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Pair-Instability Supernova Explosions and Gamma-Ray Bursts

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Abstract

Doctor of Relativistic Astrophysics

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by Andrey Baranov

According to theoretical models, massive stars with masses within the 100–250 $M_{\odot}$ range should explode as pair-instability supernovae (PISNe). Since the first stars of the Universe are believed to be very massive, these supernovae should play a significant role in the early stages of its history. Moreover because of huge mass of the progenitor PISN is the most energetic type of supernovae and it is a possible candidate for explanation of gamma-ray bursts (GRBs) phenomenon.

In this thesis PISN explosion is analyzed using various numerical codes. One-dimensional computations were performed to establish a range of masses and initial conditions where this type of explosion is possible. The role of hydrodynamical instabilities in the propagation of shockwave during the last stage of the explosion was studied using two-dimensional code based on PPML method. Such studies are needed to compute light curves and time of appearance of lines of chemical elements, so they allow to predict the observational signatures of these supernovae for future space and ground telescopes. Results of simulations are in good agreement with previous works and in general reproduce the results of other numerical codes. However a new scenario of multicore ignition explosion was proposed. This could be an “exotic scenario” and not the standard explored to now. But if the explosion is nonuniform, it could change the light curve, chemical production, and also the spectrum.

After PISN is applied for description of GRBs. Results of simulations show that pair-instability supernovae provide necessary energy budget, timescale and peak energy of emission. Moreover a correlation between total nuclear energy release and maximum temperature was found. Basing on this correlation a new physical interpretation of the empirical Amati relation is proposed. The physical model of GRBs is developed further and possible explanations of time-variability and spectra formation are proposed.
## Contents

**Abstract** i

**List of Figures** iv

**List of Tables** vi

1 **Introduction** 1

   1.1 Stellar structure 3
   1.2 Polytropic configurations 6
   1.3 Stellar evolution 9
   1.4 Supernovae 12
   1.5 Gamma Ray Bursts 16
   1.6 Thesis outline 18

2 **Pair-Instability Supernovae** 20

   2.1 Role of Electron-Positron Pairs 20
   2.2 Pair-Instability Supernova 24

3 **Pair-Instability Supernovae modeling** 27

   3.1 Physical model 27
      3.1.1 Nuclear burning 29
      3.1.2 Neutrino losses 31
      3.1.3 Equation of state 32
   3.2 Initial configuration 36
   3.3 Computational scheme 42

4 **Results of 1D computations** 45

5 **Modelling in 2D** 53

   5.1 Statement of the problem 53
   5.2 Initial configuration 54
   5.3 Results of computations in 2D 56

6 **Gamma-Ray Bursts** 59

   6.1 Observations 59
      6.1.1 Light-curve 59
      6.1.2 Spectrum 61
      6.1.3 GRB–SN connection 63
<table>
<thead>
<tr>
<th>Contents</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1.4 Host environment</td>
<td>65</td>
</tr>
<tr>
<td>6.2 Models of GRBs</td>
<td>66</td>
</tr>
<tr>
<td>6.2.1 Compactness problem</td>
<td>66</td>
</tr>
<tr>
<td>6.2.2 Fireball</td>
<td>68</td>
</tr>
<tr>
<td>6.2.3 Cannonball</td>
<td>70</td>
</tr>
<tr>
<td>6.2.4 Fireshell</td>
<td>70</td>
</tr>
<tr>
<td>6.3 GRBs: the missing link of stellar evolution</td>
<td>71</td>
</tr>
<tr>
<td>7 Gamma-Ray Burst as Pair-Instability Supernova</td>
<td>72</td>
</tr>
<tr>
<td>7.1 PISN as a possible candidate for GRBs</td>
<td>72</td>
</tr>
<tr>
<td>7.2 On compactness problem</td>
<td>73</td>
</tr>
<tr>
<td>7.3 Peak energy of emission</td>
<td>74</td>
</tr>
<tr>
<td>7.4 Possible interpretation of the Amati relation</td>
<td>74</td>
</tr>
<tr>
<td>7.5 Time-variability</td>
<td>76</td>
</tr>
<tr>
<td>7.6 Spectrum</td>
<td>78</td>
</tr>
<tr>
<td>8 Conclusions</td>
<td>81</td>
</tr>
<tr>
<td>8.1 Prospects for future work</td>
<td>82</td>
</tr>
<tr>
<td>A Source Codes</td>
<td>84</td>
</tr>
<tr>
<td>A.1 Structure of the program</td>
<td>84</td>
</tr>
<tr>
<td>A.2 main.f90</td>
<td>85</td>
</tr>
<tr>
<td>A.3 nuclear.f90</td>
<td>96</td>
</tr>
<tr>
<td>A.4 neutrino.f90</td>
<td>102</td>
</tr>
<tr>
<td>A.5 stdio.f90</td>
<td>104</td>
</tr>
<tr>
<td>Abbreviations</td>
<td>109</td>
</tr>
<tr>
<td>Physical Constants</td>
<td>110</td>
</tr>
<tr>
<td>Symbols</td>
<td>111</td>
</tr>
<tr>
<td>Bibliography</td>
<td>112</td>
</tr>
</tbody>
</table>
List of Figures

1.1 The Crab Nebula .................................................. 2
1.2 Illustration to hydrostatic equilibrium equation ............... 3
1.3 Energy balance in a thin shell at radius $r$ .................. 5
1.4 Mass-luminosity relation ...................................... 10
1.5 The Hertzsprung-Russell Diagram .......................... 12
1.6 Binding energy per nucleon as a function of atomic mass number . 13
1.7 The Pencil Nebula Supernova Shockwave .................. 14
1.8 BATSE All-Sky Plot of Gamma-Ray Burst Locations ....... 17

2.1 Adiabatic index ................................................ 22

3.1 Lagrangian mass coordinates .................................. 28
3.2 Nuclear reactions network ..................................... 30
3.3 Regions of application of different asymptotics in Nadyozhin EOS routine .............................................. 36
3.4 The density and pressure profiles for $n = 3$ polytrope .... 40
3.5 Scheme of the numerical grid .................................. 43

4.1 Dynamics of explosion of $90 \, M_\odot$ oxygen core .......... 46
4.2 Velocity profiles for different moments of time (in seconds) for $90(4)$ model .................................................. 46
4.3 Chemical composition of ejecta of the $90(4)$ model ........ 47
4.4 Trajectories in the central density–central temperature plane $(\rho_c - T_c)$ for the $60(1), 95(5)$, and $128(1)$ models .................................................. 48
4.5 Trajectories in the central density–central temperature plane $(\rho_c - T_c)$ for $90 \, M_\odot$ models .............................................. 50
4.6 Fate of a star depending on its mass, $M_{\text{tot}}$, and binding energy, $E_{\text{bind}}$ .............................................. 51
4.7 Nuclear energy release as a function of maximum temperature ................................................................. 51

5.1 Comparison between the temperature profiles inside the core, computed with Nadyozhin equation of state (black line), and with ideal gas approximation (red line). Radius is shown in the units of $R_\odot$, temperature – in $T_c = 2.36 \times 10^9$ K .................................................. 55
5.2 Initial temperature profile for the case of uniform explosion in the units of $T_c = 2.36 \times 10^9$ K .................................................. 55
5.3 Initial temperature distribution in multicore ignition scenario. Temperature is shown in units of $T_c = 2.36 \times 10^9$ K .................................................. 56
5.4 SN model with central ignition for $t = 28$ sec. Logarithm of density (a) is shown in units of $\rho_c = 2.65 \times 10^3$ g/cm$^3$. Temperature (b) is shown in units of $T_c = 2.36 \times 10^9$ K .................................................. 57
5.5 SN model with multicore ignition for $t = 28$ sec. Logarithm of density (a) is shown in units of $\rho_c = 2.65 \times 10^4$ g/cm$^3$. Temperature (b) is shown in units of $T_c = 2.36 \times 10^9$ K. .......................... 58

6.1 Duration distribution of GRBs .................................................. 60
6.2 Example of GRB light curves .................................................... 61
6.3 “Canonical” Swift X-ray light curve ............................................. 62
6.4 $E_{\text{peak, i}}-E_{\text{iso}}$ correlation ............................................... 64
6.5 Host galaxy luminosity ($M_B$) and host galaxy metallicity (in terms of oxygen abundance) at the sites of nearby broad-lined SN Ic: blue filled circles, and broad-lined SN Ic connected with GRBs: red filled squares (Modjaz et al. 2008) ................................................. 66
6.6 Fireball model ............................................................................. 69

7.1 Comparison of the maximum temperature, $T_{\text{max}}$, and total nuclear energy release, $E_{\text{nuc}}$, with the peak energy, $E_{\text{peak}}$, and isotropic equivalent energy, $E_{\text{iso}}$, of GRBs. GRB data are shown with gray dots (Amati et al. 2008), Results of earlier computations are shown with blue triangles (Ober et al. 1983; Arnett 1996). ........................................... 75
7.2 Idea of the origin of time variability ............................................ 76
7.3 Simulated light curves ................................................................. 77
7.4 Time-integrated spectra GRB 090618 ............................................ 80
List of Tables

1.1 Supernova taxonomy ........................................ 15

3.1 Coefficients for fitting formula (3.14a) for different processes of neutrino losses. .................... 32
3.2 Polytropic parameters ........................................ 39
3.3 Pre-supernova models ........................................ 41

4.1 Pre-supernova models and parameters of explosion ......................................................... 49
Chapter 1

Introduction

The Universe and the stars attracted interest of the mankind along all its history. Vast
distances make it nearly impossible to study celestial objects directly. And the scales of
the physical phenomena make it difficult to test theoretical models in direct experiments.
But the Universe is the largest available laboratory by itself. It could provide extreme
physical conditions that could be difficult to reach in man-made labs. That is why there
is continuous interest to astrophysics.

The physical description of observed astronomical phenomena for a long time was con-
 fined to application of the Newtonian theory of gravity. That theory was able to explain
the phenomenological Kepler’s laws of the planetary motion. It also allowed to make
some predictions: planet Neptune was first predicted by its influence on orbit of Uranus
and only later it was observed with telescopes.

The revolution in physics in the beginning of the XX century naturally affected ast-
rophysics. Recently developed general relativity theory and quantum mechanics were
actively applied in astrophysics, drastically changing our understanding of the Universe.
Formulation of the General Relativity theory by Einstein in 1915 made it possible to
answer questions, which were unresolved within Newtonian gravity (motion of Mercury,
deviation of light by the Sun). The first analytical solution of Einstein equations by
Schwarzschild (1916) predicted the existence of new objects in the Universe — black
holes. Now we have many objects considered as black hole candidates. And possibly at
the center of most galaxies there are super-massive black holes.

The solution of the problem of stellar energy sources was found after creation of appara-
tus of quantum mechanics. Nuclear burning was suggested as a possible mechanism of
energy production by Eddington (1926). But estimations of the temperature inside stars
showed that typical energy of protons is not enough to overcome Coulomb barrier and
to approach each other to the distance order of $10^{-12}$ cm (typical size of the proton). This limitation was removed by quantum mechanics which principles allow particles to penetrate under potential barrier. So it was understood that luminosity of most stars is supported by small fraction of high-energy protons.

Appearance of the quantum mechanics and developments in statistical physics allowed to start the investigations of stellar interiors. White dwarfs are small stars with masses order of solar mass. Their size implies that their average density should be order of $10^4$–$10^8$ g/cm$^3$. R.H. Fowler (1926) has demonstrated that the matter of white dwarf stars should be in a degenerate state. It was shown independently by several authors (Stoner 1929, 1930; Anderson 1929; Chandrasekhar 1931a) that relativistic degeneracy leads to the existence of a mass limit of stable spherical configurations. Chandrasekhar (1931b,c) considered the problem from point of view of hydrostatic equilibrium and computed value of critical mass of white dwarfs (which in now commonly called “Chandrasekhar limit”). It was the first successful attempt to build self consistent stellar configuration.

![Figure 1.1: The Crab Nebula. Credit: NASA, ESA, J. Hester and A. Loll (Arizona State University).](image)

After Chadwick (1932) has discovered the neutron, Baade & Zwicky (1934) introduced concept of neutron star (star consisting of neutrons) and suggested that neutron stars could be a result of production of supernovae, another term introduced in astrophysics...
same years to denote very luminous explosions of stars. The energy release by contraction of a star to the radius of few kilometers (characteristic size of a neutron star) corresponded to the typical energy of supernova explosion. Estimations of mass and radius of neutron stars has been done by Oppenheimer & Volkoff (1939). The existence of neutron stars has been confirmed by observations only in 1967 first as a radio pulsars (Hewish et al. 1968) and later as X-ray sources (Chodil et al. 1967; Giacconi et al. 1971). A pulsar has been found at the center of Crab Nebula which is the remnant of supernova explosion about thousand years ago. It played significant role in history of astrophysics. Thanks to many records about the “guest star” appeared on the skies in 1054 AD, it is possible to establish the precise age of the supernova remnant and the pulsar in its center.

1.1 Stellar structure

Most of their lifetime stars spend in the state of hydrostatic equilibrium. Gravitational force that contracts a star is balanced by the internal pressure. Under the assumption of spherical symmetry of a star the condition of hydrostatic equilibrium can be written as

\[
\frac{dP}{dr} = -\frac{Gm\rho}{r^2},
\]

where \(P\) is the pressure, \(r\) the radius, \(\rho\) the density, \(G\) the gravitational constant (Fig. 1.2).

![Gradient in pressure balances the gravitational force due to the mass \(m\).](image)

The mass of the star \(m\) inclosed inside radius \(r\) is

\[
m(r) = \int_0^r \rho(r') \cdot 4\pi r'^2 \, dr'
\]
or 
\[ \frac{dm}{dr} = 4\pi\rho r^2. \] 
(1.2)

This is so called mass continuity equation. Combining Eqs. (1.1, 1.2) we can obtain the following relation:
\[ \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G\rho, \]
(1.3)

which is the fundamental equation of equilibrium. The radial gradient in pressure \( P \) that supports a star is produced by a gradient in temperature \( T \) and density \( \rho \). The quantity of radiative energy per unit volume is higher for the regions with higher temperature. Therefore the temperature gradient produces radiative energy flow from the hotter central parts to the cooler outer parts. In a static case the temperature profile \( T(r) \) in much of the volume of most stars is determined by the rate at which radiative energy flows through a sphere of radius \( r \), i.e. the luminosity \( L(r) \). To find the dependence of the temperature on radius, the path of photons through the star needs to be taken into account. Photons passing through matter may be scattered or absorbed by electrons, atoms, ions, and molecules. The mean free path of a photon \( l \) is determined by the mean opacity of the matter inside the star \( \kappa \) (cm\(^2\)g\(^{-1}\) in cgs units):
\[ l = \frac{1}{\kappa\rho}. \]

Then radial gradient in temperature is defined by the equation of radiative energy transport (Eddington 1926):
\[ \frac{dT}{dr} = -\frac{3L\kappa\rho}{4\pi r^2 acT^3}, \]
(1.4)

where \( c \) is the speed of light, \( a \) is the radiation density constant related to Stefan-Boltzmann constant \( \sigma \):
\[ a = \frac{4\sigma}{c} = \frac{8\pi^5 k^4}{15c^3 h^3}. \]

The opacity \( \kappa \) is a consequence of various physical processes of absorption, and contribution of each process depends on the local density, temperature, and element abundance at every radius in the star as well as on the energy of a photon. So the absorption coefficients should be properly averaged over the spectrum of the photons (Rosseland mean) (Weiss et al. 2004). One of important processes is the Thomson scattering of photons off free electrons. The Thomson cross section does not depend on the temperature and photon energy and describes elastic scattering of low-energy photons. Scattering of high-energy X- and \( \gamma \)-photons is inelastic (Compton effect) and described by Klein-Nishina formula. In regions with relatively low temperatures when atoms are not completely ionized another processes become important sources of opacity: free-free, bound-free, and bound-bound absorption. In bound-bound (line absorption) and bound-free (photoeffect) absorption a photon is absorbed by an atom or ion which are excited to a higher
energy level, or ionized to a higher degree of ionization. In free-free absorption a photon is absorbed by a free electron and an ion, which share some of the photon’s momentum and energy. Free-free absorption is the inverse process of free-free emission, also called bremsstrahlung. All three processes depend on gas parameters as well as on photon wavelength. Depending on physical conditions other physical processes may play significant role such as:

- molecular absorption;
- Rayleigh scattering (elastic scattering on atoms or molecules);
- Raman scattering (inelastic scattering)

for low temperatures \( T \lesssim 10^4 \text{ K} \) and which may be important for stellar atmospheres; and

- pair production;
- nuclear absorption;
- Compton scattering

for high temperatures \( T \gtrsim 10^9 \text{ K} \). Number of physical processes that produce opacity is high and accurate calculation of opacity \( \kappa \) is a difficult physical and computational task (Cox et al. 1965). Interpolation formulae are used frequently for approximate work, for example mean opacity due to free-free and bound-free absorption were computed by Kramers (1923):

\[
\kappa \sim \frac{\rho}{T^{3.5}}.
\]

This relation is now called Kramers’ opacity law, which holds over limited range in temperature and density, when gas pressure is dominant.

![Figure 1.3: Energy balance in a thin shell at radius \( r \).](image)

To be in equilibrium steady state the star should be in almost perfectly energy balance: the rate at which the energy is carried away from any region must be equal to the
energy generated in this region (by nuclear energy sources for example). For a thin shell at radius $r$ with density $\rho$:

$$dL = \varepsilon dm = 4\pi r^2 \rho \varepsilon dr$$

or

$$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon, \quad (1.5)$$

where $\varepsilon$ is the power produced per unit mass of stellar matter (Fig. 1.3).

Equations (1.1, 1.2, 1.4, 1.5) define the structure of a star, but exact solution of these equations requires various information about microphysics of a star such as equation of state, opacity, sources of energy within a star, properties of heat transfer, etc. First successful attempts to build a possible mathematical description of stellar structure was done thanks to pioneering contributions of Kelvin, Emden, Lane and Ritter within the framework of polytropic gas spheres, in which the pressure $P$ and density $\rho$ are connected by the relation

$$P = K \rho^{1+1/n}, \quad (1.6)$$

where $K$ and $n$ are constants. In that case only two equations of stellar structure instead of four need to be solved: hydrostatic equilibrium condition (1.1) and mass continuity equation (1.2). Despite their simplicity polytropic models can give realistic qualitative and quantitative estimations of characteristic values of stellar structure.

### 1.2 Polytropic configurations

William Thomson better known by his later title of Lord Kelvin introduced the physical notion of convective equilibrium in 1862 in his work studying thermodynamics of the Earth’s atmosphere. He considered the behavior of portions of fluid ideally enclosed in a shell impermeable to heat, e.g. adiabatic convection. If each element of fluid moves in the stellar interior in such a way that its temperature and density equal to the temperature and density of surroundings, the star is in convective equilibrium. If these elements move without heat exchange then the thermodynamic changes are adiabatic. Assumption of adiabatic convection leads to the relation between the pressure and density throughout the star:

$$P(r) \propto \rho(r)\gamma, \quad (1.7)$$

where $\gamma$ is the ratio of the specific heats $\gamma = C_P/C_V$. The internal constitution of a gaseous sphere in adiabatic convective equilibrium in which $P$ and $\rho$ are related as in Eq. (1.7) was first considered by Lane (1870) and independently by Ritter (1878).
Equation (1.7) can be generalized to the case of non-adiabatic processes when some amount of heat $dQ$ is supplied to the element of fluid, proportional to the change of the temperature $dT = C_dQ$ (so called polytropic process; specific heat $C$ is assumed to be constant):

$$P \propto \rho^{\gamma'}, \quad \gamma' = \frac{C_p - C}{C_v - C}. \tag{1.8}$$

It is seen from Eqs. (1.6, 1.8) that the polytropic index is $n = 1/(\gamma' - 1)$. Assuming relation between the pressure and density throughout a star to be as in Eq. (1.6) it is possible to rewrite Eq. (1.3) in new variables:

$$\rho = \lambda \theta^n, \quad r = \xi \alpha,$$

where $\lambda$ is a scaling factor equal to the central density $\rho_c$ and $\alpha$ is constant:

$$\alpha = \sqrt{(n + 1)K\lambda \frac{1-n}{4\pi G}}.$$

Thus Eq. (1.3) can be written in dimensionless form:

$$\frac{d(\xi^2 d\theta/d\xi)}{d\xi} = -\xi^2 \theta^n, \tag{1.9}$$

or

$$\frac{d^2 \theta}{d\xi^2} = -\frac{2}{\xi} \frac{d \theta}{d \xi} - \theta^n. \tag{1.10}$$

This is so called Lane-Emden equation, first derived by Ritter (1878). In series of works Ritter solved this equation for several values of $n = 0; 1; 1.5; 2; 2.44; 3; 4; 5$. Ritter proved that $n = 3$ is critical value that separates stable configurations ($n < 3$) and unstable ones ($n > 3$). His works became the basis of modern theory of stellar structure. Another remarkable contribution was made by Emden. In his book (Emden 1907) he demonstrated that polytropic approximation is a powerful tool for description of selfgravitating gaseous spheres. He derived basic equations of polytropic configurations and solved them numerically for $n = 0.5; 1; 1.5; 2; 2.5; 3; 4; 4.5; 4.9; 5; 6; \infty$ (isothermal configuration).

Polytropic structure approximation can be applied not only in the case of convective equilibrium. It can be used for description of stars in which energy is transported by radiative transfer expressed by Eq. (1.4). Let’s consider a star in which equation of state can be approximated by a Maxwell-Boltzmann gas plus radiation (Clayton 1968):

$$P = P_{\text{gas}} + P_{\text{rad}},$$

$$P_{\text{gas}} = \frac{N_A k}{\mu} \rho T, \quad P_{\text{rad}} = \frac{1}{3} aT^4, \tag{1.11}$$
where \( N_A \) is the Avogadro’s number, \( \mu \) is the mean molecular weight. The ratio of the gas pressure to the total pressure is usually defined as \( \beta \):

\[
\beta = \frac{P_{\text{gas}}}{P},
\]

\[
P_{\text{gas}} = \beta P, \quad P_{\text{rad}} = (1 - \beta)P.
\]

(1.12)

Combining Eqs. (1.11, 1.12) and eliminating the temperature \( T \) we can express the pressure as

\[
P = \left[ \frac{3}{a} \left( \frac{N_A k}{\mu} \right)^4 \left( \frac{1 - \beta}{\beta^4} \right) \right]^{1/3} \rho^{4/3}.
\]

(1.13)

In general \( \beta \) changes throughout a star but in a case if \( \beta \) is constant, then the expression in square brackets in Eq. (1.13) is also constant and

\[
P = K \rho^{4/3}
\]

which corresponds to a polytrope with index \( n = 3 \). The model with the assumption of constant ratio between the gas pressure and total pressure is the Eddington standard model. Eddington was among the first who underlined importance of the radiative transfer in stars (Eddington 1916). He demonstrated that in certain conditions radiative energy transfer is more efficient than the convective one.

Another example of stellar model where polytropic approximation is applicable is the model of cold white dwarf stars. These stars are supported mainly by the pressure of degenerate electron gas, thermal pressure can be neglected. Depending on the density the pressure of degenerate electron gas can be described as:

\[
P = \frac{8 \pi}{15 h^3 m_e} \left( \frac{3 h^3 N_A}{8 \pi} \right)^{5/3} \left( \frac{\rho}{\mu_e} \right)^{5/3} = K_1 \rho^{5/3}, \quad \left( \frac{\rho}{\mu_e} \right) \ll 10^6 \text{ g/cm}^3
\]

(1.14)

for non relativistic case when energies of electrons are much smaller than the rest mass of the electron, and

\[
P = \frac{2 \pi c}{3 h^3} \left( \frac{3 h^3 N_A}{8 \pi} \right)^{4/3} \left( \frac{\rho}{\mu_e} \right)^{4/3} = K_2 \rho^{4/3}, \quad \left( \frac{\rho}{\mu_e} \right) \gg 10^6 \text{ g/cm}^3
\]

(1.15)

for extreme relativistic case (Weiss et al. 2004); \( \mu_e \) is the mean molecular weight per electron. It is seen that these equations are in the form of Eq. (1.6) and polytropic theory can be applied to description of white dwarfs and it was first done by Chandrasekhar (1931d, 1935). Particularly polytropic theory gives important relations between stellar parameters such as mass, radius, pressure and density (Arnett 1996). For example values of the pressure and density at the center of star with mass \( M \) are connected by
the relation:

\[ P_c = D_n GM^{2/3} \rho_c^{4/3}, \]  

(1.16)

where \( D_n \) is constant defined by polytropic index \( n \) (see Sec. 3.2). We can express the mass of a star as a function of values of the pressure and density at the center:

\[ M = \left( \frac{P_c}{D_n G \rho_c^{4/3}} \right)^{3/2}. \]  

(1.17)

When density is relatively low the pressure is defined by equation of state of non-relativistic degenerate gas (Eq. 1.14) and the structure of the star is defined by a polytrope with \( n = 3/2 \) (\( \gamma = 5/3 \)). Expressing the pressure in Eq. (1.17) from Eq. (1.14) we can see that the mass is proportional to the square root of the central density:

\[ M \propto \rho_c^{1/2}. \]

With increase of central density electrons become relativistic and Eq. (1.15) should be applied. From Eqs. (1.15, 1.17) it follows that with \( \rho_c \to \infty \) the mass \( M \) converges to some constant value:

\[ M_{Ch} = \left( \frac{K_2}{D_n G} \right)^{3/2} = \frac{5.75}{\mu_e^2} M_\odot, \]

where \( D_n \) is equal to \( D_3 = 0.364 \) (\( n = 3 \) polytrope). For \( \mu_e = 2 \) this expression yields the value of critical mass \( M_{Ch} = 1.44 \, M_\odot \). Chandrasekhar was among the first who computed this mass limit and it is now commonly called Chandrasekhar limit.

### 1.3 Stellar evolution

Radiating stars cannot hold conditions of equilibrium for infinitely long time. Irreversible energy losses by emission and limited total energy supply unavoidably lead to a finite lifetime of a star. There are two main sources of energy in the stars: gravitational and thermonuclear. During macroscopic contraction the potential gravitational energy transfers to the internal energy of a star, heats it up and supports emission. Thermonuclear reactions occur in central hot regions of a star and provide energy from fusion of lighter chemical elements to heavier ones. Characteristic timescales of these two mechanism are different.

