text{em}}), \]

where

\[ \mathcal{E}^{\text{em}} = -P^{\text{em}} = \frac{\mathcal{E}^2}{8\pi}, \]

and \( \mathcal{E} \) and \( P \) are the energy density and pressure of matter, and the electric field \( E(r) \) is related with the Coulomb potential \( V(r) \) and the electric charge \( Q(r) \) by

\[ E(r) = -\frac{dV}{dr} e^{-\nu(r)} = \frac{Q(r)}{r^2}. \]

With all the above definitions, the time-independent Einstein-Maxwell field equations (5.5)–(5.6) read

\[ \frac{dM}{dr} = 4\pi r^2 \mathcal{E} + 4\pi e^{\lambda/2} \mathcal{E} \rho, \]

\[ \rho = 8\pi G T^1_1, \]

\[ e^{-\lambda} \left( \frac{1}{r} \frac{dv}{dr} + \frac{1}{r^2} \right) = -8\pi G T^1_1, \]

\[ e^{-\lambda} \left( \frac{d^2 V}{dr^2} + \frac{dV}{dr} - \frac{d\lambda}{dr} \right) \left( \frac{1}{2} \frac{dv}{dr} + \frac{1}{r} \right) = -16\pi G T^2_2, \]

\[ \frac{d^2 V}{dr^2} + \frac{2dV}{r} \left( 1 - \frac{r}{4} \frac{dV}{dr} + \frac{d\lambda}{dr} \right) = -4\pi e^{\lambda/2} \rho, \]
where we have used the fact that in the static case, the electromagnetic four-current is given by

$$j^\alpha = \rho_c u^\alpha = \rho_c e^{-\nu/2} \delta_0^\alpha,$$  \hspace{1cm} (5.18)

being $\rho_c$ the charge density.

In addition, we have defined the mass of the star $M(r)$ through

$$e^{-\lambda} = 1 - \frac{2GM(r)}{r} + r^2 E^2(r) = 1 - \frac{2GM(r)}{r} + \frac{Q^2(r)}{r^2}. \hspace{1cm} (5.19)$$

In the case of a charged system, Eq. (5.19) warranties the matching with the exterior Reissner-Nordström metric, for which

$$e^{-\lambda_{RN}} = 1 - \frac{2GM(R)}{r} + \frac{Q^2(R)}{r^2}, \hspace{0.5cm} \text{for } r \geq R. \hspace{1cm} (5.20)$$

5.3 General relativistic equilibrium equations

The Einstein-Maxwell system of equations (5.14)–(5.17) is underdetermined, because we have six unknowns $\{M, \nu, E, P, V, \rho_c\}$ to be solved with four equations. If the system is composed by neutrons, protons, and electrons, the energy density and pressure can be written as a function of the particle densities $n_i$ or as a function of the free particle chemical potentials $\mu_i$ defined as usual by

$$\mu_i = \frac{\partial E}{\partial n_i}. \hspace{1cm} (5.21)$$

Although in doing this specification we have replaced the old unknowns $\{M, \nu, E, P, V, \rho_c\}$ with the new ones $\{M, \nu, n_e, n_p, n_n, V\}$, the degree of underdetermination of the Einstein-Maxwell system holds. In order to close the system of equations, two additional equations must be provided. However, the imposition of such additional constraints should be done on physical grounds and not only by mathematical convenience.

A first condition is given by the equilibrium of the configuration against direct and inverse beta decays of nuclear matter, which can be mathematically expressed as

$$\mu_n = \mu_e + \mu_p, \hspace{1cm} (5.22)$$

where $\mu_i$, the free chemical potential of the $i$-specie, is given by Eq. (5.21).

Traditionally, the equation that closes the system is represented by the condition of local charge neutrality $n_e = n_p$. Below in Section 5.4 we show that such assumption is inconsistent with any solution of the Einstein system of equations given by Eqs. (5.14)–(5.16) with $E \equiv 0$. Therefore, the global charge neutrality $N_e = N_p$ appears to be more physical and suitable constraint. The analogy with atoms, suggests that in order to impose global neutrality as well as quantum statistics on the leptonic component, the Thomas-Fermi equilibrium condition, properly generalized to general relativity must be satisfied.

The general relativistic Fermi energy of the $i$-particle specie can be written as

$$E^F_i = \sqrt{g_{00}} \mu_i + q_i V, \hspace{1cm} (5.23)$$

where $g_{00}$ is the 00 component of the metric tensor, and $q_i$ is the particle charge, i.e.

$$q_i = \begin{cases} 0, & i = n, \\ +e, & i = p, \\ -e, & i = e. \end{cases} \hspace{1cm} (5.24)$$
5.3. GENERAL RELATIVISTIC EQUILIBRIUM EQUATIONS

With the above definition (5.23) the general relativistic Thomas-Fermi equilibrium for the electron gas reads

\[ E^F_e = e^{\nu/2} \mu_e - eV = \text{constant}, \quad (5.25) \]

where

\[ \mu_e = \sqrt{\left( \frac{P^F_e}{m_e} \right)^2 + m_e^2}, \quad (5.26) \]

is the electron free chemical potential and \( P^F_e = \left( \frac{3}{\pi^2} n_e \right)^{1/3} \) is the Fermi momentum of degenerate electrons.

From Eq. (5.25) we obtain the electron number density

\[ n_e = e^{-3\nu/2} \left( \frac{\tilde{V}}{2} - m_e^2 e^{\nu} \right)^{3/2}, \quad (5.27) \]

where we have defined the shifted Coulomb potential energy

\[ \tilde{V} = eV + E^F_e. \quad (5.28) \]

Consequently, by introducing Eq. (5.27) into Eq. (5.17), and taking into account that for a gas of neutrons, protons, and electrons the charge density is given by

\[ \rho_c = e(n_p - n_e), \quad (5.29) \]

where \( e = \sqrt{\alpha} \) is the proton charge, the Einstein-Maxwell equations (5.14)–(5.17) becomes

\[ \frac{dM}{dr} = 4\pi r^2 e - \frac{4\pi^3}{e^{\nu/2}} \frac{d\tilde{V}}{dr} \left[ n_p - e^{-3\nu/2} \frac{3}{\pi^2} \left( \tilde{V}^2 - m_e^2 e^{\nu} \right)^{3/2} \right], \quad (5.30) \]

\[ \frac{1}{r} \frac{d}{dr} \left( \frac{1}{r} \frac{d\nu}{dr} + e^{-\lambda} \right) = 8\pi G e^{-\lambda} \left[ P - e^{-(\nu + \lambda)} \left( \frac{d\tilde{V}}{dr} \right)^2 \right], \quad (5.31) \]

\[ \frac{d^2\nu}{dr^2} + \left( \frac{d\nu}{dr} - \frac{d\lambda}{dr} \right) \left( \frac{1}{2} \frac{d\nu}{dr} + \frac{1}{r} \right) = 16\pi G e^{-\lambda} \left[ P - e^{-(\nu + \lambda)} \left( \frac{d\tilde{V}}{dr} \right)^2 \right], \quad (5.32) \]

\[ \frac{d^2\tilde{V}}{dr^2} + \frac{2}{r} \frac{d\tilde{V}}{dr} \left[ 1 - r \left( \frac{d\nu}{dr} + \frac{d\lambda}{dr} \right) \right] = -4\pi e^{\nu/2} e^\lambda \left( n_p - e^{-3\nu/2} \frac{3}{\pi^2} \left( \tilde{V}^2 - m_e^2 e^{\nu} \right)^{3/2} \right), \quad (5.33) \]

where the function \( \lambda(r) \) is related to the functions \( \nu(r), M(r), \) and \( \tilde{V}(r) \) through Eq. (5.19).

The Eq. (5.33) represents the general relativistic generalization of the Thomas-Fermi equation (3.4), which in the ultrarelativistic approximation of the electron gas becomes

\[ \frac{d^2\tilde{V}}{dr^2} + \frac{2}{r} \frac{d\tilde{V}}{dr} \left[ 1 - r \left( \frac{d\nu}{dr} + \frac{d\lambda}{dr} \right) \right] = -4\pi e^{\nu/2} e^\lambda \left( n_p - e^{-3\nu/2} \frac{3}{\pi^2} \tilde{V}^3 \right), \quad (5.34) \]

The full system of coupled equilibrium equations is composed by the Einstein-Maxwell equations (5.30)–(5.32), the general relativistic Thomas-Fermi equation (5.33), the beta equilibrium equation (5.22), and the general relativistic Thomas-Fermi equilibrium condition for electrons given by Eq. (5.25). The system can be completely solved, for instance numerically, providing an EOS for the baryonic component in the core and for the leptonic component of the crust.
CHAPTER 5. NEUTRON STAR EQUILIBRIUM CONFIGURATIONS

5.4 Inconsistency of local neutrality

In the case of massive nuclear density cores introduced in the Ch. 4, the state of completely charge neutrality \( N_e = N_p \) and \( n_e = n_p \) is indeed reachable for a critical value of the Fermi energy \( (E^*_F)_{\text{max}} \) obtained when \( R_{WS} = R_c \) (see Eq. (4.16) in Ch. 4 for details).

In the full general relativistic treatment such a configuration is never reachable. In correspondence of the maximum value of the Fermi energy the difference of mass between the electron and the proton induces the presence of an electric field. Indeed, we can show that even in the simpler case of a gas of degenerate electrons, protons, and neutrons obeying Fermi–Dirac statistics and interacting only through gravity, never the configuration of local charge neutrality can be reached.

In this case the equilibrium conditions given by the Einstein-Maxwell equations (5.14)–(5.17), the beta equilibrium given by Eq. (5.22), and the general relativistic Thomas-Fermi condition (5.25) reduce to the conditions

\[
E^F_i = e^{\nu_i/2} \mu_i + q_i V = \text{constant} = m_i e^{\nu_i(R)/2} + q_i V(R), \quad i = e, p, n, \tag{5.35}
\]

where \( q_i \) is the particle charge of the \( i \)-specie and we have fixed the constants at the star radius \( r = R \).

Using Eq. (5.35) for the proton and the electron, together with the relation between Fermi momentum and particle density \( P^F_i = (3\pi^2 n_i)^{1/3} \), we obtain the proton to electron density ratio

\[
\frac{n_p(r)}{n_e(r)} = \left( \frac{m_p}{m_e} \right)^{3/2}, \tag{5.36}
\]

where the function \( f(r) \) is given by

\[
f(r) = \frac{m_p e^{\nu(R)/2} + e V(R) - e V(r)}{m_e e^{\nu(R)/2} - e V(R) + e V(r)}. \tag{5.37}
\]

In the case of local neutrality we expect \( e V(R) = e V(r) \) at each point, if we impose such condition, the proton to electron density ratio (5.36) becomes

\[
\frac{n_p(r)}{n_e(r)} = \left( \frac{m_p}{m_e} \right)^3. \tag{5.38}
\]

which is clearly inconsistent with \( n_e(r) = n_p(r) \).

If we now impose the local neutrality \( n_p(r) = n_e(r) \), Eq. (5.36) implies

\[
e V(r) - e V(R) = \left( \frac{m_p - m_e}{2} \right) \left[ \frac{e^{\nu(R)} - e^{\nu(r)}}{e^{\nu(R)/2}} \right]. \tag{5.39}
\]

Then we obtain that the local neutrality condition implies a non-flat Coulomb potential and therefore an electric field, which is clearly inconsistent once again.

We have demonstrated that the local neutrality condition is not consistent with the equilibrium conditions given by Eq. (5.35), which are related not only to thermodynamic and chemical equilibrium but, being equivalent to the TOV equation, represent the conditions to satisfy hydrostatic equilibrium, and the conditions to satisfy the Einstein-Maxwell field equations from which the TOV equation comes (see Appendix C for details).
5.5 Equation of State

In order to illustrate the application of our approach to construct neutron star equilibrium configurations, we will integrate the equations of equilibrium for two different EOS. The first EOS is the one of Baym, Bethe, and Pethick (BBP) [BBP71] based on the phenomenological liquid drop model of the nucleus, which we recalled in Chapter 2. In order to show that the conclusions we will reach about the structure of the neutron star, are independent on the details of the strong interaction model adopted, we also integrate the equilibrium equations for a simplified model neglecting the strong interactions, i.e., we describe the baryonic component as a degenerate gas of neutrons and protons obeying Fermi-Dirac statistics and interacting only through gravity.

5.5.1 Baym-Bethe-Pethick model

In a classic article Baym, Bethe and Pethick [BBP71] presented the problem of matching to the crust in a neutron star a liquid core composed of \(N_n\) neutrons, \(N_p\) protons and \(N_e\) electrons. In order to do that, they proposed an EOS based on a compressible liquid drop model of the nucleus. One of the important aspects of this EOS, is that the two components of the neutron star, the core and the crust, are described using the same variables, so the same energy density and pressure functions. This means that the energy density and pressure of the uniform neutron, proton, electron gas at supranuclear density, are the same as the energy density and pressure of the non-uniform phase formed by the nuclei lattice in the background of free electrons (and neutrons when they drip out from the nucleus), ignoring the effects connected with the finite size of the nuclei, i.e. the EOS of the uniform phase it is obtained by doing a bulk approximation of the EOS of the non-uniform one.

Core

The core is the region at and above nuclear density \(\rho_0\), so it extends from the center \(r = 0\) up to the core radius \(r = R_c\), which we define as

\[
\rho(R_c) = \rho_0 \approx m_n \frac{m_n^3}{2} \approx 2.7 \times 10^{14} \text{ g cm}^{-3}.
\]  

Actually, the definition of the radius of the core can change depending on the nuclear model adopted. The reason is that the point where it is expected that the uniform phase of neutrons, protons, and electrons in the core be unstable against proton clustering, i.e. against non-uniformities, depends on the nuclear model adopted. However, for most models it is about the normal nuclear density \(\rho_0 \approx m_n m_n^3 / 2 \approx 2.7 \times 10^{14} \text{ g cm}^{-3}\). Therefore, we assume that the core formed by neutrons, protons, and electrons in a uniform phase exists up to this density.

In general, the matter energy density and the pressure can be written as the sum of the contribution due to baryons (neutrons and protons) and electrons as follows

\[
\mathcal{E} = \mathcal{E}_b + \mathcal{E}_e, \quad E_b + E_e,
\]

\[
P = P_b + P_e,
\]

where the contribution due to degenerate electrons is given by

\[
\mathcal{E}_e = \frac{2}{(2\pi)^3} \int_{0}^{p_f} \frac{p^2}{\sqrt{p^2 + m_e^2}} 4\pi p^2 dp,
\]

\[
P_e = \frac{1}{3} \frac{2}{(2\pi)^3} \int_{0}^{p_f} \frac{p^2}{\sqrt{p^2 + m_e^2}} 4\pi p^2 dp.
\]
In terms of the baryon number density
\[ n = n_p + n_n = \frac{2k^3}{3\pi^2}, \] (5.45)
and the proton fraction
\[ Y = \frac{1 - T}{2} \equiv \frac{n_p}{n}, \] (5.46)
the contribution of baryons to the energy density and to the pressure given by the BBP model can be written as (see Section 8, page 257 of [BBP71] for details)
\[ E_b = n[(1 - Y)m_n + Ym_p + E(k,Y)], \] (5.47)
\[ P_b = n^2 \frac{\partial E(k,Y)}{\partial n}, \] (5.48)
where
\[ E(k,Y) = E_k(k,Y) + \left[ E(k,1/2) - \frac{3k^2}{10m_n} \right] (1 - 3T^4 + 2T^6) + \left[ s \frac{k}{k_0^2} - \frac{k^2}{6m_n} \right] T^2(1 - T^2)^2 \]
\[ + \left[ E(k,0) - \frac{3 \cdot 2^{2/3} k^2}{10} m_n \right] (3T^4 - 2T^6) + \left[ \mu_p^{(0)} - \mu_n^{(0)} + 2^{2/3} \frac{k^2}{2m_n} \right] \frac{1}{4} (T^4 - T^6), \] (5.49)
and the parameter \( T \) is given as a function of \( Y \) through Eq. (5.46). In the above equations we have defined the quantities [BBP71]
\[ E_k(k,Y) = \frac{3 \cdot 2^{2/3} k^2}{10m_n} [Y^{5/3} + (1 - Y)^{5/3}], \] (5.50)
\[ E(k,1/2) = -w_0 + \frac{K}{2k_0^2} (k - k_0)^2, \] (5.51)
\[ E(k,0) \approx 19.74k^2 - k^3 \left( \frac{40.4 - 1.088k_0^3}{1 + 2.545k_0} \right), \] (5.52)
\[ \mu_p^{(0)} = -k^3 \left( \frac{218 + 277k}{1 + 8.57k^2} \right), \] (5.53)
\[ \mu_n^{(0)} = E(k,0) + \frac{1}{3} k \frac{\partial E(k,0)}{\partial k}, \] (5.54)
where \( \bar{k} = k/(\hbar c) \approx 5.06 \times 10^{-3} k \) is measured in fm\(^{-1} \), \( k \) is measured in MeV, and the constants of the model fixed to fit experimental data on the masses of observed nuclei, are given by \( w_0 = 16.5 \text{ MeV}, k_0 = 1.43 \text{ fm}^{-1}, K = 143 \text{ MeV}, \) and \( s = 33 \text{ MeV} \).

