Fermionic dark matter on galaxy scales

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Abbreviations

GC    Galactic center
DM    Dark matter
MW    Milky Way
Λ CDM  Λ cold dark matter
Λ WDM  Λ warm dark matter
WIMP  Weakly interacting massive particle
CDM   Cold dark matter
HDM   Hot dark matter
WDM   Warm dark matter
SM    Standard model
BH    Black Hole
HST   Hubble space telescope
IMBH  Intermediate massive Black Hole
SMBH  Super massive Black Hole
TOV   Tolman, Oppenheimer, Volkoff
2MASS Two micron all sky survey
IR    Infrared
NICMOS Near infrared camera and multi object spectrometer
MACHOs Massive compact halo objects
BBN   Big Bang nucleosynthesis
CMB   Cosmic microwave background
FRW   Friedmann Robertson Walker
LHC   Large hadron collider
CP    Charge parity
QCD   Quantum chromodynamics
2dF   Two degree field
SDSS  Sloan digital sky survey
BEC   Bose Einstein condensate
## Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
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<tbody>
<tr>
<td>LTE</td>
<td>Local thermodynamic equilibrium</td>
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<tr>
<td>DF</td>
<td>Distribution function</td>
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<td>THINGS</td>
<td>The HI nearby galaxy survey</td>
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<td>SINGS</td>
<td>Spitzer infrared nearby galaxy survey</td>
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<tr>
<td>RMF</td>
<td>Relativistic mean field</td>
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<td>GR</td>
<td>General relativity</td>
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<td>SgrA*</td>
<td>Sagittarius A*</td>
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<tr>
<td>VLBI</td>
<td>Very long baseline interferometer</td>
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<tr>
<td>RAR</td>
<td>Ruffini Argüelles Rueda</td>
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<tr>
<td>AGN</td>
<td>Active galactic nuclei</td>
</tr>
<tr>
<td>ν MSM</td>
<td>Neutrino minimal standard model extension</td>
</tr>
<tr>
<td>OV</td>
<td>Oppenheimer Volkoff</td>
</tr>
<tr>
<td>ISO</td>
<td>Isothermal sphere</td>
</tr>
<tr>
<td>IMF</td>
<td>Initial mass function</td>
</tr>
<tr>
<td>BIC</td>
<td>Bayesian information criterion</td>
</tr>
<tr>
<td>ND</td>
<td>Not constrained</td>
</tr>
<tr>
<td>LOSVD</td>
<td>Line of sight velocity dispersion</td>
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<tr>
<td>dSph</td>
<td>Dwarf spheroidal</td>
</tr>
<tr>
<td>NS</td>
<td>Neutron star</td>
</tr>
<tr>
<td>EC</td>
<td>Einstein cluster</td>
</tr>
<tr>
<td>ISCO</td>
<td>Innermost stable circular orbit</td>
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Abstract

Cosmological and astrophysical observations gathered in the last two decades shows that only about 5% of the content of the Universe consists in ordinary matter, such as protons, neutrons, electrons and photons. The rest of the matter-energy budget consists of Dark Matter (DM) (about 25%), and Dark Energy (about 70%). A rigorous theoretical study of DM requires the interconnection between particle physics (Standard Model (SM) and beyond SM Physics), within a fully relativistic treatment. Moreover, any phenomenological analysis must be always guided by the many different known astrophysical scenarios where DM plays certainly a role, i.e.: the early universe, the large structure of the Universe, structure formation, gravitational lensing in groups and clusters of galaxies, and the galaxy rotation curves. The research work of this Ph.D. thesis is centered in the study of the distribution of dark matter (DM) in galaxies, its nature, its relation with the distribution of baryonic matter and super massive dark central objects, and the role of self-interacting DM in galaxies in connection with beyond SM neutrinos.
Chapter 1

PARTICLE DM IN GALAXIES: INTRODUCTION

1.1 A brief historical review

The first evidence of the presence of a dark gravitational mass starts about one century ago on galaxy scales, more specifically, with the kinematical studies in the solar neighborhood developed by J. H. Jeans in 1922 [1] guided by his mentor J. C. Kapteyn. By studying crossings of stars (belonging to population II) around the galactic plane, Jeans was able to estimate the dynamical density of matter in the solar surroundings (as large as $0.14 \, M_\odot/pc^3$), which is only about 30% less than actual estimates (see e.g. [2]). The concept of ‘dark matter’, introduced by Jeans himself, appeared as a necessity after he realized that all known stellar populations plus the interstellar gas were unable to explain the observed motion of the stars. Within the large observational uncertainties at that time, mainly based in the estimation of the sun distance to the GC, he concluded that about 50% of DM was needed to fit the observations. However, actual constraints on the amount of DM within the radial distance between the sun and the GC, indicates a needed extra dark component of only a few percent of the total observed mass enclosed in the same radial extension (see e.g. [2],[3]). Actually, there is a general consensus among the scientific community, indicating a very small required amount of DM in the disk of the MW, as well as in any disk belonging to spiral galaxies (up to 10% of the total mass). Moreover, this dark component must be of baryonic nature (i.e. low mass stars), since non-baryonic
DM should be dissipationless\(^1\), interacting only gravitationally, and therefore cannot form highly flattened distributions (see e.g. [5]).

Before a definitive observational confirmation of the existence of DM in galaxies, a compelling evidence for the necessity of DM was provided by F. Zwicky in 1933 [6] at galaxy cluster-scales. By measuring radial velocities of galaxies belonging to the Coma cluster together with the application of the virial theorem, he calculated mean orbital velocities of galaxies of about 1000 km/s, these values being one order of magnitude larger than expected from the total amount of mass in galaxies constituting the cluster. After this discovery, the definitive confirmation of the existence of extended dark matter halos in galaxies (first referred as Galactic coronas) appeared with the work of J. Einasto in 1974 [7]. He combined photometric and kinematical data together with quite detailed information on physical and chemical evolution of galaxies to obtain mass-to-light ratios associated to different stellar populations. With the results of these analysis it was concluded that (in his own words) ...‘the only way to eliminate the conflict between photometric and rotational data was to assume the presence of an unknown almost spherical population with a very high value of the mass-to-light ratio, large radius and mass’. Specifically, he found that the ‘massive coronas’ exceeded the mass of know stellar population by one order of magnitude.

More recently, and extending the discussion at all galaxy scales, the observational presence of DM halos is safely confirmed in the smallest and less luminous \((10^3 L_\odot \lesssim L \lesssim 10^7 L_\odot)\) galactic systems, i.e.: dwarf galaxies; where DM contributes 90% or more to the total mass even much inside the half-light radius \(R_e\) (see, e.g., [8]). In the case of more luminous galaxies, both baryonic (i.e. stars) and dark matter contributes in comparable amounts to the total mass, being a big challenge to disentangle the gravitational effect of the DM component within \(r \sim R_e\) (see, e.g. [5] and refs. therein). In spiral galaxies the observations of extended HI regions in the disk structure give an important and universal dynamical tracer which, through rotation curves analysis, has provided strong evidence for the existence of DM even up to several \(R_e\) (see, e.g. [9] for recent high resolution HI data).