The idea that a star could gain its energy from the work done by gravitational contraction was suggested by Helmholz and Kelvin. If a star with mass \( M \) and luminosity \( L \) does not have other sources of energy, then it can not radiate more energy that can be liberated by gravitational binding, thus characteristic time of changes of stellar structure (Kelvin
time) is defined by its gravitational potential energy $\Omega$:

$$t_K \sim \left| \frac{\Omega}{L} \right| \sim \frac{GM^2}{RL} \sim 2 \times 10^7 \left( \frac{M}{M_\odot} \right)^2 \left( \frac{L_\odot}{L} \right) \left( \frac{R_\odot}{R} \right) \text{ years.}$$  \hspace{1cm} \text{(1.18)}$$

Kelvin time shows that gravitational energy solely can maintain luminosity of a star like our Sun only for millions of years. But geological researches show that the age of the Earth is about 4.6 billions of years, and during all this period the Sun has been shining with almost constant luminosity. Therefore thermonuclear energy source is needed to explain age of the Sun.

Observations show that most stars consist mainly of hydrogen and the dominant source of energy is the fusion of four hydrogen atoms into one helium atom with energy release of 26.73 MeV per one such conversion or $0.007c^2$ per gram of consumed hydrogen. If a star is in hydrostatic equilibrium and nearly perfect energy balance (Eq. 1.5) then the characteristic nuclear timescale $t_{\text{nuc}}$ for a star is defined by the amount of hydrogen available in the core, which is proportional to the star’s mass $M$. Assuming that about 10 per cent of the mass available for thermonuclear burning we obtain:

$$t_{\text{nuc}} \sim 0.1 \frac{0.007Mc^2}{L} \sim 10^{10} \left( \frac{M}{M_\odot} \right) \left( \frac{L_\odot}{L} \right) \text{ years,}$$ \hspace{1cm} \text{(1.19)}$$

which is enough to explain lifetime of the Sun.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{mass_luminosity.png}
\caption{The mass-luminosity relation for a sample of binary systems \cite{Torres et al. 2010}.}
\end{figure}
So the lifetime of a star is defined mainly by the amount of hydrogen: the more massive a star is, the more hydrogen is available for burning. However, as indicated by observation (Torres et al. 2010), massive stars are more luminous and faster consumes hydrogen (Fig. 1.4). Empirical correlation between mass and luminosity can be described by the following formulae (Weiss et al. 2004):

\[
\log \frac{L}{L_\odot} = 0.27 + 3.60 \log \frac{M}{M_\odot}, \quad 2 M_\odot \lesssim M \lesssim 20 M_\odot;
\]

\[
\log \frac{L}{L_\odot} = -0.13 + 4.55 \log \frac{M}{M_\odot}, \quad 0.5 M_\odot \lesssim M \lesssim 2.5 M_\odot;
\]

\[
\log \frac{L}{L_\odot} = -0.59 + 2.64 \log \frac{M}{M_\odot}, \quad 0.2 M_\odot \lesssim M \lesssim 0.6 M_\odot.
\]

Putting these relations in Eq. (1.19) we can see that more massive stars evolve faster:

\[
t_{\text{nuc}} \propto \frac{1}{M^{\sim 2.5}}, \quad (1.20)
\]

Lifetime of \(\sim 50 M_\odot\) star for example does not exceed \(10^7\) years.

Mass is the major parameter that defines stellar evolution. To better understand the role of mass one can use the Hertzsprung-Russell diagram (hereafter H-R diagram), a plot that shows luminosities of stars (or other quantities related to luminosity such as absolute bolometric magnitude, apparent visual magnitude, etc.) versus the effective surface temperature (spectral class, color index) (Clayton 1968). An example of such diagram is shown on Fig. 1.5. When a large number of stars is plotted on this diagram it becomes apparent that stars form well defined groups and most of the stars are situated along diagonal curve which is called main sequence.

Stellar evolution can be divided into three main stages: before, during and after main sequence. First stage follows the formation of a star from a cloud of interstellar matter to the moment of ignition of hydrogen burning — beginning of the main sequence stage. The initial point of a star on the main sequence is defined by its initial mass. More massive star is hotter and more luminous. Second stage follows evolution of star on main sequence. As was shown by Eq. (1.20) the length of time that star spends on the main sequence is also determined by its mass. Evolutionary studies show that stars spend about 90% of their life on main sequence. And the last stage, when star runs out of hydrogen and start to evolve rapidly leaving main sequence on H-R diagram.

During main sequence stage of evolution star gathers energy from thermonuclear burning converting lighter elements (starting from hydrogen and helium) to heavier elements. If a star has no nuclear energy sources (for example it has exhausted available nuclear fuel) or it has not yet reached conditions for ignition of thermonuclear reactions, the main
energy supply will be from the work done by gravitational contraction. Star starts to contract and increases its temperature in order to maintain luminosity. Depending on mass of the star and its composition it could reach conditions allowing ignition of new nuclear reactions to more heavier elements. For light stars ($\sim 1M_\odot$) nuclear burning stops at carbon and oxygen. Further contraction leads to degeneracy of the electrons and the star ends its life as a white dwarf. If a star is massive enough nuclear burning can proceed to the elements of iron group (Fig. 1.6) and form an iron core. Nuclear burning stops at the elements of iron group because these elements have the highest binding energy per nucleon and following reactions using $^{56}$Fe are endothermic and absorb energy from a star, leading to loss of stability and result in a dynamical collapse and explosion of the star as a supernova.

1.4 Supernovae

The term 'supernova' was introduced to denote a very bright (super) new star (nova) appearing on skies. Supernovae have been observed for a long time, but up to the beginning of the XXth century the actual scale of this phenomena has been disguised by
Figure 1.6: Binding energy per nucleon as a function of atomic mass number $A$. Several chemical elements are marked. The highest binding energy per nucleon is achieved for the iron-group elements with atomic numbers $A \approx 56$. Therefore nuclear fusion of these elements into heavier elements consumes thermal energy (Maoz 2007).

nearby novae stars. Only after estimation of the distance to Andromeda Galaxy M31, it was understood that the nova star, observed there in 1885, was much brighter than previously thought. It was about 10 000 times brighter than a typical nova, a supernova.

Contrary to its name a supernova does not accompany birth of a new star, but rather marks its death, when at final stage of evolution the star explodes. It is not a spontaneous catastrophic event but natural completion of stellar evolution. The studying of mechanism of supernovae explosions since that time became wide branch of modern theoretical astrophysics.

Supernova phenomenon has strong influence on the history of the Universe. It is responsible for chemical enrichment of Inter-Stellar Medium (ISM) and production of heavy elements. The chemical composition of the Universe after the Big Bang was primarily hydrogen and helium. So the first stars of the Universe (so called Population III stars, PopIII) also had very simple chemical composition. During their evolution these stars burned hydrogen and helium producing heavier elements. If at final stage of evolution star explodes it expels to ISM chemical elements, that were produced. So next generations of stars would have more complicated chemical composition with mixtures of metals. Particularly, presence of heavy chemical elements on the Earth indicates that the Sun was formed from remnants of older stars, exploded as supernova.

Presence of metals strongly affects the process of formation of new stars. Absorption lines in atoms of metals increase opacity of the matter, also increasing the role of radiation
pressure. As a result it changes characteristic masses of stars forming by condensation of initial molecular cloud, thus the next populations of stars have different distribution by their masses (so called initial mass function, IMF).

Another important role of supernovae is that their remnants are considered as sites of acceleration of cosmic rays. According to the supernova-remnant cosmic-ray hypothesis, protons are accelerated by the shocks created in a supernova (Fig. 1.7) and then further accelerated by the magnetic fields until they gain enough energy to escape and become a cosmic ray. These energetic protons sometimes collide with other protons and can produce a neutral pion, which decays into two gamma-ray photons. Recent observations with Fermi satellite detected the characteristic pion-decay signature in two supernova remnants within our galaxy — IC 443 and W44 (Ackermann et al. 2013).

Figure 1.7: The Pencil Nebula Supernova Shockwave. This shockwave is known as the Pencil Nebula, or NGC 2736, and is part of the Vela supernova remnant, an expanding shell of a star that exploded about 11,000 years ago. Credit: Hubble Heritage Team (STScI/AURA), W. Blair (JHU) & D. Malin (David Malin Images), NASA

Current classification of supernovae is based on their spectral characteristics (Table 1.1). If there are no lines of hydrogen in the spectrum of a supernova, then this supernova is called Type I (SN I). Otherwise it is the Type II supernova (SN II). Presence of hydrogen lines is an evidence that the star preserves its hydrogen envelope to the moment of explosion. Each of these types has subclasses, distinguished by spectral and light curve
Table 1.1: Supernova taxonomy

<table>
<thead>
<tr>
<th>Supernova</th>
<th>Type I</th>
<th>Type II</th>
</tr>
</thead>
<tbody>
<tr>
<td>No hydrogen</td>
<td>Type Ia</td>
<td>Shows hydrogen lines</td>
</tr>
<tr>
<td>Type Ib</td>
<td>Helium poor</td>
<td>Type IIb</td>
</tr>
<tr>
<td>Type Ic</td>
<td>Helium rich</td>
<td>Helium dominant, spectrum becomes like Type Ib</td>
</tr>
<tr>
<td>No silicon</td>
<td>Type III</td>
<td>Type IIb</td>
</tr>
<tr>
<td>Type IIIb</td>
<td>Linear decrease in light-curve</td>
<td>Plateau in light-curve</td>
</tr>
<tr>
<td>Type IIIc</td>
<td>Narrow lines in spectrum</td>
<td>Linear decrease in light-curve</td>
</tr>
</tbody>
</table>

features. But physical processes leading to explosion are different. As we have seen stellar evolution and its final stage is mainly defined by its mass.

SNe Ia represent quite uniform class of thermonuclear explosion triggered in low-mass white dwarfs (with masses close to the Chandrasekhar limit $M_{\text{Ch}} = 1.44 M_\odot$). In such stars nuclear reactions has stopped after formation of carbon-oxygen core. But if such star increases its mass (for example by accretion from a companion star) and overcomes critical mass $M_{\text{Ch}}$ it loses stability and starts to contract. Increase of the temperature causes explosive thermonuclear burning of the carbon–oxygen mixture which disrupts the star (Ivanova et al. 1974). Because of similar mass and structure of white dwarfs these supernovae have a single pattern of the light curve and they are used as standard candles for cosmology.

In more massive stars ($\gtrsim 8 M_\odot$) the mechanism of explosion is different, the main trigger is the gravitational instability in the iron core. In such cores nuclear reactions stop because of exhaustion of nuclear fuel. Internal pressure can not compensate gravitational forces and the core collapses, forming a neutron star or a black hole. Falling envelope bounces and shockwave propagates outward giving rise to supernova.

But in some cases of very massive stars ($100 M_\odot \lesssim M \lesssim 200 M_\odot$) instability could occur prior to formation of the iron core on the stage of oxygen burning (Barkat et al. 1967; Arnett 1972). A star could become unstable because of copious electron-positron pairs production in the core. Pair-formation reduces internal pressure and leads to rapid contraction of a star, but because of profusion of nuclear fuel the star explodes as Pair-Instability Supernovae (PISN). This type of instability was predicted for very massive stars by Rakavy & Shaviv 1967).
As was shown by recent evolutionary calculations (Yoon et al. 2012) configuration of such very massive star prior to explosion strongly depends on rotation and initial chemical composition. As a result such stars can have hydrogen envelope or can be chemically homogeneous without hydrogen. Thus PISN can be observed as both type-I and type-II supernovae.

Because of huge mass of a star that encounters pair-instability, energy release during PISN explosion is tremendous. It is probably the most energetic type of supernovae, known up to now. It is one of possible explanations of super-luminous SNe that about order of magnitude brighter that usual SNe (Chatzopoulos & Wheeler 2012b). But it is also a probable candidate for the role of the sources of another extremely energetic phenomena, Gamma-ray Bursts.

However with new observations there are possible difficulties in the theory of supernovae explosions. For example SN 1987A that exploded in Large Magellanic Cloud as Type-II supernova is the first Local Group supernova in the era of modern observations. The progenitor of this supernova is estimated to have a mass of \( \sim 20 M_\odot \) (Woosley 1988; Graves et al. 2005) and thus according to evolutional picture the core of this star should have been collapsed to a neutron star or black hole (Fryer 1999). But up to now there is no direct evidence for existence of a compact object at the center of the remnant (Baranov & Chechetkin 2011; Chevalier 2011; Burrows et al. 2012), no central point source has been detected.

### 1.5 Gamma Ray Bursts

*Gamma Ray Bursts (GRBs)*, has been discovered in 1967 by Vela satellites. These satellites were launched by the United States in order to monitor the Partial Test Ban Treaty – the ban on experiments with nuclear explosions by other countries (especially by the USSR). But what they have discovered was completely unexpected - the instantaneous flashes of gamma radiation lasting few seconds. By triangulation method (few satellites have been launched) it was found that sources of these flashes were not on the Earth or on the Sun but in Space and it was realized that these flashes have non-artificial origin (Klebesadel et al. 1973; Strong et al. 1974).

Triangulation method gave very rough angular resolution, so it was impossible to establish sources of GRBs and to determine whether the sources are in the halo of Solar system, in the Milky Way galaxy or has an extragalactic origin. Lack of observational data gave rise to many physical models of this phenomena and new instruments optimized for the GRBs survey were needed to provide detailed data. And such mission
was launched in 1991. BATSE (the Burst and Transient Source Experiment) detectors on board of the Compton Gamma Ray Observatory used the coded mask technique to determine the direction of coming gamma-quants. This technology allowed to reach an accuracy of $\sim 2$ degrees and it was enough to demonstrate isotropic distribution of GRB sources on the sky (Fig. 1.8), which ruled out galactic models, such as accreting neutron stars and other (Meegan et al. 1992; Briggs 1996).

2704 BATSE Gamma-Ray Bursts

The point in the debates about the cosmological origin was put by the BeppoSAX satellite. The detectors of this instrument had an accuracy of $\sim 20''$ and it was possible to redirect them to required point in the sky within a few hours. And this allowed to follow the source of GRB 970228 and to discover fading X-ray emission from the source of GRB — the afterglow (Costa et al. 1997). It gave a possibility to transfer the location of the burst to ground based telescopes and optical emission also has been observed (van Paradijs et al. 1997). With discovery of the afterglow it became possible to detect the host of GRBs and determine the redshift and first measurements were done soon after for GRB 970508 (Djorgovski et al. 1997; Metzger et al. 1997; Frail et al. 1997). Extragalactic origin of GRBs was recognized and the scale of energy, emitting in gamma trays, was realized. Gamma ray bursts are the most energetic explosive phenomena known, with energy release $\sim 10^{54}$ ergs (in case of isotropic source), which is 1000 times higher than typical energy of a supernova.
Intensive investigation of GRB phenomenon was continued in following years both observationally and theoretically. Fermi, Swift, and other satellites were launched, providing huge amount of data. But the origin of GRBs and their production mechanism are still uncertain.

Chardonnet et al. (2010) proposed a new interpretation of gamma-ray bursts phenomenon inside a global picture of stellar evolution. Their conception of GRB phenomenon is very different from other scenarios where ad hoc mechanisms are built in order to explain the tremendous energy released in X-γ radiation (Piran 1999; Ruffini et al. 2001a,b, 2002). Contrary to the first appearance, GRBs do not need such specific mechanism but can be explained using standard stellar physics. An interpretation of GRB phenomenon within PISN-model is presented in this work.

1.6 Thesis outline

The main purpose of this work was the investigation of PISN phenomenon and its possible connection with GRBs. The pair-instability model of GRBs (Chardonnet et al. 2010) is developed further to explain qualitatively and quantitatively observational properties of GRBs.

In Chapter 2 description of pair-instability supernova phenomenon is presented and effects of pair-production on stability of stellar configuration are discussed. This chapter also contains review of the history of the theoretical investigation of PISN.

Chapter 3 is dedicated to the physical and mathematical model of PISN explosion. It contains description of all physical processes under consideration: equation of state, nuclear reaction network, etc. It also describes numerical approach and methods that were used to simulate PISN explosion. Physical analysis and discussion of results of the computations are presented in Chapter 4.

Chapter 5 describes two dimensional (2D) simulations of PISN explosion. These simulations are simplified as they follow only hydrodynamical aspect of explosion and do not contain physics of nuclear burning inside, the energy is deposited artificially. Two possible scenarios of explosion are investigated: uniform and nonuniform explosions.

Description of GRB phenomenon is presented in Chapter 6. This chapter contains description of characteristic features of GRB emission as well as properties of host galaxies and environment. Major theoretical models are described and discussed. An interpretation of GRBs within pair-instability model is presented in Chapter 7. It is shown that the energy budget and timescale of PISN explosion is in perfect agreement with values
required in GRBs. Moreover, the possible physical interpretation of the empirical Amat- 
ti correlation for GRBs is proposed. The explanation of variability of prompt emission 
of GRBs is presented basing on 2D simulations of PISN explosion. For explanation of 
formation of the non-thermal spectrum of GRBs a possible model is suggested, which 
can be realized in PISN scenario.

Chapter 8 summarizes the results and proposes the prospects for future works.
Chapter 2

Pair-Instability Supernovae

2.1 Role of Electron-Positron Pairs

When central temperature in stellar interiors reaches sufficiently high values (order of a few times $10^9$ K) a significant part of photons of the Plankian spectrum have energies higher than $2m_e c^2$ ($m_e$ is the electron mass), which is the minimum energy required for the creation of the electron-positron pairs ($e^\pm$). Electrons and positrons annihilate but since the cross-sections of pair-creation and annihilation are finite, the number of the pairs reaches some dynamical equilibrium. Thus the kinetic energy of photons transforms to the rest mass energy of the pairs, altering the equation of state. The influence of $e^\pm$-pairs on the pressure and the internal energy should be taken into account in computations of the stellar interiors. The detailed description of the effects of the electron-positron pairs on the equation of state can be found for example in [Weiss et al. (2004)].

First computations of the equilibrium concentration of $e^\pm$-pairs from statistical mechanics were carried out by [Koppe (1948)] and now it can be found in many textbooks (e.g. [Cox & Giuli (1968)]). One of the first works, that has considered electron-positron pairs creation inside stars was the paper of [Fowler & Hoyle (1964)]. They showed that this process could play a significant role in the evolution of massive stars, because these stars reach high central temperature at relatively low central densities. This fact can be understood quantitatively by the order-of-magnitude estimations. The central density $\rho_c$ can be estimated as

$$\rho_c \sim \frac{M}{R^3},$$

(2.1)
where $M$ is the mass of a star and $R$ is its radius. The value of a typical interior pressure can be obtained from the hydrostatic equilibrium equation:

$$\frac{dP}{dr} = -G\frac{m(r)}{r^2}\rho,$$

where $m(r)$ is the mass of matter within the radius $r$, $G$ is the gravitational constant. Substituting $dP/dr$ by $-P/R$, $M(r)$ by $M$, $\rho$ by $M/R^3$ and $r$ by $R$ we can obtain:

$$P_c \sim \frac{GM^2}{R^4}.$$  \hspace{1cm} (2.2)

Let’s assume a perfect gas equation of state $P = (R/\mu\beta)\rho T$, where $R$ is the gas constant, $\mu$ is the mean molecular weight, $\beta$ denotes the ratio of the gas pressure to the total pressure. Eliminating $P_c$ and $R$ from Eqs. (2.1, 2.2) we can find

$$\rho_c \propto \frac{1}{(\mu\beta)^3} T_c^3 M^{-2},$$

where $T_c$ is the central temperature. This relation shows that $\rho_c$ decreases as $1/M^2$ for a given $T_c$ and $(\mu\beta)$. Accurate calculations by Fowler & Hoyle (1964), taking into account the variation of $(\mu\beta)$ due to effects of electron-positron pairs, yield

$$\rho_c \propto \frac{T_c^3}{M^{1/2}},$$

so these effects do not change the tendency of decreasing $\rho_c$ with increasing $M$. Fowler & Hoyle computed the density–temperature relations for massive stars ($30\, M_\odot$ and higher) using polytropic structure approximation and the assumption that a star evolves through series of quasi-equilibrium states. Variable number of electron–positron pairs was taken into account by the authors. They discovered that when central temperature of a star reaches values $\sim 2 \times 10^9 K$ intensive electron–positron pairs creation occurs. It increases energy losses by neutrino through pair annihilation process $e^+ + e^- \rightarrow \nu_e + \bar{\nu}_e$ and accelerates the contraction, raising central temperature. Rise of temperature creates new pairs and, at some point, energy supply from the contraction becomes insufficient to maintain production of additional pairs, it means that energy release $dQ$ per change of temperature $dT$ becomes negative, $dQ/dT < 0$.

From thermodynamical point of view, the electron–positron pairs creation reduces adiabatic index $\gamma = (\partial \ln P/\partial \ln \rho)_S$, where $P$ is the pressure, $\rho$ is the density and derivative is taken keeping entropy $S$ constant.\footnote{In general $\Gamma_1 \equiv (\partial \ln P/\partial \ln \rho)_S$ is not equal to the ratio of specific heats $\gamma \equiv C_p/C_v$. Equality of gammas is possible only in the case of ideal gas (non-degenerate, not self interacting). But in following discussion effects of degeneracy and self-interaction are neglected since they are negligible in conditions of interest.} Index $\gamma$ can decrease to values less than $4/3$.
Figure 2.1: Dependence of the adiabatic index $\gamma$ on temperature $T$ and density $\rho$.
The region where $\gamma$ is less than $4/3$ is shown by shading.

(Fig. 2.1), which is known lower critical value for existence of stable polytropic configurations (Zeldovich & Novikov 1971). Using Virial theorem it is possible to show that the drop of $\gamma$ in significant part of a star leads to the loss of hydrostatic equilibrium of the star and causes dynamical collapse. This theorem expresses the balance between different forms of energy within a star. Assuming spherical symmetry of a star, let’s introduce the kinetic energy of mass motion of a fluid, $K_{\text{mass}}$, and the moment of inertia, $I$:

$$K_{\text{mass}} \equiv \frac{1}{2} \int_{M} \dot{r}^2 \, dm, \quad I \equiv \int_{M} r^2 \, dm; \quad (2.3)$$

where $r$ is the distance from the center to the element with mass $dm$ and integrals are taken over the entire mass of a star $M$. The gravitational potential energy $\Omega$ can be defined as:

$$\Omega = -G \int_{0}^{M} \frac{M(r)}{r} \, dm. \quad (2.4)$$

Then one of simplified forms of Virial theorem in the absence of magnetic fields and under the assumption that pressure vanishes on the surface of a star can be written as (Weiss et al. 2004):

$$\frac{1}{2} \ddot{I} = 2K_{\text{mass}} + 3 \int_{V} P \, dV + \Omega, \quad (2.5)$$
where $V$ is the total volume of a star.

Relating the total pressure $P$ to the total internal energy per unit volume, $u$:

$$P = (\gamma - 1)u,$$

it is possible to express the integral over in Eq. (2.5) through the total internal energy $U = \int u \, dV$:

$$\frac{1}{2} \ddot{I} = 2K_{\text{mass}} + 3(\gamma - 1)U + \Omega.$$  \hspace{1cm} (2.6)

When a star is in hydrostatic equilibrium the kinetic energy of motion, $K_{\text{mass}}$, and the term $\ddot{I}$ can be neglected and the expression (2.6) is reduced to:

$$3(\gamma - 1)U + \Omega = 0.$$  \hspace{1cm} (2.7)

Defining the total energy of a star, $E_{\text{tot}}$, as a sum of internal and gravitational potential energies: $E_{\text{tot}} = U + \Omega$; we can obtain:

$$U = -\frac{1}{3(\gamma - 1)}\Omega,$$  \hspace{1cm} (2.8)

$$E_{\text{tot}} = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega.$$  \hspace{1cm} (2.9)

It is seen from Eq. (2.9) that if $\gamma > \frac{4}{3}$ then the total energy $E_{\text{tot}}$ is negative (the gravitational potential energy $\Omega$ is negative). If $\gamma = \frac{4}{3}$ then $E_{\text{tot}}$ is equal to zero and it does not depend on change of gravitational energy, $\Omega$. It means the star is marginally stable to contraction or expansion.

If in past of a star the index $\gamma$ was higher than $\frac{4}{3}$, the total energy $E_{\text{tot}}$ of the star should be negative. If due to electron-positron pairs creation $\gamma$ drops down to the value $\frac{4}{3}$ (relatively fast, so the total energy $E_{\text{tot}}$ does not change drastically), Eq. (2.9) is not valid anymore. The star is not in hydrostatic equilibrium and we cannot neglect the kinetic terms. Because of the energy conservation law new expression for the total energy should take into account the kinetic energy of mass motion $E_{\text{tot}} = U + \Omega + K_{\text{mass}}$.

Then defining $U$ from Eq. (2.6) we obtain:

$$E_{\text{tot}} = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega + \frac{3\gamma - 5}{3(\gamma - 1)}K_{\text{mass}} + \frac{1}{6(\gamma - 1)}\ddot{I}.$$  \hspace{1cm} (2.10)

Rapid decrease of $\gamma$ changes the first term in Eq. (2.10) from a negative value the order of $\Omega$ to zero. It means that the rest two terms must obtain large negative value. If
in the beginning the star was in hydrostatic equilibrium, then $K_{\text{mass}}$ can be neglected, resulting in negative $\ddot{I} = \partial^2/\partial t^2 \int m r^2 dm$, which means that the star would contract with acceleration, i.e. will be in a state of dynamical collapse.

### 2.2 Pair-Instability Supernova

For very massive stars (heavier than 30 $M_\odot$) lose of stability by pair-creation takes place at early stage of oxygen burning. This combination physical factors could lead to a catastrophic event — an explosion of entire star.

One of the first evolutionary calculations of stars with masses of 20–30 $M_\odot$ were performed by Rakavy & Shaviv (1967). The pair-creation process leads to development of dynamical instability and as it was shown by computations of the 30 $M_\odot$ oxygen star, the significant part of the star enters inside this region. It makes star softer and softer, increases neutrino losses and speeds up the contraction. When the temperature in the center reaches high enough values, the rate of nuclear reactions increases very rapidly by a few orders of magnitude, which could lead to explosive nuclear burning and reverse the collapse. Uncovered effect was independent of other physical parameters of the star such as regime of energy losses (photon or neutrino), convection, nuclear reactions: “The dynamical instability always occurred at approximately the same central conditions. Only the time scale of evolution is affected.”

First dynamical calculations of explosion triggered by pair-creation instability were performed by Barkat et al. (1967). The authors used the 40 $M_\odot$ oxygen star model and found that after reaching dynamical instability the star started to contract with acceleration. Explosive nuclear burning in the center released energy corresponding to consumption of several solar masses of oxygen (the obtained value was $\sim 6 \times 10^{51}$ ergs). This reversed the collapse and leads to total disruption of the star without any remnant. “This is despite the fact that during collapse the unbalance between the gravitational forces and the thermal pressure gradients is only about 1 % and thus induces very small accelerations.”