**Crust**

Starting from the radius of the star \( R \) where the total internal pressure is zero, i.e., \( P(R) = 0 \), the crust is composed up to a density \( \rho \approx \rho_{drip} \approx 4.33 \times 10^{11} \text{ g cm}^{-3} \) by white-dwarf-like material: a nuclei lattice in a background of electrons [BBP71]. Usually this region is called outer crust. Indeed it can be described by a sequence of Wigner-Seitz cells each one characterized by its radius and Fermi momentum which can be calculated through the global equilibrium condition (see [LL80], [JARX09a] for details)
\[ \mu_e e^{i/2} + \xi m_e e^{i/2} = \text{constant}, \] (5.55)
where $\xi = A/N_p$ is the mass to charge ratio of the nuclei forming the lattice (we will assume $\xi = 2$ for numerical calculations). Fixing the constant on the right-hand side of Eq. (5.55) at the radius of the star $r = R$ we obtain the equilibrium condition

$$\mu_e v^2 + \xi m_e v^2 = v^2 m_e + \xi m_e v^2.$$  \hfill (5.56)

Using the general relativistic definition of Fermi energy given by Eq. (5.25), we obtain the Fermi energy of the electrons of a local Wigner-Seitz cell localized at a distance $r$

$$E_{\text{crust}}^e(r) = m_e v^2 + \xi m_e [v^2(r)/2 - v^2(r)/2] \simeq \xi m_e [v^2(r)/2 - v^2(r)/2].$$  \hfill (5.57)

As we should expect at $r = R$ we have $E_{\text{crust}}^e(R) = m_e v^2$ while at $r = R_{\text{WS}}$

$$E_{\text{crust}}^e(R_{\text{WS}}) \equiv E_c \simeq \xi m_e [v^2(R)/2 - v^2(R_{\text{WS}})/2].$$  \hfill (5.58)

The Newtonian limit of the above Eq. (5.58) reads

$$E_{\text{crust}}^e(R_{\text{WS}}) \equiv E_c \simeq \xi m_e \left[ -\frac{GM(R)}{R} + \frac{GM(R_c)}{R_c} \right].$$  \hfill (5.59)

From Eqs. (5.57) and (5.58), it can be explicitly seen the compressional effects of the crust determining the Fermi energy of the electrons at the different layers of the neutron star. It is precisely on this kind of systems, where our analytic results on compressed massive nuclear density cores become relevant and acquires great interest (see Chapter 4).

If the electron Fermi energy exceeds some critical value $E_{\text{drip}}^e$, the density at the edge of the crust will be larger than $\rho_{\text{drip}} \simeq 4.33 \times 10^{11} \text{ g cm}^{-3}$, and a component of neutrons outside the nuclei appears \cite{BBP71}. This white-dwarf-like material with the additional gas of neutrons is usually called inner crust, which we describe through the BBP EOS.

It is possible to calculate the critical value $E_{\text{drip}}^e$ as follows. Using Eq. (5.27), the electron number density at the Wigner-Seitz cell of the core can be written as

$$n_e^{\text{crust}} \equiv n_e(R_{\text{WS}}) \simeq \frac{e^{-3\pi R_c^2}}{3\pi^2} (E_c)^3,$$  \hfill (5.60)

and putting $n_e^{\text{crust}} \simeq \rho_{\text{drip}}/(2m_n)$ we finally obtain

$$E_{\text{drip}}^e \simeq \left( \frac{3\pi^2 \rho_{\text{drip}}}{2m_n} \right)^{1/3} \simeq 0.2 \text{ m}_\pi.$$  \hfill (5.61)

The total baryon energy density in this regime within the BBP model can be written as

$$\mathcal{E}_b = n_N \mathcal{E}_N + (1 - V_N n_N) \mathcal{E}_n,$$  \hfill (5.62)

$$P_b = n^2 \frac{\partial \mathcal{E}_b}{\partial n},$$  \hfill (5.63)

where we have defined

$$E_N = A[(1 - Y)m_n + Y m_p] + E(k, Y) + E_C + E_{\text{surf}},$$  \hfill (5.64)

$$\mathcal{E}_n = n_n [E(k_n, 0) + m_n].$$  \hfill (5.65)

The energy $E(k, Y)$ is given by Eq. (5.49), and $E_C$ and $E_{\text{surf}}$ denote the Coulomb energy (including the lattice energy) and the surface energy of the nucleus. The number of nucleons and protons of the nucleus is denoted by $A$ and $N_p$ respectively, $n_N$ is the number density of nuclei, $V_N$ is the volume.
of the nucleus, the factor $1 - V_{NN}n$ is the fraction of the total volume occupied by the neutron gas with wave number denoted by $k_n$.

The equilibrium conditions of the system in this regime can be summarized as

\begin{align}
E_{\text{surf}} &= 2E_C, \quad (5.66) \\
\mu_n^{(N)} - \mu_p^{(N)} &= \mu_e - (m_n - m_p), \quad (5.67) \\
\mu_n^{(N)} &= \mu_n^{(G)}, \quad (5.68) \\
p^{(N)} &= p^{(G)}, \quad (5.69)
\end{align}

where $\mu_n^{(N,G)}$ is the neutron chemical potential in nuclei and in the gas outside nuclei, and $\mu_p^{(N)}$ is the proton chemical potential inside nuclei. Here the chemical potential of protons and neutrons does not include their rest-masses, whereas the electron chemical potential $\mu_e$ does. The pressure on a nucleus is denoted by $p^{(N)}$ and $p^{(G)}$ is the outside neutron pressure.

The Eq. (5.66) comes from the condition that the energy per nucleon inside nuclei be a minimum, while Eq. (5.67) enforces the beta equilibrium between neutrons, protons, and electrons. The Eqs. (5.68) and (5.69) establish the equilibrium condition between neutrons outside and inside nuclei. Additional details can be found in [BBP71] while a brief summary can be found in [ST83].

In Tables 1–3 of [BBP71] it can be found the general properties of this gas as a function of the rest-density of the system. In Table 5.1 we reproduce the Table 1 of [BBP71], which includes the most relevant information to our purpose.

It is useful to construct a polytropic-like form for the EOS of the inner crust. Denoting the pressure at nuclear density $\rho_0$ by $P_0$, and the pressure at neutron drip density $\rho_{\text{drip}}$ by $P_{\text{drip}}$ an accurate formula describing the $P-\rho$ relation for the inner crust within the BBP model is

\begin{align}
P &= K_1 + K_2 \rho^\Gamma, \quad K_1 = P_{\text{drip}} - K_2 \rho_{\text{drip}}^\Gamma, \quad K_2 = \frac{P_0 - P_{\text{drip}}}{\rho_0 - \rho_{\text{drip}}} \approx 1.6926. \quad (5.70)
\end{align}

In Fig. 5.1 we compare the above analytic expression (5.70) with the numerical integration of the BBP equations summarized in Table 5.1.

### 5.5.2 Non-interacting Fermi liquid model

The energy density and pressure given by a relativistic degenerate gas of fermions are amply known. They are obtained from the Fermi-Dirac statistics in the approximation of zero temperature. A detailed discussion of the properties of these systems can be found in [LL80, ST83].

Core

In this case, the contribution to the matter energy density and to the pressure inside the core due to baryons is simply given by [LL80]

\begin{align}
E_b &= \frac{2}{(2\pi)^3} \sum_{i=p,n} \int_0^{\rho_i} \frac{p^2}{\sqrt{p^2 + m_i^2}} 4\pi p^2 dp, \quad (5.71) \\
P_b &= \frac{1}{3} \frac{2}{(2\pi)^3} \sum_{i=p,n} \int_0^{\rho_i} \frac{p^2}{\sqrt{p^2 + m_i^2}} 4\pi p^2 dp. \quad (5.72)
\end{align}


5.6. NEUTRON STARS WITHOUT CRUST

We start our analysis from the less complicated problem of a neutron star without crust. In this way we will be able to compare and contrast the features of configurations with and without crust, leading to a better understanding of the crustal effects on the global structure of a neutron star. We then assume that the neutron star composition is the same from the center all the way up to the radius of the configuration.

Crust

Also in this case, we will use for the inner crust the BBP model, whose details can be found in Tables 1–3 of [BBP71]. The properties of the outer crust will be obtained using again the global equilibrium conditions for the sequence of Wigner-Seitz cells studied for the first EOS (see Eqs. (5.56)–(5.61) for details.)

Table 5.1: Properties of the BBP EOS for the inner crust region. The energetically favorable nucleus at each density $\rho$ is characterized by $A$ and $N_p$. $Y$ is the proton fraction inside the nuclei. The neutron chemical potential is $\mu_n$, $\mu_p$ is the chemical potential of protons, and $\mu_e$ is the electron chemical potential. Finally, $P$ denotes the total internal pressure.

<table>
<thead>
<tr>
<th>$\rho$ (g cm$^{-3}$)</th>
<th>$A$</th>
<th>$N_p$</th>
<th>$Y$</th>
<th>$\mu_n$ (MeV)</th>
<th>$\mu_p$ (MeV)</th>
<th>$\mu_e$ (MeV)</th>
<th>$P$ (MeV fm$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4.66 \times 10^{11}$</td>
<td>127</td>
<td>40</td>
<td>0.313</td>
<td>0.14</td>
<td>-24.89</td>
<td>26.31</td>
<td>$5.00 \times 10^{-4}$</td>
</tr>
<tr>
<td>$6.61 \times 10^{11}$</td>
<td>130</td>
<td>40</td>
<td>0.310</td>
<td>0.37</td>
<td>-25.33</td>
<td>26.98</td>
<td>$5.68 \times 10^{-4}$</td>
</tr>
<tr>
<td>$8.79 \times 10^{11}$</td>
<td>134</td>
<td>41</td>
<td>0.307</td>
<td>0.55</td>
<td>-25.67</td>
<td>27.51</td>
<td>$6.42 \times 10^{-4}$</td>
</tr>
<tr>
<td>$1.20 \times 10^{12}$</td>
<td>137</td>
<td>42</td>
<td>0.304</td>
<td>0.75</td>
<td>-26.08</td>
<td>28.13</td>
<td>$7.60 \times 10^{-4}$</td>
</tr>
<tr>
<td>$1.47 \times 10^{12}$</td>
<td>140</td>
<td>42</td>
<td>0.302</td>
<td>0.91</td>
<td>-26.38</td>
<td>28.58</td>
<td>$8.73 \times 10^{-4}$</td>
</tr>
<tr>
<td>$2.00 \times 10^{12}$</td>
<td>144</td>
<td>43</td>
<td>0.299</td>
<td>1.15</td>
<td>-26.88</td>
<td>29.33</td>
<td>$1.11 \times 10^{-3}$</td>
</tr>
<tr>
<td>$2.67 \times 10^{12}$</td>
<td>149</td>
<td>44</td>
<td>0.295</td>
<td>1.42</td>
<td>-27.44</td>
<td>30.15</td>
<td>$1.47 \times 10^{-3}$</td>
</tr>
<tr>
<td>$3.51 \times 10^{12}$</td>
<td>154</td>
<td>45</td>
<td>0.291</td>
<td>1.71</td>
<td>-28.05</td>
<td>31.05</td>
<td>$1.98 \times 10^{-3}$</td>
</tr>
<tr>
<td>$4.54 \times 10^{12}$</td>
<td>161</td>
<td>46</td>
<td>0.286</td>
<td>2.01</td>
<td>-28.72</td>
<td>32.02</td>
<td>$2.69 \times 10^{-3}$</td>
</tr>
<tr>
<td>$6.25 \times 10^{12}$</td>
<td>170</td>
<td>48</td>
<td>0.280</td>
<td>2.45</td>
<td>-29.69</td>
<td>33.43</td>
<td>$4.04 \times 10^{-3}$</td>
</tr>
<tr>
<td>$8.38 \times 10^{12}$</td>
<td>181</td>
<td>49</td>
<td>0.273</td>
<td>2.91</td>
<td>-30.78</td>
<td>34.98</td>
<td>$6.02 \times 10^{-3}$</td>
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<tr>
<td>$1.10 \times 10^{13}$</td>
<td>193</td>
<td>51</td>
<td>0.266</td>
<td>3.41</td>
<td>-31.98</td>
<td>36.68</td>
<td>$8.81 \times 10^{-3}$</td>
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<tr>
<td>$1.50 \times 10^{13}$</td>
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<td>54</td>
<td>0.256</td>
<td>4.07</td>
<td>-33.64</td>
<td>39.00</td>
<td>$1.38 \times 10^{-2}$</td>
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<tr>
<td>$1.99 \times 10^{13}$</td>
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<td>57</td>
<td>0.246</td>
<td>4.77</td>
<td>-35.50</td>
<td>41.56</td>
<td>$2.09 \times 10^{-2}$</td>
</tr>
<tr>
<td>$2.58 \times 10^{13}$</td>
<td>257</td>
<td>60</td>
<td>0.234</td>
<td>5.51</td>
<td>-37.57</td>
<td>44.37</td>
<td>$3.09 \times 10^{-2}$</td>
</tr>
<tr>
<td>$3.44 \times 10^{13}$</td>
<td>296</td>
<td>65</td>
<td>0.220</td>
<td>6.41</td>
<td>-40.34</td>
<td>48.10</td>
<td>$4.77 \times 10^{-2}$</td>
</tr>
<tr>
<td>$4.68 \times 10^{13}$</td>
<td>354</td>
<td>72</td>
<td>0.202</td>
<td>1.67</td>
<td>-43.99</td>
<td>52.95</td>
<td>$7.62 \times 10^{-2}$</td>
</tr>
<tr>
<td>$5.96 \times 10^{13}$</td>
<td>421</td>
<td>78</td>
<td>0.186</td>
<td>8.77</td>
<td>-47.49</td>
<td>57.56</td>
<td>$0.111$</td>
</tr>
<tr>
<td>$8.01 \times 10^{13}$</td>
<td>548</td>
<td>89</td>
<td>0.163</td>
<td>10.36</td>
<td>-52.66</td>
<td>64.32</td>
<td>$0.176$</td>
</tr>
<tr>
<td>$9.83 \times 10^{13}$</td>
<td>683</td>
<td>100</td>
<td>0.146</td>
<td>11.66</td>
<td>-56.86</td>
<td>69.81</td>
<td>$0.243$</td>
</tr>
<tr>
<td>$1.30 \times 10^{14}$</td>
<td>990</td>
<td>120</td>
<td>0.121</td>
<td>13.77</td>
<td>-63.52</td>
<td>78.58</td>
<td>$0.384$</td>
</tr>
<tr>
<td>$1.72 \times 10^{14}$</td>
<td>1640</td>
<td>157</td>
<td>0.096</td>
<td>16.39</td>
<td>-71.16</td>
<td>88.84</td>
<td>$0.616$</td>
</tr>
<tr>
<td>$2.00 \times 10^{14}$</td>
<td>2500</td>
<td>210</td>
<td>0.081</td>
<td>18.11</td>
<td>-75.79</td>
<td>95.39</td>
<td>$0.803$</td>
</tr>
<tr>
<td>$2.26 \times 10^{14}$</td>
<td>4330</td>
<td>290</td>
<td>0.067</td>
<td>19.59</td>
<td>-79.69</td>
<td>100.57</td>
<td>$0.988$</td>
</tr>
<tr>
<td>$2.39 \times 10^{14}$</td>
<td>7840</td>
<td>445</td>
<td>0.057</td>
<td>20.37</td>
<td>-81.92</td>
<td>103.57</td>
<td>$1.09$</td>
</tr>
</tbody>
</table>
Figure 5.1: Equation of state for the inner crust of the BBP model. The solid curve represents the analytic expression given by Eq. (5.70) while the points are the results of the numerical integration of the BBP equations (see Table 5.1).