\(^1\)Nevertheless, partially interacting DM particles representing a subdominant fraction of the total DM, have been recently proposed in [4]. This self-interacting DM is built to mimic dissipation effects of baryonic matter, allowing for the formation of a double-disk, together with the observed disk of baryonic matter.
1.1. A brief historical review

However, in the case of big elliptical and early-type galaxies, most of them containing super massive dark objects at their centers (see e.g. [10] and references therein), there is no definitive evidence for the existence of DM halos. The low surface brightness beyond $R_e$ makes it a hard task to obtain reliable spectra to determine dispersion velocities. Among the several methods available to prove the mass distribution beyond $R_e$ in elliptical galaxies, such as integrated stellar light spectrum, globular cluster and planetary nebulae kinematics, diffuse X-ray observation or weak gravitational lensing; no evidence for DM halos has been found even out to few $R_e$ in many ellipticals by the use of kinematical methods ([11, 12]). Meanwhile, by using X-ray observations in an small sample of nearby ellipticals, as for example in [13], a clear evidence for considerable amounts of DM at radii $r \sim 10R_e$ was given. In any case, the more interesting constraints on DM in early-type galaxies are restricted to the more massive systems, which are placed near the center of group or clusters. This implies to be a difficult task to confirm whether the existence of extensive halos is an inherent property of the galaxy itself, or corresponds to the DM content at the group-scale (see e.g. [5] chapter 4.9.2).

On theoretical and numerical grounds, the paradigm regarding the nature and spatial distribution of the DM particles in large and small distance-scales is centered in Newtonian N-body simulations within Lambda Cold Dark Matter ($\Lambda$CDM) cosmologies [14, 15], being the beyond Standard Model (SM) elementary particle WIMP (for Weakly Interacting Massive Particles), the preferred DM candidate. Nonetheless, despite the good agreement of this model with the large scale structure of the Universe, some subtle problems remains at galactic scales such as the lost satellites problem and the core-cusp discrepancy (see, e.g., [16]).

An alternative and very promising beyond SM particle which has received increasing attention in cosmology and structure formation in the last decade, is the right handed sterile neutrino with masses of $\sim$ keV (see, e.g., [17–21]). Moreover, in [22] (see also refs. therein), halos has been obtained from numerical simulations in Warm Dark Matter (WDM) cosmologies, solving the discrepancies arising at galaxy scales in the CDM paradigm, being again the sterile neutrino a plausible DM candidate.

Continuing on theoretical grounds and from a different and complementary perspective, the problem of modeling the distribution of dark matter in galaxies in terms of equilibrium configurations of collisionless self-gravitating fermionic particles, has already been considered in [23, 24]. More recently in [25–34], this
approach was developed in a fully relativistic treatment with applications to dark matter in normal galaxies being the spin 1/2 fermion with masses \( m \sim \text{few keV} \) the preferred DM candidate in excellent agreement with halo observations [29, 32]. Again the sterile neutrino appears as an appealing candidate.

Indeed, the results presented in the above citations of the last paragraph represent the main content of this Thesis (see section 1.3 for details). As I will detail along this presentation, an interesting characteristic in the density profile solutions of this kind of models with \( m \gtrsim 1 \text{ keV} \), is that they present a novel core-halo morphology composed by: i) a compact degenerate core of constant density at sub-parsec scales governed by quantum statistics (i.e. Pauli principle); ii) an intermediate region with a sharply decreasing density distribution followed by an extended plateau; iii) a decreasing \( \rho \propto r^{-2} \) leading to flat rotation curves fulfilling classical Boltzmann statistics (see sections 2.1-2.2). Moreover, in [32], and for \( m \sim 10 \text{ keV} \), using observations from typical dwarf galaxies up to typical big spirals allowing for DM halo characteristics, it is clearly shown how the corresponding dense central core arising in each case, may work as an alternative to intermediate (\( M_c \sim 10^4 M_\odot \)), and super massive Black Holes (BH) (up to \( M_c \sim 10^7 M_\odot \)) at their centers. It is further shown in [32], out of first principles, how a possible universal correlation between the DM halos and the massive dark objects arises for \( m = 10 \text{ keV} \). Interestingly, a very similar correlation law to the one theoretically found in [32] in the range \( M_c \in (10^6, 10^7) M_\odot \) (with correspondent halo masses \( M_h \) from \( \sim 10^{11} M_\odot \) up to \( \sim 10^{12} M_\odot \)), has been found from observations in [35], relating the dark halo masses to central mass concentrations, these last however identified by Ferrarese as black hole masses.

The distribution of DM at the center of galaxies is an interesting open issue in astrophysics. For example, numerical simulations developed to understand the formation of DM halos, either in \( \Lambda \text{CDM} \) or in \( \Lambda \text{WDM} \) cosmologies, predict, in the first case, cuspy DM density profiles with \( \rho \propto r^{-1} \) inside the virial radius\(^2 \) \( r_{200} \) (see, e.g., [14]); and, in the second case, DM density profiles which are in better consistency with cored profiles inside the same radial extent (see, e.g., [22]). The fact that N-body simulations can barely resolve any structure below that radial scales, ensues a debate on how would be the distribution of DM at sub-parsec scales. Of course, at that small distances, the angular resolution power of modern telescopes imposes a big challenge to constrain the DM

\(^2\)The virial radius is usually defined as the radius at which the density is 200 times the critical density of the Universe, and is of order \( \sim 10^2 \) kpc for typical spiral galaxies.
density at galactic centers. Nevertheless, with the technological revolution of the HST, many dispersion velocities at $\sim 10^{-1} - 10^1$ pc scales has been obtained in some nearby galaxies, showing that, in addition to the baryonic component surely present at this short-scales, the dark component could well be order of magnitudes higher than the typical halo densities of $10^{-1} - 10^{-2} M_\odot / pc^3$. Moreover, in the literature it is generally accepted that an additional important effect in stellar dynamics below sub-parsec scales is caused by the so-called SMBHs (believed to be ubiquitous at galactic nuclei, see [10] for an excellent and full review on this topic), and therefore, subsequent possible constraints on the DM density have to account for its presence (see, e.g., [36] for a didactic review).

Nevertheless, I propose here as one of the main objectives of this Thesis (see also section 1.3), an alternative and rather natural way to constraint the amount of DM through the center of the galaxies from first principle physics. This is, by solving in the more general way the TOV equations for hydrostatic equilibrium of a thermal and semi-degenerate fermion gas including relativistic effects; I show in chapter 2, how the dense quantum cores at sub-parsec scales (inherent of those equilibrium solutions for keV fermions), imply DM densities of $\sim 10^{11} - 10^{13}$ orders of magnitude higher than typical DM halo densities in spirals. This large dark fermionic concentrations enclosed in such short scales, automatically imply the existence of super-massive dark objects of $\sim 10^6 - 10^7 M_\odot$, changing dramatically any possible prediction on DM constraints at the center of galaxies when compared with the central BH paradigm.