The authors have checked the influence of admixture of carbon, effect of outer layers of lighter matter, but behavior remained the same as for pure oxygen 40 $M_\odot$ star. The authors also have made the attempt to investigate conditions in which pair-instability is possible; and they have found the limit on the mass of the oxygen core of 30 $M_\odot$. The cores lighter than this limit passed pair-instability zone and evolved up to Fe-He transition region. The 30 $M_\odot$ oxygen core entered into regime of decaying oscillations.

In the following years the investigation of stellar evolution and dynamical behavior of pair-unstable stars was continued. Evolution of carbon-oxygen stars with different
masses has been studied by numerical calculations (Rakavy et al. 1967) and isentropic analytical models (Rakavy & Shaviv 1968). The authors concluded that stars heavier than $25M_\odot$ are dynamically unstable because of pair-production. Instability occurs before ignition of oxygen. Fraley (1968) performed numerical calculations of the oxygen cores with masses of 45, 52 and 60$M_\odot$. Oscillations were found for the 45$M_\odot$ core, while the other two exploded without remnant. Wheeler (1977) performed computations for $10^3$ and $10^4 M_\odot$ oxygen cores. Both of them collapsed to black holes without mass ejection despite of nuclear burning during collapse.

First detailed evolutional calculations of helium cores were performed by Arnett (1972). He has demonstrated that the cores consist mainly of oxygen when reaching the pair-instability zone. Dynamical calculations of helium cores with masses 64$M_\odot$ and 100$M_\odot$ were presented by Arnett (1973, 1974). The cores exploded and the lighter one left a 2.2$M_\odot$ silicon remnant.

All these computational experiments, disagreeing in detailed values, have demonstrated the main feature: with low masses of oxygen core stars oscillate, don’t disrupt totally; heavy oxygen cores collapse to black holes without mass ejection; in the middle mass range stars produce very energetic explosion that reverses the collapse and disrupts the star.

El Eid et al. (1983) studied detailed evolution of stars with initial masses of 80-500$M_\odot$ that consist of hydrogen and helium (model of Population III stars). They demonstrated that at the end of helium burning stage stars consist mainly of carbon and oxygen ($\sim 90\%$ of the mass). Stars with initial mass higher than 100$M_\odot$ evolve very fast after helium burning and enter the electron-positron pair creation region. The authors used these carbon-oxygen cores as initial configurations for further hydrodynamical calculations in the following work (Ober et al. 1983). They found that carbon-oxygen cores with mass within 48–112$M_\odot$ explode and heavier cores collapse. This result is in agreement with the upper mass limit of 100$M_\odot$ for explosion of oxygen cores obtained by Bond et al. (1984).

Effect of rotation on evolution of massive carbon-oxygen cores in rigid-body approximation has been considered by Glatzel et al. (1985). In this paper authors showed that rotation could expand the region of explosion to higher masses.

Recent observations of super-luminous supernovae and observation of SN 2007bi as a possible PISN candidate (Gal-Yam et al. 2009) renewed interest to this subject. Series of work study evolution of very massive stars and role using modern evolutionary code MESA (Paxton et al. 2010, 2013). Chatzopoulos & Wheeler (2012a,b) showed that rotation highly affects evolution of a star changing its chemical evolution. Induced
mixid can lead to homogeneous chemical evolution when hydrogen from the envelope is mixing with matter of the core and is consumed during nuclear burning. Thus at the moment when the star reaches pair instability it does not have hydrogen envelope at all. This could drastically change the light curve during PISN explosion.

Many recent articles (Chen et al. 2011; Joggerst & Whalen 2011) address the problem of multidimensional simulations of a PISN explosion. Such studies are needed to compute light curves and time of appearance of lines of chemical elements, so they allow us to predict the observational signatures of these supernovae.
Chapter 3

Pair-Instability Supernovae modeling

3.1 Physical model

To study the dynamics of unstable massive oxygen cores the system of hydrodynamical equations was solved numerically. For simplicity the spherical symmetry was assumed. Under this assumption it is possible to write the system of equations in Lagrangian coordinates and it is convenient to choose as an independent Lagrangian variable, $m$, the mass of the matter inside a sphere of radius $r$:

$$m = \int_0^r \rho(r') \cdot 4\pi r'^2 \, dr', \quad dm = 4\pi \rho r^2 \, dr,$$

where $\rho(r)$ is the density of the matter at the distance $r$ from the center. Then the equations for the motion of an element of fluid are:

$$\frac{\partial r}{\partial t} = v,$$  \hspace{1cm} (3.2)

$$\frac{\partial v}{\partial t} = -\frac{Gm}{r^2} - 4\pi r^2 \left( \frac{\partial (P + Q)}{\partial m} \right),$$  \hspace{1cm} (3.3)

$$\frac{\partial T}{\partial t} = \left( -4\pi \frac{\partial (r^2 v)}{\partial m} \left( T \frac{\partial P}{\partial T} \rho + Q \right) + \varepsilon_{\text{nucl}} - \varepsilon_{\nu} \right) \left( \frac{\partial E}{\partial T} \right) \rho.$$  \hspace{1cm} (3.4)

In Equations (3.2)–(3.4) $t$ is the time, $v$ velocity of the element of fluid, $P$ total pressure, $T$ temperature and $E$ the specific internal energy. $Q$ is the momentum flux density due to artificial viscosity introduced to damp out numerical oscillations \cite{Caramana et al. 1998}. \footnote{\textit{Caramana et al. 1998}.}
It is a combination of the linear and quadratic viscosities:

\[
Q = \begin{cases} 
\mu_0 \rho \left( \Delta r \frac{1}{r^2} \frac{\partial (r^2 v)}{\partial r} \right)^2 - \mu_1 \rho c_s \Delta r \frac{1}{r^2} \frac{\partial (r^2 v)}{\partial r}, & \frac{\partial (r^2 v)}{\partial r} < 0; \\
0, & \frac{\partial (r^2 v)}{\partial r} > 0
\end{cases}
\]  
(3.5)

Here \( \Delta r \) is the smearing length, usually equal to the interval of a numerical grid, coefficients of quadratic and linear viscosity \( \mu_0 \) and \( \mu_1 \) are order of unity so the shock fronts is smeared over a few numerical cells; \( c_s \) is the sound speed. Values of \( P, (\partial P/\partial T)_{\rho}, (\partial E/\partial \rho)_{\rho} \) and \( c_s \) are determined by the equation of state and depend on density, temperature and chemical composition of the matter:

\[
P(\rho, T, Y_j) = EOS(\rho, T, Y_j),
\]  
(3.6)

where \( Y_j = X_j/A_j \) is the abundances of the isotopes (the number of isotopes per nucleon), \( X_j \) the mass fraction of the \( j \)-th isotope with the number of nucleons \( A_j \). The equation of state should take the electron-positron pair-creation process into account.

For that purpose the routine was used, which is based on Nadyozhin’s asymptotic relations [Blinnikov et al. 1996] for an equilibrium mixture of electrons, positrons and photons (Sec. 3.1.3).

The terms \( \varepsilon_{\text{nuc}} \) and \( \varepsilon_\nu \) in Eq. (3.4) correspond to the changes of the energy per unit of mass per unit of time [erg s\(^{-1}\) g\(^{-1}\)] owing to nuclear burning (Sec. 3.1.1) and neutrino losses (Sec. 3.1.2) accordingly. Change of the chemical composition \( dY_j/dt \) during nuclear burning should be taken into account and the balance equation for each isotope should be added to the system of equations.
Usage of the Lagrangian approach to hydrodynamics is quite convenient for the computation of nuclear burning. Since the Eqs. (3.2)–(3.4) are written down not for a spatial coordinates but for the elements of fluid, the chemical composition changes only due to nuclear reactions and not affected by the motion of the fluid. So there are no terms related to hydrodynamical mixing in equations for $dY_j/dt$. This fact drastically simplifies the computational task.

### 3.1.1 Nuclear burning

Nuclear burning was computed using the simplified network of reactions (Fig. 3.2.), consisting of the 13 isotopes from $^4$He to $^{56}$Ni connected by reactions with the capture of $\alpha$-particles and photodissociation processes:

$$ (Z, A) + \alpha \rightleftharpoons (Z + 2, A + 4) + \gamma; $$

and the “heavy-ion” reactions:

\begin{align*}
12^C + 12^C & \rightarrow \quad 24^{Mg} + \gamma, \\
& \quad \rightarrow \quad 20^{Ne} + \alpha, \\
12^C + 16^O & \rightarrow \quad 28^{Si} + \gamma, \\
& \quad \rightarrow \quad 24^{Mg} + \alpha, \\
16^O + 16^O & \rightarrow \quad 32^{S} + \gamma, \\
& \quad \rightarrow \quad 28^{Si} + \alpha.
\end{align*}

The rate of change of the number density $N_j$ of the isotope $j$ is described by the differential equation [Lang 1980]:

$$ \frac{dN_j}{dt} = N_kN_l \langle \sigma v \rangle_{kl,j} - N_jN_l \langle \sigma v \rangle_{jl,m} + ... + N_n\lambda_{n,j} - N_j\lambda_{j,k} + ... \quad (3.7) $$

$$ j, k, ... = 1, 13 $$

The term $\langle \sigma v \rangle_{kl,j}$ is the averaged product of cross section and relative velocity of the isotopes $k$ and $l$ in the center-of-mass system in the reaction $k + l \rightarrow j + ...$; $\lambda_{j,k}$ is the probability of spontaneous decay of the isotope $j$ per second $j \rightarrow k + \alpha$ (photodissociation).
For computations it is convenient to deal with the abundances (mole fractions) of the elements $Y_j$:

$$Y_j = \frac{X_j}{A_j} = \frac{N_j}{\rho N_A},$$  

(3.8)

where $N_A$ is the Avogadro’s number. Unlike the number density $N_j$ the mole fraction $Y_j$ does not change during hydrodynamical expansion or contraction (does not depend on density $\rho$). Then Eq. (3.9) becomes:

$$\frac{dY_j}{dt} = Y_k Y_l \rho N_A \langle \sigma v \rangle_{kl,j} - Y_j Y_l \rho N_A \langle \sigma v \rangle_{jl,m} + \ldots + Y_n \lambda_{n,j} - Y_j \lambda_{j,k} + \ldots ,$$  

(3.9)

or by introducing reaction rates $R_{kl,j} = N_A \langle \sigma v \rangle_{kl,j}$:

$$\frac{dY_j}{dt} = \rho Y_k Y_l R_{kl,j} - \rho Y_j Y_l R_{jl,m} + \ldots + Y_n \lambda_{n,j} - Y_j \lambda_{j,k} + \ldots$$  

(3.10)

The rates of nuclear reactions are the subject of extensive investigations in theory and experiments. Direct measurements of cross sections are available only in the limited range of energies. Theoretical synthesis of experimental data and statistical models are used for conditions not accessible to experiment. One of the most recent compilations of the rates for a wide range of nuclear reactions is the JINA REACLIB database (Cyburt et al. 2010). The rate of the forward reaction $R$ and the rate of the reverse reaction of
photodissociation $\lambda$ in this database are interpolated by the formulae:

$$R_{\lambda} = \exp \left( a_0 + \frac{a_1}{T_9} + \frac{a_2}{T_9^{1/3}} + a_3 \cdot T_9^{1/3} + a_4 \cdot T_9 + a_5 \cdot T_9^{5/3} + a_6 \cdot \log T_9 \right), \quad (3.11)$$

$$T_9 = \frac{T}{10^9 \text{K}}$$

where $a_i$ are the adjustable parameters.

If the reaction $k + l \rightarrow j + m$ releases the amount of energy $Q_{kl}$ (usually defined in MeV) then the rate of nuclear energy release $\varepsilon_{kl}$ per unit of mass is:

$$\varepsilon_{kl} = \frac{Q_{kl}}{\text{MeV nucleon} \cdot \text{s}} = 0.964844 \times 10^{18} \frac{Q_{kl}}{\text{erg g} \cdot \text{s}} \quad (3.12)$$

The rate of total energy release $\varepsilon_{\text{nucl}}$ is the sum of contributions $\varepsilon$ from all considered nuclear reactions.

### 3.1.2 Neutrino losses

Neutrino processes in massive stars were examined by Fowler & Hoyle [1964] and they found that the neutrino emission by the electron-positron pair annihilation process $e^- + e^+ \rightarrow \nu + \bar{\nu}$ is the most part of neutrino losses. Their computations showed that neutrino losses highly affect evolution of a star at high temperatures.

In this work the neutrino losses from pair annihilation, photo production and plasma were taken into account:

$$\varepsilon_{\nu} = \varepsilon_{\text{pair}} + \varepsilon_{\text{photo}} + \varepsilon_{\text{plasma}}. \quad (3.13)$$
Table 3.1: Coefficients for fitting formula (3.14a) for different processes of neutrino losses.

<table>
<thead>
<tr>
<th>Process</th>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_0$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$c_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plasma</td>
<td>2.146(−7)</td>
<td>7.814(−8)</td>
<td>1.653(−8)</td>
<td>2.581(−2)</td>
<td>1.734(−2)</td>
<td>6.990(−4)</td>
<td>0.56457</td>
</tr>
<tr>
<td></td>
<td>10^8 K ≤ T ≤ 10^{10} K</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Photo</td>
<td>3.897(10)</td>
<td>5.906(10)</td>
<td>4.693(10)</td>
<td>6.290(−3)</td>
<td>7.483(−3)</td>
<td>3.061(−4)</td>
<td>1.5654</td>
</tr>
<tr>
<td></td>
<td>10^{10} K ≤ T ≤ 10^{11} K</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pair</td>
<td>5.026(19)</td>
<td>1.745(20)</td>
<td>1.568(21)</td>
<td>9.383(−1)</td>
<td>−4.141(−1)</td>
<td>5.829(−2)</td>
<td>5.5924</td>
</tr>
</tbody>
</table>

Schinder et al. (1987) computed the rates of neutrino losses and interpolated them with the fitting formulae:

\[
\varepsilon_p = K(\rho, \lambda) \, e^{-c_0 \xi} \left( \frac{a_0 + a_1 \xi + a_2 \xi^2}{\xi^3 + b_1/\lambda + b_2/\lambda^2 + b_3/\lambda^3} \right) ,
\]

(3.14a)

\[
\xi = \left( \frac{\rho/\mu_e}{10^9} \right) ,
\]

(3.14b)

\[
\lambda = \frac{T}{5.9302 \times 10^9 K} ,
\]

(3.14c)

\[
K(\rho, \lambda) = \begin{cases}
(\rho/\mu_e)^3 & \text{plasma}, \\
(\rho/\mu_e)^5 & \text{photo}, \\
g(\lambda)e^{-2/\lambda} & \text{pair},
\end{cases}
\]

(3.14d)

\[
g(\lambda) = 1 - 13.04 \lambda^2 + 133.5 \lambda^4 + 1534 \lambda^6 + 918.6 \lambda^8 ,
\]

(3.14e)

where $\mu_e$ is the number of nucleons per electron, which is equal to 2 for the chemical composition considered (Sec. 3.1.1). Adjustable parameters $a_0, a_1, a_2, b_1, b_2, b_3, c_0$ for different processes are presented in the Table 3.1.

### 3.1.3 Equation of state

To solve Eqs. (3.2)–(3.4) it is necessary to compute relations between thermodynamical quantities, which are determined by the equation of state of the stellar matter. There are several equation-of-state routines available (e.g. Iben 1963; Arnett 1969; Weaver, Zimmerman, & Woosley 1978; Nadyozhin 1974a,b). The choice of the routine to use in the computations was done basing on the analysis of Timmes & Arnett (1999). The authors
analyzed five different equations of state and compared their accuracy, thermodynamical consistency and execution speed. Since there are possibly large uncertainties in stellar physics (nuclear burning, convection, etc.) absolute accuracy was not the primary concern in our research. For that case authors suggest: “When an optimal balance between accuracy, thermodynamic consistency, and speed is desirable, then the Nadyozhin EOS is a very good choice”. Therefore the Nadyozhin EOS was chosen (Nadyozhin 1974a,b; Blinnikov et al. 1996).

The routine has the following input parameters: the temperature $T$, density $\rho$, the mean number of nucleons per isotope $\bar{A}$ and the mean charge per isotope $\bar{Z}$. Then it returns values of the pressure $P$, specific internal energy $E$ and other quantities such as partial derivatives of thermodynamical values, specific heats and indexes $\Gamma$. The pressure and specific energy are computed as a sum of contributions from electrons, positrons, ions and photons:

$$P = P_\text{e} + P_\text{e^+} + P_\text{ion} + P_\text{rad},$$  \hspace{1cm} (3.15)
$$E = E_\text{e} + E_\text{e^+} + E_\text{ion} + E_\text{rad}. $$  \hspace{1cm} (3.16)

The pressure and specific energy of the photons are computed as blackbody radiation in local thermodynamic equilibrium:

$$P_\text{rad} = \frac{aT^4}{3}, \quad E_\text{rad} = \frac{aT^4}{\rho},$$  \hspace{1cm} (3.17)

where $a$ is the radiation density constant related to the Stefan-Boltzmann constant $\sigma$ as $a = 4\sigma/c$.

The portion of ions is described as an ideal gas:

$$P_\text{ion} = \frac{N_A \rho}{A} kT, \quad E_\text{ion} = \frac{3 N_A}{2} kT,$$  \hspace{1cm} (3.18)

where $N_A$ is the Avogadro’s number and $k$ is Boltzmann’s constant. The mean number of nucleons per isotope $\bar{A}$ and the mean charge per isotope $\bar{Z}$ for a mixture different chemical elements should be computed as:

$$\bar{A} = \left( \sum_i \frac{X_i}{A_i} \right)^{-1}, \quad \bar{Z} = \bar{A} \left( \sum_i \frac{X_i Z_i}{A_i} \right),$$  \hspace{1cm} (3.19)

where $X_i$ is the mass fraction of the isotope with the number of nucleons $A_i$ and charge $Z_i$. 

Electrons and positrons are fermions and the number densities of non-interacting electrons \( n_- \) and positrons \( n_+ \) are described by Fermi-Dirac statistics:

\[
\begin{align*}
n_- &= \frac{8\pi}{\hbar^3} \int_0^\infty \frac{p^2 dp}{e^{-\eta_- + \mathcal{E}/kT} + 1}, \\
n_+ &= \frac{8\pi}{\hbar^3} \int_0^\infty \frac{p^2 dp}{e^{-\eta_+ + \mathcal{E}/kT + 2m_e c^2} + 1}.
\end{align*}
\] (3.20, 3.21)

Here \( h \) is the Planck’s constant, \( p \) momentum of the particle, \( \mathcal{E} \) the kinetic energy of the particle, \( \eta_- = \mu_- / kT \) and \( \eta_+ = \mu_+ / kT \) are normalized chemical potentials. The zero of the energy scale is chosen by subtracting the rest mass of the electron \( m_e c^2 \) from the total energy of a free electron. So the total energy of a free electron is just \( \mathcal{E} \):

\[
\mathcal{E} = m_e c^2 \left( \sqrt{\left( p/m_e c \right)^2 + 1} - 1 \right). \tag{3.22}
\]

This subtraction is taken into account explicitly in the term for the energy of a free positron in Eq. 3.21, where additional term \( 2m_e c^2 \) appears.

The \( e^\pm \)-pair can be considered as the “dissociated” state of a photon, which is treated as a “composite particle”. Then pair creation can be presented by the reaction:

\[
e^+ + e^- \leftrightarrow 2\gamma,
\]

and the thermodynamical equilibrium implies that “chemical equilibrium” among electrons, positrons and photons is established. Since the chemical potential of blackbody photons \( \eta_\gamma = 0 \), the condition for thermodynamical equilibrium is:

\[
\eta_- + \eta_+ = 0 \quad \text{or} \quad \eta_- = -\eta_+ = \eta.
\]

Then Equations 3.20, 3.21 can be written as:

\[
\begin{align*}
n_- &= \frac{8\pi}{\hbar^3} \int_0^\infty \frac{p^2 dp}{e^{-\eta_- + \mathcal{E}/kT} + 1}, \\
n_+ &= \frac{8\pi}{\hbar^3} \int_0^\infty \frac{p^2 dp}{e^{\eta_+ + \mathcal{E}/kT + 2m_e c^2} + 1}.
\end{align*}
\] (3.23, 3.24)

By introducing dimensionless variables

\[
x = \mathcal{E} / kT,
\] (3.25)
\[ \beta = \frac{kT}{m_e c^2}, \] (3.26)

and using Eq. 3.22 it is possible to express number density or electrons and positrons in more explicit way by extracting all physical quantities:

\[ n_- = \frac{8\pi \sqrt{2}}{h^3} m_e^3 c^3 \beta^{3/2} \int_0^\infty \frac{x^{1/2}(1 + (1/2)\beta x)^{1/2}(1 + \beta x)dx}{e^{-\eta + x} + 1}, \] (3.27)

\[ n_+ = \frac{8\pi \sqrt{2}}{h^3} m_e^3 c^3 \beta^{3/2} \int_0^\infty \frac{x^{1/2}(1 + (1/2)\beta x)^{1/2}(1 + \beta x)dx}{e^{+\eta + 2/\beta + x} + 1}. \] (3.28)

Integrals in Eqs. 3.27, 3.28 can be expressed as sum of Fermi-Dirac integrals \( F_k(\eta, \beta) \):

\[ F_k(\eta, \beta) = \frac{x^k(1 + (1/2)\beta x)^{1/2}dx}{e^{-\eta + x} + 1}, \] (3.29)

\[ n_- = \frac{8\pi \sqrt{2}}{h^3} m_e^3 c^3 \beta^{3/2} \left[ F_{1/2}(\eta, \beta) + \beta F_{3/2}(\eta, \beta) \right], \] (3.30)

\[ n_+ = \frac{8\pi \sqrt{2}}{h^3} m_e^3 c^3 \beta^{3/2} \left[ F_{1/2}(-\eta - \frac{2}{\beta}, \beta) + \beta F_{3/2} \left( -\eta - \frac{2}{\beta}, \beta \right) \right]. \] (3.31)

In case of complete ionization, the number density of free electrons related to the matter is:

\[ n_-^{(m)} = \frac{\bar{Z}}{A} N_A \rho, \]

and the requirement of electrical charge neutrality leads to

\[ n_-^{(m)} = n_- - n_+ . \] (3.32)

Equation 3.32 allows to compute normalized chemical potential \( \eta \) as a function of the matter density \( \rho \) and the temperature \( T \). Once \( \eta \) is known it is possible to compute the pressure and specific internal energy of the electrons and positrons:

\[ P_- = \frac{16\pi \sqrt{2}}{3h^3} m_e^4 c^5 \beta^{5/2} \left[ F_{3/2}(\eta, \beta) + \frac{1}{2} \beta F_{5/2}(\eta, \beta) \right], \] (3.33)

\[ P_+ = \frac{16\pi \sqrt{2}}{3h^3} m_e^4 c^5 \beta^{5/2} \left[ F_{3/2}(-\eta - \frac{2}{\beta}, \beta) + \frac{1}{2} \beta F_{5/2}(-\eta - \frac{2}{\beta}, \beta) \right]. \] (3.34)
$E_- = \frac{8\pi \sqrt{2}}{\rho h^3} m_e^4 c^5 \beta^{5/2} \left[ F_{3/2}(\eta, \beta) + \beta F_{5/2}(\eta, \beta) \right]$, \quad (3.35)

$E_+ = \frac{8\pi \sqrt{2}}{\rho h^3} m_e^4 c^5 \beta^{5/2} \left[ F_{3/2}(\eta - 2\beta, \beta) + \beta F_{5/2}(\eta - 2\beta, \beta) \right] + \frac{2m_e c^2 n_+}{\rho}$. \quad (3.36)

To compute the thermodynamic quantities the Nadyozhin EOS uses various asymptotics and series expansions for different regions of the temperature–density plane (see Fig. 3.3): relativistic asymptotics, expansion for a degenerate gas (Chandrasekhar 1939), perfect gas approximation with the first-order corrections for degeneracy, expansions of the half-integer Fermi-Dirac functions ($F_{D1/2}$) and Gaussian quadrature for the thermodynamic quantities. Special attention given to the transitions between regions of different asymptotics making them continuous and smooth.

![Figure 3.3: Regions of application of different asymptotics in Nadyozhin EOS routine from Blinnikov et al. (1996).](image)

### 3.2 Initial configuration

One of the key questions of supernova modeling is the choice of the initial configuration of a star prior to supernova explosion (pre-supernova). A very common approach is to compute the pre-supernova configuration by evolutionary codes. But evolutionary codes
usually assume spherical symmetry, and any asymmetric process, such as convection or rotation, can only be taken into account in some effective way with approximation. Also detailed computation of stellar evolution requires considering many physical processes that have uncertain descriptions. This lead to precise pre-supernova configurations possibly being unknown.

The question of the presence of massive hydrogen envelope in very massive stars is under debate. It is not clear that such stars keep this envelope up to the moment of explosion since it is very weakly bound, for example, it was shown by Woosley et al. (2007) that the envelope could be expelled in previous stages of evolution. In some works (e.g. Nozawa et al. 2003) the dynamics of very massive stars was considered by taking the massive envelope into account. In others (e.g. Kasen et al. 2011), only the cores were considered. In this work we only followed the dynamics of the cores without an envelope.