For sake of simplicity, we will integrate the general relativistic equations for the simplest model of neutron star. We then assume the neutron star as composed by neutrons, protons, and electrons, obeying Fermi-Dirac statistics and ignoring the strong interaction between nucleons but considering gravitational and electromagnetic interactions.

In this case, we need to integrate the equilibrium equations composed by Eqs. (5.30)–(5.33) in addition to the beta equilibrium (5.22) and the Thomas-Fermi equilibrium condition (5.25) for the EOS given by Eqs. (5.41), (5.43), and (5.71).

### 5.6.1 Boundary conditions

Now we describe the general steps to construct the equilibrium configurations.

1. We start picking a value for the central rest-mass density of the configuration

   \[ \rho(0) = \sum_{i=e,p,n} m_i n_i(0). \]  

2. From the boundary and regularity conditions at the center,

   \[ M(0) = 0, \]  
   \[ n_e(0) = n_p(0). \]
together with Eq. 5.73 and the beta equilibrium condition 5.22 we can evaluate the central chemical potentials \( \mu_e(0) \), \( \mu_p(0) \), and \( \mu_n(0) \), or equivalently, the central number densities \( n_e(0) \), \( n_p(0) \), and \( n_n(0) \).

3. We pick any negative value for the gravitational potential at the center \( \nu(0) \leq 0 \).

4. Using Eq. 5.25 we fix the shifted Coulomb potential energy at the center of the star \( \tilde{V}(0) \) through
\[
\tilde{V}(0) = e^{\nu(0)/2} \mu_e(0).
\] (5.76)

5. Having defined all the initial conditions, now it is possible to integrate the system of equilibrium equations until the surface of the star defined by the condition
\[
P(R) = 0.
\] (5.77)

Without the presence of a crust, this system has a net positive charge as a consequence of the mass difference between protons and electrons as demonstrated in Section 5.4. This implies that at the neutron star radius \( r = R \), we will have a matching condition with the exterior Reissner-Nordström spacetime which imposes the boundary condition
\[
e^{\nu(R)/2} = \sqrt{1 - \frac{2GM(R)}{R} + R^2E^2(R)}.
\] (5.78)

6. Reached the radius of the star, we verify if the above boundary condition 5.78 is satisfied by the gravitational potential \( \nu \). If this last boundary condition is not satisfied, we shift the solution that we have obtained for the gravitational potential in the following form
\[
\nu^{\text{new}}(r) = \nu^{\text{old}}(r) - \nu^{\text{old}}(R) + 2 \ln \sqrt{1 - \frac{2GM(R)}{R} + R^2E^2(R)},
\] (5.79)
which determines the correct shifted Coulomb potential energy
\[
\tilde{V}^{\text{new}}(r) = \frac{\tilde{V}^{\text{old}}(r)}{e^{\nu^{\text{old}}(R)/2}} \sqrt{1 - \frac{2GM(R)}{R} + R^2E^2(R)},
\] (5.80)

7. and consequently, the correct Coulomb potential energy
\[
eV^{\text{new}}(r) = \tilde{V}^{\text{new}}(r) - E^F, 
\] (5.81)
where the Fermi energy of the electrons is given by
\[
E^F_e = m_e \sqrt{1 - \frac{2GM(R)}{R} + R^2E^2(R) - R^2E^2(R)} < 0.
\] (5.82)

This Fermi energy \( E^F_e \) in this case is negative, which means that the system has a net positive charge.

### 5.6.2 Results of the numerical integration

In Fig. 5.2 we show the results of the numerical integration for \( \rho(0) \approx 9.8 \times 10^{14} \, \text{g cm}^{-3} \).

The small electric field inside the configuration is the self-consistent field due to the proton and electron mass difference. The charge separation \( n_p - n_e \) is so small that it is not appreciable on the scale of the plot. In fact, inside the configuration we have
\[
n_e \simeq n_p \neq n_e,
\] (5.83)
which does not mean that for simplicity one can assume \( n_e = n_p \), in that case one obtains an inconsistent solution as demonstrated in Section 5.4. In addition, we will see that the boundary conditions related to the existence of the Coulomb potential which lower the Fermi energy of the electrons are crucial for the determination of the structure of globally neutral neutron stars with crust.

We can estimate the Coulomb potential energy difference between the center and the radius of the star as follows. Because the condition \( n_p(0) = n_e(0) \) must be satisfied at the center of the star, we can evaluate Eq. (5.39) at \( r = 0 \) to obtain

\[
e V(0) - e V(R) = \left( \frac{m_p - m_e}{2} \right) \left[ \frac{e^{\nu(R)} - e^{\nu(0)}}{e^{\nu(R)/2}} \right].
\]  

Denoting by \( g(r) \) the Newtonian gravitational potential, the above formula (5.84) in the weak field limit becomes

\[
e V(0) - e V(R) \approx (m_p - m_e) [g(R) - g(0)].
\]  

If we consider a configuration at constant nuclear density \( n \approx m_\pi^3/2 \) with a mass numbers of the order of \( A \approx (m_{\text{Planck}}/m_n)^3 \sim 10^{57} \), the Eq. (5.85) gives

\[
e V(0) - e V(R) \approx \frac{(2\pi/3)^{1/3}}{2} m_\pi \approx 0.64 m_\pi,
\]  

which agrees with the results obtained before (see Fig. 5.2) from the full numerical integration of the general relativistic equations of equilibrium.

5.7 Neutron star with crust

We turn now to the analysis of neutron stars with crust. The key point here are the boundary conditions, which now must be modified in order to take into account the compressional effect of the crust on the core. In particular, now we will able to accomplish the condition of global neutrality, contrary to the case of the neutron star without crust, where neutrons, protons, and electrons go smoothly up to the surface of the star.

Here the situation is much more complex, and in fact the matter phase in the crust is completely different to the one existing inside the core. However, the equilibrium equations are the same, namely, the Einstein-Maxwell equations (5.30)-(5.32), the general relativistic Thomas-Fermi equation (5.33), the beta equilibrium (5.22), and the Thomas-Fermi equilibrium condition (5.24). The crucial point is the treatment of the boundary conditions in the core-crust transition surface, which will be the key for the overall neutrality of the star.

5.7.1 Boundary conditions

As before, we start with the description of the main steps that must be follow in order to construct globally neutral neutron star configurations with crust. We will find that contrary to the previous case, where the neutron star equilibrium configurations can be parametrized by the central density \( \rho(0) \), here for each central density, there exists an entire family of solutions to the equilibrium equations, parametrized by the Fermi energy of the electrons of the core.

1. At the neutron star radius \( r = R \), all the electrodynamical quantities must be zero as a consequence of the global charge neutrality condition. We then have a matching condition with the
exterior Schwarzschild spacetime which imposes the boundary condition
\[ e^{\nu(R)/2} = \sqrt{1 - \frac{2GM(R)}{R}}. \] (5.87)

2. Now we take a value for the central rest-mass density of the configuration
\[ \rho(0) = \sum_{i=e,p,n} m_i n_i(0), \] (5.88)

3. and as before, from the boundary and regularity conditions at the center,
\[ M(0) = 0, \] (5.89)
\[ n_e(0) = n_p(0), \] (5.90)
together with Eq. (5.88) and the beta equilibrium condition \[5.22\] we obtain the central chemical potentials \[\mu_e(0), \mu_p(0), \text{and } \mu_n(0)\], or equivalently, the central number densities \[n_e(0), n_p(0), \text{and } n_n(0)\].

4. We pick any negative value for the gravitational potential at the center, namely \(\nu(0) \leq 0\).

5. Using Eq. \[5.25\] we fix the shifted Coulomb potential energy at the center of the star \(\tilde{\nu}(0)\) through
\[ \tilde{\nu}(0) = e^{\nu(0)/2} \mu_e(0). \] (5.91)

6. While in the case of a neutron star without crust, the Fermi energy of the electrons is completely determined by its value at the surface of the star (see Eq. \[5.35\]), here we need to specify both the shifted potential \(\tilde{\nu}(0)\) and the Fermi energy \(E^F_e\), or equivalently, the potential \(eV(0)\) and \(E^F_e\). Such a value of the Fermi energy of the electrons of the core must be positive, namely
\[ E^F_e \geq 0. \] (5.92)

7. We can now calculate the central Coulomb potential energy
\[ eV(0) = \tilde{\nu}(0) - E^F_e. \] (5.93)

8. Having determined the boundary conditions at infinity and at the center, we turn now to the matching conditions at the surface of the core. In order to take into account the effect of the compression of the crust on the leptonic component of the core we solve the equilibrium conditions for the core within a Wigner-Seitz cell \[\text{[MRX09]}\]. The radius \(R_{WS}\) of this cell determines the Fermi energy of the electrons of the core which has to be matched with the Fermi energy of the leptonic component of the crust. Global charge neutrality is specified by
\[ \int_0^{R_{WS}} e^{\lambda/2} n_p d^3r = \int_0^{R_{WS}} e^{\lambda/2} n_e d^3r. \] (5.94)

From Eqs. \[5.22\] and \[5.94\] we can determine self-consistently the proton, neutron, electron fractions inside the core as well as the radius \(R_{WS}\) of the Wigner-Seitz cell of the core \[\text{[MRX09]}\].

9. Following BBP \[\text{[BBP71]}\], the neutron profile at the core-crust interface is given by
\[ n_n(z) = n_n^{\text{crust}} + (n_n^{\text{core}} - n_n^{\text{crust}}) f(z/b). \] (5.95)
We have defined \( n_n^{\text{core}} = n_n(R_c) \) and \( n_n^{\text{crust}} = n_n(R_{\text{WS}}) \). Here \( R_c \) is the radius of the core defined as the point where the rest-mass density reaches normal nuclear density [BBP71] (see Eq. (5.40)). The function \( f(z/b) \) satisfies \( f(-\infty) = 1, f(\infty) = 0 \), where the length \( b \) is of the order of \([BBP71]\)

\[
b \approx \frac{1}{(n_n^{\text{core}} - n_n^{\text{crust}})^{1/3}} \approx \frac{1}{m_{\pi}}.
\]

This estimate of the characteristic length \( b \) determining the diffuseness of the baryon density around nuclear density, could be seen as a phenomenological estimate. However, we strongly expect that a dynamical calculation of this length will lead to similar results. For instance, the length \( b \) within the RMF model is related to the decay length of the effective potential acting on nucleons given by the combination of the attractive potential of the scalar field \( \sigma \) and the repulsive potential of the vector field \( \omega \) (see Appendix A for details). The self-consistent attractive and repulsive potentials, are obtained from explicit integration of the Klein-Gordon equations of these fields.

As proposed by BBP, an appropriate choice for the function \( f(z/b) \) is the Woods-Saxon profile

\[
f(z/b) = \frac{1}{1 + e^{z/b}},
\]

where the \( z \)-coordinate is perpendicular to the sharp surface separating two semi-infinite regions (core and crust) in the planar approximation [BBP71]. The neutron profile then satisfies

\[
\lim_{z \to -\infty} n_n = n_n^{\text{core}}, \quad \lim_{z \to +\infty} n_n = n_n^{\text{crust}}.
\]

10. The matching between the core and the crust occurs at the radius \( R_{\text{WS}} \), where we have

\[
\left. \frac{d \bar{V}}{dr} \right|_{r = R_{\text{WS}}} = 0, \quad \bar{V}(R_{\text{WS}}) = E^F_c,
\]

by virtue of the global neutrality condition given by Eq. (5.94), and fixing the level of reference of the Coulomb potential at the Wigner-Seitz cell radius

\[
V(R_{\text{WS}}) = 0.
\]

From the electron chemical potential \( \mu_e(R_{\text{WS}}) \) at the edge of the crust, we calculate the corresponding neutron chemical potential \( \mu_n(R_{\text{WS}}) \). If \( \mu_n(R_{\text{WS}}) - m_n > 0 \), neutron drip occurs. In this case, the pressure is due to the neutrons as well as to the leptonic component, so we have the inner crust (see Table 5.2 and [BBP71], [ARX09a] for details). For larger values of the radii, i.e., for \( r > R_{\text{WS}} \) the condition \( \mu_n(r) - m_n < 0 \) is reached at \( \rho_{\text{drip}} \approx 4.3 \times 10^{11} \text{ g cm}^{-3} \) and there the outer crust starts, with the pressure only determined by the leptonic component. If \( \mu_n(R_{\text{WS}}) - m_n < 0 \), only the outer crust exists. Summarizing we have

\[
\text{Crust} = \begin{cases} 
\text{inner crust + outer crust}, & \text{if } \mu_n(R_{\text{WS}}) - m_n > 0, \\
\text{only outer crust}, & \text{if } \mu_n(R_{\text{WS}}) - m_n < 0.
\end{cases}
\]

11. We then integrate the entire set of equations until reach the surface of the star, defined by

\[
P(R) = 0.
\]
5.8. CONCLUSIONS

12. Reached the radius of the star, we verify if the boundary condition (5.87) is satisfied by the gravitational potential \( \nu \). If the boundary condition is not satisfied, we shift the gravitational potential that we have obtained as

\[
\nu_{\text{new}}(r) = \nu_{\text{old}}(r) - \nu_{\text{old}}(R) + 2 \ln \sqrt{1 - \frac{2GM(R)}{R}},
\]

which defines also the correct shifted Coulomb potential energy

\[
\hat{V}_{\text{new}}(r) = \frac{\hat{V}_{\text{old}}(r)}{e^{\nu_{\text{old}}(R)/2}} \sqrt{1 - \frac{2GM(R)}{R}},
\]

and consequently the correct Coulomb potential energy

\[
eV_{\text{new}}(r) = \hat{V}_{\text{new}}(r) - E_F^e.
\]

Therefore, we obtain a unique structure of the neutron star for each pair \((\rho(0), E_F^e)\). This means that for each central density we obtain an entire new family of equilibrium configurations depending on the value of the Fermi energy of the electrons of the core, and each one characterized by a unique electromagnetic structure.

5.7.2 Results of the numerical integration

For a fixed central rest-mass density \( \rho(0) \approx 9.8 \times 10^{14} \text{ g cm}^{-3} \) and selected values of \( E_F^e \), we have integrated the system of equations composed by the general relativistic Thomas-Fermi equation (5.33), the beta equilibrium condition (5.22), the Einstein-Maxwell equations (5.30)–(5.32), with the constraint of overall neutrality (5.94).

In Table 5.2 we show the general properties of the equilibrium configurations for the BBP and the non-interacting EOS described in Section 5.5. In order to compare and contrast the differences of this novel treatment, we also show the corresponding properties of the neutron star without crust described in Section 5.6 as well as the neutron star constructed under the condition of local neutrality with the BBP EOS (see [BBP71] and Section 5.5).

We found that although the electrodynamical properties of the core are very sensitive to the Fermi energy of the electrons, as can be seen from Table 5.2, the bulk properties of the core like its mass and radius are not sensitive to the value of \( E_F^e \). This is perfectly in line with the analytic results of Ruffini et al. in [MRX09] on compressed massive nuclear density cores studied in Chapter 4.