Regarding the above mentioned novel morphology of the fermionic DM density profiles, it is important to recall the causes which led to such particular DM distribution. Much after the relativistic primordial decoupling of our WDM particles, their temperature cools down due to the expanding Universe until gravity starts to bound the system with a consequent virialization of the DM halo. Now, if we assume an underlying Fermi-Dirac distribution in a very low-degenerate initial state condition (i.e. Boltzmannian-like) at the moment of the structure formation, depending on the microscopic model of the DM particle at decoupling, it exist the possibility that at this formation epoch ($z \sim 5$), the DM temperature has decreased low enough for a gravitational phase-transition to take place. It has been indeed demonstrated in [37], that this gravitational phase-transition exist and leads to the emergence of a condensed phase through the center of the configuration, forming central dark compact objects. This last result clearly supports the astrophysical application of the model presented here. Indeed, in
1. PARTICLE DM IN GALAXIES: INTRODUCTION

Chapter 3 I show a clear example on how it is possible to put constraints on the particle mass and on the amount DM at sub-parsec scales, even in the presence of baryonic matter for well resolved nearby dwarf galaxies, as in the case of NGC205. While in Chapter 4, I investigate an important extension of the model presented in Chapter 2, in which the role of self-interactions among the fermions is analyzed.

1.2 Particle DM candidates and its nature

1.2.1 Baryonic DM

The first attempts to try to account for the needed DM in galaxies were by means of ordinary matter, i.e. the one involving protons, neutrons, electrons and photons; usually named baryonic DM. Already since the ’70s, low massive stars \((0.01M_\odot < M < 0.08M_\odot)\) as brown dwarfs\(^3\), as well as extended halos of neutral gas or very hot X-ray emitting gas were proposed as forms of DM. However, at the beginning of the 21st century, with the technological revolution on Infrared astronomy conducted mainly by the 2MASS IR survey or the HST IR NICMOS camera, it was measured an amount of brown dwarfs only as large as twice the amount of normal stars (i.e., stars with \(M > 0.08M_\odot\)); implying a negligible contribution to the DM budget. Regarding a possible DM candidate in halos in the form of neutral gas or even very hot gas, already in 1975 [38], it was understood that the first cannot be the main responsible for the dark component, since it would be easily ionized by the intergalactic hot gas, and therefore would have been observed. On the other hand, if the hot gas itself\(^4\) is proposed as the main DM component, it is not sufficient to account for the kinematics of the observed orbits in the different structures (see e.g. [39]).

The low massive stars, as brown dwarfs or white dwarfs, enter in a wider category of DM candidates named MAssive Compact Halo Objects (MACHOs), which also includes primordial BHs. The primordial BHs were postulated by S. W. Hawking in 1975 [40], and they would have been produced in the early Universe under an special initial perturbation spectrum. The fact that these law mass

\(^3\)The brown dwarfs are massive enough stars which never acquire high enough central temperature to start the hydrogen burning

\(^4\)A hot extended gas, had been observed through X-ray emissions, and was detected around many galaxies, groups and clusters.
BHs would be actually evaporated due to Hawking radiation for BH masses lower than $\sim 10^{-17} M_\odot$, sets an absolute lower limit for the masses of these DM candidates. Stronger constraints on the masses of MACHOs, appeared in the ’90s with the MACHO project, developed to search for gravitational microlensing events caused by the MACHOs as they pass nearby background stars. Even if this pioneering technique showed to be sensitive to a wide range of masses of $10^{-7} M_\odot < M < 100 M_\odot$; after several years of observations, in [41], it was concluded by means of statistical estimates that the mean mass of the MACHOs is between $0.15 M_\odot < M < 0.9 M_\odot$. Moreover, it was there shown that this objects can only correspond to about 20% of the DM in a typical halo, at 95% confidence level.

An independent and very strong limit to the total amount of baryon-composite DM, comes from primordial nucleosynthesis. The standard Big Bang model accounting for the first minutes of the life of the Universe, predicts with high exactitude, the abundance of light elements such as $^4$He, $^3$He, D, etc; being these elements a sensitive tracer of the mean baryon density of the Universe. Actually, in [42] the amount of these nuclides as obtained from recent observational data in different astrophysical environments (i.e., stars, galaxies, etc), is compared with its theoretically expected amount from BBN. Moreover, in [42] it is clearly shown the remarkably good agreement among primordial nucleosynthesis theory, observational data of light nuclides, and the temperature fluctuation power spectrum of the CMB; all of them indicating a baryonic contribution to the total matter density of the Universe, roughly of only a 15%. This is, $\Omega_B \approx 0.045$.

1.2.2 Non-baryonic DM: HDM and CDM

The fact that baryonic matter can not be the responsible for the missing mass problem in the structures of the Universe, forced the scientific community to start thinking in non-baryonic composite DM. Already since the beginning of the ’80s (see e.g. [23] and references therein), after the original proposal by B. Pontecorvo in 1957 for the possible existence of neutrino oscillations, heavy neutrinos were proposed as an appealing candidate to the DM problem. Within the SM extension with a mass generation mechanism, the neutrino flavor oscillations among the three known types implies a mass variation ($\Delta m^2$) between two oscillating types. This variation is explained through the $\Delta m^2$ dependence at a given time, on the mixing states probability (see e.g. [43]). Nevertheless, it is a hard task to
determine the exact rest mass of an SM neutrino. Double beta-decay laboratory experiments, imposes upper limits somewhat about a few $10^{-1}$ eV, while uppermost bounds from cosmology implies $m_\nu \lesssim$ few eV (see [43] for a pedagogical review).

Considering the fundamental relevance of the role of massive neutrinos (either SM or beyond SM) in galaxy structures, structure formation, and its corresponding early Universe counterpart; I will briefly discuss in next two pillars on cosmological neutrino mass bounds: the Gershtein & Zeldovich-like bound for ordinary neutrinos (i.e. $\nu_e, \nu_\mu, \nu_\tau$) [44, 45], and the Lee & Weinberg bound for beyond SM weakly-interacting neutrinos [46]. The first gave rise to the so called hot dark matter (HDM) paradigm, and the second to the so called cold dark matter (CDM) paradigm.