Instead of taking initial conditions from evolutionary calculations a polytropic structure approximation was used. Although this approach simplifies description of the stellar structure comparing to evolutionary calculations, but it gives a possibility to explore a wider variety explosion scenarios that depend on initial configuration. Under this assumption the pressure $P$ and the density $\rho$ vary through a star as:

$$P = K \rho^{1+1/n}$$

where $n$ and $K$ are constants. To construction initial configuration it is needed to integrate the system of equations of hydrostatic equilibrium and mass conservation:

$$\begin{align*}
\frac{dP}{dr} &= -\frac{Gm\rho}{r^2}, \\
\frac{dm}{dr} &= 4\pi\rho r^2.
\end{align*}$$

Boundary conditions for Eqs. (3.38) is that the pressure vanishes at the surface of a star:

$$P|_{\text{surface}} = 0, \quad m|_{\text{surface}} = M_{\text{tot}},$$

where $M_{\text{tot}}$ is the total mass of the core. Equations (3.37)–(3.39) completely define structure of a star and it is possible to integrate them from the center to the surface for a given $n$. For analysis of this equations it is convenient to introduce new variables:

$$\rho = \lambda \theta^n, \quad r = \xi \alpha,$$
where $\lambda$ is a scaling factor equal to the central density $\rho_c$, $\alpha$ is constant:

$$\alpha = \sqrt{\frac{(n + 1)K\lambda^{1/n}}{4\pi G}}.$$ 

Then Eqs. (3.38) can be written as:

$$\frac{d(\xi^2d\theta/d\xi)}{d\xi} = -\xi^2\theta^n,$$

or

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi} \frac{d\theta}{d\xi} - \theta^n;$$

this is so called Lane-Emden equation. Boundary conditions for this equation are:

$$\theta = 1, \text{ for } \xi = 0,$$

At the surface when $\xi$ is equal to some value $\xi_1$:

$$\theta = 0, \text{ for } \xi = \xi_1.$$

The mass of a star $m(r)$ contained within radius $r$ is:

$$m(r) = 4\pi \int_0^r \rho(r') \cdot 4\pi r'^2 \, dr' = 4\pi \left( \frac{(n + 1)K}{4\pi G} \right)^{3/2} \lambda^{3-n/2} \mu;$$

$$\mu = \left( -\xi^2 \frac{d\theta}{d\xi} \right), \quad \mu_1 = \left( -\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi_1}.$$

Lane-Emden equation allows to obtain important relations between physical quantities in a star without actually solving them. Let’s fix the mass of a star $M_{\text{tot}}$ and parameters of a gas $K, n$. Then the ratio of the average density $\langle \rho \rangle$

$$\langle \rho \rangle = \frac{3M_{\text{tot}}}{4\pi R^3},$$

(where $R$ is the radius of a star) to the central density $\rho_c$ is

$$\frac{\langle \rho \rangle}{\rho_c} = \frac{3M_{\text{tot}}}{4\pi R^3 \rho_c} = \frac{3M_{\text{tot}}}{4\pi \xi_1^3 \alpha^3 \lambda} = \frac{3\mu_1}{\xi_1^3}. $$

The expressions for the central density $\rho_c$ and central pressure $P_c$ are:

$$\rho_c = \lambda = \left\{ \frac{M_{\text{tot}}}{\mu_1} \left( \frac{4\pi G^3}{K^3(n + 1)^3} \right)^{1/2} \right\}^{\frac{2n}{3-n}}.$$


$$P_c = \frac{GM_{\text{tot}}^2}{R^4} \frac{\xi_1^4}{4\pi(n+1)\mu_1^4}.$$  (3.47)

It is possible to show that the central pressure $P_c$ and central density $\rho_c$ are connected by the relation:

$$P_c = D_n GM_{\text{tot}}^{2/3} \rho_c^{4/3},$$  (3.48)

where $D_n$ is a function of the polytropic index $n$. Using Eq. (3.37):

$$K = D_n GM_{\text{tot}}^{2/3} \rho_c^{1/3} \mu_1^{1-n/3}.  \tag{3.49}$$

The quantities $\xi_1, \mu_1, D_n$ and $\langle\rho\rangle/\rho_c$ for different values of the polytropic index $n$ are presented in the Table 3.2.

As was shown by evolutionary calculations, structure of very massive stars can be approximated by polytrope with index $n = 3$. It is a special case for selfgravitating polytropic configurations. There is no analytical solution of Eqs. (3.38) for $n = 3$. For a given $K$ there is only one possible value of $M_{\text{tot}} = M_3$ (from Eq. (3.49)). For this value of $M_{\text{tot}}$, the central density $\rho_c$ and radius $R$ can take any values that satisfy the relation:

$$\rho_c^4 \pi^3 R^3 \left(\frac{3\mu_1}{\xi_1^3}\right) = M_3.$$  (3.50)

This allows to built several configurations for each core having mass $M_{\text{tot}}$, by choosing different values of the central density, $\rho_c$ (see Table 3.3). The density $\rho$ and pressure $P$ as functions of mass coordinate $m$ for the case of $n = 3$ polytrope are presented in Fig. 3.4. Choice of $\rho_c$ changes the value of the radius and the initial binding energy of a star, $E_{\text{bind}}$:

$$E_{\text{bind}} = \Omega + U = \int_{0}^{M_{\text{tot}}} \left(-\frac{Gm}{r} + E\right) dm,$$  (3.51)

which is equal to the total energy. At initial moment of time a star is assumed to be in hydrostatic equilibrium, so there is no kinetic term in this expression.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\xi_1$</th>
<th>$\mu_1$</th>
<th>$\rho_c/\langle\rho\rangle$</th>
<th>$D_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.45</td>
<td>4.90</td>
<td>1.00</td>
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Figure 3.4: The density $\rho$ and pressure $P$ in units of the central values are shown as functions of mass coordinate $m$ for $n = 3$ polytrope.

For a star entirely made of a gas with adiabatic index $\gamma = 4/3$ and polytropic index $n = 3$ the total energy is zero. But in a realistic situation adiabatic index is not a constant through a star, thus $E_{\text{bind}}$ can have some non-zero value. Thermodynamical quantities at the center that we have chosen and values of binding energy are very close to the results of evolutionary calculations (Waldman 2008).
Table 3.3: Pre-supernova models.

<table>
<thead>
<tr>
<th>Model</th>
<th>$M_{\text{tot}}$</th>
<th>$E_{\text{bind}}$</th>
<th>$\rho_c$</th>
<th>$T_c$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[M$_\odot$]</td>
<td>[10$^{51}$ erg]</td>
<td>[10$^5$ g/cm$^3$]</td>
<td>[10$^9$ K]</td>
<td>[R$_\odot$]</td>
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<tr>
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</tr>
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<tr>
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<td>1.70</td>
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<td>0.265</td>
</tr>
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<td>0.245</td>
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<td>2.23</td>
<td>0.242</td>
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<td>2.43</td>
<td>0.243</td>
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<td>0.76</td>
<td>1.70</td>
<td>0.424</td>
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</table>

**Note.** $M_{\text{tot}}$ is the mass of the oxygen core, $E_{\text{bind}}$ the initial binding energy, $\rho_c$ the initial central density, $T_c$ the initial central temperature, $R$ the initial radius.
3.3 Computational scheme

Final system of equations that describes behavior of unstable oxygen cores is obtained by combining the equations of hydrodynamics (3.2–3.4) with the equations of nuclear burning (3.10):

\[
\begin{aligned}
\frac{\partial r}{\partial t} &= v, \\
\frac{\partial v}{\partial t} &= -\frac{Gm}{r^2} - 4\pi r^2 \left( \frac{\partial (P + Q)}{\partial m} \right), \\
\frac{\partial T}{\partial t} &= \left( -4\pi \frac{\partial (r^2 v)}{\partial m} \left( T \left( \frac{\partial P}{\partial T} \right) + Q \right) + \varepsilon_{\text{nuc}} - \varepsilon_{\nu} \right) / \left( \frac{\partial E}{\partial T} \right)_{\rho},
\end{aligned}
\]

(3.52)

\[
\begin{aligned}
\frac{dY_{ij}}{dt} &= Y_kY_i\rho R_{jk,l} - Y_jY_i\rho R_{jl,m} + Y_n\lambda_{n,j} - Y_j\lambda_{j,k}, \quad j, k, ... = 1, 13, \\
P^i &= EOS(\rho^i, T^i, Y_{ij}^i).
\end{aligned}
\]

To solve Eqs. (3.52) numerically the stellar model was divided into \( N \) intervals by mass by introducing the grid of values of \( m \): \([m^i], \ i = 0, N, \ m^0 = 0, \ m^N = M_{\text{tot}}\). Variables \( r^i \) and \( v^i \) are the radius and velocity at the right boundary of \( i \)th layer between \( m^{i-1} \) and \( m^i \). Other values with index "i" are attributed to the center of this layer (Fig. 3.5). The derivatives with respect to \( m \) are substituted by finite differences, with \( \Delta m = m^i - m^{i-1} \). Thus the problem is reduced to the system of ordinary differential equations with respect to the time, \( t \):

\[
\begin{aligned}
\frac{\partial r^i}{\partial t} &= v^i, \\
\frac{\partial v^i}{\partial t} &= -\frac{Gm^i}{r^i v^i} - 4\pi r^i v^i \left( \frac{P^i + Q^i - P^{i-1} - Q^{i-1}}{\Delta m} \right), \\
\frac{\partial T^i}{\partial t} &= \left( \frac{r^{i+1} v^{i+1} - r^{i} v^{i}}{4\pi \Delta m} \right) \left( T^i \left( \frac{\partial P}{\partial T} \right) - Q^i \right) + \varepsilon_{\text{nuc}} = \varepsilon_{\nu} / \left( \frac{\partial E}{\partial T} \right)_{\rho},
\end{aligned}
\]

(3.53)

\[
\begin{aligned}
\frac{dY_{ij}^i}{dt} &= Y_kY_i^j\rho R_{jk,l} - Y_j^iY_i^j\rho R_{jl,m} + Y_n\lambda_{n,j} - Y_j^i\lambda_{j,k}, \quad j, k, ... = 1, 13, \\
P^i &= EOS(\rho^i, T^i, Y_{ij}^i).
\end{aligned}
\]
Expression (3.5) for viscosity is approximated as:

\[ Q^i = \begin{cases} 
\mu_0 \rho^i \left[ \frac{1}{r_{i+1/2}^2} (r_{i+1/2} v_{i+1} - r_i^2 v^i) \right]^2 - \mu_1 \rho^i c_s^i \frac{1}{r_{i+1/2}^2} (r_{i+1/2} v_{i+1} - r_i^2 v^i), \\
0, & \text{if } (r_{i+1/2} v_{i+1} - r_i^2 v^i) > 0.
\end{cases} \]  

(3.54)

To solve the system of equations (3.53) implicit numerical methods were used. Since the change of the chemical composition is not connected directly with hydrodynamical motion the following simplification was made: computations of hydrodynamics and nucleosynthesis were performed by turns. First the values of the radius, velocity and temperature on the next time step, \( \hat{r} \), \( \hat{v} \) and \( \hat{T} \), are computed, while chemical composition of matter, \( Y_j \), is fixed. However energy production from nucleosynthesis is taken into account at this step, since there are terms \( \varepsilon_{\text{mc}} \) in the equations for \( T \). On the next step the changes in composition are computed with the new values of the density, \( \rho \), and the temperature, \( T \).

For convenience of programming and data storage in our code we used different numerical solvers for computation of hydrodynamics and nucleosynthesis. For the calculations of hydrodynamics we have used implicit Gear’s method which is implemented in the Livermore Solver for Ordinary Differential Equations (LSODE) ([Hindmarsh 1983][Radhakrishnan & Hindmarsh 1993]). Then for nucleosynthesis we have used RADAU5 solver.
This solver uses implicit Runge-Kutta method of 5th order. These methods are suitable for solving the stiff differential problems, such as nuclear burning problem, when the rates of reactions extremely sensitive to the change temperature.
Chapter 4

Results of 1D computations

Hydrodynamical simulations of PISN explosion were performed for several models of stars with different masses $M_{\text{tot}}$ of the core. Only the cores were considered with initial composition assumed to be pure oxygen. Initial configurations were computed numerically from the hydrostatic equilibrium condition with the polytropic index $n = 3$.

In general the results of computations are in agreement with previous works (e.g., El Eid et al. [1983]). An example of the calculations of the 90 $M_\odot$ with initial central density $2.7 \times 10^5$ g/cm$^3$ (model 90(4)) is presented in Fig. 4.1. It shows the radius of the core $R$, absolute value of the total neutrino losses, $\varepsilon_\nu$, and nuclear energy production rate, $\varepsilon_{\text{nuc}}$. Rate of nuclear release reaches values up to few times $10^{51}$ erg/s and the total energy release is equal to $E_{\text{nuc}} = \int \varepsilon_{\text{nuc}} \, dt = 4.77 \times 10^{52}$ erg which is comparable with typical GRB energy. Also it is seen that most part of nuclear energy is released within time interval of 20–40 seconds and it takes the same order of time for the core to expand from the minimum to its initial radius. Thus the explosion of the core solely is a fast process with the same characteristic time as duration of GRBs.

Figure 4.2 demonstrates the evolution of the velocity profile for 90(4) model. In the beginning contraction accelerates reaching values up to few times $10^8$ cm/s (curves 1 and 2). Then the deflagration wave from the oxygen burning reaches outer layers of the core accelerating them to velocity of $\sim 5 \times 10^8$ cm/s (3 and 4). These layers start to expand reducing pressure on the central part which also starts to expand with acceleration (5). In the end the core is disrupted leaving no remnant. The ejected matter is expanding with the maximum velocity of $1.7 \times 10^9$ cm/s (8).

Chemical composition of the ejecta of this model is presented in Fig. 4.3. Explosive burning of oxygen took place in the inner $\sim 25$ $M_\odot$ of the core resulting in production of silicon and sulfur. In the central part where higher temperature is reached these
Chapter 4. Results of 1D computations

Figure 4.1: Dynamics of explosion of 90 $M_{\odot}$ oxygen core; $R$ is the radius of the core, $\varepsilon_{\text{nuc}}$ is the rate of total nuclear energy release, $\varepsilon_\nu$ is the total neutrino losses.

Figure 4.2: Velocity profiles for different moments of time (in seconds) for 90(4) model.
elements are transformed by further reactions of capturing of \( \alpha \)-particles to the elements of the iron group up to \( {^{56}}Ni \). The total amount of nickel produced is about \( 0.1 \, M_{\odot} \). It is unstable isotope which half-life time of 6 days. The decay of this isotope by the reaction:

\[
^{56}Ni \rightarrow ^{56}Co \rightarrow ^{56}Fe
\]

contributes to the luminosity of the following supernova.

![Chemical composition of ejecta of the 90(4) model.](image)

Figure 4.3: Chemical composition of ejecta of the 90(4) model.

It is important to find out the conditions in which oxygen cores explode and in which they collapse. Examples of the trajectories in the central density–central temperature plane (\( \rho_c-T_c \)) for different stellar models (60(1), 95(5), and 128(1)) are presented in Fig. 4.4. The cores with masses of \( 60 \, M_{\odot} \) and \( 95 \, M_{\odot} \) explode and the \( 128 \, M_{\odot} \) core collapses. In the beginning any core contracts almost homogeneously along \( T \propto \rho^{1/3} \) line. More massive cores move along trajectories with higher temperature for the same value of the density (see Sec. 2.1). When the central temperature \( T_c \) reaches values about \( 3 \times 10^9 \, K \) the contraction shifts to higher trajectory owing to energy release from nuclear burning. Oxygen burns explosively and a deflagration wave is initiated and propagates outward. At temperatures about \( 4 \times 10^9 \, K \) the photodissociation of heavy nuclei becomes effective and changes the slope of the trajectory. Photodissociation is an endothermic process and reduces the internal energy and pressure, thus decelerates the growth of the central temperature. If at this point nuclear energy release has not affected the contraction the core proceeds to collapse.

The summary of results of computations of different models are presented in Table 4.1. It shows values of initial central density \( \rho_c \) and initial binding energy \( E_{\text{bind}} \) for each model.
Figure 4.4: Trajectories in the central density–central temperature plane ($\rho_c$–$T_c$) for the 60(1), 95(5), and 128(1) models. The first two explode, while the 128(1) collapses.

For the models that explode it also contains maximum temperature that was reached at the point of reversal of collapse $T_{\text{max}}$ and total nuclear energy release $E_{\text{nuc}} = \int \varepsilon_{\text{nuc}} \, dt$. It is seen that the fate of cores with the same total masses $M_{\text{tot}}$ could be different (collapse or explosion) depending on initial configuration. The trajectories in the $\rho_c$–$T_c$ plane for the cores having the same mass but different initial binding energy are presented in Fig. 4.5. These are trajectories for two 90 $M_\odot$ models, one of them collapses (90(1)), another explodes (90(4)). Orange trajectory shows the test computations performed for the 90(4) model with photodissociation reactions turned off and it illustrates the role of these reactions in behavior of a star. Photodissociation changes the slope of the trajectory and causes contraction to higher densities and temperatures. Circles in Fig. 4.5 mark the values of $\rho_c$ and $T_c$ for each second. It is seen that to the moment when the photodissociation becomes important ($T_c \sim 4 \times 10^9$ K) the model 90(1) contracts faster. Thus outer layers of the core do not have enough time to bounce and expand. Therefore the pressure on the central part could not be reduced. Photodissociation dramatically drops the pressure in the center, and the core collapses. A detailed scenario of the dynamical stage depends on the balance between the rate of neutrino losses, speed of contraction, and rate of nuclear energy release.

The important fact was established that for each core mass, $M_{\text{tot}}$, there is some critical value $E_{\text{bind}}^*$ of initial binding energy that determines the fate of the core: explosion
Table 4.1: Pre-supernova models and parameters of explosion.

<table>
<thead>
<tr>
<th>Model</th>
<th>$M_{\text{tot}}$</th>
<th>$E_{\text{bind}}$</th>
<th>$\rho_c$</th>
<th>$T_{\text{max}}$</th>
<th>$E_{\text{nuc}}$</th>
<th>Fate</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$[M_{\odot}]$</td>
<td>$[10^{51} \text{ erg}]$</td>
<td>$[10^5 \text{ g/cc}]$</td>
<td>$[10^9 \text{ K}]$</td>
<td>$[10^{51} \text{ erg}]$</td>
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<td>0.76</td>
<td>—</td>
<td>—</td>
<td>CC</td>
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</table>

**Note.** $M_{\text{tot}}$ is the mass of the oxygen core, $E_{\text{bind}}$ the initial binding energy, $\rho_c$ the initial central density, $T_{\text{max}}$ the maximum temperature at the center at the moment of reversal of collapse, $E_{\text{nuc}}$ the total nuclear energy release. PISN – the model explodes without any remnant, CC – collapse at the center of the core.
Chapter 4. Results of 1D computations

(PISN) if $E_{\text{bind}} > E_{\text{bind}}^*$ or core-collapse (CC) for $E_{\text{bind}} < E_{\text{bind}}^*$. Two regions could be seen clearly in the $M_{\text{tot}}-E_{\text{bind}}$ diagram (Fig. 4.6). This behavior could be explained by the fact that models with lower $E_{\text{bind}}$ (higher absolute value of $E_{\text{bind}}$) gain higher kinetic energy to the moment of oxygen ignition and proceeds faster to Fe-He transition zone (photodissociation). The critical value $E_{\text{bind}}^*$ tends to zero with a growth of $M_{\text{tot}}$.

Considering that for a stable nonrotating configuration the binding energy should be negative, we can propose the mass limit for the explosion of nonrotating oxygen core at a value of about 110 $M_\odot$. This value is in good agreement with results of the previous works (Ober et al. 1983; Bond et al. 1984). Rotation could push the mass limit of explosion to higher values (Glatzel et al. 1985).

Total nuclear energy release, $E_{\text{nuc}}$, reaches values up to few times $10^{52}$ ergs which is the order of characteristic GRB energy. An interesting correlation has been found for the models that explode: value of total nuclear energy release, $E_{\text{nuc}}$, increases with maximum temperature $T_{\text{max}}$ reached at the center (Fig. 4.7). This correlation is more evident if we take into account results of other computations (Ober et al. 1983; Arnett 1996). The rough estimation of this dependence could be done in a similar way to Ayasli & Joss 1982. For the case of thermonuclear flash in neutron star they have shown that the mass of consumed nuclear fuel is proportional to the square of burning temperature.

\footnote{Collapse of the central part of the core with possible ejection of outer layers.}
Figure 4.6: Fate of a star depending on its mass, $M_{\text{tot}}$, and binding energy, $E_{\text{bind}}$. Explosion is marked by diamonds and collapse is marked by circles.

Figure 4.7: Nuclear energy release as a function of maximum temperature (diamonds). The slope $E \propto T^2$ is shown. For comparison data from Arnett (1996) (stars) and Ober et al. (1983) (triangles) are shown.
The nuclear energy release $E_{\text{nuc}}$ is proportional the rate of the energy release $L_{\text{nuc}}$ and the timescale of explosion $\delta t$ and it is also is proportional the mass of consumed fuel $\Delta M$:

$$E_{\text{nuc}} \propto L_{\text{nuc}} \delta t \propto \Delta M.$$  

(4.1)

The radiative temperature gradient within the star is defined as

$$\frac{dT}{dr} = -\frac{3}{16\pi} \frac{\kappa \rho L}{ac r^2 T^3},$$

where $L$ is the luminosity, $\kappa$ is the opacity. Substituting $dT/dr$ by $-T_{\text{max}}/R$, $r$ by $R$ and $\rho$ by $\Delta M/R^3$ we can obtain:

$$T^4 \sim \frac{3}{16\pi} \frac{\kappa \Delta M L}{ac R^4}.$$  

Taking into account Eq. (4.1) we obtain

$$E_{\text{nuc}} \propto T_{\text{max}}^2 \delta t.$$  

(4.2)

So the nuclear energy release and the temperature are naturally related. This fact can help us to explain on of the important features of GRBs, so called Amati relation (see Sec. 6.1.2).
Chapter 5

Modelling in 2D

5.1 Statement of the problem

Hydrodynamic simulations were performed with our own numerical code based on the Piecewise Parabolic Method on a Local stencil (PPML) (Popov & Ustyugov 2007; Popov 2012). The key PPML-procedure written on FORTRAN could be found in Ustyugov et al. (2009). PPML is an improvement over the popular Piecewise Parabolic Method (PPM) suggested by Colella & Woodward (1984) for compressible flows with strong shocks and extended by Dai & Woodward (1994) for magnetohydrodynamics. PPM was implemented in many modern hydrodynamic codes and is widely used in computational practice now. The PPML method has demonstrated high accuracy on both smooth and discontinuous solutions. It could be more suitable for problems where the requirement of low dissipation is crucial. For example, PPML was successfully implemented for direct turbulence simulation (Kritsuk et al. 2009). In our case low dissipation of PPML is needed to describe shocks and contact discontinuities with higher accuracy.

There are a few recent modelizations of PISNe in 2D (Chen et al. 2011; Joggerst & Whalen 2011). In both cases a modern astrophysical code, CASTRO (Almgren et al. 2010; Zhang et al. 2011, 2013), has been used. This code is based on a high-order radiation-hydrodynamics solver, includes self-gravity, and allows following nucleosynthesis and energy output from nuclear burning during the explosion. Extension of PPML code to full 3D case to study the 3D hydrodynamic effects on the explosion of a star with realistic physics will be made in future. This includes implementation of full equation of state of the stellar matter, self-gravity computations, radiation transfer implementation. In the current research we focused only on the hydrodynamics of explosion applying a new high-quality hydrodynamic PPML-solver. We neglected the energy feed-back from nuclear reactions and gravity changes in the source of explosion trying to obtain the
principal possibility of total disruption of the stellar core to many fragments in the case of very massive progenitor. Additional energy supply from the nucleosynthesis and gravity changes will favor this scenario.

In the current version of PPML-code we used a cylindrical coordinate system where all the variables depend on the vertical coordinate $z$ and the distance $r$ to $z$-axis, i.e. we assume the rotational symmetry. This allowed us to construct a simple model of a supernova explosion and to perform many simulations with different parameters to reveal the basic physics without the requirement of high computational performance of the computer cluster and with less memory requirement for data storage. The sequential description of the PPML-algorithm in cylindrical geometry is presented in Popov (2012). Reflecting boundary conditions have been imposed along the cylindrical axis and the equatorial plane, allowing free outflow across the outer boundaries.

5.2 Initial configuration

With our hydrodynamic SN model we investigated a Pop III star with a 100 $M_{\odot}$ oxygen core assuming rotational symmetry. The oxygen core has the radius $r = 0.29 R_{\odot}$, beyond which a uniform density of $\sim 1$ g/cm$^3$ was set for the uniform atmosphere. We used a polytropic model of a star with index $\gamma = 4/3$ ($n = 3$). The initial configuration was calculated from the condition of hydrostatic equilibrium of a nonrotating ideal gas sphere with a polytropic equation of state (EOS) $P = K\rho^{\gamma}$, where $K = 8.225 \times 10^{15}$, $P$, and $\rho$ are expressed in CGS units. Temperature was computed as $P = \rho RT/\mu$, where $R$ is the universal gas constant and $\mu$ the mean molecular weight, which was assumed to be $\mu = 0.37143$. This value of $\mu$ gives the ideal gas approximation, which is very close to the full EOS in the considered range of density and temperature. Figure 5.1 shows the comparison between the temperature profiles inside the stellar core, computed with full equation of state (Nadyozhin 1974a,b) and with our ideal gas approximation.

We simulated the explosion by depositing the thermal energy in the central region in the two variants. In the first case the energy $E = 5 \times 10^{52}$ ergs was uniformly deposited within the radius $r_e = 0.1 R_{\odot}$, which contains $60 M_{\odot}$ (Fig. 5.2).

In the second variant the energy was inserted by the series of ten ignition bubbles at $t = 0$ sec. All of the bubbles had different energy values and sizes distributed in a stochastic way. The bubbles were randomly put within 0.06 $R_{\odot}$ of the center. The total energy deposited was also $E = 5 \times 10^{52}$ ergs (Fig. 5.3). This nonuniformity could present some inhomogeneities in the core that occur prior to explosion.
Figure 5.1: Comparison between the temperature profiles inside the core, computed with Nadyozhin equation of state (black line), and with ideal gas approximation (red line). Radius is shown in the units of $R_\odot$, temperature – in $T_c = 2.36 \times 10^9$ K.

\[ p = \frac{\rho RT}{\mu} \]
\[ \mu = 0.37143 \]

Figure 5.2: Initial temperature profile for the case of uniform explosion in the units of $T_c = 2.36 \times 10^9$ K.
Nuclear burning in the center of a star could cause the development large-scale convection (Arnett & Meakin [2011]). If convection occurs prior to the moment of pair instability, the contraction and explosion could be nonsymmetrical. Inhomogeneities in temperature and density could cause ignition spots to occur in the core.

5.3 Results of computations in 2D

The results of computing of the uniform explosion are presented in Fig. 5.4. It shows density on a logarithmic scale and temperature for the moment $t = 28$ sec. The solid line represents the initial position of the boundary of the core. The shock, produced by the explosion, is split on two fronts that are propagating through the rarefied matter and heating it. In the central part of the core, there is a region with a Rayleigh-Taylor instability. The radius at which this instability occurs is very close to the value obtained in Chen et al. (2011). The development of the instability requires more precise computations with a more physical configuration of the envelope, but we suppose that this instability could result in the formation of large-scale nonuniform structures on a large time scale.

To cure the pathological “carbuncle” behavior that could appear in a region where the shock is almost but not exactly parallel to the grid edges, the artificial dissipation was
Figure 5.4: SN model with central ignition for $t = 28$ sec. Logarithm of density (a) is shown in units of $\rho_c = 2.65 \times 10^5$ g/cm$^3$. Temperature (b) is shown in units of $T_c = 2.36 \times 10^9$ K.
Section 5. Modelling in 2D

Figure 5.5: SN model with multicore ignition for $t = 28$ sec. Logarithm of density (a) is shown in units of $\rho_c = 2.65 \times 10^5$ g/cm$^3$. Temperature (b) is shown in units of $T_c = 2.36 \times 10^9$ K.