Particularly interesting are the electrodynamical structure and the distribution of neutrons, protons, and electrons as the surface of the core is approached (see Figs. 5.3). It is interesting to compare and contrast these results with the preliminary ones obtained in the simplified model of massive nuclear density cores [MRX09]. The values of the electric field are quite close and are not affected by the constant proton density distribution assumed there. In the present case, the proton distribution is far from constant and increases outward as the core surface is approached.

5.8 Conclusions

For any given value of the central density an entire new family of equilibrium configurations exists. Each configuration is characterized by a strong electric field at the core-crust interface. Such an electric field extends over a thin shell of thickness \( \sim 1/m_e \) and becomes largely overcritical in the limit of decreasing values of the crust mass and size (see Table 5.2 and Figs. 5.3).

These configurations endowed with overcritical electric fields are indeed stable against the quantum instability of pair creation because of the Pauli blocking of the degenerate electrons [RVX09]. It
is expected that during the gravitational collapse phases leading to the formation of a neutron star, a large emission of electron-positron pairs will occur prior to reaching a stable ground state configuration. Similarly during the merging of two neutron stars or a neutron star and a white-dwarf leading to the formation of a black hole, an effective dyadotors \cite{CGJR09} will be formed leading to very strong creation of an electron-positron plasma. In both cases the basic mechanism which makes gravitational collapse depart from a pure gravitational phenomena is due to the electrodynamical process introduced in this work.

Finally, it is appropriate to recall that the existence of overcritical fields on macroscopic objects of $M \sim M_{\odot}$ and $R \sim 10$ km was first noted in the treatment of quark stars \cite{Wit84, Ito70, AFO86, KWW95}. In that case the ultrarelativistic Thomas-Fermi equations were also considered (see Chapter 4, Section 4.1 for a direct comparison with the treatment of massive nuclear density cores). However, in all of these investigations, a hybrid combination of general and special relativistic treatments was adopted, resulting in an inconsistency in the boundary conditions. The treatment given here is the first self-consistent treatment of the general relativistic Thomas-Fermi equations, the beta equilibrium condition and the Einstein-Maxwell equations. Critical fields are indeed obtained on the surface of the neutron star core involving only neutrons, protons, and electrons, their fundamental interactions, and with no quarks present.

Indeed, the existence of neutron stars with huge crusts, i.e., with both inner and outer crusts,
is mainly a consequence of assuming no electrodynamical structure (i.e., assuming local neutrality) and of allowing electrons to have larger values of their Fermi energy $E_F^e$. It can also be demonstrated that no consistent solution of the Einstein-Maxwell equations satisfying the local $n_e = n_p$ condition exists, even as a limiting case \cite{ARX09a}.

5.9 Perspectives

Recently, an interesting observational problematic has emerged from the Chandra X-ray Observatory observations of the central compact object in the center of the supernova remnant Cassiopeia A \cite{PL09, HH09}. It is with a similar steadily emitting and non-pulsating neutron star that our theoretical predictions can be tested. In particular, the existence for each central density of a new family of neutron stars with a smaller crust than the one obtained when the local neutrality condition is adopted.

In order to show how this approach should be used to construct neutron star equilibrium configurations satisfying global neutrality, we have used two different models for the EOS: the BBP EOS and the non-interacting EOS. However, none of these models can represent the correct behavior of matter at high densities. The BBP EOS is based on the liquid drop model of the nucleus, which has been constructed to fit properties of ordinary nuclei at moderate conditions of energy, which do not represent the conditions in the internal layers of a neutron star for instance in the core of a neutron star. Although we know the results we obtained from these models are quite general, and independent on their details, it would be interesting to construct neutron star configurations following our approach with a model for the EOS that let us to calculate all the properties of the neutron star dynamically, without any additional assumption that the equilibrium conditions. As we suggested before, a model based on the RMF model of the nucleus satisfy this requirement (see Appendix A for details).

In this case the equations of equilibrium are determined by:

1. the Einstein-Maxwell equations \eqref{equations1}–\eqref{equations2},
2. the general relativistic Thomas-Fermi equation \eqref{equation3},
3. the general relativistic Thomas-Fermi equilibrium condition \eqref{equation4},
4. the beta equilibrium condition \eqref{equation5},
5. and the Klein-Gordon equations for the meson fields $\sigma$, $\omega$, and $\rho$ \eqref{equation6}–\eqref{equation7}.
CHAPTER 5. NEUTRON STAR EQUILIBRIUM CONFIGURATIONS

Figure 5.2: Results of the numerical integration of the equilibrium equations for a neutron star without crust using the non-interacting model of neutrons, protons, and electrons. The central density of this example is \( \rho(0) \simeq 9.8 \times 10^{14} \) g cm\(^{-3}\). We show the mass function \( M(r) \) in solar masses, the Coulomb potential energy \( eV \) in units of the pion mass, the electric field \( E(r) \) in units of the critical field \( E_c \), the charge \( Q(r) \) in Coulombs, the pressure and rest-mass density profiles normalized to their central values, and the particle number densities also normalized to their central values.
Figure 5.3: Electric field (left column) and particle density (right column) in the core-crust transition surface for selected values of $E_f^F$. First row: $E_f^F = 0.20m_π$, second row $E_f^F = 0.35m_π$. For the plots on the right column, the solid curve is the neutron density, the short-dashed curve is the proton density, and the long-dashed curve is the electron density. The electric field is given in units of the critical field $E_c = m_e^2 / \sqrt{\alpha}$ while the densities are normalized to the nuclear density $n_0 \simeq m_π^3 / 2$. 
Chapter 6

Critical fields in Astrophysics

The discovery of gamma ray bursts has offered an extreme example of the most energetics \( E \lesssim 10^{55} \text{ ergs} \) and the fastest transient \( \Delta t \lesssim 10^{-3} - 10^{-4} \text{ s} \) phenomena ever observed in the universe [RBB+07]. The dynamics of such astrophysical process is dominated by an \( e^+ - e^- \) plasma [RVX09]. A theoretical model based on vacuum polarization process [DR75] occurring in a Kerr-Newman geometry can indeed explain such enormous energetics and the sharp time variability. What is most important is that such a model is based on explicit analytic solutions of well-known ultra-relativistic field theories. The formation of such black holes is expected from a large variety of binary systems composed of neutron stars, white dwarfs and massive stars at the end point of their thermonuclear evolution [KJR08] in all possible combinations.

In particular, in the merging of two neutron stars and in the final process of gravitational collapse to a black hole is expected the occurrence of electromagnetic fields with strength larger than the critical value of vacuum polarization \( E_c = m_e^2/\alpha \) [RBB+07]. In this work, and in particular in Chapter 5, we have reexamined the electrodynamical processes of a neutron star through the solution of the general relativistic equilibrium equations of neutron stars, including self-consistently the electromagnetic interaction, in order to identify a possible physical origin of this overcritical electric field [VPX09, MRX09, JARX09b, JARX09a].

The time evolution of the gravitational collapse (occurring on characteristic gravitational time scales \( \tau = GM/c^3 \simeq 5 \times 10^{-5} M/M_\odot \text{ s} \)) and the associated electrodynamical process are too complex for a direct description. We address here a more confined problem: the polarization process around an already formed Kerr-Newman black hole. This is a well defined theoretical problem which deserves attention. It may represent a physical state asymptotically reached in the process of gravitational collapse. We expect such an asymptotic configuration be reached when all the multipoles departing from the Kerr-Newman geometry have been radiated away either by process of vacuum polarization or electromagnetic and gravitational waves. What it is most important is that by performing this theoretical analysis we can gain a direct evaluation of the energetics of the spectra and dynamics of the \( e^+ - e^- \) plasma created on the extremely short time scales due to the quantum phenomena of \( \Delta t = \hbar / (m_e c^2) \simeq 10^{-21} \text{ s} \). Similarly, this entire transient phenomena, starting from an initial neutral condition, undergoes the formation of the Kerr-Newman black hole by the collective effects of gravitation, strong, weak, electromagnetic interactions during a fraction of the above mentioned gravitational characteristic time scale of collapse.

In this chapter we will explore the initial condition for such a process to occur generalizing to the Kerr-Newman geometry the concept of the “dyadosphere” previously introduced in the case of the spherically symmetric Reissner-Nordström geometry [PRX98], which will lead us to the concept of “dyadotorus” for a Kerr-Newman spacetime (see [CGJR99] for details).

Hereafter we adopt geometric units \( G = c = 1 \) and metric signature \((-+,+,+,+\)).
6.1 Dyadosphere of a Reissner-Nordstrom black hole

In this section we recall the definition of dyadosphere and its main properties in the field of a Reissner-Nordström black hole as derived in [PRX98]. In standard Schwarzschild-like coordinates the Reissner-Nordström black hole metric is given by

\[
\begin{align*}
\text{ds}^2 &= - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2 ). \\
\end{align*}
\]

(6.1)

The associated electromagnetic field is given by

\[
F = - \frac{Q r}{r^2} dt \wedge dr.
\]

(6.2)

The horizons are located at \( r_{\pm} = M \pm \sqrt{M^2 - Q^2} \); we consider the case \(|Q| \leq M\) and the region \( r > r_+ \) outside the outer horizon. For an extremely charged hole we have \(|Q| = M\) and the two horizons coalesce.

Let us introduce an orthonormal frame adapted to the static observers

\[
e_t = \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1/2} \partial_t, \quad e_r = \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{1/2} \partial_r, \quad e_\theta = \frac{1}{r} \partial_\theta, \quad e_\phi = \frac{1}{r \sin \theta} \partial_\phi.
\]

(6.3)

The electric field as measured by static observers with four-velocity \( U = e_t \) is purely radial

\[
E(U) = \frac{Q}{r^2} e_r.
\]

(6.4)

The radius \( r_{\text{ds}} \) at which the electric field strength \(|E| = |E'|\) reaches the critical value \( E_c \) has been defined in [PRX98] as the outer radius of the dyadosphere, which extends down to the horizon and within which the electric field strength exceeds the critical value

\[
r_{\text{ds}} \simeq 1.12 \times 10^8 \sqrt{\lambda \mu} \text{ cm},
\]

(6.5)

where the dimensionless quantities \( \lambda = Q/M \) and \( \mu = M/M_\odot \) have been introduced. The critical electric field \( E_c \) in geometric units is given by \( E_c \simeq 2.68 \times 10^{-11} \text{ cm}^{-1} \).

The electromagnetic energy contained inside the dyadosphere has been evaluated by Vitagliano and Ruffini [RV02]

\[
E(\xi)_{(r_+, r_{\text{ds}})} = \frac{Q^2}{2r_+} \left( 1 - \frac{r_+}{r_{\text{ds}}} \right) ,
\]

(6.6)

by using a “truncated version” of the definition of energy in terms of the conserved Killing integral

\[
E(\xi) = \int_\Sigma T_{\mu\nu}^{(\text{em})} \xi^\mu d\Sigma^\nu ,
\]

(6.7)

where \( \xi = \partial_t \) is the timelike Killing vector. We refer to Section 6.2 for a detailed discussion on this point. Fig. 6.1 shows the behavior of the electromagnetic energy (6.6) as a function of the mass parameter \( \mu \) for fixed values of the charge parameter \( \lambda \).

Ruffini and collaborators estimated also the total energy of pairs converted from the “static electric energy” (6.6) and deposited within the dyadosphere

\[
E_{\text{pairs}} = \frac{Q^2}{2r_+} \left( 1 - \frac{r_+}{r_{\text{ds}}} \right) \left[ 1 - \left( \frac{r_+}{r_{\text{ds}}} \right)^4 \right] .
\]

(6.8)
Figure 6.1: The behavior of the electromagnetic energy (6.6) in solar mass units is shown as a function of the mass parameter $\mu$ for selected values of the charge parameter $\lambda = [0.1, 0.5, 1]$, from bottom to top. The straight lines (dashed) correspond to the maximum energy extractable from a Reissner-Nordström black hole given by $Q^2/(2r_+)$.

Figure 6.2: The total energy of pairs (6.8) is plotted as a function of the two mass and charge parameters $\mu$ and $\lambda$. The different curves correspond to selected values of the energy (in ergs). Only the solutions below the solid line are physically relevant. The configurations above the solid line correspond instead to unphysical solutions with $r_{ds} < r_+$. The plot is reproduced from [RBC+03] with the kind permission of the authors.
Its behavior as a function of the charge and mass parameters $\lambda$ and $\mu$ is shown in Fig. 6.2.

The rate of pair creation per unit four-volume is given by the Schwinger formula \[\text{Sch51}\]

\[
2\text{Im} L = \frac{1}{4\pi} \left( \frac{|E|e}{\pi \hbar} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi E_c/|E|}.
\] (6.9)

The leading term $n = 1$ agrees with the WKB results obtained by Sauter \[\text{Sau31}\] and Heisenberg-Euler \[\text{HE36}\]

\[
2\text{Im} L = \frac{1}{4\pi} \left( \frac{|E|e}{\pi \hbar} \right)^2 e^{-\pi E_c/|E|}.
\] (6.10)

The dyadosphere has been defined by Ruffini and collaborators \[\text{PRX98}\] by the condition $|E| = E_c$. One might better define it by requiring the electric field strength to be such that the rate of pair creation is suppressed exactly by a factor $1/e$, leading to the condition $|E| = \pi E_c$. However, from Eq. (6.9) it is clear that no sharp threshold exists for electron-positron pair creation, so that the definition $|E| = kE_c$ (6.11) appears to be more appropriate and should be explored for different values of the constant parameter $k$, even for $k < 1$. Consequently, we shall define in the following both dyadosphere and dyadotorus as the locus of points where the electric field satisfies the condition (6.11).

### 6.2 Dyadotorus of a Kerr-Newmann black hole

The Kerr-Newman metric in standard Boyer-Linquist type coordinates writes as \[\text{MTW73}\]

\[
ds^2 = -\left(1 - \frac{2Mr - Q^2}{\Sigma}\right)dt^2 - \frac{2a \sin^2 \theta}{\Sigma} \left(2Mr - Q^2\right)dt d\phi + \frac{\Sigma}{\Delta} dr^2
+ \Sigma d\theta^2 + \left[r^2 + a^2 + \frac{a^2 \sin^2 \theta}{\Sigma} \left(2Mr - Q^2\right)\right] \sin^2 \theta d\phi^2,
\] (6.12)

with associated electromagnetic field

\[
F = \frac{Q}{\Sigma \Delta} (r^2 - a^2 \cos^2 \theta) dr \wedge [dt - a \sin^2 \theta d\phi]
+ 2\frac{Q}{\Sigma} ar \sin \theta \cos \theta d\theta \wedge [(r^2 + a^2) d\phi - adt],
\] (6.13)

where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2Mr + a^2 + Q^2$. Here $M$, $Q$, and $a$ are the total mass, total charge and specific angular momentum respectively characterizing the spacetime. The (outer) event horizon is located at $r_+ = M + \sqrt{M^2 - a^2 - Q^2}$.