**HDM**

In the first fractions of a second in the life of the Universe, at typical temperatures of about $kT \sim 1$ MeV, the time-scale of the weak-interactions $t_{weak}$, which prevented neutrinos escaping from the plasma of $\nu, e^\pm, \gamma$, equals the time scale of the expansion of the Universe $t_H$ (see, e.g., [47] for a detailed derivation). This condition, i.e. $t_H \sim t_{weak}$, implies the condition for decoupling of the neutrinos from the relativistic plasma. When the time-scale of the interaction becomes greater than the expansion rate (i.e. $t_{weak} > t_H$), the neutrinos are ‘freeze out’ and therefore become fundamental relics which will play an essential role in the structure formation of the Universe. At this early epoch, at the aforementioned decoupling temperature $kT \sim 1$ MeV, it can be shown that the ratio between the particle number density of neutrinos and anti-neutrinos to the photon number density is: $^5 \frac{n_\nu}{n_\gamma} = 6/11$. Thus, after the neutrino decoupling, and under the assumption that this ratio holds until now (i.e. at redshift $z \sim 0$), the photon number density can be directly calculated from the temperature of the CMB, $T \approx 2.7$ K, to obtain the actual neutrino mass-energy density, $\rho_0(\nu\bar{\nu}) = m_\nu n_0(\nu\bar{\nu})$. Now, if this neutrino density is demanded to account for the total mass-energy density of the Universe today, $\rho_0 = 3H_0/(4\pi G)$ (with $H_0 \approx 70$ km/sMpc$^{-1}$ the

[^5]: For this calculation it has been assumed an ultra-relativistic approximation for the neutrinos (all types), zero chemical potential in the distribution functions of the bosons ($\gamma$ particles) and fermions, and a complete annihilation of the $e^\pm$ plasma into photons which re-hit the pre-existing photon radiation. This last assumption implies the following relation between the photon and neutrino-antineutrino temperatures $T_\gamma = T_{\nu\bar{\nu}}(11/4)^{1/3}$. 
1.2. Particle DM candidates and its nature

Hubble constant today, i.e. \( \rho_{\nu\bar{\nu}} \lesssim \rho_0 \), it is found the following upper bound for the neutrino (and anti-neutrino) mass

\[
m_\nu \lesssim 15 \text{eV} \quad \text{(Gershtein & Zeldovich-like).} \tag{1.1}
\]

Similar upper bounds based on basically the same hypothesis used here, were originally found in [44, 45].

Cosmological DM in the form of ordinary neutrinos with masses up to few eV is known as HDM. However, if the rest mass of the neutrinos as estimated by laboratory experiments is \( \sim 0.1 \) eV (or even lower), then they cannot answer for all the DM present of the Universe. Nevertheless, since late ’70s until early ’80s, there was not any reliable result for the neutrino mass from laboratories experiments, and therefore HDM in the form of eV-ish neutrinos was seriously considered as the main candidate for DM in the Universe. Its role in the large scale structures (cluster and group of galaxies) was analyzed with some detail in numerical N-body simulations of dissipationless gravitational clustering by that time. The first results presented impressive similarities with respect to observations on the large scale structure of the Universe, which had been recently obtained by J. Einasto and his group in a fantastic and pioneering research developed in late ’70s, by following the innovative ideas of Y. B. Zeldovich. In the early ’70s, Zeldovich developed the pancake theory of structure formation, i.e., a hierarchical top-down scenario in which larger structures form first and smaller structures appears as the fragmentation of the former. By that time a different bottom-up hierarchical clustering scenario had been proposed by J. Peebles. Thus, a clear contrast with the observations on the distribution of all the known structures was needed to discriminate between both theories. After a detailed analysis of all the available catalogs of galaxies, groups, clusters and super cluster of galaxies, M. Jõeveer, J. Einasto and others [48], it was shown that the Universe, rather than being structureless, forms a cosmic web. This web consists in a filamentary structure of superclusters, composed by galaxies, groups and clusters of galaxies; with voids in between the filaments. The overall feature of this network resembled more the original idea of Zeldovich, further supported by the numerical simulation of test particles developed at the beginning of the ’80s. All these similarities among the different approaches seemed to be in favor the HDM paradigm (see [49] for the detailed history).

Interestingly enough, already in the early 1980s, a Universe in which the
structures are formed of elementary particles as neutrinos, allowed J. Silk (among others) to make a link between the microscopic matter density perturbations in the early Universe, with the features of the macroscopic large scale structures. In his own words (1982): ‘It seems that the large-scale structure of the Universe is intimately related to its microscopic structure on elementary particle scales. This is perhaps not surprising if one recalls that it is the initial seed of fluctuations at the Planck epoch that are likely to determine the asymptotic growth of irregularities in the expanding Universe’.

However, still in early ’80s, more refined numerical simulations within the HDM paradigm, dealing with both, linear and non-linear clustering (i.e. at co-moving length scales of $> 10 \text{ Mpc}$ and $\lesssim 10 \text{ Mpc}$ respectively) showed that superclusters should have formed quite recently at $z \sim 1$ (see [50]). Then, when this result was contrasted with galaxy ages estimators from globular clusters and observation of distant quasars, both indicated galaxy formation at $z \gtrsim 3$; and therefore the top-down scenario was in serious problems.

Soon after, the scientific community in the field started to take more seriously the idea that the DM could be in the form of novel hypothetical heavy leptons of $m \gg eV$, created in the very first moments of the life of the Universe. This idea started with the pioneering work of Lee & Weinberg which I present in next.

CDM

In 1977 B. W. Lee and S. Weinberg [46] realized that, if an hypothetical neutrino decoupled form the primordial plasma of $\nu, e^\pm, \gamma$ in a non-relativistic manner, oppositely as ordinary neutrino did, then, a completely different bound for the particle mass can be obtained. For this, they introduced an hypothetical massive stable neutral lepton $L^0 \equiv L$, such that $m_L \gg \text{MeV}$. Because of its large mass, such a neutrino species must decouple in a non-relativistic regime implying that the correspondent decoupling temperature from the primordial plasma can no longer be $\sim 1 \text{ MeV}$, but higher. Otherwise, being so heavy, this hypothetical neutrinos would have much less number density at typical MeV decoupling temperatures (i.e. would reach a particle abundance $n_L \propto e^{-m/kT} \ll 1$). Thus, in order to treat the decoupling properly, they studied the evolution of particle’s phase space distribution function with the Boltzmann equation in a Friedam-Robertson-Walker (FRW) model. This is, they solved the equation $\hat{L}[f] = C[f]$ ($\hat{L}$ the covariant Liouville operator and $C$ the collision operator), for
1.2. Particle DM candidates and its nature

\( n_L \propto \int d^3p f(E,t) \), with \( f(E,t) \) following the Maxwell-Boltzmann statistics. To deal with the collision term, they treated the annihilations cross section in the ‘V-A’ model of charged weak interactions (i.e. \( \langle \sigma v \rangle = N_A G_F^2 m_L^2 / (2\pi) \)), taking the fudge factor \( N_A = 14 \) subject to all possible annihilation channels of \( L^0 \) for such a massive particle. While, to deal with the energy density of the Universe in the high temperature primordial plasma \(^6\), they took \( N_F = 4.5 \) which accounts for the different particle species that contributes effectively to \( \rho \). Once with all these considerations they solve numerically the Boltzmann equation for the time dependent particle abundance \( n_L(t) \), which after fulfilling the ‘freeze out’ condition (i.e. the interaction rate \( \sim \) expansion rate), they obtain for the present heavy neutrino and antineutrino density \( \rho_0(L) \propto m_L^{-1.85} \). They finally compared this neutrino density to the actual matter density in the Universe (i.e. \( \rho_0(L) \lesssim \rho_0 \)), to obtain

\[
m_v > 2 \text{GeV}/c^2 \quad \text{(Lee & Weinberg).} \tag{1.2}
\]