Inserted according to Loh & Jorgenson (2009). This is the reason for the instability development in Fig. 5.4 being smoothed near the z-axis.

Figure 5.5 presents the density and temperature for multicore ignition. In this scenario the front of the shockwave is not as smooth as in the previous case. Many fragments of hot matter appear behind the shockwave. This could lead to disrupting of the star in many fragments. As a result the light curves of such supernova could be more complex than in a spherically symmetric case.
Chapter 6

Gamma-Ray Bursts

6.1 Observations

The Gamma Ray Bursts (GRBs) are sudden, intense flashes of gamma radiation characterized by very fast variability with time scale as short as milliseconds. These intense flashes of gamma radiation are at cosmological distances and as consequence their energy could be as high as $10^{54}$ ergs. It is equivalent that a star like our sun is converted into pure energy within a few seconds: GRBs are the most powerful explosions in the Universe.

Although a lot of progress has been done last few years, the origin of these explosions is still the great mystery in physics. This topic has focused the attention of the most powerful earth and space observatories. Any study of GRBs requires consideration of the factors that produce such exceptional events.

6.1.1 Light-curve

Gamma ray bursts usually are divided in two classes by their duration: short and long (Kouveliotou et al. 1993). The duration of a burst is defined by $T_{90}$, the time in which 90% of the total fluence is emitted. The burst with $T_{90} \leq 1$ s are identified as short and the bursts with $T_{90} \geq 1$ s as long. The duration distribution ($T_{90}$) of 2041 GRBs detected by BATSE is shown in Fig. 6.1 where these two classes can be seen. Long bursts are in general softer in their spectra than short bursts (Dezalay et al. 1992, Kouveliotou et al. 1993).

Temporal profiles of the prompt emission of GRBs can have very complex behavior, no two GRBs look the same. Some examples of light curves of GRBs are presented
Figure 6.1: Distribution of duration ($T_{90}$) of 2041 GRBs detected by BATSE.

It is seen that there is no any common template, which makes it very difficult to build a classification of GRBs basing on the morphology of light curves.

But there are features common to many X-rays afterglows. Swift mission has discovered the transition from the prompt emission dominated phase to the afterglow (Fig. 6.3): after the prompt phase the afterglow decays very steeply before flattening out to what is called the plateau phase. Then the light curve breaks to steeper decay. Striking features superposed on the steadily fading light, called X-ray flares, can be seen. X-ray flares can be very intense - once a giant flare containing almost as much energy as in the GRB itself was observed (Falcone et al. 2006).

The X-ray afterglow is followed by emission in optical and radio wavelengths lasting for days or even months. But about a half of GRBs with detected X-ray afterglow are “dark bursts”: they don’t have an optical afterglow. This could be due to the fact that these bursts are faint or the emission could be absorbed by dust or matter in vicinity of the source.

In some bursts (not in all) the power-law decay of the afterglow exhibits a steepening in all energy bands – achromatic break. This break of light curve was interpreted as a “jet break” predicted by Rhoads (1999). If an emitting region moves relativistically with the Lorentz factor $\Gamma$ than the emission is beamed within a half-angle of $1/\Gamma$. If the GRB has geometrical structure of the jet with half-angle $\theta > 1/\Gamma$, than an observer would first detect emission from a region beamed within $1/\Gamma$ angle. As the source decelerates and $\Gamma$ becomes order of $\sim 1/\theta$ the observed flux starts to decay steeper. In that case the break in light curve should be achromatic and occur in all wavelengths. But there are examples of chromatic breaks when steepening in the X-ray afterglow is not accompanied by a break in the optical emission (Panaitescu et al. 2006). The explanation of this behavior
was the suggestion that the optical emission is produced by a different mechanism than the X-ray emission.

6.1.2 Spectrum

Contrary to light curves of the prompt emission, spectra of GRBs are quite similar. Spectra of most GRBs can be fitted successfully by the empirical function (the “GRB function”; Band et al. (1993)). This function consists of two smoothly joined power laws:

\[
N_E(E) = \begin{cases} 
    CE^\alpha e^{-E/E_0}, & (\alpha - \beta)E_0 \geq E, \\
    C'e^{\beta}, & (\alpha - \beta)E_0 < E,
\end{cases}
\]

(6.1)

where \(N_E(E)\) is the photon number spectrum, \(E\) is the photon energy in keV, \(E_0\) is the e-folding energy, \(\alpha\) and \(\beta\) are the asymptotic power-law indices, \(C\) is the amplitude in photons s\(^{-1}\) cm\(^{-2}\) keV\(^{-1}\) and \(C'\) is chosen to make \(N_E(E)\) a continuous and continuously
differentiable function:

\[ C' = C \left( (\alpha - \beta)E_0 \right)^{\alpha - \beta} e^{-(\alpha - \beta)} \].

Spectral hardness is usually measured by the peak energy \( E_{\text{peak}} \) of the \( E\,E\,E \) spectrum \( (F_{\text{E}} \equiv E\,N_{\text{E}}) \). This energy is related to \( E_0 \) as \( E_{\text{peak}} = (2 + \alpha)E_0 \). For bright long GRBs typical values of the low-energy spectral index \( \alpha \) is \( \sim -1 \) and for the high energy index \( \beta \) is \( \sim -2 - 3 \). Although the GRB function was proposed to fit time-integrated spectra, it adequately fits time-resolved GRB spectra as well. Despite tremendous energy release there are not so many bursts with MeV and GeV photons. Analysis of the complete BATSE catalog (more than 2000 of burst!) by Kaneko et al. (2006) demonstrated that \( E_{\text{peak}} \) of the most burst is quite narrowly distributed around few hundreds of keVs.

While in general the spectrum of GRBs is non thermal, there are evidences of the presence of thermal radiation in some bursts (Ryde 2004). GRB 060218 was detected by Swift at \( z=0.033 \). This burst was unusually long. The UVOT telescope found an emission peaking in a broad plateau first at ultraviolet wavelengths and later in the optical. A few days after the Swift observations pointed out the presence of rising SN 2006aj. The most striking feature exhibited by this gamma event is the presence of a thermal component observed in the XRT data up to 10 ks and in the UVOT up to about 100 ks (Campana et al. 2006). This black body component shows a decreasing temperature accompanied by an increasing luminosity which implies an increase emission radius from an initial \( 5 \times 10^{11} \) to \( 3 \times 10^{14} \) cm in about 100 ks. This corresponds to expansion velocity of the
order of 30,000 km/s.

Another striking feature of GRBs is the correlations between various pairs of parameters of emission, for example the power-law correlation between the intrinsic peak energy in cosmological rest frame $E_{\text{peak},i}$ (which is connected with the observed peak by the relation $E_{\text{peak},i} = (1 + z)E_{\text{peak,obs}}$, $z$ is the redshift) and the isotropic equivalent energy of emission $E_{\text{iso}}$ (Amati et al. 2002):

$$E_{\text{peak},i} = K \times E_{\text{iso}}^m,$$

where $K$ is constant and $m \sim 0.5$. Some researches (Nakar & Piran 2005; Band & Preece 2005) questioned the validity of this relation by analyzing BATSE GRBs without known redshift. These studies concluded that the whole GRB population can not satisfy the correlation for any values of redshift and selection effects are introduced in the analysis of GRB spectra and determination of the redshift $z$. Probably the redshift is only measured for those bursts that follow this correlation. However these conclusions were contested by numerous works (Ghirlanda et al. 2005; Bosnjak et al. 2006; Pizzichini et al. 2006) that showed that accurate accounting of the observed dispersion of the correlation and the uncertainties in the observed $E_{\text{peak}}$ and fluence values significantly reduce the number of possible outliers.

First reported for the sample of 12 BeppoSAX bursts $E_{\text{peak},i}$–$E_{\text{iso}}$ correlation remains valid for most long GRBs (up to now there is only one outlying burst GRB 980425) observed with modern missions such as Swift and Fermi (Amati 2008; Amati et al. 2009) (Fig. 6.4).

### 6.1.3 GRB–SN connection

A connection between long duration GRBs and supernovae has been established long ago. The association was made observationally between GRB 980425 and a bright and energetic supernova. Initially it was considered as a coincidence, but in the spectra of subsequent long GRBs: GRB 030329, GRB 031203, GRB 060218, GRB 060218, further supernova signatures were apparent (e.g. Galama et al. 1998; Hjorth et al. 2003; Price et al. 2003; Stanek et al. 2003; Uemura et al. 2003; Cobb et al. 2004). These supernovae were classified as core-collapse supernovae, type Ib/c. This provided the proof that the origin of at least some long GRBs is in the core collapse of very massive stars. While all long GRBs could be in principle accompanied to supernovae, the reverse is not true. It is estimated that the core collapse supernovae are 1000–10000 times more frequent than GRBs.
Figure 6.4: Location of bursts in $E_{\text{peak},i}$–$E_{\text{iso}}$ plane for different instruments: BeppoSAX, Swift/BAT and Fermi/GBM from Amati et al. (2009).
No supernovae have been so far associated with short-duration GRBs. GRB 060614 had duration of 102 seconds, which classifies it as a long-duration GRB, but its temporal lag and peak luminosity fall entirely within the short-duration GRB subclass. It does not fit into any class. This combination of a long-duration event without an accompanying supernova poses a challenge to both the collapsar and merging neutron stars interpretations and opens the door to a new classification scheme that straddles both long and short-duration bursts.

### 6.1.4 Host environment

Long GRBs tend to lie in small faint blue galaxies. Afterglow spectroscopy has shown that these galaxies have a lower ratio of metals to hydrogen than does our local neighborhood in the Milky-Way, has possibly reaching as metal-poor content as one 1/100 of the metallicity of the Sun [Starling et al. 2005]. Thone et al. (2008) deduced from the VLT observations at high resolution with FORS2 that the site of the GRB 060505 is considerably different from the rest of the galaxy. It has intense star formation, low metallicity and young age. Host galaxies of long-duration GRBs have been found often to be faint irregular galaxies [Fruchter et al. 2006 Christensen et al. 2004].

Modjaz et al. (2008) have compared metallicities at the site of nearby type Ic supernovae which had no GRBs with the chemical abundances of nearby galaxies at the site of GRBs where SNe Ic were seen. They found that the progenitors of GRBs have systematically lower metallicity (Fig. 6.5).

A major related issue which is under discussion is if GRBs are unbiased tracers of star formation. Kistler et al. (2008) found an evidence of enhanced evolution in the GRB rate with \( \sim 4 \) times as many GRBs observed at \( z \sim 4 \) than expected from star formation measurements. It is consistent with theoretical expectations from metallicity effects. The fact that such measurement contradicts that GRB rate traces the star formation history is another prove to the fact that the GRB-SN association is different from the one it is usually expected. GRBs are the result of formation of very massive stars and therefore the star formation rate should be different. An increase of GRB rate observed at high \( z \) is directly related to the prediction that we expect more GRBs at high \( z \).

Thank to these many observations, we have now a more clear picture of this enigmatic phenomenon: it is presumably the explosive end of a very massive star in a very specific environment. The theoretical interpretation of GRBs should incorporate all these elements of the puzzle in a coherent way.
6.2 Models of GRBs

6.2.1 Compactness problem

Non-thermal spectrum of GRBs and their cosmological origin raise the question of mechanism of emission. Typical fluence $F$ observed in GRBs is the order of $10^{-7}$ ergs/cm$^2$. Being at cosmological distance, $D$, the isotropic source emits the total isotropic energy $E_{\text{iso}}$:

$$E_{\text{iso}} = 10^{50} \left( \frac{D}{3000 \text{ Mpc}} \right)^2 \left( \frac{F}{10^{-7} \text{ ergs/cm}^2} \right) \text{ ergs.} \quad (6.2)$$

The observed spectrum of GRBs is non-thermal and contains a large fraction of photons with energies higher than pair-production threshold. If we assume that typical size of the source is $R \sim c\delta t \approx 10^{10}$ cm, where $\delta t$ is the timescale of variability order of few milliseconds, in that case the source should be very compact. We can estimate the
optical depth, $\tau_{\gamma\gamma}$ as (Piran 2005):

$$\tau_{\gamma\gamma} \sim R\sigma_T n_{e^+e^-},$$

where $\sigma_T$ is the Thomson cross section and $n_{e^+e^-}$ is the number density of electrons and positrons. It can be estimated as

$$n_{e^+e^-} \sim \frac{E_{\text{iso}}}{m_e c^2} \frac{1}{R^3}$$

where, $f_{e^+e^-}$ is a portion of photons with energies sufficient for pair-production, $m_e c^2$ is a typical photon energy. Than we can obtain an expression for $\tau_{\gamma\gamma}$:

$$\tau_{\gamma\gamma} \sim f_{e^+e^-} \frac{\sigma_T E_{\text{iso}}}{m_e c^2 R^2}.$$  \hspace{1cm} (6.3)

For typical parameters of GRBs the resulting depth could be very high, order of $10^{13}$. High energy photons will produce large number of $e^\pm$-pairs which will Compton scatter with low energy photons. The resulting optical depth becomes very high for all photons, which is incompatible with observed non thermal spectrum, that requires optically tho source. So there is a difficulty to explain such energetic emission from a compact source, which is called “compactness problem”.

In order to solve this paradox, it was suggested that the emitting object expands ultra-relativistically toward an observer (Fenimore et al. 1993). Let’s consider the relativistic motion with a bulk Lorentz factor $\Gamma$. The observed photons have been blue-shifted by a factor of $\Gamma$ so the energy of the photons in the rest frame is $h\nu = h\nu_{\text{obs}}/\Gamma$. It reduces the number of photons with energies above the pair-production limit at the source by a factor of $\Gamma^{-2|\beta|+2}$, where $\beta$ is high-energy photon index of the observed spectrum (Lithwick & Sari 2001). Also due to the relativistic contraction effect the size of the emitting source can now be estimated as $R \sim \Gamma^2 c \delta t$. So the total optical depth is proportional to a factor of $\Gamma^{-2|\beta|+2}$. For a typical GRB spectral index $\beta \sim 2$ it implies that the Lorentz factor $\Gamma$ should be higher than $\sim 100$ for the source to be optically thin. Such extreme relativistic motion has never been observed in any other celestial objects.

The idea of the ultra-relativistic motion is the corner stone of the most of physical models of GRBs but it is not enough to explain the creation of the $\gamma$-ray emission. The way to transform the kinetic energy to the electromagnetic energy of emission should be found. Several physical models were proposed to explain two major factors of GRBs: the engine that provides energy (progenitor) and the mechanism of the prompt and afterglow emission.
6.2.2 Fireball

The original spherically symmetric Fireball model was suggested in the eighties by Paczynski (1986) and Goodman (1986). They have shown that sudden release of high energy photons in the compact source creates $e^\pm$-pairs. This mixture of the pairs and photons concentrated within a small region and having initial energy much higher than its rest mass was called a “fireball”. Paczynski (1986) and Goodman (1986) considered only photon-lepton fireballs in which there are no baryons. They demonstrated that a fireball can self-accelerate and expand relativistically.

If the initial compact fireball has high enough temperature, pairs are created by energetic photons, which make the plasma opaque. Radiation is trapped and fireball expands under its own radiation pressure. It expands until temperature drops down below the pair-production limit. Exciting pairs annihilate and the plasma becomes transparent, photons can escape freely towards an observer.

Admixture of baryons changes the behavior of fireball. As was shown by Paczynski (1990) and Shemi & Piran (1990), baryons are accelerated and transform initial energy of the fireball into kinetic energy. Depending on initial mass of the baryon loading the baryons can be accelerated relativistically or sub-relativistically. In order to keep ultra relativistic flow the baryonic mass should be low enough (so called “baryonic contamination” problem, Piran (1999)). Also electrons associated with the baryons change the optical depth of the fireball increasing radius of transparency and decreasing characteristic energy of emission.

Since the time when the fireball model was proposed it actively evolved (e.g. Meszaros & Rees 1993, 1997; Panaitescu & Meszaros 1998). The most commonly accepted model is the fireball-shock model in which the fireballs are strongly collimated and the internal and external shocks are incorporated to explain the prompt emission and the afterglow. According to this model the relativistically expanding fireball interacts with interstellar media (ISM) and creates so-called external shocks (Rees & Meszaros 1992, Meszaros & Rees 1993). The kinetic energy of the bulk motion of the fireball is transferred to the electrons of ISM. It is assumed that the electrons of ISM are accelerated by the Fermi mechanism at external shocks which creates a non-thermal power-law distribution. These electrons subsequently emit synchrotron radiation which is observed as afterglow. It is important to say that the afterglow was predicted by Meszaros & Rees (1997) before the discovery of BeppoSAX.

In order to explain fine structure of the prompt GRB emission another type of shocks was introduced – internal shocks (Rees & Meszaros 1994). In this model the fireball is represented by a series of shells which are ejected with different Lorentz factors $\Gamma$. The
shells collide and the shocks occur inside the fireball. In a similar way as in external shocks the bulk kinetic energy is transformed to the internal energy of the electrons, and after fireball becomes transparent it is emitted in \( \gamma \)-rays.

\[
\begin{array}{c}
\text{GRB} \\
\text{Prompt emission} \\
\gamma\text{-rays} \\
\text{Internal shocks}
\end{array}
\quad
\begin{array}{c}
\text{Afterglow} \\
\text{X-rays} \\
\text{optical} \\
\text{radio} \\
\text{ISM} \\
\text{External shocks}
\end{array}
\]

**Figure 6.6:** Fireball model.

To conclude the fireball-shock model explains the prompt emission by internal shocks and the low-energy long duration afterglow by external shocks (Fig. 6.6).

The fireball-shock model does not describe the engine that creates the fireball and what is the progenitor. But the progenitor should satisfy certain requirements: it should be compact and be able to produce strongly collimated ultra-relativistic outflows.

Despite the fact that the fireball models is the most popular one and it was considered as a standard, it faces some serious difficulties in explanation of observational facts. First of all as mentioned above Lorentz factors \( \Gamma \) of \( \sim 100 \) have never been observed in any astrophysical objects. Lorentz factors of jets in active galactic nuclei are \( \lesssim 10 \). Also internal shocks are very inefficient mechanism of conversion of the kinetic energy of electrons into electromagnetic emission \( \text{(Kobayashi et al. 1997)} \). Another difficulties arise in production of a Band-like spectrum by synchrotron emission. The spectral slope produced by synchrotron emission cannot exceed value \(-2/3\), while there is significant fraction of bursts with low-energy slope higher than this limit \( \text{(Preece et al. 1998)} \). Recent observations with the IceCube experiment \( \text{(Abbasi et al. 2012)} \) have shown an absence of energetic neutrinos associated with GRBs, that were predicted within the fireball model \( \text{(Waxman & Bahcall 1997)} \). These and other unresolved questions make it necessary to look for alternative approaches and models in order to successfully describe GRBs.
6.2.3 Cannonball

In the cannonball model the engine of GRBs is usual core-collapse supernova explosion (Shaviv & Dar 1995; Dado et al. 2009). It is assumed that after formation of a compact object (neutron star or black hole) an accretion disk is produced either by stellar matter that was left in vicinity of the compact object or by a fallback of stellar matter that was ejected during explosion. When accreting material falls onto the compact object bipolar jets of highly relativistic plasmoids (cannonballs) made of ordinary atomic matter are produced with high bulk Lorentz factor $\Gamma$. The electrons of cannonballs up-scatter photons scattered and/or emitted by the pre-supernova wind blown from the progenitor star to typical GRB energies. Owing to ultra-relativistic motion up-scattered photons are collimated into a narrow beam along the directions of motion of cannonballs. Each spike of GRB emission corresponds to individual cannonball. Diversity of GRB light curves represents chaotic process of accretion.

The beamed radiation of the cannonballs ionizes the wind/ejecta emitted by pre-supernova. The electrons of the ionized gas are swept into the cannonballs and are Fermi accelerated by turbulent magnetic field, emitting synchrotron radiation that dominates the broad-band afterglow emission.

The cannonball models requires Lorentz factor $\Gamma$ to be even higher than in the fireball model: $\Gamma > 1000$.

6.2.4 Fireshell

In the fireshell scenario GRB is produced at the moment of formation of electromagnetic black hole (Ruffini et al. 2001a,b). During the formation phase of a black hole so called dyadosphere (or dyadotorus in case of a rotating black hole) occurs — a region with electric field stronger than the critical field for $e^\pm$-pair production. Vacuum polarization is reached and $e^\pm$-plasma in thermal equilibrium is created (Ruffini et al. 2010). This plasma self-accelerates owing to its internal pressure and expands relativistically as an optically thick and spherically symmetric “fireshell” having a constant width in the laboratory frame, the frame in which the black hole is at rest. Baryonic matter which left after the process of collapse is trapped by the $e^\pm$-plasma and baryons are thermalized with electrons and positrons. The plasma is characterized by two parameters: the total energy of $e^\pm$-plasma, $E_{\pm}^{\text{tot}}$, and baryon loading — ration of baryons’ rest mass, $M_Bc^2$, to the total energy of the plasma $B = M_Bc^2/E_{\pm}^{\text{tot}}$. In order to expand relativistically plasma should have value baryon loading $B < 10^{-2}$ (Ruffini et al. 2000).
When relativistically expanding plasma becomes transparent it emits a burst of gamma radiation which in the fireshell scenario is called proper GRB (P-GRB). The energy emitted as P-GRB is a fraction of the total energy and it is uniquely determined by the baryon loading. The P-GRB emission is expected to have a significant fraction of thermal radiation \cite{Penacchioni et al. 2012}. Decelerating plasma then interacts with circumburst medium (CBM) giving rise to multi-wavelength emission: the so called “extended afterglow”. While in the fireball model temporal structure of GRB is explained by internal shocks, in the fireshell model variability of GRB come from the inhomogeneities in CBM, thus density distribution of CBM, \( n_{\text{CBM}} \), is needed in order to fully describe GRB. It is important to underline that in the fireshell model emission of the P-GRB does not always coincide with what is called prompt emission in the fireball scenario. Within the fireshell model, this prompt emission corresponds to the gamma-ray emission, which addresses not only the P-GRB, but also the peak of the extended afterglow. The relative energetics and observed time separation of the P-GRB and the extended afterglow are functions of the parameters of the plasma \( E_{\text{tot}}^\pm, B \), and density of CBM, \( n_{\text{CBM}} \).

### 6.3 GRBs: the missing link of stellar evolution

\cite{Chardonnet, Chechetkin, & Titarchuk 2010} proposed a new paradigm of gamma-ray bursts phenomenon as a missing link of stellar evolution. The first element which characterizes a GRB as an explosive event is huge amount of hard radiation released in a very short time. Paradoxically, such event exists inside the stellar evolution: it is the total disruption of a very massive star ongoing pair instability. The mass is a key parameter that determines the fate of a star. The authors assumed that gamma-ray burst, as a star, follows also this principle and the main element of this concept is related to the mass of the progenitor. In the next chapter results of our PISN simulations will be applied within this model.
Chapter 7

Gamma-Ray Burst as Pair-Instability Supernova

7.1 PISN as a possible candidate for GRBs

In order to successfully explain the GRB phenomena any physical model should explain its exceptional features of emission and take into account specific environment of GRBs. In this chapter GRBs will be analyzed within PISN model (Chardonnet et al. 2010). There are some considerations of PISN as a possible source of GRB (Yoon et al. 2012), but the authors considered collapsing pair-unstable stars. Instead Chardonnet et al. (2010) were focused on exploding pair-unstable stars suggesting that total disruption of a very massive star can produce an energetic flashes of $\gamma$-rays, that can be observed as GRBs.

Very massive stars are believed to be formed in a low-metallicity environment (Abel et al. 2000; Bromm & Larson 2004), which is in direct agreement with properties of GRBs’ hosts. Since there are GRBs detected at low redshifts, a question may arise that is it possible to form a very massive star at low redshift? Tornatore et al. (2007) performed numerical simulations of metal mixing as a function of the redshift, pointing out that even at a redshift of 3, the uniform metal mixing has not yet been completed. The free-metal pocket may still exist and PISN explosion can be observed in a nearby Universe.

Since pair instability occurs only in very massive stars, this type of supernovae is very energetic. One dimensional simulations have shown that nuclear energy release is comparable with characteristic isotropic equivalent energy of emission $E_{\text{iso}}$ of GRBs (see Table 4.1). It is seen that the maximum nuclear energy release is $4.5 \times 10^{52}$ ergs for
the model with the mass of 100 $M_\odot$. In simulations performed by Arnett (1996) maximum energy release was even higher: $\sim 8 \times 10^{52}$ ergs. More massive cores that could potentially produce more energetic explosion collapsed. But spherically symmetric simulations can not take into account effects of rotation. According to the works by Glatzel et al. (1985) and Yoon et al. (2012) rotating cores explode while non-rotating cores with the same mass collapse. Therefore rotation extends the range of masses in which explosion is possible, increasing amount of available thermonuclear fuel and potentially increasing total nuclear energy release. Another impact of rotation is that explosion is not spherically symmetric and emission is not isotropic. It reduces constrains on the total energy budget by a factor of $\Omega/4\pi$, where $\Omega$ is the collimation angle. It is seen that in order to reproduce isotropic equivalent energy of $\sim 10^{54}$ ergs observed in some GRBs with total energy release of $10^{53}$ ergs collimation should not be very strong, $\Omega \sim 1$. It is also worth noting that energy is released during quite short period of time about few tens of seconds. For example it takes $\sim 20$ sec for the 90 $M_\odot$ model to increase its radius from the minimum to initial value (Chapter 4, Fig. 4.1). So the timescale of explosive process is in agreement with typical GRB duration.

7.2 On compactness problem

As mentioned above the compactness problem rises from non-thermal spectrum and short time variability of emission of GRBs (Sec. 6.2.1). Estimations of optical depth result in expression (Eq. (6.3)):

$$\tau_{\gamma\gamma} \sim f_{e^+e^-} \frac{\sigma_T E_{\text{iso}}}{m_e c^2 R^2},$$

where $E_{\text{iso}}$ is the total isotropic equivalent energy of GRB, $R$ is the typical size of the source, $f_{e^+e^-}$ is a portion of photons with energies sufficient for pair-production, $m_e c^2$ is a typical photon energy. To reduce optical depth ultra-relativistic motion of emitting plasma is usually introduced. However there is another way to decrease $\tau_{\gamma\gamma}$ — to increase $R$ and distribute emission over a more extended area. Taking into account geometrical structure of the emitting region it is still possible to keep fast time variability. Assuming that emission is produced by an expanding shockwave in exploding star let’s put the typical width to be $L \sim 10^{10}$ cm and typical radius is of the order of $R \sim 10^{15}$ cm. Then the timescale of variability will be $\delta t \sim L/c \sim 0.1$–1 sec and one can obtain optical depth $\tau_{\gamma\gamma}$ to be order of 1:

$$\tau_{\gamma\gamma} \sim f_{e^+e^-} \frac{\sigma_T E_{\text{iso}}}{m_e c^2 4\pi R^2} \sim 1.$$
It is important also to underline that not every burst show millisecond variability, so the constrains on the size could be less strong. Moreover modern observational data, resolving spectra with higher accuracy, indicates that thermal emission could play significant role. In some cases fraction of energy emitted in black-body component is higher than 50\% \cite{Ryde2004}. Thus there is a possibility for a source to be optically thick.