Let us introduce the Carter’s observer family \[\text{Car68}\], whose four-velocity is given by

\[
U_{\text{car}} = \frac{r^2 + a^2}{\sqrt{\Delta \Sigma}} \left[ \partial_t + \frac{a}{r^2 + a^2} \partial \phi \right].
\] (6.14)

An observer adapted frame to $U_{\text{car}}$ is then easily constructed with the triad

\[
e_r = \frac{1}{\sqrt{8\pi}} \partial_r, \quad e_\theta = \frac{1}{\sqrt{8\pi}} \partial_\theta, \quad \hat{U}_{\text{car}} = \frac{a \sin \theta}{\sqrt{\Sigma}} \left[ \partial_t + \frac{1}{a \sin^2 \theta} \partial \phi \right].
\] (6.15)

The Carter observers measure parallel electric and magnetic fields $E$ and $B$ \[\text{DR75}\], with components

\[
E(U_{\text{car}})^{[a} = F^{a}{}_{[b} U_{\text{car}}^{b]}, \quad B(U_{\text{car}})^{[a} = {^*F}^{a}{}_{[b} U_{\text{car}}^{b]},
\] (6.16)
where \( \mathcal{F} \) is the dual of the electromagnetic field (6.13). Both \( E \) and \( B \) are directed along \( e_\perp \) and assuming as usual \( Q > 0 \), the strength of electric and magnetic fields are given by

\[
|E| = |E\perp| = \frac{Q}{\sqrt{2}} \left( r^2 - a^2 \cos^2 \theta \right), \quad |B| = |B\perp| = \frac{Q}{\sqrt{2}} r \cos \theta .
\] (6.17)

It is worth noting that the Carter orthonormal frame is the unique frame in which the flat spacetime Schwinger discussion can be locally applied. This is due both to the meaning of the Carter orthonormal frame and its relation to the geometry of the Weyl curvature tensor and the spacetime itself, as well as to the fact that the invariantly described Schwinger process demands this unique frame for its application. An alternative but equivalent derivation of this result is presented in Appendix D, where the electric and magnetic field strengths are obtained in terms of the electromagnetic invariants by using the Newman-Penrose formalism, hence showing more clearly the invariant character of the dyadotorus.

The Schwinger formula generalized to include both electric and magnetic fields, i.e.

\[
2 \text{Im} \mathcal{L} = \frac{1}{4 \pi} \left( \frac{|E| e}{\pi \hbar} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left( n \pi \frac{|B|}{|E|} \right) \coth \left( n \pi \frac{|B|}{|E|} \right) e^{-n \pi E_\perp / |E|} ,
\] (6.18)

has been used by Damour and Ruffini [DR75] for the case of a Kerr-Newman geometry.

We are interested in the region exterior to the outer horizon \( r \geq r_+ \). Solving Eq. (6.11) for \( r \) and introducing the dimensionless quantities \( \lambda = Q / M, \kappa = a / M, \mu = M / M_\odot \) and \( \epsilon = k E_\perp M_\odot \approx 1.873 \times 10^{-6} \) (with \( M_\odot \approx 1.477 \times 10^5 \) cm) we get

\[
\left( \frac{r^d_+}{M} \right)^2 = \frac{1}{2} \frac{\lambda}{\mu \epsilon} - a^2 \cos^2 \theta \pm \left[ \frac{1}{4} \frac{\lambda^2}{\mu^2 \epsilon^2} - 2 \frac{\lambda}{\mu \epsilon} \cos^2 \theta \right]^{1/2}
\] (6.19)

where the ± signs correspond to the two different parts of the surface. They join at the particular values \( \theta^* = \pi - \theta^* \) of the polar angle given by the condition of vanishing argument of the square root in Eq. (6.19)

\[
\theta^* = \arccos \left( \frac{1}{2 \sqrt{2\lambda}} \sqrt{\frac{\lambda}{\mu \epsilon}} \right).
\] (6.20)

The requirement that \( \cos \theta^* \leq 1 \) can be solved for instance for the constant parameter \( k \), giving the range of allowed values for which the dyadotorus appears indeed as a torus-like surface (see Figs. 6.4 (b), (c) and (d))

\[
k \geq \frac{\lambda}{8 E_\perp M_\odot \mu \epsilon} \approx 6.6 \times 10^4 \frac{\lambda}{\mu a^2} ;
\] (6.21)

for lower values of \( k \) the dyadotorus consists instead of two disjoint parts, one of them (corresponding to the branch \( r^d_+ \)) always external to the other (corresponding to the branch \( r^d_- \)), and has rather the shape of an ellipsoid (see Fig. 6.4 (a)). Therefore, the use of the term dyadoregion should be more appropriate in this case.

In terms of the dimensionless quantities \( \lambda \) and \( a \) the horizon radius is then given by

\[
\frac{r_+}{M} = 1 + \sqrt{1 - \lambda^2 - a^2} .
\] (6.22)

The condition for the existence of the dyadotorus is given by \( r^d_+ \geq r_+ \). The allowed region for the pairs \( (\lambda, \mu) \) (with fixed values of the rotation parameter \( a \) and the polar angle \( \theta \)) satisfying this condition is shown in Fig. 6.3.

Figure 6.3 shows the shape of the projection of the dyadotorus on a plane containing the rotation axis for an extreme Kerr-Newman black hole with fixed \( \mu \) and \( \lambda \) and different values of the
Figure 6.3: The space of parameters $(\lambda, \mu)$ is shown for different values of the rotation parameter $\alpha = a/M = [0, 0.4, 0.6, 0.8, 0.9, 0.99]$ and fixed value of the polar angle $\theta = \pi/3$. The region below each curve represents the allowed region for the existence of the dyadoregion with fixed $\alpha$. The configurations above each line correspond to unphysical solutions where $r^2_{\pm} < r_+$ for the selected set of parameters. The value of the parameter $k$ has been set equal to one.
6.2. DYADOTORUS OF A KERR-NEWMANN BLACK HOLE

Figure 6.4: The projection of the dyadotorus on the $X - Z$ plane ($X = r \sin \theta$, $Z = r \cos \theta$ are Cartesian-like coordinates built up simply using the Boyer-Lindquist radial and angular coordinates) is shown for an extreme Kerr-Newman black hole with $\mu = 10$, $\lambda = 1.49 \times 10^{-4}$ and different values of the parameter $k$: (a) $k = 0.9$ (orange), (b) $k = 1.0$ (red), (c) $k = 1.1$ (light blue), (d) $k = 1.5$ (blue). The boundary of the dyadoregion becomes a torus-like surface for $k \approx 0.998$, according to Eq. (6.21). The black disk represents the black hole horizon.
parameter $k$ using Cartesian-like coordinates $X = r \sin \theta$, $Z = r \cos \theta$, built up simply by taking the Boyer-Lindquist coordinates $r$ and $\theta$ as polar coordinates in flat space.

A “dynamical” view of topology change in the shape of the dyadoregion is shown in Fig. 6.3, where the case of a Reissner-Nordström black hole with the same total mass and charge is also shown for comparison. We point out some interesting qualitative differences between dyadotorus and dyadosphere which can be seen clearly from these plots. In particular, the dyadotorus appear for instance the maximum electric field for the Kerr-Newman black hole in contrast with the Reissner-Nordström one. We can compare for instance the maximum electric field $E_{\text{max}} = Q/r_+^2$, of an extreme Kerr-Newman black hole while that one of a Reissner-Nordström black hole goes to $r_+ \sim 2M$. This fact is crucial because it leads to the presence of stronger electric fields for the Kerr-Newman black hole in contrast with the Reissner-Nordström one. We can compare for instance the maximum electric field $E_{\text{max}} = Q/r_+^2$, of an extreme Kerr-Newman black hole while that one of a Reissner-Nordström black hole, which is obtained for $r = r_+ = \pi/2$ in the former case and $r = r_+$ in the latter case, in the limit of small charge to mass ratio

$$E_{\text{max}}^\text{KN} = \frac{Q}{M^2} = 4E_{\text{max}}^\text{RN}. \quad \text{(6.23)}$$

The total electromagnetic energy distributed in a stationary spacetime can be determined by evaluating the conserved Killing integral (see e.g. [RV02])

$$E(\xi) = \int_\Sigma T_{\mu \nu}^{\text{em}} \xi^\mu d\Sigma^\nu, \quad \text{(6.24)}$$

where $\xi = \partial_t$ is the timelike Killing vector, $T_{\mu \nu}^{\text{em}}$ is the electromagnetic energy-momentum tensor of the source, $d\Sigma^\nu = n^\nu d\Sigma$ is the surface element vector with $n$ the unit timelike normal to the smooth compact spacelike hypersurface $\Sigma$. The integration is meant to be performed through the whole spacetime occupied by the electromagnetic field, i.e. by allowing $\Sigma$ to extend up to the spatial infinity. Evaluating the electromagnetic energy stored inside a finite region with boundary $r = \text{const}$ of spacetime would require instead the introduction of the concept of “quasilocal energy.” However, it is interesting to compare the results of the quasilocal treatment with the expression of the electromagnetic energy contained in the portion of spacetime with boundary $r = \text{const}$ obtained simply by truncating the integration over $r$ at a given $R$ in Eq. (6.24)

$$E(\xi)_{(r_+, R)} = \int_0^R \int_0^{2\pi} \int_0^\pi E(\xi) \sqrt{h_n} r_+ d\theta d\phi$$

$$= \frac{Q^2}{4r_+} \left(1 - \frac{r_+}{R}\right) + \frac{Q^2}{4r_+} \left[\frac{\arctan(a/r_+)}{a/r_+} - \frac{r_+}{R} \left(\frac{a^2}{R^2} \right) \frac{\arctan(a/R)}{a/R}\right], \quad \text{(6.25)}$$

where

$$E(\xi) = T_{\mu \nu}^{\text{em}} \xi^\mu h^\nu = \frac{Q^2}{8\pi \Sigma^3/2} \sqrt{\Delta} \sqrt{r^2 - a^2 \cos^2 \theta + 2a^2 \sin^2 \theta} \sin^2 \theta \quad \text{(6.26)}$$

can be interpreted as the electromagnetic energy density, $h_n$ is the unit normal to the time coordinate hypersurfaces and $h_n = (\Sigma/\Delta)/[(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta] \sin^2 \theta$ is the determinant of the induced metric. It is interesting to note that the same results can be obtained by using the theory of pseudotensors [ACV96] (see Appendix E). In the limit of vanishing rotation parameter Eq. (6.25) becomes

$$E(\xi)_{(r_+, R)} = \frac{Q^2}{2r_+} \left(1 - \frac{r_+}{R}\right), \quad \text{(6.27)}$$

which is just the expression for the electromagnetic energy obtained by Vitagliano and Ruffini [RV02] for the Reissner-Nordström geometry. Eq. (6.24) can be actually considered as a possible quasilocal
Figure 6.5: The projections of the dyadotorus on the $X-Z$ plane corresponding to different values of the ratio $|E|/E_c \equiv k$ are shown in Fig. (a) for $\mu = 10$ and $\lambda = 1.49 \times 10^{-4}$. The corresponding plot for the dyadosphere with the same mass energy and charge to mass ratio is shown in Fig. (b) for comparison.
definition of energy [Sza04], although it strongly depends on the existence of certain spacetime symmetries, i.e. the existence of a timelike Killing vector, which characterizes stationary spacetimes. In addition, we can see that since the current \( J^\mu(\xi) = T^\mu_{(em)}(\xi)^v \) is a conserved vector, the resulting energy does not depend on the chosen cut through spacetime. In contrast, in any given spacetime one can always introduce a physically motivated congruence of observers \( U \) measuring the energy irrespective of spacetime symmetries. But the current \( J^\mu(U) = T^\mu_{(em)}(U)^v \) is not a conserved vector in general. Therefore, in this case the energy has an observer dependent meaning; in addition, the results of the measurement could be different for different cuts through spacetime.

Such an approach consists of using the definition [KLB06, LKB07]

\[
E_{\Sigma}(U) = \int_{\Sigma} T^\mu_{(em)}(U)^v d\Sigma^\nu ,
\]

where \( \Sigma \) is now a bounded hypersurface containing only a finite portion of spacetime, and \( U \) is the 4-velocity of the observer measuring the energy. In general the flux integral of the current \( J^\mu(U) = T^\mu_{(em)}(U)^v \) depends on the hypersurface, because this is not connected with the spacetime symmetries. In particular, the vector field \( U \) can be chosen to be the unit timelike normal \( n \) of \( \Sigma \). Therefore, generally we may always evaluate \( E_{\Sigma}(U) \) with respect to any preferred observer \( U \), but should not expect to get an answer independent of the chosen cut. In the case of axially symmetric spacetimes in practice there is normally a good time coordinate such as Boyer-Lindquist in Kerr and cuts are chosen to be at constant time. The current \( J^\mu(U) \) will be conserved both for static observers and ZAMOs (Zero Angular Momentum Observers), since their 4-velocities are aligned with Killing vectors.

Due to the spacetime symmetries it is indeed quite natural to consider in the Kerr-Newman spacetime two families of observers which are described by two geometrically motivated congruences of curves: 1) static observers, at rest at a given point in the spacetime, whose 4-velocity \( m = 1/\sqrt{g_{tt}} \partial_t \) is aligned with the Killing temporal direction; 2) ZAMOs, a family of locally nonrotating observers with 4-velocity \( n = N^{-1}(\partial_t - N^\phi \partial_\phi) \), where \( N = (-g^{tt})^{-1/2} \) and \( N^\phi = g_{t\phi}/g^{t\phi} \) are the lapse and shift functions respectively, characterized as that normalized linear combination of the two given Killing vectors which is orthogonal to \( \partial_t \) and future-pointing, and it is the unit normal to the time coordinate hypersurfaces. Since the static observers do not exist outside the ergosphere, the ZAMOs seem to be the best candidates to construct the energy (6.28). However, their 4-velocity diverges at the horizon, since the lapse function goes to zero there.

In order to obtain a finite energy at the horizon one can then chose a family of infalling observers as the Painlevé-Gullstrand observers, which move radially with respect to ZAMOs and form a congruence of geodesic and irrotational orbits, whose 4-velocity is given by \( U_{PG} = N^{-1}(n - \sqrt{1 - N^2} \dot{t}) \). Since they do not follow the spacetime symmetries the current \( J^\mu(U_{PG}) = T^\mu_{(em)}(U_{PG}) \) is not conserved, so the corresponding energy \( E_{\Sigma}(U_{PG}) \) depends on the hypersurface. The result is that the expression (6.25) of the electromagnetic energy contained in the dyadoregion constructed by means of the (not normalized) timelike Killing vector agrees with the electromagnetic energy assessed by the Painlevé-Gullstrand geodesic family of infalling observers through the \( T = \text{const cut} \) of the Kerr-Newman spacetime, where \( T \) denotes the Painlevé-Gullstrand time coordinate, i.e.

\[
E_{n}(\xi) = \int_{\Sigma} T^\mu_{(em)}(\xi)^v d\Sigma^\nu = \int_{\Sigma} N^\mu d\Sigma^\nu \equiv E_{\Sigma}(N) ,
\]

BL coordinates, Killing vector, \( t = \text{const cut} \)  \quad \text{PG coordinates, PG 4-velocity, } T = \text{const cut}

with \( N \) the timelike normal to the chosen cut. Details can be found in Appendix E.

From Eq. (6.25), a rough estimate of the electromagnetic energy stored inside the “dyadoregion”

\[
E(\xi)_{r_+, R} \simeq 5.5 \times 10^{-3} \text{ cm } \simeq 6.7 \times 10^{40} \text{ ergs by assuming } R = 2r_+ \text{ with the}
\]
same parameters as in Fig. 6.4 (d), and \( E(\xi)_{(r_+, R)} \approx 1.9 \times 10^{-2} \text{ cm} \approx 2.3 \times 10^{47} \text{ ergs} \) if \( R = 3r_+ \) with the same choice of parameters as in Fig. 6.4 (a). We note that an exact analytic expression for the electromagnetic energy can also be obtained by taking the actual shape \( r = r_+^d \) given by Eq. (6.19) instead of the approximate expression \( r = R = \text{const} \) in the evaluation of the integral (6.25). However, this only complicates matters by introducing a nontrivial dependence on the polar angle \( \theta \) which makes the integration procedure more involved, even if it can be analytically performed (not shown here for the sake of brevity). Furthermore, the numerical values of the energy corresponding to the above choice of parameters agree with previous estimates.