Where it can be shown that the decoupling temperature in correspondence to this bound is \( \gtrsim 140 \text{ MeV} \), this is, well before the relativistic decoupling of the ordinary neutrinos. It is important to note that the Lee and Weinberg bound \((2.6)\) is a lower limit, while the Gershtein and Zeldovich-like bound is an upper limit. Indeed, by solving the Boltzmann equation in a FRW model for stable and weakly interacting massive neutrinos which decouples in thermal equilibrium, either in a non-relativistic regime or a relativistic one, it can be more generally shown that when the energy density of the Universe is neutrino dominated, the only two possible mass bounds are given by equations \((2.3)\) and \((2.6)\) (see e.g. \cite{51} Fig. 5.2). This is, only HDM and CDM paradigms are possible within a weakly interacting massive neutrino thermal decoupling context.\(^7\)

Different candidates within physics beyond SM has been proposed up to now. The Weakly Interacting Massive Particles (WIMPS), as for example the hypothetical fermions considered by Lee & Weinberg, are among the most cited

\(^6\)With \( \rho = N_F a T^4 \propto H(T)^2 \), being \( \rho \) dominated by the highly relativistic particles \( v, \gamma, e^+ \) and \( e^- \).

\(^7\)It is important to mention that within this framework higher upper mass bounds (i.e. in the keV regime), can be found for hypothetical fermions decoupling relativistically and in thermodynamic equilibrium, by setting the energy scale of decoupling at electro-weak scales, i.e. at \( kT_d \sim 100 \text{ GeV} \) (see e.g. \cite{52}, equation 2.49).
candidates in the literature. In particular, the supersymmetric partners of the neutrino, i.e. the neutralino, can be a reasonable DM relic for rest masses of the order of 50 to 500 GeV. Other supersymmetric particles as the gravitino or photino has been considered as DM within this WIMP framework (see, e.g., [47]). Despite the attributed relevance to WIMPs in the realm of ΛCDM cosmologies since the ’90s, a strong tension appeared in the last few years regarding the validity of the assumed supersymmetry as a fundamental symmetry of nature. Specifically, in [53] it is shown that most natural supersymmetric models are ruled out from the gathered data at the actual accessible energies in the LHC, leaving still some hope for a potential discovery of this hypothesized symmetry in the future, but only in the range of energies above TeV physics. Another potential candidate in the context of CDM, which does not enter in the former category, is the axion, proposed in [54] as possible solution to the strong CP violation problem in QCD. In order to work as a viable cosmological DM particle, it should have a very light rest mass (i.e., $m \sim 10^{-5}$ eV). Despite being so light, this particle is associated with a ‘cold’ relic because is supposed to be born in a boson condensate and decoupling in a non-relativistic regime at a temperature of about 100 MeV (see, e.g., [51] for a discussion of its role in cosmology and structure formation).

Unfortunately, the different candidates for CDM particles beyond the SM physics, can not be distinguished within the free parameters of the ΛCDM cosmological models. Nonetheless, these models gave certainly quite successful results regarding the formation of the large scale structure of the Universe from numerical simulations (see, e.g., [55] for a recent paper), but in clear contrast with HDM cosmologies. Specifically, the success of the standard ΛCDM model relies in the consistency when contrasted with independent observational data sources as: i) The observed distribution of galaxies in the nearby ($\sim 150$ Mpc) Universe as obtained in [56]; ii) The baryon acoustic peaks in the power spectrum of galaxies as obtained by the the teams of the 2dF Galaxy Redshift Survey and the Sloan Digital Sky Survey (SDSS) [57, 58]; iii) The temperature fluctuations in the full CMB radiation spectrum as obtained by the Planck satellite [59]. See for example the book of M. S. Longair (2008) [47], for an excellent didactic discussion regarding the compatibility of the ΛCDM model against observations.

Nevertheless, as already commented in the section 1.1, some important problems remains at small scale structures ($\sim 10^2 - 10^3$ pc). Among the most relevant discrepancies between the effects of CDM in structure formation at these
small scales, and observations, I quote: i) a predicted overabundance of satellite
galaxies; ii) a prediction of cuspy halos which gives rise to the often called core-
cusp discrepancy, and iii) a prediction of too high dispersion velocities in dwarf
satellite galaxies. All these related effects can be understood mainly in terms of
the relatively too small free streaming length of the cold particles, which where
non-relativistic since their decoupling and therefore implying too many and too clumped
gravitationally bounded structures when compared with observations. See for example [16, 52, 60, 61], for discussion on all these problems.

1.2.3 WDM: a promising alternative

Since the ’80s, and contemporaneously to the two proposed forms of DM dis-
cussed before, another proposal for the dark candidates with masses in the keV
region appeared as an intriguing and appealing candidate, but this time from
astrophysical constraints. As we are dealing here with collisionless massive par-
ticles, constraints on its phase space density can be set by knowing its precise
evolution from the time of decoupling up to the approximate time of virializa-
tion of a DM halo. The starting point to this discussion is the famous work of S.
Tremaine and J. Gunn in 1979 [62]. They used the Liouville’s theorem plus the
concept of violent relaxation introduced in [63] to argue that, 1) the primordial
coarse-grained phase-space density must never increase, but only decrease, im-
plying therefore a maximum value for the late-time coarse-grained phase space
density ending in a DM halo. Just after decoupling, the microscopic fine-grained
phase space DF $f(p)$ coincides with the coarse-grained DF due to the absence
of phase mixing effects. Thus, they considered massive neutrinos which decou-
pled in a relativistic regime with an initial fine-grained phase space density,
denoted here as $Q_{f g}$, which at its maximum is given by $Q_{\text{max}} = 4(g_{\nu}h^{-3}/2)$, for the whole two neutrinos species with the corresponding anti-neutrinos (i.e.

\begin{enumerate}
\item[$^8$] A coarse-grained DF is defined as the average of the (microscopic) fine-grained DF $f(p)$ over
a finite volume, containing a measurable large enough number of particles.
\item[$^9$] The primordial fine-grained occupation number for neutrinos is given by the Fermi-Dirac
distribution function $f(p) = \left[\exp \left(\frac{e(p) - \mu}{kT} + 1\right)\right]^{-1}$, and its fine-grained phase-space density
by $Q_{f g} = \left(\frac{g_{\nu}}{h^3}\right)f(p)$; with $e^2 = \left(pc^2 + m^2c^4\right)$ and $g_{\nu}$ the spin degeneracy. The Fermi-Dirac
occupation number acquires its maximum at $p = 0$ (i.e., $f_{\text{max}} = \left[\exp \left(mc^2 - \mu/kT + 1\right)\right]^{-1}$). In
the case of $\mu = 0$ and for ordinary light neutrinos decoupling at $T_d \sim 1$ MeV (i.e. $mc^2/kT_d \to 0$)
as considered in [62], the resulting maximum occupation number is 1/2. Thus, the maximum
phase-space density is $Q_{f g,\text{max}} \equiv Q_{\text{max}} = 4(g_{\nu}h^{-3}/2)$, where the factor 4 overall corresponds to
the four kinds of neutrino considered.
\end{enumerate}
1. PARTICLE DM IN GALAXIES: INTRODUCTION