### 7.3 Peak energy of emission

The key question of explosion of very massive star is the presence of extended hydrogen envelope. If a stars preserves its envelope up to the moment of explosion the from the central parts of a star will be scattered by the envelope making the light curve to be long lasting plateau-type \cite{Dessart2013}. But possible absence of the envelope can drastically change the behavior of light curve and spectrum. In that case there is a possibility for an observer to detect the emission directly from hot material of the stellar interiors.

Since a very massive star can explode without any remnant we expect hot material of the core to be expelled outside during explosion. In that case typical energy of emission should correspond to the temperature of the matter. As it can be seen from 1D simulations typical maximum temperatures $T_{\text{max}}$ reached in the center of exploding core is $\sim 3-5 \times 10^9$ K or in the units of keV $\sim 300-500$ keV, which corresponds precisely to the typical value of the spectral peak energy $E_{\text{peak}}$ of most of the bursts. Of course a real peak energy is not equal to $T_{\text{max}}$ since there is a distribution of the temperature in a star and peak energy depend on mechanism of emission (for black body radiation for example $E_{\text{peak}} \approx 3kT$), but we expect that they should be related and the 1D computations show order of magnitude correspondence. Narrow distribution of peak energy of emission of GRBs around 300 keV is naturally explained in the PISN model. This is a direct consequence of the temperature of nuclear burning of the matter.

### 7.4 Possible interpretation of the Amati relation

If we assume that spectral peak energy, $E_{\text{peak}}$, is related to the maximum temperature of the matter in the core, it is also reasonable to suppose that the total energy of emission is proportional to the total nuclear energy release, $E_{\text{nuc}}$. Certainly there should be a physical process that transforms nuclear energy to the energy of gamma rays. But we expect the efficiency of such transformation to be high enough, because there are no intermediate processes of acceleration and deceleration of particles and subsequent
emission in this model. The electromagnetic emission is produced directly by the heated material of the stellar interiors during complete disruption of a star.

Taking into account the association between $E_{\text{peak}} - T_{\text{max}}$ and between $E_{\text{iso}} - E_{\text{nuc}}$ it is possible to check if PISN explosion satisfies in the first approximation the empirical $E_{\text{peak}} - E_{\text{iso}}$ correlation (Amati relation, Sec. 6.1.2). Figure 7.1 shows the values obtained in 1D computations in comparison with the GRB data taken from Amati et al. (2008). The computed points perfectly fit inside the area of observational data. The staggering fact is that no fitting of models of stars or parameters of computations were done in order to obtain these points. To support this fact points obtained by earlier computations are also presented (Ober et al. 1983; Arnett 1996). Authors of these works did not apply their computations of PISN explosion to GRBs, so there are no bias in their computations.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.1.png}
\caption{Comparison of the maximum temperature, $T_{\text{max}}$, and total nuclear energy release, $E_{\text{nuc}}$, with the peak energy, $E_{\text{peak}}$, and isotropic equivalent energy, $E_{\text{iso}}$, of GRBs. GRB data are shown with gray dots (Amati et al. 2008). Results of earlier computations are shown with blue triangles (Ober et al. 1983; Arnett 1996).}
\end{figure}

In our interpretation the source of this correlation is in the origin of energy production mechanism. Since the main energy reservoir in PISN model is thermonuclear burning it is natural that emission of GRBs obey the same type of correlation as nuclear energy and temperature within a star (Eq. (4.2)): \[ E_{\text{nuc}} \propto T_{\text{max}}^2 \delta t. \]
The total nuclear energy release during PISN explosion is related to the amount of nuclear fuel available, which is in turn defined by the mass of a progenitor. So the key parameter of GRB is the mass of a star exploding as PISN. Then scattering of points in $E_{\text{peak}} - E_{\text{iso}}$ within this interpretation has not statistical nature but rather physical one. It could be related to the diversity of stellar configurations having different rate of rotation, chemical composition, etc.

### 7.5 Time-variability

In order to explain time variability of the prompt emission of GRBs we need to consider a multi-dimensional picture of explosion. Two scenario of explosion were considered in 2D (Chapter 5): uniform and nonuniform ignition scenarios. As was shown by 2D simulations in each of scenarios a shockwave propagating outwards undergo development of the Rayleigh-Taylor instability. In non-uniform explosion structure of exploding star is even more complicated: it has large-scale structure which leads to non-spherical explosion of a star, which could be a possible explanation of time variability of emission.

![Figure 7.2: Hot spots randomly appear on the surface of exploding star. Each spot produces spike of emission. Signals from these spots arrive at different times, so an observer sees the superposition of the spikes.](image)

Assuming that star exploded without any remnant and was disrupted in fragments we can propose the following scenario of explosion and emission. Shockwave propagates outward having non spherical shape. Regions of growing instabilities breaks out of the star earlier than the rest of the shockwave front. Reaching surface these fragments of hot matter produce hot spots on the surface of exploding star. These spots start to expand and become transparent. Each of these spots will produce a spike of emission.
Due to the different time of appearance on the surface and different distance to observer these spikes will drive at different time. In our interpretation GRB is superposition of these individual spikes (Fig. 7.2). Since growth of instabilities and fragmentation of a star is a stochastic process, therefore number of spots, their position on the surface and sizes are randomly distributed, which can be an explanation of the diversity of GRB light curves.

To illustrate this idea a simple computer code was developed. It stochastically distributes spots on a star of a given radius. It is assumed that distribution of spots on the surface is homogeneous. Sizes of spots are assumed to have lognormal distribution. Subsequently each spot with index $i$ produces a spike of radiation, which has Gaussian time profile $^{1}$

$$I_i = I_{0,i} e^{-(t-t_{0,i})^2}$$

where $I_i$ is the intensity, $I_{0,i}$ is normalization factor, which is assumed to be proportional to the size of a spot $i$, i.e. also has lognormal distribution, $t_{0,i}$ is the time delay of arrival of a signal to observer, which depends on position of a spot on the surface. And the

---

1 The time profile can be any, for example a fast rise and an exponential decay. For the purpose of simple illustration of an idea it is not so important.
observed light curve is just a superposition of contribution from each spot:

\[ I = \sum_i I_i. \]

Results of few runs of this code with different realizations of spots distribution are presented on Fig. 7.3. It is seen that variety of light curves is very high and corresponds to observed one for GRBs (Fig. 6.2).

### 7.6 Spectrum

Non-thermal spectrum of GRBs is one of the most striking features of this phenomenon. In order to explain it different models were proposed assuming ultra-relativistic motion of the emitting plasma. However recent studies show presence of significant thermal component in the spectra of GRBs [Ryde et al. 2010]. In order to explain non-thermal spectrum typically described by Band function (Eqs. 6.1) we propose an idea that different physical processes are responsible for different part of the spectrum. Regions with lower density radiate thermal bremsstrahlung spectrum owing to interaction of ionized electrons with protons and ions. Thermal bremsstrahlung is responsible for formation of low-energy part of GRB spectrum. Central part of GRB spectrum in our model is dominated by thermal emission from the regions with higher density. This idea was proposed only recently and requires a lot of work to be done in order to construct full physical description of spectrum formation. Here I just present preliminary result on possible fitting of GRB spectra and will focus only on low-energy part. The high-energy part could be formed by inverse-Compton scattering of soft photons on electrons in hotter regions [Titarchuk et al. 2012].

Thermal bremsstrahlung as a possible process responsible for spectra of GRBs was proposed soon after their discovery [Anzer & Boerner 1976]. The spectrum of the radiated energy can be described as (Greene 1959):

\[ I_E = \frac{n_e n_i}{3} \frac{32}{3} \left( \frac{\pi}{3} \right)^{1/2} \frac{e^4}{m_e^2 c^4} \left( \frac{\text{Ry}}{kT} \right)^{1/2} e^{-E/kT} \bar{g}, \]  

(7.1)

where \( I_E \) is the power emitted per unit volume per unit of time per unit energy, \( n_e \) and \( n_i \) are the number densities of electrons and protons, \( E \) is the photon energy, \( T \) is the electron temperature, \( \text{Ry} \) is the Rydberg energy constant, \( \bar{g} \) is the Gaunt factor. For the task of spectrum fitting it is convenient to express \( I_E \) in physical units:

\[ I_E = 2 \times 10^{-20} \frac{n_e n_i}{\sqrt{T}} e^{-E/kT} \bar{g} \frac{\text{erg}}{s \cdot \text{cm}^3 \cdot \text{keV}}, \]  

(7.2)
Then the photon number spectrum measured by the observer is:

\[ N^B_E = \frac{I_{E V}}{4\pi E D^2}, \tag{7.3} \]

where \( V \) is the volume of the emitting region, \( D \) is the distance towards the observer. Therefore the photon number density is proportional to \( E^{-1} e^{-E/kT} \):

\[ N^B_E = A^B \frac{1}{E} e^{-E/kT}, \tag{7.4} \]

where \( A^B \) is a normalization factor that can be computed from Eqs. (7.2, 7.3).\(^2\) Thermal bremsstrahlung naturally explains \( N_E \propto E^{-1} \) low-energy slope observed in many bursts (Preece et al. 1998).

The spectrum of thermal emission is described by a Planck function (blackbody emission):

\[ N^{BB}_E = A^{BB} \frac{E^2}{e^{E/kT} - 1}, \tag{7.5} \]

where \( A^{BB} \) is a normalization factor which depends on the area of the emission surface and the distance to observer. The resulting spectrum is thus the sum of two contributions:

\[ N_E = N^{BB}_E + N^B_E = A^{BB} \frac{E^2}{e^{E/kT} - 1} + A^B \frac{1}{E} e^{-E/kT}. \tag{7.6} \]

In order to check the validity of this idea a well studied GRB 090618 (Izzo et al. 2012) was chosen to perform a spectral fitting. Figure 7.4 shows the fit of the time-integrated \( E^2 \cdot N_E \) spectra (the energy spectra per logarithm of the energy interval, measured in photons \( \cdot \) keV \( \cdot \) cm\(^{-2} \cdot \) s\(^{-1} \)) for the time interval from 50 to 59 s after the trigger time with the Band function (Eq 6.1) and blackbody + bremsstrahlung (Eq. 7.6). Since we didn’t consider high-energy part of the spectrum the difference between these to fittings is significant for the photon energies above 500 keV, but our fitting is still within error boxes of the data. The central and low-energy parts are reproduced very well by both functions. The value of the temperature that we obtain from the fitting is \( kT = 138.715 \) keV which can be easily obtained in PISN explosion. Of course it is only preliminary estimations and further analysis has to be done: fitting should be done for high-energy part too, more GRBs should be analyzed.

\(^2\)In keV–MeV energy range Gaunt factor \( \bar{g} \) is a slowly varying function of the photon energy and can be assumed to be constant order of 10 (Brussaard & van de Hulst 1962; Anzer & Boerner 1976).
Figure 7.4: Time-integrated spectra for the time interval from 50 to 59 s after the trigger time of GRB 090618. Blue line shows the fit with the Band function (Eq 6.1), orange line shows the fit with blackbody + bremsstrahlung (Eq 7.6).
Chapter 8

Conclusions

In the thesis explosions of very massive stars ongoing pair instability was studied in one and two-dimensional simulations. Although there are lot of one-dimensional simulations of the hydrodynamics of PISN explosions to date, in this work PISNe were analyzed in connection with gamma-ray bursts phenomenon, basing on the model originally proposed by Chardonnet et al. (2010). This model is further developed in the thesis, providing more detailed description of properties of GRB emission.

For 1D-simulations of PISN explosion an original code was developed, the physical and mathematical model were presented in Chapter 3 and the source code can be found in Appendix A. This code is based on Lagrangian approach and includes equation of state with variable number of electron-positron pairs, thermonuclear burning, neutrino losses and self-gravity. Computations were performed for several configurations with masses within 60–128 $M_\odot$ range. Results of 1D-simulations were presented and analyzed in Chapter 4. In general they are in good agreement with previous works: heavier cores collapsed and lighter ones exploded without any remnant. However in the middle of this range the fate of the core depends on its initial configuration which in our computations was defined by initial central density. We analyzed behavior of stellar models depending on initial configuration and found that the initial binding energy of the core can be a criteria of the final fate of a star: core-collapse or explosion. An interesting correlation between total nuclear energy release and maximum temperature was found, which is related to the specific temperature distribution in a star and to the fact that the nuclear energy release is proportional to the mass of available fuel. This correlation can be an explanation of the Amati relation for GRBs.

Two dimensional simulations under the assumption of rotational symmetry were presented in Chapter 5. Computations were performed with a numerical code based on the PPML solver. While the spherically symmetric explosion in general reproduces the
results of other numerical codes, newly proposed multicore ignition scenario shows different picture of explosion. Many fragments of hot matter appear behind the shockwave. This could lead to disruption of the star in many fragments. As a result the light curves of such supernova could be more complex than in a spherically symmetric case.

In Chapter 7 GRB phenomenon was analyzed within the framework of PISN model. It was shown that 1D simulations of PISN can reproduce characteristic values of energy budget and timescale of GRBs. Maximum value of the total nuclear energy release that was obtained in our computations is $4.5 \times 10^{52}$ ergs which is comparable with the most GRBs. Basing on the assumption that the energy of emission, $E_{\text{iso}}$, is close to the total nuclear energy release and the spectral peak energy, $E_{\text{peak}}$, is defined by the temperature of the matter, we were able to reproduce empirical $E_{\text{peak}}-E_{\text{iso}}$ correlation for GRBs (Amati relation).

A possible explanation of time-variability of GRBs was proposed basing on inhomogeneous PISN explosion. It was suggested that each GRB is a superposition of individual spikes coming from different hot fragments of exploding star. Using simple Monte Carlo code it was shown that random distribution of hot fragments during the explosion can explain the variety of observed GRB light curves. But this problem requires additional investigation by radiation-transfer computations.

A model to explain non-thermal spectra of GRBs was suggested. It was proposed that different physical processes are responsible for low- and high-energy part of the spectrum. Low energy emission could be explained by a combination of thermal bremsstrahlung process in regions with lower density and blackbody emission from more dense regions. This combination naturally explains photon number spectrum with power-law index $-1$ observed in many bursts. The high-energy part is formed by inverse-Compton scattering of soft photons on electrons in hot regions. It was shown for the case of GRB 090618 that with this approach and taking into account only thermal bremsstrahlung and blackbody emission it is possible to reconstruct most part of the spectrum.

8.1 Prospects for future work

To investigate PISN explosion further we need to improve numerical simulations. First of all more detailed presupernova models can be obtained from evolutionary codes. It will provide a possibility to explore larger variety of scenarios of explosion, to study the effects of metallicity and rotation. These presupernova models can be used as initial conditions for one-dimensional and multi-dimensional simulations.
In order to improve multi-dimensional simulations we need to include more physical processes in our code based on PPML: computations of thermonuclear burning (at least oxygen burning, as the main source of energy production), neutrino losses and dynamic treatment of gravitational field. Of course there are codes that implement all this physics inside (for example CASTRO, \cite{Almgren2010}, but multi-dimensional simulations of PISN are still rare \cite{Chen2011,Joggerst2011}, and less well explored than 1D. So it is interesting to perform simulations of PISN, using an independent code based on a particular version of a Riemann solver (PPML, \cite{Popov2012}). These more detailed multi-dimensional computations can then be used for further development and better understanding of PISN-GRB model.
Appendix A

Source Codes

A.1 Structure of the program

The program for one-dimensional simulations is written in Fortran-90 language. The modules are used to keep global variables and parameters. To separate different functional tasks the code is stored in several files.

- **main.f90** — contains main body of the program and description of modules. Also contains subroutines and functions needed to define systems of equations that will be solved with LSODE and RADAU5 solvers.

- **nuclear.f90** — contains subroutines that computes nuclear reaction rates and the rate of energy production.

- **neutrino.f90** — contains the function that compute neutrino losses basing on fitting formulae from Schinder et al. (1987).

- **stdio.f90** — list of subroutines to perform input/output operations with files. Results of computations are saved in formatted ASCII files.

All additional routines that were used in this code: LSODE, RADAU5 and NADOS, have their own source files. These files can be found on corresponding web-sites and are not present in the Appendix.


RADAU5: [http://www.unige.ch/~hairer/software.html](http://www.unige.ch/~hairer/software.html)

Appendix A. Source Codes

A.2 main.f90

module constants
    ! All units in cgs system
    real(8), parameter :: solar_radius = 6.95d10
    real(8), parameter :: solar_mass = 2.0d33
    !------new units------
    real(8), parameter :: length0 = 1.0d0
    real(8), parameter :: mass0 = 1.0d34
    real(8), parameter :: temperature0 = 1.0d9
    !
    real(8), parameter :: velocity0 = length0
    real(8), parameter :: density0 = mass0 / length0 ** 3
    real(8), parameter :: pressure0 = mass0 / length0
    real(8), parameter :: energy0 = mass0 * length0 ** 2
    !
    ! Gravitational constant:
    real(8), parameter :: Grav_const = 6.67d-8 * mass0 / length0 ** 3
    ! Boltzmann constant:
    real(8), parameter :: k_Boltzmann = 1.380658d-16 * temperature0 / energy0
    real(8), parameter :: amu = 1.6605402d-24 / mass0
    real(8), parameter :: Pi = 4.0d0 * datan(1.0d0)
end module constants

module nuclear
    integer, parameter :: Nisotopes = 13
    real(8), dimension (1:Nisotopes) :: A, Z
    real(8), dimension (1:Nisotopes) :: PF ! Partition functions
    ! Reaction rates, energy release, etc.
    real(8) f21, f22, f23, f33, f31, f41, f51, f61, f71, f81, f91
    real(8) f101, f111, f121
    real(8) q21, q22, q23, q33, q31, q41, q51, q61, q71, q81, q91,
    real(8) q101, q111, q121, q22alpha, q23alpha, q33alpha
    real(8) v101, v111, v121
    real(8) lambda3, lambda4, lambda5, lambda6, lambda7, lambda8, lambda9
    real(8) lambda10, lambda11, lambda12, lambda13
    real(8) rc41, rc51
    real(8) current_rho, current_T
end module nuclear

module global
    use nuclear, only: Nisotopes
    ! Number of steps in central part and in outer part:
    integer, parameter :: Ncentral = 0, Nouter = 2000
    ! Number of steps of mass-grid:
    integer, parameter :: Ntot = Ncentral + Nouter
    real(8), parameter :: Mtot = 120.0d33
    real(8), parameter :: Mcentral = 0.0d0
    real(8), parameter :: Mouter = Mtot - Mcentral
    real(8) :: Now = 0.0d0
Appendix A. Source Codes

! velocity, radius, energy, density, pressure, concentration
! internal energy, entropy, derivatives
real(8), dimension(0:1,1:Ntot):: v, r, temp, rho, p, xi
real(8), dimension(0:1,1:Ntot):: e, s, Cp, DeDt, DeDrho, DpDt, sound_speed
! mole fraction of elements:
real(8), dimension(1:Nisotopes,1:Ntot):: Ymf
! artificial viscosity, adiabatic index, mass coordinate:
real(8), dimension(1:Ntot):: g, gamma
! artificial viscosity coefficients:
real(8) mu0, nui
real(8) r_init, p_init
end module global

Program main
use constants
use global
use nuclear implicit none
integer i, j, k, l ! counters
! mass step, radius step, time step, output time interval:
real(8) hm, hr, tau, tau_out
! current time, End time:
real(8) :: tbegin, Tend, tout
integer :: Next=0 ! number of file
character(40) filename ! name of the folder for results
logical(4) rslt
integer(4) istatus
real(8) temp1, yyyy
real(8) neutrino_loss
real(8), dimension(1:Nisotopes):: YmfTemp
integer, dimension(1:5):: Err ! Error control
real(8) nuclear_energy

! LSODE!
integer, parameter:: NEQ=Ntot*3
external f, jac
integer :: iopt=0, istate=1, itask=2, itol=1, iwork
integer, parameter :: ml=10, mu=10
! for banded Jacobian (i-ml <= j <= i+mu)
real(8) atol, rtol, rwork
integer, parameter :: meth=2
integer, parameter :: miter=5

integer, parameter :: mf=10* meth + miter
integer, parameter :: lrw=22+10*NEQ+(2*ml+mu)*NEQ
integer, parameter :: liw=20+NEQ
dimension rwork(lrw), iwork(liw)
real(8), dimension(1:NEQ) :: y
integer :: mxstep=500

!---------------------------------------------------------/LSODE-----------------------------------------------
!
!---------------------------------------------------------RADAU-----------------------------------------------
integer, parameter :: NEQ_Ymf=nisotopes,LJAC_Ymf=nisotopes
integer, parameter :: LMAS_Ymf=0,LE_Ymf=nisotopes
integer, parameter :: LWORK_Ymf=NEQ_Ymf*(LJAC_Ymf+LMAS_Ymf+3*LE_Ymf+12)+20
integer, parameter :: LIWORK_Ymf=3*NEQ_Ymf+20
real(8), dimension(LWORK_Ymf) :: work_Ymf
integer, dimension(LIWORK_Ymf) :: iwork_Ymf
real(8) :: h_ymf=1.d-3
integer :: ITOL_Ymf=0
real(8) :: RTOL_Ymf,ATOL_Ymf
real(8) :: mu0=16.0
real(8) :: tau_out=1.d0
integer :: IJAC_Ymf=1,MLJAC_Ymf=NEQ_Ymf,MUJAC_Ymf=NEQ_Ymf
integer :: IMAS_Ymf=0,IOUT_Ymf=0
integer :: MLMAS_Ymf=NEQ_Ymf,MUMAS_Ymf=NEQ_Ymf
integer :: IPAR
integer :: IDID

iwork(1)=ml
iwork(2)=mu

Err(1)=0
Err(2)=0
Err(3)=0
Err(4)=0
Err(5)=0

! Error tolerances:
rtol = 1.d-8
atol = 1.d-10

!---------------------------------------------------------init-----------------------------------------------
Tend=1.d3  !End time
tau_out=1.d0  !output timestep
mu0=16.0d0  !viscosity coefficient 1
mu1=1.d0  !viscosity coefficient 2

!Atomic numbers and charges:
A(1)=4.d0  ;  Z(1)=2
A(2)=12.d0  ;  Z(2)=6
A(3)=16.d0  ;  Z(3)=8
A(4)=20.d0  ;  Z(4)=10
A(5)=24.d0  ;  Z(5)=12
A(6)=28.d0  ;  Z(6)=14
A(7)=32.d0  ;  Z(7)=16
A(8)=36.d0  ;  Z(8)=18
\[ A(9) = 40.0; Z(9) = 20 \]
\[ A(10) = 44.0; Z(10) = 22 \]
\[ A(11) = 48.0; Z(11) = 24 \]
\[ A(12) = 52.0; Z(12) = 26 \]
\[ A(13) = 56.0; Z(13) = 28 \]

\[ Ymf(1: Nisotopes, 1: Ntot) = 0.0 \]

\[ \text{!call stdinput()} \]
\[ \text{!call continue_input()} \]
\[ \text{call polytropic_init()} \]
\[ \text{write(*,*) 'input OK'} \]

! keeping all variables in one array y
! \( y = (r(1), v(1), temp(1), r(2), v(2), temp(2), ...) \):

\[ \text{do} \]
\[ i = 1, Ntot \]
\[ y(-2+i*3) = r(0, i) \]
\[ y(-1+i*3) = v(0, i) \]
\[ y(0+i*3) = temp(0,i) \]
\[ \text{enddo} \]

! Description file with parameters of computation:
open (335, file='.._readme.txt')
write (335, '(A, e10.4)') 'Total mass', Mtot/solar_mass
write (335, '(A, e10.4)') 'Central density, g/cc', rho(0,1)*density0
write (335, '(A, 1110)') 'Ntot', Ntot
write (335, '(A, e10.4)') 'tau_out', tau_out
write (335, '(A, e10.4)') 'Tend', Tend
write (335, '(A)') '-----------Viscosity-----------'
write (335, '(A, e10.4)') 'Mu0', mu0
write (335, '(A, e10.4)') 'Mu1', mu1
write (335, '(A)') '-----------LSODE-----------'
write (335, '(A, 1110)') 'meth', meth
write (335, '(A, 1110)') 'miter', miter
write (335, '(A, 1110)') 'ml', ivork(1)
write (335, '(A, 1110)') 'msu', ivork(2)
write (335, '(A, e10.4)') 'rtol', rtol
write (335, '(A, e10.4)') 'atol', atol
close (335)

open (333, file='.._log.txt')
call stdio_init()
call stdoutput(Next)
Next=Next+1
call check_cnsrv()

!Main cycle of the program
!---------------------------/init--------------------------

\[ \text{do while (Now < Tend)} \]
\[ \text{call timestep(tau)} \]
\[ \text{tau} = \min (\text{tau_out, tau}) \]
\[ \text{tbegin} = \text{Now} \]
\[ \text{tout} = \text{Now} + \text{tau} \]
rwork(1)=tout
if (istate==−1) then
  iopt=1
  mxstep=mxstep∗2
  iwork(6)=mxstep
  istate=1
  write(*,'(A,16)') ' Increasing MXSTEP to ', iwork(6)
else
  iopt=0
  mxstep=500
  iwork(6)=mxstep
endif
! istate=1
! Call of LSODE solver. Computation of r, v, temp on the next timestep
 caller dlsode(f, neq, y, tbegin, tout, itol, rtol, atol, itask, istate, iopt, rwork, ←
lrw, iwork, liw, jac, mf)
 write(*,*) Now
 do i=1,Ntot
    r(1,i) =y(−2+3∗i)
    v(1,i) =y(−1+3∗i)
    temp(1,i)=y( 0+3∗i)
 enddo
 do i=1,Ntot−1
    rho(1,i)=3*(m(i+1)+m(i))/(4*Pi*(r(1,i+1)**3−r(1,i)**3))
    call EOS(temp(1,i),rho(1,i),Ymf(1:Nisotopes,i), &
           p(1,i),s(1,i),Cp(1,i),DeDt(1,i),DeDrho(1,i),DpDt(1,i←
           ),sound_speed(1,i),gamma(i))
 enddo
!
!Computations of nucleosynthesis in each sell
 do i=1,Ntot−1
    IJAC_Ymf=0
    MLJAC_Ymf=NEQ_Ymf
    MJAC_Ymf=NEQ_Ymf
    IMAS_Ymf=0
    IDOUT_Ymf=0
    ITOL_Ymf=0
    h_ymf=1.d−3
    rtol_Ymf  = 1.d−10
    atol_Ymf  = 1.d−15
    current_rho=rho(1,i)
    current_T=temp(1,i)
    tbegin=0.d0
    tout=WORK(11)
    y_Ymf(1:NEQ_Ymf)=ymf(1:Nisotopes,i)
    do j=1,20
       IWORK_Ymf(j)=0
       WORK_Ymf(j)=0.DO
       enddo
    CALL RADAU5(neq_ymf,f_ymf,tbegin,y_Ymf,tout,H_Ymf, &
                 rtol_Ymf, atol_Ymf, itol_Ymf, JAC_Ymf, IJAC_Ymf, &
                 MLJAC_Ymf, MJAC_Ymf, Mas_Ymf, IMAS_Ymf, MLMAS_Ymf, &
                 MUMAS_Ymf, SOLOUT_Ymf, IDOUT_Ymf, WORK_Ymf, &
                 ...)
source_codes