It is interesting to compare the electromagnetic energy (6.25) of an extreme Kerr-Newman black hole contained in the portion of spacetime with boundary \( R = \text{const} \) and that of a Reissner-Nordström black hole (6.27) with the same total mass and charge in the limit of small charge to mass ratio. In this limit we have

\[
E_{\text{RN}} \approx \frac{Q^2}{4M} \left( 1 - \frac{2M}{R} \right),
\]

\[
E_{\text{KN}} \approx \frac{Q^2}{4M} \left( 1 - \frac{2M}{R} \right) + \frac{Q^2}{4M} \left( \frac{\pi}{2} + \frac{M}{R} - \left( 1 + \frac{M^2}{R^2} \right) \arctan(M/R) \right). \quad (6.30)
\]

A comparison between energies is meaningful only at infinity, where the radial coordinates of a Kerr-Newman and a Reissner-Nordström geometry can be identified (both with an ordinary radial coordinate in flat space). For \( R \to \infty \) we thus have

\[
E_{\text{KN}} - E_{\text{RN}} \to \frac{Q^2 \pi}{4M} > 0. \quad (6.31)
\]

### 6.3 Conclusions

Vacuum polarization processes can occur in the field of a Kerr-Newman black hole inside a region we have called dyadotorus, whose properties have been investigated here. Such a region has an invariant character, i.e. its existence does not depend on the observer measuring the electromagnetic field: therefore, it is a true physical region.

Some pictorial representations of the boundary surface similar to those commonly used in the literature have been shown employing Cartesian-like coordinates (i.e. ordinary spherical coordinates built up simply using the Boyer-Lindquist radial and angular coordinates) as well as Kerr-Schild coordinates. The dyadotorus has been also shown on the corresponding embedding diagram, which gives the correct geometry allowing to visualize the spacetime curvature.

We have then estimated the electromagnetic energy contained in the dyadotorus by using three different approaches, which give rise to the same final expression for the energy. The first one follows the standard approach consisting of using the (not normalized) timelike Killing vector through the Boyer-Lindquist constant time cut of the Kerr-Newman spacetime (see e.g. [RV02]), the second one follows a recent observer dependent definition by Katz, Lynden-Bell and Bičák [KLB06, LKB07] for axially symmetric asymptotically flat spacetimes, for which we have used the Painlevé-Gullstrand geodesic family of infalling observers through the Painlevé-Gullstrand constant time cut, and the last one adopts the pseudotensor theory (see e.g. [ACV96]). We have found by rough estimates that the extreme Kerr-Newman black hole leads to larger values of the electromagnetic energy as compared with a Reissner-Nordström black hole with the same total mass and charge.
6.4 Perspectives

Although the vacuum polarization process due to overcritical fields is a quantum phenomenon, the definition of dyadosphere and dyadotorus have been done using solutions to the “classical” Einstein-Maxwell equations. Here we use the expression “classical” in the sense that we do not use to define them a theory of quantum gravity. It would be interesting to explore the quantum corrections to the Einstein-Maxwell field equations and consequently to the metric, of an additional non-linear Euler-Heisenberg term. In such case we need to solve the quantum-corrected Einstein-Maxwell field equations obtained from the action

$$S_{EEH} = -\frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} R + S_{EH}, \quad (6.32)$$

where

$$S_{EH} = \int d^4x \sqrt{-\tilde{g}} L_{EH}, \quad L_{EH} = L_S + \Delta L, \quad (6.33)$$

and $L_S$ is the classic Maxwell action, and $\Delta L$ the correction due to strong fields.
Chapter 7

General Conclusions

1. We have first generalized the treatment of heavy nuclei by enforcing the condition of beta equilibrium in the relativistic Thomas-Fermi equation, avoiding the imposition of \( N_p \approx A/2 \) or any other relation based on phenomenological assumptions between \( N_p \) and \( A \) traditionally assumed in the literature. In doing so we have obtained (see Fig. 2.2) an \( N_p-A \) relation which extends the ones adopted in the literature.

2. By enforcing the beta equilibrium between neutrons, protons, and electrons, we have evidenced the effect of the screening of the positive charged nucleus due to penetration of the electrons on the charge to mass ratio of the nucleus. In particular, it appears to be a major contribution for systems with high mass numbers \( A \gtrsim 10^3 \) (see Fig. 2.2).

3. By constructing a very simple model based on an incompressible Fermi liquid model of the nucleus, we confirmed the above statements, and obtained an analytic Weizsacker-like \( N_p-A \) relation for mass numbers of the order of ordinary nuclei (see Eqs. (2.33) and (2.36)). This charge to mass relation is constructed from first principles by imposing the beta equilibrium of matter. Consequently, it depends only on the potential \( U_{pn} = U_p - U_n \), where \( U_{p,n} \) is the potential energy of protons, neutrons. When applied to a pure electrostatic model \( U_{pn} = eV \), the analytic formula obtained from this simple incompressible model of the nucleus fits quite well the charge to mass ratio obtained from full numerical integration of the relativistic Thomas-Fermi equation done by Ruffini et al. in [VPX09] (see Fig. 2.5). By including the effects due to the strong interaction given by the Relativistic Mean Field model of the nucleus (see [Wal74, Rin96] and Appendix A for details) we obtained once again a Weizsacker-like \( N_p-A \) relation, but now it has adjustable parameters related to the Dirac nucleon effective mass and to the \( \rho \)-meson mass and coupling constant (see Eq. (2.36)). We then have found the values of these parameters in order to fit the Weizsacker stability curve. The welcome result was that the values we found for these parameters perfectly agree with those found in literature through much more complicate calculations. Therefore we had success in obtaining from first principles a stability curve in complete agreement with what we know from experimental data of ordinary nuclei, avoiding the necessity of invoking any phenomenological assumption.

4. We considered the problem of a compressed atom described by a relativistic Thomas-Fermi equation. As in the previous works [FRS80, RS81, RRX07] the protons in the nuclei were assumed to be at constant density, the electron distribution has been derived by the Thomas-Fermi relativistic equation and the neutron component has been derived by the beta equilibrium between neutrons, protons and electrons. The effect of compression has been described by constraining the system in a Wigner-Seitz cell. In doing so we have generalized the classic results obtained in the non-relativistic treatment by Feynman, Metropolis and Teller. In the non-relativistic treatment the Fermi energy of electrons can vary from zero to infinity in view
of the point-like structure of the nucleus. In the relativistic Thomas-Fermi equation, a perfectly finite maximum value of the Fermi energy is reached. The relativistic generalization introduce corrections with two major results:

- The softening of the dependence of the electron Fermi energy on the compression factor.
- The reaching of a limiting value of the electron Fermi energy.

5. It is also appropriate to remark that the correct treatment via a relativistic Thomas-Fermi equation, essential in determining the electron distribution in a compressed atom, is not equivalent to current treatments which have been often adopted in the literature using a variety of approximations (see e.g. [BMG07]). We then have established the energetic range of validity of the uniform approximation of the electron gas.

6. We analyzed the effect of the nuclear model adopted to describe strong nucleon interactions on the behavior of the proton fraction and of the Fermi energy of the electrons with compression. For sake of simplicity, we perform the calculation for a simple incompressible Fermi liquid model adopting a Relativistic Mean Field model for the nucleon strong interaction. We concluded that the effect of the interaction between nucleons due to strong force can be neglected at small electron Fermi energies $<0.1m_{\pi}$ while at larger energies, it becomes important allowing the electrons to possess larger Fermi energies.

7. We extrapolated the results obtained for compressed atoms to the case of massive neutron density cores for $A \simeq (m_{\text{planck}}/m_{n})^3 \sim 10^{57}$. In both systems of the compressed atoms and of the massive nuclear density cores a maximum value of the Fermi energy has been reached corresponding to the case of Wigner-Seitz cell radius $R_{WS}$ coincident with the core radius $R_c$. The results generalize the considerations presented in the article corresponding to a massive nuclear density core with null Fermi energy of the electrons [VPX09].

8. For the case of compressed massive nuclear density cores, an entire family of configurations exist with values of the Fermi energy of the electrons ranging from $E_F^e = 0$ to the maximum value $(E_F^e)_{\text{max}}$. The configuration with $E_F^e = (E_F^e)_{\text{max}}$ fulfills both the global and the local charge neutrality and correspondingly no electrodynamical structure is present in the core. The configuration with $E_F^e = 0$ has the maximum value of the electric field at the core surface, well above the critical value $E_c$ (see Fig. 4.3).

9. The stability of massive nuclear density cores against electron-positron pair creation due to overcritical fields is warranted by the Pauli blocking of the degenerate electrons [RVX09].

10. We have compared and contrasted our treatment of Thomas-Fermi ultrarelativistic solutions to the corresponding one addressed in the framework of strange stars [AFO86]. We have pointed out an inconsistency in the definition of the Coulomb potential energy in [AFO86]. There, by increasing the compression on the electrons of the core due to the crust, the depth of the Coulomb potential energy remains unchanged, which is clearly incorrect as can be seen from Fig. 4.2.

11. We have discussed the problem about the ground-state configuration of a massive nuclear density core by calculating their energetics for selected values of the electron Fermi energy. We found that the configuration with null Fermi energy of the electrons represent the ground-state of the system.

12. We have also discussed the effect of the proton density profile on the electromagnetic structure of the cores. We then integrated the relativistic Thomas-Fermi equation for massive nuclear density cores assuming a proton distribution fulfilling a Woods-Saxon dependence. We have
confirmed the existence of overcritical fields on the core surface also in this case. However, the field is less intense. We analyzed the effect of the bulk value of the proton density and the effect of the diffuseness parameter of the proton distribution on the intensity of the electric field.

13. Exploiting the analytic solution to the ultrarelativistic Thomas-Fermi equation, we discuss the stability of massive nuclear density cores against nuclear fission and against Coulomb repulsion. We found that these systems are indeed stable against both fission and repulsion. The criterion of stability against Coulomb repulsion determines the existence of a lower limit for the mass number $A$ below which the gravitational interaction cannot balance the Coulomb repulsion on the protons at the surface of the core.

14. We compare the pressure due to surface tension and the neutron pressure on the core surface in order to establish if such an abrupt density decrease at nuclear density can indeed exist on these massive nuclear density cores. We found that assuming the surface tension per nucleon to be of the same order of the one acting on ordinary nuclei, such sharp density profile can occur.

15. We formulated a novel approach to the construction of neutron star equilibrium configurations satisfying global charge neutrality instead of local neutrality. The corresponding equilibrium equations then follow from the self-consistent solution of the general relativistic Thomas-Fermi equation, the Einstein-Maxwell equations and the beta equilibrium condition.

16. For any given value of the central density an entire new family of neutron star equilibrium configurations exists. Each configuration is characterized by a strong electric field at the core-crust interface. Such an electric field extends over a thin shell of thickness $\sim 1/m_e$ and becomes largely overcritical in the limit of decreasing values of the crust mass and size.

17. It is appropriate to recall that the existence of overcritical fields on macroscopic objects of $M \sim M_\odot$ and $R \sim 10$ km was first noted in the treatment of quark stars [Wit84, Ito70, AFO86, KWWG95]. In that case the ultrarelativistic Thomas-Fermi equations were also considered (see Chapter 4, Section 4.1 for a direct comparison with the treatment of massive nuclear density cores). However, in all of these investigations, a hybrid combination of general and special relativistic treatments was adopted, resulting in an inconsistency in the boundary conditions. Our treatment indeed represents the first self-consistent formulation of the general relativistic Thomas-Fermi equations, the beta equilibrium condition and the Einstein-Maxwell equations. Critical fields are indeed obtained on the surface of the neutron star core involving only neutrons, protons, and electrons, their fundamental interactions, and with no quarks present.

18. Indeed, the existence of neutron stars with huge crusts, i.e., with both inner and outer crusts, is mainly a consequence of assuming no electrodynamical structure (i.e. assuming local neutrality) and of allowing electrons to have larger values of their Fermi energy $E_F$.

19. We demonstrate analytically that no consistent solution of the Einstein-Maxwell equations satisfying the local neutrality condition exists, even as a limiting case.

20. We generalized the concept of dyadotorus introduced for Reissner-Nordström geometries to the case of a Kerr-Newman geometry. In this case we introduced the concept of dyadotorus, identifying the region around electromagnetic Kerr-Newman black holes where the vacuum polarization processes can occur.

21. We have found by rough estimates that the extreme Kerr-Newman black hole leads to larger values of the electromagnetic energy as compared with a Reissner-Nordström black hole with the same total mass and charge.
22. We estimated the electromagnetic energy contained in the dyadotorus by using three different approaches, which give rise to the same final expression for the energy. The first one follows the standard approach consisting of using the (not normalized) timelike Killing vector through the Boyer-Lindquist constant time cut of the Kerr-Newman spacetime (see e.g. [RV02]), the second one follows a recent observer dependent definition by Katz, Lynden-Bell and Bičák [KLB06, LKB07] for axially symmetric asymptotically flat spacetimes, for which we have used the Painlevé-Gullstrand geodesic family of infalling observers through the Painlevé-Gullstrand constant time cut, and the last one adopts the pseudotensor theory (see e.g. [ACV96]).
Appendix A

Relativistic Mean Field Theory

The Relativistic Mean Field (RMF) description of nuclear matter that we present here, is based mainly on the classic article of Ring “Relativistic mean field theory in finite nuclei” [Rin96]. Consequently, we will follow the so-called Ring formulation (convention) of the RMF model, which is an extension of the pioneer formulation given by Walecka in their classic article “A theory of highly condensed matter” [Val74]. Often, this description of nuclei and nuclear matter is referred as Quantum Hadrodynamics (QHD).

The RMF model describe phenomenologically the nuclei. The nucleons are treated as point-like particles described by Dirac spinors $\psi$, interacting only through exchange of point-like particles called mesons $\phi_i$, which are characterized by their quantum numbers, their mass $m_i$, and by coupling constants $g_i$. Being a phenomenological theory, within the RMFT the number of these mesons, their quantum numbers such as spin ($J$), parity ($P$) and isospin ($T$), and the values of their masses and coupling constants are determined to reproduce as well as possible the experimental data.

In classical field theory, denoting the Lagrangian by $\mathcal{L}(q, \partial_\mu q, t)$, the dynamics of the fields $q_i$ is determined from the variational principle

$$\delta \int dt L = \delta \int d^4x L(q, \partial_\mu q, t) = 0, \quad (A.1)$$

which leads to the classical Euler-Lagrange equations of motion

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu q_j)} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0, \quad \text{for all } j. \quad (A.2)$$

The energy-momentum tensor is given by

$$T^{\mu\nu} = -g^{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu q_j)} \partial^\nu q_j. \quad (A.3)$$

As a consequence of Eq. (A.2) the energy-momentum tensor satisfies the continuity equation

$$\partial_\mu T^{\mu\nu} = 0, \quad (A.4)$$

and the four-momentum

$$P^\nu = \int d^3r T^{0\nu}, \quad (A.5)$$

is conserved. The energy is given by

$$E = P^0 = \int d^3r \mathcal{H}, \quad (A.6)$$

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where $H$ is the Hamiltonian density defined by

$$H = T^{00} = \frac{\partial L}{\partial \dot{q}_j} - L,$$  \hspace{1cm} (A.7)

where as usual a dot denotes time derivative.