$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu$). They further proposed the simplest late-time velocity distribution for the formed DM halos: the Maxwellian distribution, leading to an isothermal-sphere-like solution for the hydrostatic equilibrium condition. More specifically, they further assumed: 2) that the central region of these bound systems of self-gravitating massive neutrinos are described by classical Newtonian solutions resembling isothermal gas spheres; 3) that their velocity distribution is Maxwellian and the maximum phase-space density of that self-gravitating configuration is given by

$$Q_{\text{iso}}^{\text{max}} = \rho_0 m^{-4}(2\pi \sigma^2)^{-3/2}$$

$m$ being the neutrino mass, $\rho_0$ the central density, $\sigma$ the one dimensional velocity dispersion and the King radius being $r_K^2 = 9\sigma^2 / 4\pi G \rho_0$. Then, relying in the fact that the primordial phase-space can only decrease as stated in 1), they made the comparison $Q_{\text{iso}}^{\text{max}} < Q_{\text{max}}$, to get the lower limit in the particle mass:

$$m > 101 \left( \frac{1 \, 100 \, \text{km/s}}{g_\nu} \frac{1 \, \text{kpc}}{\sigma} \frac{r_K}{r_0^2} \right)^{1/4} \text{eV}/c^2. \quad (1.3)$$

In the case of typical spiral galaxies with $\sigma \sim 100$ km/s and $r_K \sim 1$ kpc, the expression (1.3) sets a keV-ish lower bound for the massive neutrino (with i.e. $g_\nu = 2$), excluding in an elegant way (from phase-space constraints applied to DM halos) the eV mass-scales as obtained in the above section. More important is the fact that, if this new keV mass scale particles are pretended to provide the main DM budget in the Universe, the particles has to decouple either out of thermodynamic equilibrium or at electro-weak scales, as can be concluded from the former section. Otherwise only the eV-ish or GeV-sih mass scales are allowed.

The fact that the phase-space density plays a central role in structure formation with the subsequent setting of particle mass bounds, makes mandatory the need of a detailed and deep understanding on: i) the conditions to be fulfilled by

---

The physical meaning of the maximum phase-space density of a cored isothermal-like sphere $Q_{\text{iso}}^{\text{max}}$ can be easily understood by considering the usual definition of a coarse-grained phase-space density $Q = \langle n(t) / (p^2)^{3/2} \rangle$ (see e.g. [52]), with $n(t)$ the particle number density and $p$ the physical momentum both sensitive to the cosmological redshift. Then, at the time of structure formation $t_0 >> t_d$ (with $t_d$ the decoupling time) the particles are in a non-relativistic regime, and then the DM density reads $\rho_0 = m n_0$ and $\langle p^2 \rangle = m \langle v_0^2 \rangle$, with $\sigma \equiv \langle v_0^2 \rangle^{1/2}$ the root-mean-square velocity or dispersion velocity. Then, the phase-space density in terms of $\rho_0, m$ and the one dimensional $\sigma$ reads: $Q = \rho_0 / (m^4 \sigma^3)$, where $\rho_0$ is the maximal density at the center of the cored isothermal-like sphere implying therefore a maximum for $Q$ (i.e. $Q_{\text{iso}}^{\text{max}}$).
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the phase-space density at the moment of decoupling, and the functional form it can take; ii) the cosmological evolution on the phase-space density until the time of structure formation; iii) the functional form $f(p)$ of the late-time microscopic phase-space density of DM in a virialized halo.

The Gunn & Tremaine work ‘opened’ a profound insight into this matter. Indeed, in the last three decades all these points have been extensively studied. For example, recently in [52] the Gunn & Tremaine bound has been generalized for particles (fermions as well as bosons) decoupling from the primordial plasma either in a relativistic or non-relativistic regime, and either in or out of local thermodynamic equilibrium. Moreover, for particles decoupling in LTE, in that work it is clearly shown the fact that for light relics decoupling in a relativistic regime at temperatures $T_d \gg m$ (i.e. the case of ordinary neutrinos), the primordial coarse-grained DM phase space $Q_{DM} = n(t)/\langle p^2 \rangle^{3/2}$ does not depend on the particle mass (which is the framework considered in [62]). Instead, for heavy relics (or light bosons in a BEC as analyzed in [52]) decoupling in a non-relativistic regime, $Q_{DM}$ do depend on the particle mass, but also in the particle-anti particle annihilation cross section and in the effective number of degrees of freedom at decoupling $g_d$ (see [52] for details).

Thus, as first pointed out in [62], in order to obtain realistic particle DM mass bounds, the primordial coarse-grained DM phase space $Q_{DM}$ has to be compared with the one corresponding to an already collapsed galaxy structure (i.e. can never exceed the later), in which observations can be performed. Now, the accessible observational quantity is the following phase-space density $\rho / \sigma^3$, where $\rho$ is the matter density (at the inner halo region) and $\sigma$ is the line-of-sight velocity dispersion, both well constrained in DM dominated dSph galaxies (see e.g. [64] for recent data). On the other hand, as already known since the work of Lynden-Bell in 1967 [63], during collisionless gravitational dynamics, complex processes as phase mixing and violent relaxation occurs (see [5] chapter 4.10 for an introduction to these issues). These phase space mixing mechanisms makes the late-time coarse-grained phase space to diminish. Therefore, an important question to be answered is how much can diminish the phase space during this relaxation processes. Modern N-body numerical simulations (see e.g. [65] and more recently in WDM cosmologies [61]) indicate that the phase space density can decrease within one order (and up to two orders) of magnitude due to gravitational relaxation during structure formation (i.e. at $z \lesssim 10$).

Therefore in [52], by properly comparing the late-time coarse-grained DM
1. PARTICLE DM IN GALAXIES: INTRODUCTION

phase space $Q_{DM} \sim 10^{-1} \rho_{DM}/(\sigma_{DM}^3)$, where the factor $10^{-1}$ accounts for relaxation processes during galaxy formation obtained in numerical simulations, with the observed phase space density $\rho/\sigma^3$ in DM dominated dSph (i.e. $0.1\rho_{DM}/\sigma_{DM}^3 \sim \rho/\sigma^3$), it was concluded that:

- For DM fermions of $m \sim$ few keV which decouple relativistically (either in or out LTE) leads to primordial phase-space densities (at the moment of structure formation) $\rho_{DM}/\sigma_{DM}^3$, of the same order of magnitude as observed in inner halo regions of dSph galaxies, in compatibility with cored profiles instead of cuspy ones.

- CDM relics with masses $m \gtrsim 100$ GeV which decouple non-relativistically at weak scale temperatures $kT_d \sim 10$ MeV, are at odds with observed phase space densities in cored DM dominated dSph galaxies, i.e., implies phase-space density values many orders of magnitude larger than the observed ones. Moreover, these results showed that cold relics have to be associated with cuspy profiles.