Appendix A. Source Codes

LWORK_Ymf, iwork_Ymf, LIWORK_Ymf, RPAR, IPAR, IDID)
ymf(1:Nisotopes,i)=y_Ymf(1:NEQ_Ymf)
xi(i,i)=Ymf(3,i)*A(3)
if (IDID==−3) then
  Err(1)=1
  Err(2)=Err(2)+1
  Err(3)=i
endif
enddo
! write(*,*) 'NS end'
Now=Now+RWORK(11)
call boundaries()
do i=1,Ntot
  r(0,i)=r(1,i)
p(0,i)=p(1,i)
rho(0,i)=rho(1,i)
temp(0,i)=temp(1,i)
v(0,i)=v(1,i)
DeDrho(0,i)=DeDrho(1,i)
DeDt(0,i)=DeDt(1,i)
xi(0,i)=xi(1,i)
e(0,i)=e(1,i)
DpDt(0,i)=DpDt(1,i)
s(0,i)=s(1,i)
Cp(0,i)=Cp(1,i)
endo
diff (Now>Next*tau_out) then
  Next=floor(Now/tau_out)
call stdout(Next)
  Next=ceiling(Now/tau_out)
call check_cnsrv()
endif
endo
! ---/next step------
close(333)
END program main

subroutine boundaries()
use constants
use global
implicit none
!------left border------
v(1,1)=0.d0
r(1,1)=r(0,1)
!
!------right border------
p(1,Ntot)=p(0,Ntot)*r(0,Ntot)/r(1,Ntot)**4.d0
xi(1,Ntot)=xi(0,Ntot)
rho(1,Ntot)=rho(0,Ntot−1)
temp(1,Ntot)=temp(0,Ntot−1)
e(1,Ntot)=e(0,Ntot−1)
g(Ntot)=0.d0
s(1,Ntot)=s(0,Ntot)

end subroutine boundaries

subroutine f ( neq, t, y, ydot)
!RHS of hydrodynamic system of equations for LSODE
use constants
use global
use nuclear
implicit none
integer neq
real(8) t
real(8), dimension(1:neq) :: y, ydot
real(8) F_conv
real(8) delta_grad_T, velocity_conv, length_mix
integer i
real(8) neutrino_loss, nuclear_energy
real(8) l, dr2vdr3, dvdr, dr3gdr3

do i=1,Ntot
   r(1,i) =y(-2+3*i)
   v(1,i) =y(-1+3*i)
   temp(1,i)=y(0+3*i)
dendo
call boundaries
! density:
do i=1,Ntot-1
   rho(1,i)=3*s(m(i)+1-m(i))/(4*Pi*(r(1,i+1)**3-r(1,i)**3))
call EOS(temp(1,i),rho(1,i),Ymf(1:Nisotopes,i), &
p(1,i),e(1,i),s(1,i),Cp(1,i),DeDt(1,i),DeDrho(1,i),DpDt←
(1,i),sound_speed(1,i),gamma(i))
dendo
! viscosity:
! introducing artificial viscosity to smooth breaks in solution
! otherwise we will obtain nonphysical oscillations

   do i=1,Ntot-1
      if ((r(1,i+1)**2+v(1,i+1)-r(1,i)**2+v(1,i))<0.d0) then
         g(i)=mu0*rho(1,i)*((r(1,i+1)**2+v(1,i+1)-r(1,i)**2+v(1,i))/r←
(1,i+1)**2)*2.0 &
         -mu1*rho(1,i)*sound_speed(1,i)*((r(1,i+1)**2+v(1,i+1)-r←
(1,i)**2+v(1,i))/r(1,i+1)**2)
      else
         g(i)=0.d0
      endif
endo
! boundary cells:
ydot(1)=0.d0
ydot(2)=0.d0
ydot(3*Ntot)=0.d0
! rhs of equations for radius:
do i=2,Ntot
   ydot(-2+3*i)=v(1,i)
dendo
Appendix A. Source Codes

! rhs of equations for velocity:
do i=2,Ntot
   ydot(-1+3*i)= -4*Pi*(p(1,i)+g(i)-p(1,i-1)-g(i-1))*r(1,i)**2/(m(i)-m(i-1)-m(i)) + Grav_const/r(1,i)**2
enddo

! rhs of equations for temperature:
do i=1,Ntot-1
   ydot( 0+3*i)= -temp(1,i)*dPdT(1,i)+g(i))/dEdT(1,i)*4*Pi*(r(1,i)**2+v(1,i))/dEdT(1,i)
call nuclear_reaction_rate(temp(1,i))
ydot( 0+3*i)= ydot( 0+3*i)+nuclear_energy(rho(1,i),temp(1,i),Ymf(1:Nisotopes,i))/DeDt(1,i)
ydot( 0+3*i)= ydot( 0+3*i)-neutrino_loss(temp(1,i),rho(1,i))/DeDt(1,i)
enddo

return
end subroutine f

subroutine jac (neq, t, y, ml, mu, pd, nrowpd)
! empty Jacobian because it is calculated numerically
return
end

subroutine EOS(temp, rho, Ymf, p, e, s, Cp, DeDt, DeDrho, DpDt, sound_speed, gamma)
! interface to call Nadyozhin EOS
use constants
use nuclear
include 'vector_eos.dek'
integer, parameter:: ionmax=Nisotopes
real(8), dimension(1:Nisotopes):: Ymf
real(8) temp, rho, xi, p, e, s, Cp, DeDt, DeDrho, DpDt, sound_speed, gamma
real(8), dimension(ionmax):: xmass, aion, zion
real(8) abar, zbar
integer i, j

do i=1,Nisotopes
   xmass(i)=Ymf(i)*A(i)
aion(i)=A(i)
zion(i)=Z(i)
endo
dobar=1.0d0/sum(xmass(1:ionmax)/aion(1:ionmax))
zbar=abar*sum(xmass(1:ionmax)*zion(1:ionmax)/aion(1:ionmax))
temp_row(1)=temp*temperature0
den_row(1)=rho*density0
abar_row(1)=abar
zbar_row(1)=zbar
jlo_eos=1
jhi_eos=1
call nados !Call of Nadyozhin EOS routine

p=ptot_row(1)/pressure0
DeDrho=ded_row(1)*density0*mass0/energy0
DeDt=det_row(1)*temperature0*mass0/energy0
e=etot_row(1)*mass0/energy0
s=stot_row(1)*mass0*temperature0/energy0

Cp=cp_row(1)*mass0*temperature0/energy0
DpDt=dpt_row(1)*temperature0/pressure0
sound_speed=cs_row(1)/length0
gamma=gm1_row(1)

end subroutine EOS

subroutine timestep(tau)
use constants
use global
real(8) tau
real(8), parameter :: cur=1.d0
real(8) delta_t,delta_r,v_sound,min_t
integer j,min_j

min_t=cur*(r(0,2)-r(0,1))/sound_speed(0,1)
delta_t=min_t

do j=1,Ntot-1
   delta_t=cur*(r(0,j+1)-r(0,j))/sound_speed(0,j)
   if (delta_t<min_t) then
      min_t=delta_t
   endif
endo do

end subroutine timestep

subroutine polytropic_init()
use constants
use global
use nuclear
implicit none
integer i
real(8) hm,tmp
real(8) k_eos
real(8) T_poly

! rho(0,1)=3.16d5/density0 !3.16
rho(0,1)=1.5d5/density0 !3.16
r(0,1)=0.d0
xi(0,1)=.50d0
Ymf(3,1)=xi(0,1)/A(3)
Ymf(2,1)=0.0d0/A(2)
Ymf(6,1)=0.50d0/A(6)
p(0,1)=0.364*Grav_const*(Mtot/mass0)**(2.d0/3.d0)*rho(0,1)**(4.d0/3.d0)
temp(0,1)=T_poly(rho(0,1),p(0,1),Ymf(1:Nisotopes,1))

end subroutine polytropic_init()
Appendix A. Source Codes

\[ p(0,1), e(0,1), s(0,1), C_p(0,1), D_e(0,1), \text{DeDrho}(0,1), D_p(0,1), \leftrightarrow \]
\[ \text{sound_speed}(0,1), \gamma(1) \]

\[ k_{eos} = \frac{p(0,1)}{\rho(0,1)} \times (4.d0/3.d0) \]

do i=2,Ntot

\[ \text{hm} = \text{Mouter}/\text{mass0}/\text{Nouter} \]
\[ m(i) = m(i-1) + \text{hm} \]
\[ v(0,i) = 0.d0 \]
\[ x_i(0,i) = x_i(0,1) \]
\[ \text{Ymf}(2,i) = \text{Ymf}(2,1) \]
\[ \text{Ymf}(3,i) = \text{Ymf}(3,1) \]
\[ \text{Ymf}(6,i) = \text{Ymf}(6,1) \]

!radius from mass and density:
\[ r(0,i) = \frac{r(0,i-1) \times 3 + 3 \times (m(i)-m(i-1))}{(\rho(0,i-1) \times 4 \times \pi)} \times (1.d0/3.d0) \]

!hydrostatic condition itself:
\[ p(0,i) = p(0,i-1) - \text{Grav_const} \times m(i) \times (m(i)-m(i-1)) / (4 \times \pi \times r(0,i) \times 4) \]
\[ \rho(0,i) = \frac{p(0,i) / k_{eos}}{(3.d0/4.d0)} \]

!\rho(0,i) = \rho_\text{poly}(p(0,i), k_{eos})
\[ \text{temp}(0,i) = T_{poly}\text{poly}(0,i), p(0,i), Ymf(1:1:\text{Nisotopes},i)) \]
\[ \text{call } \text{EOS}(\text{temp}(0,i), \rho(0,i), \text{Ymf}(1:1:\text{Nisotopes},i), \& \]
\[ p(0,i), e(0,i), s(0,i), C_p(0,i), \text{DeDt}(0,i), \text{DeDrho}(0,i), \& \]
\[ D_p(0,i), \text{sound_speed}(0,i), \gamma(i)) \]

enddo
\[ p_{\text{init}} = p(0,Ntot) \]
\[ r_{\text{init}} = r(0,Ntot) \]
end subroutine polytropic_init

\[ \text{real}(8) \text{ function } \text{rho}_\text{poly}(p,k_{eos}) \]

!------------------------
! equation of state for density
!------------------------
\[ \text{real}(8) \ p, k_{eos} \]
\[ \text{rho}_\text{poly} = (p/k_{eos}) \times (3.d0/4.d0) \]
end function \text{rho}_\text{poly}

\[ \text{real}(8) \text{ function } T_{\text{poly}}(r_{\text{ho}}, p, Ymf) \]
\[ \text{use constants} \]
\[ \text{use nuclear} \]
\[ \text{implicit none} \]
\[ \text{real}(8) r_{\text{ho}}, p \]
\[ \text{real}(8) \text{ dimension}(1:1:\text{Nisotopes}) :: \ Ymf \]
\[ \text{real}(8) \text{ p}, \text{tst1}, \text{tst2}, \text{tst3}, \text{tst4}, \text{tst5}, \text{tst6}, \text{tst7}, \text{tst8} \]
\[ \text{real}(8) \text{ tempL}, \text{ tempR}, \ T_{\text{tmp}} \]
\[ \text{integer } j \]
\[ \text{tempR} = 5.d12/\text{temperature0} \]
\[ \text{tempL} = 0.d0 \]
do j=1,100
\[ T_{\text{tmp}} = (\text{tempL} + \text{tempR}) \times .5d0 \]
\[ \text{call } \text{EOS}(T_{\text{tmp}}, r_{\text{ho}}, Ymf, p, \text{tst1}, \text{tst2}, \text{tst3}, \text{tst4}, \text{tst5}, \text{tst6}, \leftrightarrow \]
\[ \text{tst7}, \text{tst8}) \]
if (p_tmp > p_f) then
    tempR = T_tmp
else
    tempL = T_tmp
endif
enddo
T_poly = T_tmp
end function T_poly

subroutine f_ymf (neq_ymf, t, Ymf, Ymfdo, RPAR, IPAR)
! RHS of nuclear actions equations for RADAU
use constants
use nuclear
implicit none
integer neq_ymf
real(8) t
real(8), dimension(1:neq_ymf) :: Ymf, Ymfdo
integer IPAR
real(8) rho, sum, Vtt33
integer i

call nuclear_reaction_rate(current_T)
rho = current_rho*density0

v21 = f21 + Ymf(2) * Ymf(1)
v31 = f31 + Ymf(3) * Ymf(1)
v41 = f41 + Ymf(4) * Ymf(1)
v51 = f51 + Ymf(5) * Ymf(1)
v61 = f61 + Ymf(6) * Ymf(1)
v71 = f71 + Ymf(7) * Ymf(1)
v81 = f81 + Ymf(8) * Ymf(1)
v91 = f91 + Ymf(9) * Ymf(1)
v101 = f101 + Ymf(10) * Ymf(1)
v111 = f111 + Ymf(11) * Ymf(1)
v121 = f121 + Ymf(12) * Ymf(1)
v33 = f33 + Ymf(3) * Ymf(3)
v22 = f22 + Ymf(2) * Ymf(2)
v23 = f23 + Ymf(2) * Ymf(3)

Ymfdo(1) = rho*(-0.30*d0 * 0.5*d0 * v33 + 0.50*d0 * 0.5*d0 * v22 + 0.40*d0 * v23 - v21 - v31)
Ymfdo(2) = rho*(-v21 - v22 - v23)
Ymfdo(3) = rho*(v21 - v31 - v33)
Ymfdo(4) = rho*(v31 - v41 + 0.50*d0 * 0.5*d0 * v22)
Ymfdo(5) = rho*(v41 - v51 + 0.50*d0 * 0.5*d0 * v22 + 0.40*d0 * v23)
Ymfdo(6) = rho*(v51 - v61 + 0.30*d0 * 0.5*d0 * v33 + 0.60*d0 * v23)
Ymfdo(7) = rho*(v61 - v71 + 0.70*d0 * 0.5*d0 * v33)
Ymfdo(8) = rho*(v71 - v81)
Ymfdo(9) = rho*(v81 - v91)
Ymfdo(10) = rho*(v91 - v101)
Ymfdo(11) = rho*(v101 - v111)
Ymfdo(12) = rho*(v111 - v121)
Ymfdo(13) = rho*(v121)
\begin{verbatim}
! Photodissociation
Ymf(l) = Ymf(l) + \lambda_3 \cdot Ymf(3) + \lambda_4 \cdot Ymf(4) + \lambda_5 \cdot Ymf(5) + \lambda_6 \cdot Ymf(6) + \lambda_7 \cdot Ymf(7) + \lambda_8 \cdot Ymf(8) + \lambda_9 \cdot Ymf(9) + \lambda_10 \cdot Ymf(10) + \lambda_11 \cdot Ymf(11) + \lambda_12 \cdot Ymf(12) + \lambda_13 \cdot Ymf(13)
Ymf(2) = Ymf(2) + \lambda_3 \cdot Ymf(3)
Ymf(3) = Ymf(3) - \lambda_3 \cdot Ymf(3) + \lambda_4 \cdot Ymf(4)
Ymf(4) = Ymf(4) - \lambda_4 \cdot Ymf(4) + \lambda_5 \cdot Ymf(5)
Ymf(5) = Ymf(5) - \lambda_5 \cdot Ymf(5) + \lambda_6 \cdot Ymf(6)
Ymf(6) = Ymf(6) - \lambda_6 \cdot Ymf(6) + \lambda_7 \cdot Ymf(7)
Ymf(7) = Ymf(7) - \lambda_7 \cdot Ymf(7) + \lambda_8 \cdot Ymf(8)
Ymf(8) = Ymf(8) - \lambda_8 \cdot Ymf(8) + \lambda_9 \cdot Ymf(9)
Ymf(9) = Ymf(9) - \lambda_9 \cdot Ymf(9) + \lambda_10 \cdot Ymf(10)
Ymf(10) = Ymf(10) - \lambda_10 \cdot Ymf(10) + \lambda_11 \cdot Ymf(11)
Ymf(11) = Ymf(11) - \lambda_11 \cdot Ymf(11) + \lambda_12 \cdot Ymf(12)
Ymf(12) = Ymf(12) - \lambda_12 \cdot Ymf(12) + \lambda_13 \cdot Ymf(13)
Ymf(13) = Ymf(13) - \lambda_13 \cdot Ymf(13)
return
end subroutine f_ymf

subroutine jac_ymf (n, t, y, DFY, LDFY, RPAR, IPAR)
! --- JACOBIAN OF THE PROBLEM (DUMMY)
IMPLICIT REAL*8 (A-H, O-Z)
REAL*8 Y(N), DFY(LDFY, N)
RETURN
end subroutine jac_ymf

SUBROUTINE MAS_ymf (N, AM, LMAS, RPAR, IPAR)
! --- MATRIX "M" FOR THE PROBLEM (DUMMY)
IMPLICIT REAL*8 (A-H, O-Z)
REAL*8 M(N, N)
RETURN
END

SUBROUTINE SOLOUT_Ymf (NR, XOFLD, X, Y, CONT, LRC, N, RPAR, IPAR, IRTRN)
IMPLICIT REAL*8 (A-H, O-Z)
DOUBLE PRECISION X, Y(N), CONT(LRC)
end subroutine SOLOUT_Ymf

A.3 nuclear.f90

subroutine nuclear_reaction_rate(T)
use constants
use nuclear
implicit none
real(8) T
real(8) t9, t912, t913, t914, t915, t916
real(8) t923, t927, t932, t934, t935, t938, t943, t945, t947, t952, t953, t954, t956--
t957, t958
\end{verbatim}
Appendix A. Source Codes

```fortran
real(8) t9m1, t9m13

t9=temperature0*1.d-9
if(t9>10.d0) then
    t9=10.d0
endif

t912  = t9**(1.00 d/2.00 d)  ! t9**1/2

t913  = t9**(1.00 d/3.00 d)  ! t9**1/3

t914  = t9**(0.25 d)         ! t9**1/4

t915  = t9**(0.20 d)         ! t9**1/5

t916  = t9**(1.00 d/6.00 d)  ! t9**1/6


t923  = t9**(2.00 d/3.00 d)  ! t9**2/3

t927  = t9**(2.00 d/7.00 d)  ! t9**2/7


t932  = t9**(1.50 d)         ! t9**3/2

t934  = t9**(0.75 d)         ! t9**3/4

t935  = t9**(0.60 d)         ! t9**3/5

t938  = t9**(3.00 d/8.00 d)  ! t9**3/8


t943  = t9**(4.00 d/3.00 d)  ! t9**4/3

t945  = t9**(0.80 d)         ! t9**4/5

t947  = t9**(4.00 d/7.00 d)  ! t9**4/7


t952  = t9**(2.50 d)         ! t9**5/2

t953  = t9**(5.00 d/3.00 d)  ! t9**5/3

t954  = t9**(5.00 d/4.00 d)  ! t9**5/4

t956  = t9**(5.00 d/6.00 d)  ! t9**5/6

t958  = t9**(5.00 d/8.00 d)  ! t9**5/8


t9m13=t9**(1.00 d/3.00 d)
t9m1=t9**(1.00 d)

PF(2) =1.d0
PF(3) =1.d0
PF(4)=partition_function(0.9979 d0, 0.01081 d0, -0.01211 d0, 0.0038 d0, -1.79787 d-4)
PF(5)=partition_function(0.99778 d0, 0.01322 d0, -0.01743 d0, 0.00651 d0, -3.69384 d-4)
PF(6)=partition_function(0.99811 d0, 0.00933 d0, -0.00992 d0, 0.00293 d0, -1.32736 d-4)
PF(7)=partition_function(0.9995 d0, 0.00216 d0, -0.00178 d0, 2.45145 d-4, -4.5.12018 d-5)
PF(8)=partition_function(0.99912 d0, 0.00408 d0, -0.00383 d0, 8.17877 d-4, -4.4.25145 d-5)
PF(9)=partition_function(1.00047 d0, -0.00235 d0, 0.00268 d0, -9.91726 d-4, -4.1.44716 d-4)
PF(10)=partition_function(1.00066 d0, 0.00277 d0, -0.01258 d0, 0.0075 d0, -2.37428 d-4)
PF(11)=partition_function(1.01673 d0, -0.07045 d0, 0.04794 d0, 0.00209 d0, -1.69378 d-4)
PF(12)=partition_function(1.01151 d0, -0.04488 d0, 0.02525 d0, 0.00387 d0, -2.36258 d-4)
```
PF(13) = partition_function(1.00056d0, 0.00286d0, 0.00339d0, -0.00134d0, -1.73196d-4)

! O16 + O16 -> S32 + gamma
q33=16.542d0
go33alpha=9.5928d0
f33 = 7.10d+36/T923*exp(-135.93d0/T913 - 0.629d0*T923 - 0.445d0*T943 + 0.0103d0*T9 + 2.0)

! C12 + C12 -> Mg24 + gamma
q22=13.933d0
go22alpha=4.617d0
T9A=T9/(1.0d0 + 0.0396d0 + T9)
T9A13=T9A**(1.0d0 / 3.0d0)
T9A56=T9A**((5.0d0 / 6.0d0)
f22 = 4.27d+20*T9A56/T923*exp(-84.165d0/T9A13 - 2.12d0 - 0.03*T9**3.0d0)

! C12 + O16 -> Si28 + gamma
q23=16.755d0
go23alpha=6.7697d0
T9A=T9/(1.0d0 + 0.055d0 + T9)
T9A13=T9A**((1.0d0 / 3.0d0)
T9A56=T9A**((5.0d0 / 6.0d0)
T9A23=T9A**((2.0d0 / 3.0d0)
f23 = 1.72d+31*T9A56/T932*exp(-106.594d0/T9A13)/dexp(-0.180d0*T9A**2.0)
d0) +1.06d0 - 0.03*dexp(2.562d0*T9A23)
endif

! C12 + He4 -> O16 + gamma
q21=7.162d0
f21=ffREACLIB(1.76346d01, 0.0000000d0, -3.30044d1, 3.33580d0, 3.071220d0, -1.0, -4.827930d0, -2.621470d001)
lambda23=ffREACLIB(4.229510d01, -8.311050d01, -3.30044d01, 3.33580d0, 3.071220d0, -1.0, -4.827930d0, -2.621470d001)

! O16 + He4 -> Ne20 + gamma
q31=4.734d0 !/(A(3)+A(1))
f31 = 9.37e+09/T923*exp(-39.757/T913 - (T9/1.586)**2) + 6.21e+01/T932*exp(-10.297/T9) &
+5.38e+02/T932*exp(-12.226/T9) + 1.30e+01*T9**2*exp(-20.093/T9)
lambda4 = f31 + 5.65e+10*T932*exp(-54.937/T9)
lambda4 = lambda4 + PF(3)/PF(4)

! Ne20 + He4 -> Mg24 + gamma
q41=9.312d0
f41=ffREACLIB(1.3354d02, -2.5044d0, 7.3517d01, -2.2172d02, 1.3148d01, -7.4756d0, -0.9, 0.6027d01)
lambda5 = ffREACLIB(1.58364d02, -2.5044d0, 7.3517d01, -2.2172d02, 1.3148d01, -7.4756d0, -0.9, 0.6027d01, q41)
lambda5 = lambda5 + PF(4)/PF(5)