In the RMF model the fields $q_j(x)$ are given by the wave functions of the nucleons, the meson fields and the electromagnetic potentials. The nucleons are described by Dirac spinors $\psi_i(x, s, t)$ where $x = (t, r)$ is the spacetime coordinate, $s$ labels the four Dirac components and $t = p, n$ characterizes isospin. The index $i$ runs over all the $A$ nucleons in the system. The mesons are characterized by their quantum numbers $(J, T, P)$, i.e. angular momentum, isospin and parity.

$$L = L_N + L_M + L_{\text{int}},$$  \hspace{1cm} (A.8)

where $L_N$ is the free-nucleons Lagrangian given by

$$L_N = \bar{\psi}(\gamma^\mu \partial_\mu - m)\psi,$$  \hspace{1cm} (A.9)

where $m$ is the nucleon mass and $\psi$ is a Dirac spinor

$$\psi = \left( \begin{array}{c} \psi_p \\ \psi_n \end{array} \right).$$  \hspace{1cm} (A.10)

The Lagrangian $L_M$ for the free-mesons can be written as

$$L_M = L_\sigma + L_\pi + L_\omega + L_\rho + L_\gamma,$$  \hspace{1cm} (A.11)

with

$$L_\sigma = \frac{1}{2}(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2),$$  \hspace{1cm} (A.12)

$$L_\pi = \frac{1}{2}(\partial_\mu \pi \partial^\mu \pi - m_\pi^2 \pi^2),$$  \hspace{1cm} (A.13)

$$L_\omega = -\frac{1}{2} \left( \frac{1}{2} \Omega_{\mu\nu} \Omega^{\mu\nu} - m_\omega^2 \omega_\mu \omega^\mu \right),$$  \hspace{1cm} (A.14)

$$L_\rho = -\frac{1}{2} \left( \frac{1}{2} \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} - m_\rho^2 \bar{\rho}_\mu \bar{\rho}^\mu \right),$$  \hspace{1cm} (A.15)

$$L_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$  \hspace{1cm} (A.16)

where $m_\sigma, m_\pi, m_\omega,$ and $m_\rho$ are the rest masses of the mesons, and

$$\Omega^{\mu\nu} = \partial^\mu \omega^\nu - \partial^\nu \omega^\mu,$$  \hspace{1cm} (A.17)

$$\bar{R}^{\mu\nu} = \partial^\mu \bar{\rho}^\nu - \partial^\nu \bar{\rho}^\mu,$$  \hspace{1cm} (A.18)

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$  \hspace{1cm} (A.19)

are the field tensors, and arrows denote vectors in isospin space. Finally, we have the interaction between the nucleons and the mesons. In the simplest way, this is given by the ansatz of minimal coupling

$$L_{\text{int}} = -g_\omega \bar{\psi} \gamma_5 \sigma \psi - ig_\pi \bar{\psi} \gamma_5 \tau_3 \pi \psi - g_\omega \bar{\psi} \gamma_\mu \omega^\mu \psi - g_\rho \bar{\psi} \gamma_\mu \bar{\rho}^\mu \psi - e \left( \frac{1}{2} + \tau_3 \right) \bar{\psi} \gamma_\mu A^\mu \psi,$$  \hspace{1cm} (A.20)
with the coupling constants $g_σ, g_π, g_ω, \text{ and } g_ρ$, and the Pauli isospin matrices $\vec{τ}$.

In order to fit appropriately the surface properties of the nucleus, often it is also included a non-linear self-interaction of the $σ$-meson given by

$$U(σ) = \frac{1}{3}g_2σ^3 + \frac{1}{4}g_3σ^4.$$  \hspace{1cm} (A.21)

The equation of motion for the Dirac fields gives us the Dirac equation

$$[\gamma^μ(\partial^μ - V^μ) - m - S - iγ_5P]ψ_i = 0,$$  \hspace{1cm} (A.22)

with the relativistic fields

$$S = g_σσ,$$  \hspace{1cm} (A.23)
$$P = g_π\vec{τ}π,$$  \hspace{1cm} (A.24)
$$V^μ = g_ωω^μ + g_ρ\vec{τ}_ρ^μ + e\frac{1 + τ_3}{2}A^μ,$$  \hspace{1cm} (A.25)

while the equations of motions for the meson fields read

$$(\partial^ν\partial_ν + m_σ^2)σ = -g_σρ_s - \frac{dU(σ)}{dσ},$$  \hspace{1cm} (A.26)
$$(\partial^ν\partial_ν + m_π^2)π = -g_π\vec{ρ}_ps,$$  \hspace{1cm} (A.27)
$$(\partial^ν\partial_ν + m_ω^2)ω^μ = g_ωj^μ,$$  \hspace{1cm} (A.28)
$$(\partial^ν\partial_ν + m_ρ^2)ρ^μ = g_ρj^μ,$$  \hspace{1cm} (A.29)
$$(\partial^ν\partial_ν)A^μ = e_j^μc,$$  \hspace{1cm} (A.30)

with the scalar density

$$ρ_s = \sum_{i=1}^A \bar{ψ}_iψ_i,$$  \hspace{1cm} (A.31)

the pseudo-scalar density

$$\vec{ρ}_ps = \sum_{i=1}^A \bar{ψ}_iγ_5\vec{τ}ψ_i,$$  \hspace{1cm} (A.32)

the baryon current

$$j^μ = \sum_{i=1}^A \bar{ψ}_iγ^μψ_i,$$  \hspace{1cm} (A.33)

the isocurrent

$$\vec{j}^μ = \sum_{i=1}^A \bar{ψ}_iγ^μ\vec{τ}ψ_i,$$  \hspace{1cm} (A.34)

and the electromagnetic current

$$j^μ = \sum_{i=1}^A \bar{ψ}_i\left(\frac{1 + τ_3}{2}\right)γ^μψ_i,$$  \hspace{1cm} (A.35)

where

$$τ_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (A.36)

In the static case, we assume time-independence for the meson fields and the single particle wave functions are written as $ψ_i = ψ_i(r) \exp(\imath E_i t)$. If one assumes furthermore systems with time reversal
invariance and good parity, as for instance in the ground-state of even-even nuclei, the space-like components of all currents \( j, \vec{j}, j_c \) and the pion field vanish. In this case the Lagrangian becomes

\[
\mathcal{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi + \frac{1}{2} \left[ -(\nabla \sigma)^2 - m_\sigma^2 \sigma^2 \right] - \frac{1}{2} \left[ -(\nabla \omega^0)^2 - m_\omega^2 (\omega^0)^2 \right] - \frac{1}{2} \left[ -(\nabla \rho^0_3)^2 - m_\rho^2 (\rho^0_3)^2 \right] - \frac{1}{2} \left( \nabla A^0 \right)^2 - g_\sigma \rho_s - g_\omega \omega^0 \rho - g_\rho \rho^0_3 \rho_3 - eA^0 \rho_c,
\]

(A.37)

where the superscript 0 indicates the temporal component of the vector-meson, which is the only that survives within this approximation.

The stationary RMF equations are

\[
\begin{align*}
-ia \nabla + \beta(m + S) + V \psi_i &= E_i \psi_i, \\
( - \nabla^2 + m_\sigma^2 ) \sigma &= -g_\sigma \rho_s - \frac{dU(\sigma)}{d\sigma}, \\
( - \nabla^2 + m_\omega^2 ) \omega^0 &= g_\omega \rho, \\
( - \nabla^2 + m_\rho^2 ) \rho_3^0 &= g_\rho \rho_3, \\
- \nabla^2 A^0 &= e \rho_c,
\end{align*}
\]

(A.38) - (A.42)

where \( \gamma^0 = \beta \) and \( \gamma^i = \beta \alpha^i \). Here \( \rho_s \) is the scalar density

\[
\rho_s = \sum_{i=1}^{A} \bar{\psi}_i \psi_i,
\]

(A.43)

\( n \) is the usual baryon density

\[
\rho = \sum_{i=1}^{A} \bar{\psi}_i \psi_i,
\]

(A.44)

\( \rho_3 \) is the isovector density

\[
\rho_3 = \sum_{i=1}^{A} \bar{\psi}_i \tau^3 \psi_i,
\]

(A.45)

and \( \rho_c \) is the charge density

\[
\rho_c = \sum_{i=1}^{A} \bar{\psi}_i \left( \frac{1 + \tau^3}{2} \right) \psi_i.
\]

(A.46)

In the Dirac equation (A.38) appears two potentials, the so-called vector potential

\[
V(r) = g_\omega \omega^0(r) + g_\rho \rho^0_3(r) + e \left( \frac{1 + \tau^3}{2} \right) A^0(r),
\]

(A.47)

and the scalar potential

\[
S(r) = g_\sigma \sigma(r),
\]

(A.48)

which contributes to the effective Dirac mass (see Eq. (A.38)) in the form

\[
m^* (r) = m + S(r).
\]

(A.49)

The total energy density of the system is given by

\[
\mathcal{E} = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi + \frac{1}{2} \left[ -(\nabla \sigma)^2 - m_\sigma^2 \sigma^2 \right] - \frac{1}{2} \left[ -(\nabla \omega^0)^2 - m_\omega^2 (\omega^0)^2 \right] - \frac{1}{2} \left[ -(\nabla \rho^0_3)^2 - m_\rho^2 (\rho^0_3)^2 \right] - \frac{1}{2} \left( \nabla A^0 \right)^2 - g_\sigma \rho_s - g_\omega \omega^0 \rho - g_\rho \rho^0_3 \rho_3 - eA^0 \rho_c,
\]

(A.50)
where we have used the definition (A.7).

The Fermi energy of nucleons can be calculated as usual by

\[ E_F^i = \frac{\partial E}{\partial n_i}, \quad i = n, p, \]  

(A.51)

where \( n_i \) denotes particle density. By using the Klein-Gordon equations for the fields (A.38)–(A.41) the Fermi energy (A.51) becomes

\[ E_F^i = \sqrt{p_i^2 + (m^*)^2 + V_i}, \]  

(A.52)

where

\[ V_i = \begin{cases} g_\omega \omega_0(r) + g_p \rho_3^0(r) + eA^0(r), & i = p, \\ g_\omega \omega_0(r) - g_p \rho_3^0(r), & i = n. \end{cases} \]  

(A.53)

Using the above formulas, we obtain the beta equilibrium condition

\[ E_{F n} = E_{F p} + E_{F e}, \]  

(A.54)

which using Eq. (A.52) becomes

\[ \sqrt{(P_{Fn})^2 + (m^*)^2 - g_\rho \rho_0^0(r)} = \sqrt{(P_{Fp})^2 + (m^*)^2 + g_\rho \rho_3^0(r) + eA^0(r) + E_{F e}}. \]  

(A.55)

Assuming a uniform approximation for the \( \sigma, \omega, \) and \( \rho \) mesons, from Eqs. (A.39)–(A.41) we have

\[ m^2 \sigma = -g_\sigma \rho_0 \frac{dU(\sigma)}{d\sigma}, \]  

(A.56)

\[ m^2 \omega = g_\omega \rho, \]  

(A.57)

\[ m^2 \rho_3 = g_\rho \rho_3, \]  

(A.58)

and consequently the beta equilibrium equation (A.55) becomes

\[ \sqrt{(P_{Fn})^2 + (m^*)^2 - C_\rho (n_p - n_n)} = \sqrt{(P_{Fp})^2 + (m^*)^2 + C_\rho (n_p - n_n) + eA^0(r) + E_{F e}}, \]  

(A.59)

where

\[ C_\rho = \left( \frac{g_\rho}{m_p} \right)^2, \]  

(A.60)

and \( n_p \) and \( n_n \) are the proton and neutron density.
Appendix B

General anisotropic energy-momentum tensor

We start with the Schwarzschild-like metric
\[ ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2. \] (B.1)

Let us to write the above using tetrad formalism as \[ [\text{Cha83}] \]
\[ g_{\alpha\beta} = e^{(a)}_{\alpha} e^{(b)}_{\beta} \eta_{(a)(b)}, \] (B.2)
where \( e^{(a)}_{\alpha} \) is an orthonormal tetrad that we will define as
\[ e^{(0)}_{\alpha} = u_{\alpha}, \quad e^{(1)}_{\alpha} = \chi_{\alpha}, \quad e^{(2)}_{\alpha} = \theta_{\alpha}, \quad e^{(3)}_{\alpha} = \phi_{\alpha}. \] (B.3)

Then, the four-vectors must satisfied the conditions
\[ u_{\alpha} \chi^\alpha = u_{\alpha} \theta^\alpha = u_{\alpha} \phi^\alpha = 0, \quad u_{\alpha} u^\alpha = 1, \quad \chi_{\alpha} \chi^\alpha = \theta_{\alpha} \theta^\alpha = \phi_{\alpha} \phi^\alpha = -1. \] (B.4)

In order to describe the physical quantities in the energy-momentum tensor, we choose \( \eta_{(a)(b)} \) as the Minkowski metric. We then obtain the orthonormal tetrad
\[ e^{(a)}_{\alpha} = L^{(a)}_{\beta} e^{(\beta)}_{\alpha}, \quad e^{(\beta)}_{\alpha} = \delta^{(\beta)}_{\alpha}, \] (B.5)
where
\[ L^{(a)}_{\beta} = \text{diag}(e^{\nu/2}, e^{\lambda/2}, r, r \sin \theta), \] (B.6)
and \( e^{(a)}_{\alpha} \) denotes the base of the Minkowski spacetime.

In the Minkowski spacetime, we can write the general energy-momentum tensor as
\[ T^{(a)}_{\alpha\beta} = \rho u^{(a)} u_{\alpha} + S^{(a)}_{\alpha\beta} + Q^{(a)}_{\alpha\beta}, \] (B.7)
where \( S^{(a)}_{\alpha\beta} \) is the stress tensor and \( Q^{(a)}_{\alpha\beta} \) is the dissipation tensor which we take as zero because we are interested in describing systems in thermodynamic equilibrium.

In general, the stress tensor can be written as
\[ S^{(a)}_{\alpha\beta} = P_{xx} x^{(a)} x_{\beta} + P_{yy} y^{(a)} y_{\beta} + P_{zz} z^{(a)} z_{\beta}, \] (B.8)
where the base \( \{ x_\alpha, y_\alpha, z_\alpha \} \) represent the principal directions of stresses, and \( P_i \) the principal pressures as can be seen from the eigenvalues equation

\[
S_{\alpha \beta} x^\beta = -P_{xx} x_\alpha, \\
S_{\alpha \beta} y^\beta = -P_{yy} y_\alpha, \\
S_{\alpha \beta} z^\beta = -P_{zz} z_\alpha.
\]

Therefore, in spherical symmetry the above stress tensor becomes

\[
S_{\alpha \beta} = P_r \chi_\alpha \chi_\beta + P_\perp (\theta_\alpha \theta_\beta + \phi_\alpha \phi_\beta).
\]

Using the fact that

\[
\theta_\alpha \theta_\beta + \phi_\alpha \phi_\beta = h_{\alpha \beta} - \chi_\alpha \chi_\beta,
\]

where

\[
h_{\alpha \beta} = u_\alpha u_\beta - \eta_{\alpha \beta},
\]

is the projection tensor, we obtain the stress tensor

\[
S_{\alpha \beta} = P_\perp u_\alpha u_\beta + (P_r - P_\perp) \chi_\alpha \chi_\beta - P_\perp \eta_{\alpha \beta}.
\]

Correspondingly, the energy-momentum tensor (B.7) in the Minkowski spacetime

\[
T_{\alpha \beta} = (\rho + P_\perp) u_\alpha u_\beta - P_\perp \eta_{\alpha \beta} + (P_r - P_\perp) \chi_\alpha \chi_\beta,
\]

and applying the general covariance principle we finally obtain

\[
T_{\alpha \beta} = (\rho + P_\perp) u_\alpha u_\beta - P_\perp g_{\alpha \beta} + (P_r - P_\perp) \chi_\alpha \chi_\beta.
\]
Appendix C

Equilibrium conditions and the Tolman-Oppenheimer-Volkoff equation

The equation of hydrostatic equilibrium $T^\alpha_\beta = 0$ for the metric (5.8) for a general energy-momentum tensor

$$T^\alpha_\beta = \text{diag}(T_0^0, T_1^1, T_2^2, T_3^3),$$  
(C.1)

with $T_2^2 = T_3^3$ reads

$$-\frac{d T_1^1}{dr} = -\frac{1}{2} \frac{d \nu}{dr} (T_0^0 - T_1^1) - \frac{2}{r} (T_2^2 - T_1^1).$$  
(C.2)

The above Eq. (C.2) can be called the generalized Tolman-Oppenheimer-Volkoff equation (TOV).

For the combined energy-momentum tensor of the matter and fields $T_{\mu \nu}$ given by Eq. (5.11) the generalized TOV given by Eq. (C.2) becomes

$$\frac{d P}{dr} = -\frac{1}{2} \frac{d \nu}{dr} (E + P) - \left( \frac{4 P_{em}}{r} + \frac{d P_{em}}{dr} \right),$$  
(C.3)

where $P_{em} = -E^2/(8\pi)$, with $E$ being the electric field, which is related to the Coulomb potential $V$ by

$$E = -\frac{d V}{dr} e^{-(\nu + \lambda)/2}.$$  
(C.4)

The energy density and pressure of a gas of neutrons, protons, and electrons can be written as

$$E = \sum_{i=e,p,n} E_i, \quad P = \sum_{i=e,p,n} P_i,$$  
(C.5)

where $E_i$ and $P_i$ are the contributions to the energy density and pressure of the $i$-specie.