Nevertheless, a more detailed comparison between high resolution data in galaxies and theory involving phase-space DFs, can be performed. Indeed, a comparison between observed galaxy rotation curves and its matter density profiles against the ones arising from an underlying equilibrium phase space DF $f(r,p)$, has been extensively studied in the literature in the last few decades. This approach faces a central subtlety: actually, it is an unknown thing how to determine the functional form $f(r,p)$ for a late-time microscopic phase-space density of DM in a virialized halo. I mention in next two different important attempts to attack this problem: I) To propose a fundamental physical principle from which to obtain $f(r,p)$; II) To propose a phenomenological expression for $f(r,p)$ (either first principle physics supported or not) from which the corresponding density profiles fit as best as possible the observations, and further compare it with the fitted profiles arising from numerical N-body simulations.

I) In some analogy to the maximization of the entropy reached by an ideal gas composed of collisional particles, a fundamental physical principle of maximum entropy for self-gravitating collisionless particles was proposed by Lynden-Bell in [63], out of which the most likely DF can be obtained at equilibrium. For this he introduced the concept of violent relaxation, which describes the most extreme relaxation process that can undergo a galaxy during its formation due to violent changes in the gravitational field of the system (see, e.g., [5]). As the
famous Boltzmann’s H-function \( H = - \int (F \log F) dx dv \), with \( F(x, v) \) a coarse-grained phase space DF, he interpreted the entropy as a particular H-function in the context of violent relaxation\(^{11}\). The strong result of this statistical approach is that the resulting coarse-grained equilibrium distribution \( F(r, p) \) is precisely of the form of the Fermi-Dirac DF.

As clearly shown in [67], the main importance of a DF of this form relays in the fact that it is able to prevent the ‘gravothermal catastrophe’ (see e.g. [5] chapter 7.3), which is an inherent fate of finite and isolated self-gravitating systems described by Maxwellian statistics. More specifically, in [67] it is demonstrated that for semi-degenerate equilibrium solutions described by a non-relativistic Fermi-Dirac DF, the central degeneracy (subject to the Pauli principle) is able to stop the gravitational collapse, and a global maximum of the entropy exists, contrary to the case of Maxwellian configurations. Moreover, for finite isothermal sphere solutions described by the Maxwell DF, it is important to mention that even if a local maximum entropy can be achieved under certain circumstances, solutions which present a density contrast between the center and the border of the configuration \( (n(0)/n(R) > 709) \) as observed in some normal galaxies (see, e.g., [67]), are unstable tending to the singular spheres \( n \propto r^{-2} \). This problem is naturally bypassed in the context of a Fermi-Dirac DF as detailed in [67]. Even if these results has been originally applied to the distribution of stars, the Lynden-Bell theory can be applied to DM due to the dissipationless nature of it [68]. Indeed it has been done already within this statistical framework in [69], for a composite system of neutrinos plus stars, concluding that the mass scale of the DM candidate has to be \( m \sim \text{few keV} \) to be in agreement with a gravitationally bounded system. This last result is in a remarkable coincidence with the results presented in the third chapter of this Thesis, this last nonetheless, achieved from a different approach.

Some interesting discussions about the extent of applicability of the statistical theory of violent relaxation in the context of galaxy formation (even in the context of DM) is presented in [68], where again it is argue the importance of taking into account degeneracy in Fermi-Dirac solutions contrary to historical expectations supporting Maxwellian distributions.

II) The first strong results obtained by modeling the distribution of DM in galaxies in terms of equilibrium configurations of collisionless self-gravitating

\(^{11}\)See [66] for an interesting discussion on the conditions to be fulfilled by the H-function, and the meaning of entropy in this framework.
1. PARTICLE DM IN GALAXIES: INTRODUCTION

particles, starts with the work of Gunn & Tremaine and its generalizations [52, 62], under the assumption of an underlying Maxwellian phase space DF. This DF can lead to isothermal spheres equilibrium configurations with cored density profiles characterized by the King radius and ending in the power law $r^{-2}$, in relatively good consistency with observations of some DM halos\(^{12}\) (see e.g. [5] chapter 4.3).

In the ’80s a generalization of the king model for fermions (in a low degenerate state) was developed by R. Ruffini and L. Stella [23], with corresponding density profiles similar to the King ones, and with phase space mass bounds differing in less than 10% compared with the Gunn & Tremaine limit (Eq. 1.3). Continuing with fermionic particles, in 1990, an extension of the former model was developed in [24], and done for any degree of fermion degeneracy at the center, within general relativity.

With the aim of interconnect my research with the historical contextualization of the role of WDM on galaxy scales given above, in next I briefly explain the scientific contribution provided in this Thesis.

1.3 OBJECTIVES

Continuing the discussions started in the point II) of the above issue, i.e. on how a proper knowledge of the late-time microscopic phase-space density of DM in a virialized halo $f(r, p)$, is essential to understand the role and distribution of DM in galaxies. I present in next the four main objectives of this Thesis which are detailed as follows:

1. To solve (numerically) in the more general way, the TOV equations for hydrostatic equilibrium of a thermal and semi-degenerate fermion gas including relativistic effects. After this is done, I demonstrate the existence of a novel core-halo DM profile composed by: 1) an inner core of almost constant density governed by degenerate quantum statistics; 2) an intermediate region with a sharply decreasing density distribution followed by an extended plateau, implying quantum corrections; 3) a decreasing density

\(^{12}\)Even more sophisticated phase space DFs similar to Maxwellian has been obtained by R. W. Michie and I. R. King in the ’60s [70, 71], by means of particular solutions of the Boltzmann equation, accounting for finite mass systems and escape velocities. Although these last DFs has been extensively used only to fit the surface brightness of some galaxies, i.e. applied to light profiles and not to DM.
1.3. OBJECTIVES

distribution $\rho \propto r^{-2}$ leading to flat rotation curves fulfilling the classical Boltzmann statistics. Different applications of this model are presented in CHAPTER 2. In sections 2.1 - 2.5, the following new objectives are reached:

**a)** To give a review of the former works in the field, before presenting the novel core-halo DM density profiles (as described above in 1), which have a definite halo mass $M_h$ and circular velocity $v_h$ at the halo radius $r_h$ of typical spiral galaxies; and to analyze the fermion mass constraints associated with these solutions. This is given in CHAPTER 2.1, based on the paper of C. R. Argüelles et. al. (2014) [28].