! Mg24 + He4 -> Si28 + gamma
q51=9.9840d0
\begin{verbatim}
  f51=ffREACLIB(1.4291d02, -3.2886d0, 1.0427d02, -2.6505d02, 1.3919d01, -6.9995d01, 1.2162d02)
  lambda6=rrREACLIB(1.6777d02, -3.2886d0, 1.0427d02, -2.6505d02, 1.3919d01, -6.9995d01, 1.2162d02, q51)
  lambda6=lambda6*PF(5)/PF(6)
  ! S128 + He4 --> S32 + gamma
  q61=6.9480d0
  f61=ffREACLIB(9.7101d01, -3.3244d0, 5.3585d01, -1.6568d02, 7.2000d0, -2.8284d01, 7.9339d01)
  lambda7=rrREACLIB(1.2199d02, -3.3244d0, 5.3585d01, -1.6568d02, 7.2000d0, -2.8284d01, 7.9339d01, q61)
  lambda7=lambda7*PF(6)/PF(7)
  ! S32 + He4 --> Ar36 + gamma
  q71=6.6390d0
  f71=ffREACLIB(-1.9170d02, 6.7979d0-01, -3.3847d02, 5.5016d02, -3.8813d01, 2.5300d0, -2.4324d02)
  lambda8=rrREACLIB(-1.6679d02, 6.7979d0-01, -3.3847d02, 5.5016d02, -3.8813d01, 2.5300d0, -2.4324d02, q71)
  lambda8=lambda8*PF(7)/PF(8)
  ! Ar36 + He4 --> Ca40 + gamma
  q81=7.0400d0
  f81=ffREACLIB(-1.2901d02, 3.2520d-01, -3.3228d02, 4.6877d02, -2.9137d01, 1.7660d0, -2.2245d02)
  lambda9=rrREACLIB(-1.0408d02, 3.2520d-01, -3.3228d02, 4.6877d02, -2.9137d01, 1.7660d0, -2.2245d02, q81)
  lambda9=lambda9*PF(8)/PF(9)
  ! Ca40 + He4 --> Ti44 + gamma
  q91=5.1270d0
  f91=ffREACLIB(-7.4806d02, 1.2359d01, -1.3601d03, 2.1991d03, -1.3308d02, 7.7346d0, -1.0340d03)
  lambda10=rrREACLIB(-7.3211d02, 1.2359d01, -1.3601d03, 2.1991d03, -1.3308d02, 7.7346d0, -1.0340d03, q91)
  lambda10=lambda10*PF(9)/PF(10)
  ! Ti44 + He4 --> Cr48 + gamma
  q101=7.6920d0
  f101=ffREACLIB(-9.1308d02, 1.5949d01, -1.6950d03, 2.7118d03, -1.5754d02, 8.8564d0, -1.2926d03)
  lambda11=rrREACLIB(-8.8812d02, 1.5949d01, -1.6950d03, 2.7118d03, -1.5754d02, 8.8564d0, -1.2926d03, q101)
  lambda11=lambda11*PF(10)/PF(11)
  ! Cr48 + He4 --> Fe52 + gamma
  q111=7.9390d0
  f111=ffREACLIB(-9.2910d02, 2.0573d01, -2.0397d03, 3.0780d03, -1.7157d02, 9.3883d0, -1.5093d03)
  lambda12=rrREACLIB(-9.0413d02, 2.0573d01, -2.0397d03, 3.0780d03, -1.7157d02, 9.3883d0, -1.5093d03, q111)
  lambda12=lambda12*PF(11)/PF(12)
  ! Fe52 + He4 --> Ni56 + gamma
\end{verbatim}
Appendix A. Source Codes

real(8) function ffREACLIB(a0, a1, a2, a3, a4, a5, a6)
   real(8) a0, a1, a2, a3, a4, a5, a6
   ffREACLIB = dexp( a0 + a1* T9 + a2* T9*m1 + a3* T9*m13 + a4* T9 + a5* T953 + a6* dlog(T9) )
end function ffREACLIB

real(8) function rrREACLIB(a0rev, a1, a2, a3, a4, a5, a6, q)
   real(8) a0rev, a1, a2, a3, a4, a5, a6, q
   airev = a1 - 11.6045d0 + q
   a6rev = a6 + 1.50d0
   rrREACLIB = dexp( a0rev + airev* T9 + a2* T9*m1 + a3* T9*m13 + a4* T9 + a5* T953 + a6rev* dlog(T9) )
end function rrREACLIB

real(8) function partition_function(b0, b1, b2, b3, b4)
   real(8) b0, b1, b2, b3, b4, x
   partition_function = b0 + b1* T9 + b2* T9*m1 + b3* T9*m13 + b4* T9**2.0 + b5* T9**3.0 + b6* T9**4.0
end function partition_function

end subroutine nuclear_reaction_rate

real(8) function nuclear_energy(rho, T, Ymf)
use constants
use nuclear
implicit none
real(8) T, rho
real(8), dimension(1:Nisotopes) :: Ymf

v21 = f21* Ymf(2) * Ymf(1)
v31 = f31* Ymf(3) * Ymf(1)
v41 = f41* Ymf(4) * Ymf(1)
v51 = f51* Ymf(5) * Ymf(1)
v61 = f61* Ymf(6) * Ymf(1)
v71 = f71* Ymf(7) * Ymf(1)
v81 = f81* Ymf(8) * Ymf(1)
v91 = f91* Ymf(9) * Ymf(1)
v101 = f101* Ymf(10) * Ymf(1)
v111 = f111* Ymf(11) * Ymf(1)
v121 = f121* Ymf(12) * Ymf(1)
v33 = f33* Ymf(3) * Ymf(3)
v22 = f22* Ymf(2) * Ymf(2)
v23 = f23* Ymf(2) * Ymf(3)
nuclear_energy = 0.964844d18*mass0/energy0*density0*rho* &
   (q21* v21 + q31* v31 + q41* v41 + q51* v51 + q61* v61 + &
   q71* v71 + q81* v81 + q91* v91 + q101* v101 + q111* v111 + q121* v121 + &
+ (0.5 * d0 * q22 + 0.5 * d0 * q22 * alpha) * 0.5 * d0 * v22 &
+ (0.6 * d0 * q23 + 0.4 * d0 * q23 * alpha) * v23 &
+ (0.7 * d0 * q33 + 0.3 * d0 * q33 * alpha) * 0.5 * d0 * v33)

nuclear_energy = nuclear_energy + 0.9648444d18 * mass0 / energy0 &
(-q21 * lambda3 * Ymf(3) - q31 * lambda4 * Ymf(4) - q41 * lambda5 * Ymf(5) &
- q51 * lambda6 * Ymf(6) - q61 * lambda7 * Ymf(7) - q71 * lambda8 * Ymf(8) &
- q81 * lambda9 * Ymf(9) - q91 * lambda10 * Ymf(10) - q101 * lambda11 * Ymf(11) &
- q111 * lambda12 * Ymf(12) - q121 * lambda13 * Ymf(13))
end function nuclear_energy

real(8) function photo_diss_loss(rho, T, Ymf)
  use constants
  use nuclear
  implicit none
  real(8) T, rho
  real(8), dimension(1:Nisotopes) :: Ymf

  v21 = f21 * Ymf(2) * Ymf(1)
  v31 = f31 * Ymf(3) * Ymf(1)
  v41 = f41 * Ymf(4) * Ymf(1)
  v51 = f51 * Ymf(5) * Ymf(1)
  v61 = f61 * Ymf(6) * Ymf(1)
  v71 = f71 * Ymf(7) * Ymf(1)
  v81 = f81 * Ymf(8) * Ymf(1)
  v91 = f91 * Ymf(9) * Ymf(1)
  v101 = f101 * Ymf(10) * Ymf(1)
  v111 = f111 * Ymf(11) * Ymf(1)
  v121 = f121 * Ymf(12) * Ymf(1)
  v33 = f33 * Ymf(3) * Ymf(3)
  v22 = f22 * Ymf(2) * Ymf(2)
  v23 = f23 * Ymf(2) * Ymf(3)

  photo_diss_loss = 0.9648444d18 * mass0 / energy0 &
(-q21 * lambda3 * Ymf(3) - q31 * lambda4 * Ymf(4) - q41 * lambda5 * Ymf(5) &
- q51 * lambda6 * Ymf(6) - q61 * lambda7 * Ymf(7) - q71 * lambda8 * Ymf(8) &
- q81 * lambda9 * Ymf(9) - q91 * lambda10 * Ymf(10) &
- q101 * lambda11 * Ymf(11) - q111 * lambda12 * Ymf(12) &
- q121 * lambda13 * Ymf(13))
end function photo_diss_loss

real(8) function nuc_only(rho, T, Ymf)
  use constants
  use nuclear
  implicit none
  real(8) T, rho
  real(8), dimension(1:Nisotopes) :: Ymf

  v21 = f21 * Ymf(2) * Ymf(1)
  v31 = f31 * Ymf(3) * Ymf(1)
  v41 = f41 * Ymf(4) * Ymf(1)
  v51 = f51 * Ymf(5) * Ymf(1)
  v61 = f61 * Ymf(6) * Ymf(1)
  v71 = f71 * Ymf(7) * Ymf(1)
  v81 = f81 * Ymf(8) * Ymf(1)
Appendix A. Source Codes

v91=f91*Ymf(9)*Ymf(1)
v101=f101*Ymf(10)*Ymf(1)
v111=f111*Ymf(11)*Ymf(1)
v121=f121*Ymf(12)*Ymf(1)
v33=f33*Ymf(3)*Ymf(3)
v22=f22*Ymf(2)*Ymf(2)
v23=f23*Ymf(2)*Ymf(3)

nuc_only=0.964844*d18*mass0/energy0*density0*rho*(
    q21*v21+q31*v31+q41*v41+q51*v51+q61*v61 &
    +q71*v71+q81*v81+q91*v91 &
    +q101*v101+q111*v111+q121*v121 &
    +(0.5*d0*q22+0.5*d0*q22*alpha)*0.5*d0*v22 &
    +(0.6*d0*q23+0.4*d0*q23*alpha)*v23 &
    +(0.7*d0*q33+0.3*d0*q33*alpha)*0.5*d0*v33)
end function nuc_only

A.4 neutrino.f90

real(8) function neutrino_loss(T,rho)
!---------------------------------!
! neutrino loss
!---------------------------------!
! P.J. Schinder, D.N. Schramm, P.J. Wiita, S.H. Margolis, D.L. Tubbs
! Neutrino emission by the pair, plasma, and photo processes in the Weinberg-Salam model,
!
    use constants
    implicit none
    real(8) T,rho
    real(8) lambda,xi,fraction,g
    real(8) photo_neutrino, pair_neutrino, plasma_neutrino
    real(8) lambda2,lambda4,lambda6,lambda8
    real(8) a0, a1, a2, b1, b2, b3, c

    if (T*temperature0<1.d8) then
        neutrino_loss=0.d0
        return
    else
        lambda=T*temperature0/5.9302d9
        xi=(rho/2.d0*energy0/1.d0)**(1.d0/3.d0)/lambda
        lambda2=lambda+lambda
        lambda4=lambda2+lambda2
        lambda6=lambda2*lambda2+lambda2
        lambda8=lambda4+lambda4
    !pair
    if (T*temperature0<1.d10) then
        a0=5.026d19
        a1=1.745d20
    !plasma
    if (T*temperature0>1.d10) then
        a0=5.026d19
        a1=1.745d20
    end if
    end if
end function neutrino_loss
a2 = 1.568d21
b1 = 9.383d-1
b2 = -4.141d-1
b3 = 5.829d-2
c = 5.592d0
d = 1.568
b1 = 9.383
d = -4.141
b2 = 5.829
d = 5.592
b1 = 0.000
c = 4.992d0

else
a0 = 5.026d19
a1 = 1.745d20
a2 = 1.568d21
b1 = 1.2383
b2 = -8.141
d = 0.00
b3 = 0.00
c = 4.992d0
endif
g = 1 - 13.04d0 * lambda2 + 133.5d0 * lambda4 + 1534.0d0 * lambda6 + 918.6d0 * lambda8
fraction = (a0 + a1 * xi + a2 * xi ** 2) / (xi ** 3 + b1 / lambda + b2 / (lambda ** 2) + b3 / (lambda ** 3))
pair_neutrino = g * dexp(-2.0d0 / lambda) * dexp(-c * xi) * fraction

! photo
a0 = 3.897d10
a1 = 5.906d10
a2 = 4.693d10
b1 = 6.290d-3
b2 = 7.483d-3
b3 = 3.061d-4
c = 1.565d0
fraction = (a0 + a1 * xi + a2 * xi ** 2) / (xi ** 3 + b1 / lambda + b2 / (lambda ** 2) + b3 / (lambda ** 3))
photo_neutrino = (rho * density0 / 2.0d0 * lambda ** 5) * dexp(-c * xi) * fraction

! plasma
a0 = 2.146d-7
a1 = 7.814d-8
a2 = 1.653d-8
b1 = 2.581d-2
b2 = 1.734d-2
b3 = 6.990d-4
c = 0.565zd0
fraction = (a0 + a1 * xi + a2 * xi ** 2) / (xi ** 3 + b1 / lambda + b2 / (lambda ** 2) + b3 / (lambda ** 3))
plasma_neutrino = (rho * density0 / 2.0d0) ** 3 * dexp(-c * xi) * fraction

neutrino_loss = (photo_neutrino + pair_neutrino + plasma_neutrino) / (rho * density0)
neutrino_loss = neutrino_loss * mass0 / energy0
return
endif

end function neutrino_loss
A.5 stdio.f90

```fortran
subroutine stdio_init()
  open(42, file='outchk.dat')
  write(42, '(7A14)') 'time', 'e_in', 'e_g', 'e_k', 'e_total', 'e_nuc', ←
  'm_burned'
  close(42)

  open(43, file='central.dat')
  write(43, '(9A14)') 'time', 'r', 'rho', 'T', 'e_nuc', 'e_neut', ←
  'e_nuc_only', 'ph_dis'
  close(43)
end subroutine

subroutine stdinput()
  !-- input from file --!
  use constants
  use global
  use nuclear
  ! use IFPORT
  implicit none
  integer i,k
  real(8) tst
  character(40) filename
  write(filename, '(''init'','I5.5, 'dat')') Ntot
  write(*,*) 'Reading data from file:', TRIM(filename)
  open(100, file=TRIM(filename))
  read(100,*) ! skip first line in file
  do i=1,Ntot-1
    read(100,*) r(0,i), m(i), rho(0,i), v(0,i), temp(0,i), p(0,i), xi(0,i), ←
    gamma(i)
    r(0,i)=r(0,i)*solar_radius/length0
    v(0,i)=v(0,i)/length0
    rho(0,i)=rho(0,i)/density0
    temp(0,i)=temp(0,i)/temperature0
    m(i)=m(i)*solar_mass/mass0
    Ymf(3,i)=xi(0,i)/A(3)
    Ymf(7,i)=(1.d0-xi(0,i))/A(7)
    call EOS(temp(0,i), rho(0,i), Ymf(1:Nisotopes,i), p(0,i), e(0,i), &
      s(0,i), Cp(0,i), DeDt(0,i), DeDrho(0,i), DpDt(0,i), ←
      sound_speed(0,i), gamma(i))
  enddo
  !-----------right border-----------
  i=Ntot
  read(100,*) r(0,i), m(i), rho(0,i), v(0,i), temp(0,i), p(0,i), xi(0,i), ←
  gamma(i)
  r(0,Ntot)=r(0,Ntot)*solar_radius/length0
  v(0,i)=v(0,i)/length0
  rho(0,Ntot)=rho(0,Ntot)/density0
  temp(0,Ntot)=temp(0,Ntot)/temperature0
  m(Ntot)=m(Ntot)*solar_mass/mass0
```

p(0,Ntot)=p(0,Ntot)/pressure0
Ymf(3,Ntot)=xi(0,Ntot)/A(3)
Ymf(7,Ntot)=(1.d0-xi(0,Ntot))/A(7)
call EOS(temp(0,i),rho(0,i),Ymf(1:Nisotopes,i),tst,e(0,i), &
   s(0,i),Cp(0,i),DeDt(0,i),DeDrho(0,i),DpDt(0,i),sound_speed←
(0,i),gamma(i))
p_init=p(0,Ntot)
r_init=r(0,Ntot)
end subroutine stdin

subroutine continue_input()

! input from file
!
    use constants
    use global
    use nuclear
    ! use IFPORT
    implicit none
    integer i,k
    real(8) tst
    character(40) filename
write(filename,'("init_","I5.5",".dat")') Ntot
write(*,*) 'Reading data from file: ',TRIM(filename)
open(100, file=TRIM(filename))
open(1042, file='YmfInit.dat')
read(100,*) ! skip first line in file
do i=1,Ntot-1
    read(100,*) r(0,i), m(i), rho(0,i),v(0,i),temp(0,i),p(0,i),xi(0,i),gamma(i)
    r(0,i)=r(0,i)*solar_radius/length0
    v(0,i)=v(0,i)/length0
    rho(0,i)=rho(0,i)/density0
    temp(0,i)=temp(0,i)/temperature0
    m(i)=m(i)*solar_mass/mass0
    read(1042,*) (Ymf(k,i),k=1,13)
    !Ymf(3,i)=xi(0,i)/A(3)
    !Ymf(7,i)=(1.d0-xi(0,i))/A(7)
call EOS(temp(0,i),rho(0,i),Ymf(1:Nisotopes,i),p(0,i),e(0,i), &
   s(0,i),Cp(0,i),DeDt(0,i),DeDrho(0,i),DpDt(0,i),←
   sound_speed(0,i),gamma(i))
enddo
!
end subroutine continue_input

!----------------------- right border -----------------------
i=Ntot
read(100,*) r(0,i), m(i), rho(0,i),v(0,i),temp(0,i),p(0,i),xi(0,i),gamma(i)
    r(0,i)=r(0,Ntot)*solar_radius/length0
    v(0,i)=v(0,i)/length0
    rho(0,Ntot)=rho(0,Ntot)/density0
    temp(0,Ntot)=temp(0,Ntot)/temperature0
    m(Ntot)=m(Ntot)*solar_mass/mass0
    p(0,Ntot)=p(0,Ntot)/pressure0
read(1042,*) (Ymf(k,i),k=1,13)
call EOS(temp(0,i),rho(0,i),Ymf(1:Nisotopes,i),tst,e(0,i), & 
   s(0,i),Cp(0,i),DeDt(0,i),DeDrho(0,i),DpDt(0,i),sound_speed←
(0,i),gamma(i))
p_init=p(0,Ntot)
r_init=r(0,Ntot)
!-------------------------right border-------------------------
close(100)
close(1042)
end subroutine continue_input

subroutine stdoutput(filenumber)
!-------------------------
! output in file
!-------------------------
use constants
use global
implicit none
integer Next,j,i
integer filenumber
character(40) filename
real(8) e_nuc, e_neutrino, etot, e_in, e_g, e_k, e_nuc_only, & 
   ph_dis_only
real(8) neutrino_loss, nuclear_energy, nuc_only, photo_diss_loss
write(filename,'(3 hout , i 4 . 4 , 4 h . dat )') filenumber
write(*,* ) '--------------------------'
write(*,* ) filename
write(*,* ) '--------------------------'
open(1, file=filename)
write(1,'(10A20)') 'r','m','rho','v','T','p','xi','gamma-4/3','s','e'
do j=1,Ntot
   write(1,'(10e20.10)') r(0,j)*length0/solar_radius, & 
      m(j)*mass0/solar_mass,rho(0,j)*density0, & 
      v(0,j)*velocity0,temp(0,j)*temperature0,p(0,j)*pressure0, & 
      xi(0,j),gamma(j)-4.d0/3.d0,s(0,j),s(0,j)*energy0/mass0
endo
close(1)
write(filename,'(3 hymf , i 4 . 4 , 4 h . dat )') filenumber
write(*,* ) '--------------------------'
write(*,* ) filename
write(*,* ) '--------------------------'
open(1, file=filename)
do j=1,Ntot
   write(1,'(13e23.15)') Ymf(i,j), i=1,Nisotopes
endo
close(1)
write(filename,'(3 hxen , i 4 . 4 , 4 h . dat )') filenumber
write(*,* ) '--------------------------'
write(*,* ) filename
write(*,* ) '--------------------------'
open(1, file=filename)
write(1, '(9A20)') 'm', 'e_tot', 'e_in', 'e_g', 'e_k', 'e_nuc', 'e_neutr', 'e_nuc_only', 'ph_dis',
etot=0.d0
do i=1,Ntot-1
e_in=e_in+e(0,i)*(m(i+1)-m(i))
e_g=e_g-Grav_const*(m(i+1)-m(i))*m(i)/r(0,i+1)
e_k=e_k+(m(i+1)-m(i))*v(0,i+1)*2.d0/2.d0
call nuclear_reaction_rate(temp(0,i))
write(1, '(9e20.10)') m(i), (e_in+e_g+e_k)*energy0, e_in*energy0, &
e-g*energy0, e_k*energy0, &
nuclear_energy(rho(0,i),temp(0,i),Ymf(1:Nisotopes,i))&
*(m(i+1)-m(i))*energy0, &
nuclonly(temp(0,i),rho(0,i))*(m(i+1)-m(i))*energy0, &
nuc_only(temp(0,i),rho(0,i),Ymf(1:Nisotopes,i))&
*(m(i+1)-m(i))*energy0, &
ph_dis_only=ph_dis_only+&
photo_diss_loss(rho(0,i),temp(0,i),Ymf(1:Nisotopes,i))&
*(m(i+1)-m(i))*energy0
enddo
close(1)
e_nuc=0.d0
e_neutrino=0.d0
e_nuc_only=0.d0
e_ph_dis_only=0.d0
do j=1,Ntot-1
call nuclear_reaction_rate(temp(0,j))
e_nuc=e_nuc+&
nuclear_energy(rho(0,j),temp(0,j),Ymf(1:Nisotopes,j))&
*(m(j+1)-m(j))
e_neutrino=e_neutrino+&
nuclonly(temp(0,j),rho(0,j))&
*(m(j+1)-m(j))
e_nuc_only=e_nuc_only+&
nuc_only(temp(0,j),rho(0,j),Ymf(1:Nisotopes,j))&
*(m(j+1)-m(j))
ph_dis_only=ph_dis_only+&
photo_diss_loss(rho(0,j),temp(0,j),Ymf(1:Nisotopes,j))&
*(m(j+1)-m(j))
enddo
open(43, file='central.dat', position='APPEND')
write(43, '(9e14.6)') now, r(0,Ntot)*length0/solar_radius, &
p(0,1)*pressure0, rho(0,1)*density0, &
temp(0,1)*temperature0, e_nuc*energy0, e_neutrino*energy0, &
e_nuc_only*energy0, dabs(ph_dis_only*energy0)
close(43)
end subroutine stdoutput

subroutine check_cnsrv()
!
! Checking of conservation laws in numerical scheme
!

use constants
use global
use nuclear
implicit none
integer i
real(8) total_m, e_in, e_g, e_k, e_bound, mass_burned, e_nuc
real(8) sum
total_m=0.d0
e_in=0.d0
e_g=0.d0
e_k=0.d0
mass_burned=0.d0
! work against boundary pressure:
! e_bound=4*Pi*p_init*r_init**4*(1/r_init - 1/r(0,Ntot))
do i=1,Ntot-1
  mass_burned=mass_burned+(m(i+1)-m(i))*(1.d0-xi(0,i))
  e_nuc=k_nuclear*mass_burned
  e_in=e_in+e(0,i)*(m(i+1)-m(i))
  e_g=e_g-Grav_const*(m(i+1)-m(i))*m(i)/r(0,i+1)
  e_k=e_k+(m(i+1)-m(i))*v(0,i+1)**2.d0/2.d0
endo
dsum=0.d0
do i=1,13
  sum=sum+A(i)*Ymf(i,1)
endo
dwrite(*,*) sum
open(42, file='outchk.dat', position='APPEND')
dwrite(42, '((7e14.6),1)') Now, e_in*energy0, -e_g*energy0, &
e_k*energy0, (e_in+e_g+e_k)*energy0, &
e_nuc*energy0, mass_burned*mass0/solar_mass
close(42)
dend subroutine check_cnsrv
Abbreviations

CBM  Circumburst medium
CC   Core Collapse
EOS  Equation of State
GRB  Gamma-Ray Burst
IMF  Initial Mass Function
ISM  Interstellar Medium
PISN Pair-Instability Supernova
SN   Supernova
Physical Constants

<table>
<thead>
<tr>
<th>Physical Constant</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed of Light</td>
<td>$c = 2.99792458 \times 10^{10}$ cm s$^{-1}$</td>
</tr>
<tr>
<td>Gravitational Constant</td>
<td>$G = 6.6742 \times 10^{-8}$ cm$^3$ g$^{-1}$ s$^{-1}$</td>
</tr>
<tr>
<td>Plank’s Constant</td>
<td>$h = 6.626 \times 10^{-27}$ erg s</td>
</tr>
<tr>
<td></td>
<td>$h = h/2\pi = 1.0546 \times 10^{-27}$ erg s</td>
</tr>
<tr>
<td>Boltzmann Constant</td>
<td>$k = 1.3807 \times 10^{-16}$ erg K$^{-1}$</td>
</tr>
<tr>
<td>Avogadro’s Number</td>
<td>$N_A = 6.022 \times 10^{23}$ mole$^{-1}$</td>
</tr>
<tr>
<td>Gas constant</td>
<td>$\mathcal{R} = 8.314 \times 10^7$ erg K$^{-1}$ mole$^{-1}$</td>
</tr>
<tr>
<td>Stefan-Boltzmann Constant</td>
<td>$\sigma = 5.670 \times 10^{-5}$ erg K$^{-4}$ cm$^{-2}$ s$^{-1}$</td>
</tr>
<tr>
<td>Radiation Density Constant</td>
<td>$a = 4\sigma/c = 7.566 \times 10^{-15}$ erg K$^{-4}$ cm$^{-3}$</td>
</tr>
<tr>
<td>Electron’s Rest Mass</td>
<td>$m_e = 9.109 \times 10^{-28}$ g</td>
</tr>
<tr>
<td></td>
<td>$m_e c^2 = 0.511$ MeV</td>
</tr>
<tr>
<td>Thomson cross section</td>
<td>$\sigma_T = 0.66524 \times 10^{-24}$ cm$^2$</td>
</tr>
<tr>
<td>Electron volt</td>
<td>$1$ eV $= 9.602 \times 10^{-12}$ erg</td>
</tr>
<tr>
<td></td>
<td>$1$ eV/$k = 11604$ K</td>
</tr>
<tr>
<td>Rydberg</td>
<td>$1$ Ry $= 13.606$ eV</td>
</tr>
<tr>
<td>Solar Mass</td>
<td>$M_\odot = 1.989 \times 10^{33}$ g</td>
</tr>
<tr>
<td>Solar Radius</td>
<td>$R_\odot = 6.955 \times 10^{10}$ cm</td>
</tr>
<tr>
<td>Solar Luminosity</td>
<td>$L_\odot = 3.846 \times 10^{33}$ erg s$^{-1}$</td>
</tr>
</tbody>
</table>
## Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>specific internal energy</td>
<td>erg $g^{-1}$</td>
</tr>
<tr>
<td>$\mathcal{E}$</td>
<td>kinetic energy of a particle</td>
<td>erg</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>energy production rate</td>
<td>erg $g^{-1} s^{-1}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>rate of spontaneous decay</td>
<td>$s^{-1}$</td>
</tr>
<tr>
<td>$m$</td>
<td>Lagrangian mass coordinate</td>
<td>g</td>
</tr>
<tr>
<td>$N_E$</td>
<td>photon number spectrum</td>
<td>photons cm$^{-2} s^{-1} \text{keV}^{-1}$</td>
</tr>
<tr>
<td>$n$</td>
<td>number density</td>
<td>cm$^{-3}$</td>
</tr>
<tr>
<td>$P$</td>
<td>pressure</td>
<td>erg cm$^{-3}$</td>
</tr>
<tr>
<td>$p$</td>
<td>momentum of a particle</td>
<td>g cm $s^{-1}$</td>
</tr>
<tr>
<td>$R$</td>
<td>thermonuclear reaction rate</td>
<td>cm$^3$ mole$^{-1}$ s$^{-1}$</td>
</tr>
<tr>
<td>$r$</td>
<td>radius</td>
<td>cm</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
<td>g cm$^{-3}$</td>
</tr>
<tr>
<td>$T$</td>
<td>temperature</td>
<td>K</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
<td>s</td>
</tr>
<tr>
<td>$v$</td>
<td>velocity</td>
<td>cm $s^{-1}$</td>
</tr>
<tr>
<td>$u$</td>
<td>internal energy density</td>
<td>erg cm$^{-3}$</td>
</tr>
<tr>
<td>$X$</td>
<td>mass fraction</td>
<td></td>
</tr>
<tr>
<td>$Y$</td>
<td>mole fraction</td>
<td>mole $g^{-1}$</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>gravitational potential energy of a star</td>
<td>erg</td>
</tr>
<tr>
<td>$U$</td>
<td>total internal energy of a star</td>
<td>erg</td>
</tr>
</tbody>
</table>
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