From the first law of thermodynamics for a gas at zero temperature we have

$$E_i + P_i = n_i \mu_i,$$  
(C.6)

where as usual $n_i$ and $\mu_i$ denote respectively the particle number density and the free chemical potential of the $i$-specie.

Using the Eq. (C.6), the TOV equation (C.3) can be rewritten as

$$\sum_{i=e,p,n} n_i \frac{d \mu_i}{dr} + \frac{1}{2} \frac{d \nu}{dr} \sum_{i=e,p,n} n_i \mu_i + e^{-\nu/2} \frac{d V}{dr} \sum_{i=e,p,n} q_i n_i = 0,$$  
(C.7)
where \( q_i = +e, 0, -e \) is the electric charge of protons, neutrons, and electrons respectively, and we have used the general relativistic Poisson equation (5.17).

Introducing the beta equilibrium condition (see Eq. (5.22))

\[
\mu_n = \mu_p + \mu_e, \tag{C.8}
\]

and the general relativistic definition of Fermi energy of the \( i \)-specie (see Eq. 5.25)

\[
E_{F_i} = e^{\nu/2} \mu_i + q_i V, \tag{C.9}
\]

the TOV equation (C.7) reduces to

\[
\sum_{j=p,n} (n_n + n_j) \frac{dE_{F_j}}{dr} = 0. \tag{C.10}
\]

Therefore, from the above equation (C.10) we conclude that as a consequence of beta equilibrium and the general relativistic Thoms-Fermi equilibrium condition for the electrons (5.25) the following conditions must be satisfied

\[
E_{F_i} = e^{\nu/2} \mu_i + q_i V = \text{constant}, \quad i = e, p, n. \tag{C.11}
\]
The existence of the “dyadotorus” has an invariant character. This fact appears more evident if the electric and magnetic field strengths are expressed in terms of the electromagnetic invariants. Let us adopt here the metric signature \((+,-,-,-)\) in order to use the Newman-Penrose formalism in its original form and then easily get the necessary physical quantities [Cha83, CBC+02]. The Kerr-Newman metric is thus given by

\[
d s^2 = \left(1 - \frac{2M r - Q^2}{\Sigma}\right) dt^2 + \frac{2a \sin^2 \theta}{\Sigma} \left(2M r - Q^2\right) d t d \phi - \frac{\Sigma}{\Delta} d r^2 \\
-\Sigma d \theta^2 - \left[r^2 + a^2 + \frac{a^2 \sin^2 \theta}{\Sigma}(2Mr - Q^2)\right] \sin^2 \theta d \phi^2 ,
\]  

with associated electromagnetic field

\[
F = \frac{Q}{\Sigma^2} (r^2 - a^2 \cos^2 \theta) dr \wedge [dt - a \sin^2 \theta d \phi] + 2 \frac{Q}{\Sigma^2} a r \sin \theta \cos \theta d \theta \wedge [(r^2 + a^2) d \phi - a d \phi] .
\]  

Introduce the standard Kinnersley principal tetrad [Kin69]

\[
l^\mu = \frac{1}{\Delta} [r^2 + a^2, \Delta, 0, a] , \\
n^\mu = \frac{1}{2\Sigma} [r^2 + a^2, -\Delta, 0, a] , \\
m^\mu = \frac{1}{\sqrt{2}(r + ia \cos \theta)} \left[i a \sin \theta, 0, 1, i \frac{1}{\sin \theta}\right] ,
\]  

which gives nonvanishing spin coefficients

\[
\rho = - \frac{1}{r - ia \cos \theta} , \quad \tau = - \frac{ia}{\sqrt{2}} \rho \rho^* \sin \theta , \quad \beta = - \frac{\rho^*}{2\sqrt{2}} \cot \theta , \\
\pi = \frac{ia}{\sqrt{2}} \rho^2 \sin \theta , \quad \mu = \frac{1}{2} \rho^2 \rho^* \Delta , \quad \gamma = \mu + \frac{1}{2} \rho \rho^* (r - M) , \quad \kappa = \pi - \beta^* ,
\]  

and the only nonvanishing Weyl scalar

\[
\psi_2 = M \rho^3 + Q^2 \rho^* \rho^3 ,
\]
showing clearly the Petrov type D nature of the Kerr-Newman spacetime, whereas the Maxwell scalars are

$$\phi_0 = \phi_2 = 0, \quad \phi_1 = \frac{Q}{2} \rho^2.$$  \hfill (D.6)

The electromagnetic invariants are given by

$$\mathcal{F} \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (B^2 - E^2) = 2 \text{Re}(\phi_0\phi_2 - \phi_1^2),$$

$$\mathcal{G} \equiv \frac{1}{4} F_{\mu\nu}^* F^{\mu\nu} = E \cdot B = -2 \text{Im}(\phi_0\phi_2 - \phi_1^2),$$ \hfill (D.7)

where \(E\) and \(B\) are the electric and magnetic fields. Requiring parallel electric and magnetic fields \cite{DR75} as measured by the Carter observer \cite{Car68}, the previous relations become

$$|B|^2 - |E|^2 = -4 \text{Re}(\phi_1^2), \quad |E| |B| = 2 \text{Im}(\phi_1^2),$$ \hfill (D.8)

taking into account Eq. (D.6). This system can then be easily solved for the magnitudes of \(E\) and \(B\) in the Kerr-Newman background, which turn out to be given by

$$|E| = \left| \frac{Q}{\Sigma} (r^2 - a^2 \cos^2 \theta) \right|, \quad |B| = \left| 2 \frac{Q}{\Sigma} \bar{a} r \cos \theta \right|,$$ \hfill (D.9)

which coincide with those of Eq. (6.17). We have thus recovered the results by Damour and Ruffini \cite{DR75}, but using a different derivation using the Newman-Penrose formalism.

Finally, the Schwinger formula for the rate of pair creation per unit four-volume in terms of the electromagnetic invariants (D.7) is given by \cite{Sch51}

$$2 \text{Im} \mathcal{L} = \frac{e^2 |\mathcal{G}|}{4 \pi^2 \hbar^2} \sum_{n=1}^{\infty} \frac{1}{n} \coth \left[ n \pi \left( \frac{(\mathcal{F}^2 + \mathcal{G}^2)^{1/2} + \mathcal{F}}{(\mathcal{F}^2 + \mathcal{G}^2)^{1/2} - \mathcal{F}} \right)^{1/2} \right] e^{-n\pi E_c/(|\mathcal{F}^2 + \mathcal{G}^2|^{1/2} - |\mathcal{F}|^{1/2})}. \hfill (D.10)$$

After introducing the Carter frame \cite{6.14}--\cite{6.15} with respect to which electric and magnetic fields are parallel, the previous formula reduces to Eq. (6.15), since

$$[(\mathcal{F}^2 + \mathcal{G}^2)^{1/2} + \mathcal{F}]^{1/2} = |B|, \quad [(\mathcal{F}^2 + \mathcal{G}^2)^{1/2} - \mathcal{F}]^{1/2} = |E|, \quad |\mathcal{G}| = |E| |B|. \hfill (D.11)$$
Appendix E

Dyadotorus electromagnetic energy

In order to evaluate the energy \( E_\Sigma(U_{PC}) \) it is useful to transform the Kerr-Newman metric (6.12) from Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) to Painlevé-Gullstrand coordinates \((T, R, \Theta, \Phi)\) [Dor00, Coo00], which are related by the transformation

\[
T = t - \int f(r) dr, \quad R = r, \quad \Theta = \theta, \quad \Phi = \phi - \int \frac{a}{r^2 + a^2} f(r) dr,
\]

where

\[
f(r) = - \sqrt{\frac{(2Mr - Q^2)(r^2 + a^2)}{\Delta}}.
\]

Let us notice that \( r \) and \( R \) are identified. This is also true for their differential \( dr = dR \) but it is no more true for the associated differentiations \( \partial_r \neq \partial_R \). Hereafter we will always use \( r \) in place of \( R \), except for the differentiation operations. In differential form, this transformation writes as

\[
\begin{align*}
&T = dt - f(r) dr, \quad R = dr, \quad \Theta = d\theta, \quad \Phi = d\phi - \frac{a}{r^2 + a^2} f(r) dr.
\end{align*}
\]

Finally, the Kerr-Newman metric in the Painlevé-Gullstrand coordinates is given by

\[
ds^2 = - \left(1 - \frac{2Mr - Q^2}{\Sigma}\right) dT^2 + 2 \sqrt{\frac{2Mr - Q^2}{r^2 + a^2}} dT dr - \frac{2a(2Mr - Q^2)}{\Sigma} \sin^2 \theta dT d\Phi
\]
\[
+ \sin^2 \theta \left[ r^2 + a^2 + \frac{a^2(2Mr - Q^2)}{\Sigma} \sin^2 \theta \right] d\Phi^2 - 2a \sin^2 \theta \sqrt{\frac{2Mr - Q^2}{r^2 + a^2}} dr d\Phi
\]
\[
+ \frac{\Sigma}{r^2 + a^2} dr^2 + \Sigma d\theta^2
\]

with associated electromagnetic field

\[
F = \frac{Q}{\Sigma^2} (r^2 - a^2 \cos^2 \theta) dr \wedge [dT - a \sin^2 \theta d\Phi]
\]
\[
+ 2 \frac{Q}{\Sigma^2} dr \sin \theta \cos \theta d\theta \wedge [(r^2 + a^2) d\Phi - adT],
\]

which has the same form as (6.13) with \( dt \to dT \) and \( d\phi \to d\Phi \).

The limit of vanishing rotation parameter \( a = 0 \) of the previous equations (E.4)–(E.5) gives rise to the Reissner-Nordström solution in Painlevé-Gullstrand coordinates

\[
ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dT^2 + 2 \sqrt{\frac{2M}{r} - \frac{Q^2}{r}} dT dr + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

\[
F = \frac{Q}{r^2} dr \wedge dT.
\]

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In the Painlevé-Gullstrand coordinates the slicing observers \( (T - \text{slicing hereafter}) \) have 4-velocity
\[
\mathcal{N} = \partial_T - \frac{\sqrt{2Mr - Q^2(r^2 + a^2)}}{\Sigma} \partial_R
\]  
\hspace{1cm} \text{(E.7)}
and associated 1-form \( \mathcal{N}^\alpha = -dT \). This family of \( T \)-slicing-adapted observers does not coincide with the \( t \)-slicing-adapted observers in Boyer-Lindquist coordinates once the coordinate transformation is performed. In fact, when expressed in Boyer-Lindquist coordinates the \( T \)-slicing-adapted observers move with respect to the \( t \)-slicing-adapted observers in the radial direction, as already pointed out in Section V.

We are now ready to evaluate the energy \( \mathcal{E}(\mathcal{N}) \) through a \( T = \text{const} \) hypersurface as measured by Painlevé-Gullstrand observers with 4-velocity \( \mathcal{N} \). The energy density turns out to be
\[
\mathcal{E}(\mathcal{N}) = T^{(\text{em})\mu}_\nu \mathcal{N}^\mu \mathcal{N}^\nu = \frac{Q^2}{8\pi \Sigma} (r^2 - a^2 \cos^2 \theta + 2a^2),
\]  
\hspace{1cm} \text{(E.8)}
where \( T^{(\text{em})\mu}_\nu \) is the Kerr-Newman electromagnetic energy-momentum tensor expressed in Painlevé-Gullstrand coordinates. Let us assume that the boundary \( S \) of \( \Sigma \) be the 2-surface \( r = R = \text{const} \) for simplicity. Therefore the energy \( \mathcal{E}(\mathcal{N}) \) turns out to be given by
\[
E(\mathcal{N})_{(r_+,R)} = 2\pi \int_{r_+}^{R} \int_0^\pi \mathcal{E}(\mathcal{N}) \sqrt{h_N} dr d\theta = \frac{Q^2}{4a} \left[ \frac{a^2}{r^2} + 2 \frac{a^2}{r^2} \arctan \frac{a}{r} - \frac{\pi}{2} \right]_{r_+}^{R} = \frac{Q^2}{4r_+} - \frac{Q^2}{4R} + \frac{1}{4a^2} \left( r_+^2 + a^2 \right) \arctan \frac{a}{r_+} - \frac{1}{4a^2} \left( R^2 + a^2 \right) \arctan \frac{a}{R},
\]  
\hspace{1cm} \text{(E.9)}
where \( h_N = \Sigma^2 \sin^2 \theta \) is the determinant of the induced metric. The total electromagnetic energy contained in the whole spacetime is obtained by taking the limit \( R \to \infty \) in the previous equation
\[
E(\mathcal{N})_{(r_+,\infty)} = \frac{Q^2}{4r_+} + \frac{1}{4a^2} \left( r_+^2 + a^2 \right) \arctan \frac{a}{r_+},
\]  
\hspace{1cm} \text{(E.10)}
which in the limiting case \( a = 0 \) reduces to
\[
E^{RN}(\mathcal{N})_{(r_+,\infty)} = \frac{Q^2}{2r_+}.
\]  
\hspace{1cm} \text{(E.11)}

It is interesting to note that the same result \( \text{(E.9)} \) for the energy assessed by Painlevé-Gullstrand observer is achieved simply by using the Killing vector \( \xi = \partial_T \), since \( \mathcal{E}(\mathcal{N}) = \mathcal{E}(\xi) \). But it is quite surprising that the same result is again obtained by taking a \( t = \text{const} \) hypersurface in Boyer-Lindquist coordinates with unit normal the ZAMO 4-velocity \( n \) with respect to “Killing observers” \( \tilde{\xi} = \partial_t \), since
\[
\mathcal{E}(\mathcal{N}) \sqrt{h_N} = \mathcal{E} \sqrt{h_n} = \frac{Q^2}{8\pi \Sigma^2} (r^2 - a^2 \cos^2 \theta + 2a^2) \sin \theta.
\]  
\hspace{1cm} \text{(E.12)}

For completeness we list here similar results presented in [ACV96] by using the standard definition of symmetric energy-momentum pseudotensor as given by Landau and Lifshitz [Lan53], although we stress that the physical interpretation of these quantities are controversial in the literature, due to their strict relation with specific coordinate sets. This fact is clearly not in the spirit of general relativity. The Landau-Lifshitz prescription for the pseudotensor is given by
\[
16\pi L_{\alpha\beta} = \Lambda^{\alpha\beta\gamma\delta} \gamma_{\gamma\delta},
\]  
\hspace{1cm} \text{(E.13)}
where comma denotes partial derivative and
\[ \lambda^{a\beta\gamma\delta} = -g \left( g^{a\beta} g^{\gamma\delta} - g^{a\gamma} g^{\beta\delta} \right). \] (E.14)

The conservation law \( L^{a\beta}_{\ ,\beta} = 0 \) implies that the total energy is given by
\[ E = \int \int \int L_{00} dx^1 dx^2 dx^3. \] (E.15)

By computing the pseudotensor in the quasi-Cartesian Kerr-Schild and requiring the integration to be performed on a Boyer-Lindquist \( r = R = \text{const} \) surface, one obtains
\[ E - M = E^m = \frac{Q^2}{4r_+} - \frac{Q^2}{4R} + \frac{1}{4} \frac{Q^2}{a r_+^2} (r_+^2 + a^2) \arctan \frac{a}{r_+} - \frac{1}{4} \frac{Q^2}{a R^2} (R^2 + a^2) \arctan \frac{a}{R}, \] (E.16)
which is the same result given by Eq. (E.9). Note that in [ACV96] the same result is obtained using plenty other different energy-momentum pseudotensors.
Bibliography


BIBLIOGRAPHY