**b)** To apply the model described in 1) to observable quantities in DM halos, ranging from dSphs, to big spiral galaxies; to further analyze up to which extent their dense central cores may provide an alternative to IMBHs to SMBHs with masses $M_c \sim 10^4$ to $M_c \sim 10^7$ respectively, when the particle mass is in the keV regime. This is given in CHAPTER 2.2, based on the paper of R. Ruffini, C. R. Argüelles and J. A. Rueda [32].

c) For a fermion mass in the keV regime, to study the different one-parameter sequence of equilibrium configurations of the solutions described in 1), up to each critical point (represented in a central density $\rho_0$ - core mass $M_c$ diagram). To further analyze if some of these critical core-halo configurations are able to explain both: the most super-massive dark-compact-objects at the center of elliptical galaxies, together with the DM halo simultaneously. This is given in CHAPTER 2.3, based on the papers of C. R. Argüelles and R. Ruffini; and C. R. Argüelles et. al. (2014) [30, 31].

d) To find scaling laws for the quantum core and diluted halo in terms of the three free parameters of the model (temperature, degeneracy and particle mass), to further constrain them, when asking the model to reproduce the constant DM halo surface density (i.e. the universality law), found by F. Donato et. al. (2009) [72]. This is given in CHAPTER 2.4, based on the papers of B. M. O. Fraga, C. R. Argüelles, et. al. (2013) [27, 34].

e) To compare and contrast the three-parametric model described in 1) with other models resulting from N-body cosmological simulations as for example: the two parametric NFW model and the three parameter Einasto-like model, by means of a statistical analysis using the THINGS and SINGS high resolution data sample for spiral galaxies. This is given in CHAPTER
1. PARTICLE DM IN GALAXIES: INTRODUCTION


2. To provide a generalization of the Jeans equations in the context of galactic dynamics, for a self-gravitating system composed by DM and stars, being the phase space DF of the DM in the galaxy the one described in 1). The well resolved dwarf galaxies are considered within this approach, to naturally explain their nucleated central light excess behaviour. This is given in CHAPTER 3, based on the paper of C. R. Argüelles et. al. (2014) [33].

3. To extend the model described in 1), by adding new effective interactions (other than gravity) among the keV fermions. The aim is to apply the tractable theoretical framework of RMF to deal with the many body interaction problem and analyze the effects on the equation of state of the semi-degenerate fermi gas. This is done with the final objective of understanding the overall effects of the new effective interactions in the distribution of DM in galaxies; in particular to focus the attention on the possible sets of free parameters of the model, for which the dense quantum core works as an alternative to the massive BH thought to be hosted in SgrA*. The applications of this novel approach are presented in CHAPTER 4, based on the paper of C. R. Argüelles et. al. (2014) [74].

4. To analyze from first principle physics, other feasible alternative to the paradigm of the massive BH thought to be at the galactic center, based on a regular and relativistic dark Einstein cluster within the S2 orbit. This is given in CHAPTER 5, based on the paper of C. R. Argüelles and R. Ruffini (2014) [75].
Chapter 2

MASSIVE FERMIONS IN GR &
THE DISTRIBUTION OF DM IN
GALAXIES

Within the realm of non-baryonic DM in the form of collisionless massive fermions, I present in this chapter a model of self-gravitating and semi-degenerate fermions including relativistic effects. The integration of the system under study will allow to deal with distance-scales well below mpc up to Mpc, being able to segregate different marked physical regimes: a degenerate quantum regime in the central part of the configurations, and a classical Boltzmannian one in the outermost part. I further show the mean features of the equilibrium configurations in terms of the free parameters, and most important, how sensitive is the particle mass when the model is asked to fulfill all the accessible observables at galaxy scales. With the different applications of this model to galaxies, ranging from dwarf to big elliptical, it will be clear the central role of an underlying Fermi-Dirac late-time phase-space density ($g/h^3 f(r,p)$) of DM in a virialized halo. It is shown in particular, a natural and novel way to constraint the DM content in the very central part of dwarf spheroidal and spiral galaxies. More interestingly, it is analyzed in some detail up to which extents this central DM content can be interpreted as an alternative to the central BH paradigm, and how, when equilibrium solutions allows for it, the keV fermionic DM particle appears as a natural candidate.
2. MASSIVE FERMIONS IN GR & THE DISTRIBUTION OF DM IN GALAXIES

2.1 First approaches: from King to quantum (fermionic) profiles

The topic of this section is the problem of modeling the distribution of dark matter in galaxies in terms of equilibrium configurations of collisionless self-gravitating quantum particles. I first summarize the pioneering model of a Newtonian self-gravitating Fermi gas in thermodynamic equilibrium developed by Ruffini and Stella (1983) [23], which is shown to be the generalization of the King model for fermions. I further review the extension of the former model, as developed by Gao, Merafina and Ruffini (1990) [24], and done for any degree of fermion degeneracy at the center \( \theta_0 \), within general relativity. Finally, I present here as an original contribution to the field, the solutions of the density profiles and rotation curves corresponding to the Gao et. al. model, which have a definite mass \( M_h \) and circular velocity \( v_h \), at the halo radius \( r_h \) of the configurations, typical of spiral galaxies. This treatment allows to determine a novel core-halo morphology for the dark matter profiles, as well as a novel particle mass bound associated with those profiles.

2.1.1 Introduction: the first models

The work of Tremaine & Gunn [62] is one of the most quoted papers in the study of dark matter distribution in galactic halos. In that work the underlying phase-space DF in a virialized halo was assumed Maxwellian, implying isothermal-like spheres for the morphology of the DM density profiles over which to impose the observational constraints (see section 1.2.3 for details). One of the preliminary works in order to prove the analogy and differences between a classical and quantum self-gravitating system was advanced within a Newtonian approach in Ruffini & Stella (1983) [23]. There, the problem of a semi-degenerate system of fermions under gravitational interaction was approached, compared and contrasted with the classical King model.

They proposed a distribution function built for non-relativistic particles, which reads:

\[
f(v) = \begin{cases} 
1 - \exp \left[ -f^2 \left( v_e^2 - v^2 \right) \right] & , v \leq v_e \\
\exp \left[ f^2 \left( v^2 - \bar{\mu} \right) \right] + 1 & , v > v_e,
\end{cases}
\]
where $v_e$ is the escape velocity. In the limit $v_e \to \infty$, the usual Fermi distribution is obtained. The other two constant parameters are $j^2 = m/(2kT)$ and $\bar{\mu} = 2\mu/m$. The relevance of this $f(v)$ is that it is an extension of the King model to the case of a Fermi gas. Moreover, if the degeneracy parameter $\theta = j^2 \bar{\mu}$ is defined, it can be easily seen that when $\theta \to -\infty$, the non-degenerate limit is reached and the distribution function used by King is recovered. Instead, when $j^2 \to \infty$ and $\bar{\mu} \to v_e^2$ the degenerate limit is obtained, and the escape velocity is associated with the Fermi energy.

The energy integral,

$$E = v^2/2 + V(r),$$

(2.1)

together with the Jeans theorem for spherical systems, allowed them to simply relate the escape velocity with the gravitational potential by $v_e^2 = -2V$ (being $V = 0$ at the surface of the configuration). They finally solved the Poisson equation for $W = -2j^2V$, being the mass density given by $\rho \propto \int f(v)v^2dv$ which is related with $W$ via the $j$ parameter.

For simplicity, in the attempt to understand the physical interpretation of the parameters, a unique value for the central degeneracy parameter was assumed for the sake of example,

$$\theta(0) \equiv \theta_0 = 0,$$

(2.2)

and no other values for $\theta_0$ were explored at the time. The different normalized mass density solutions were given for different values of $W(0) \equiv W_0$ as shown in Fig. 2.1.

Under these special conditions the analogy between the self-gravitating system of fermions and the King model was proven, as well as a first attempt to justify the Tremaine & Gunn limit.

It soon became clear that these solutions, although interesting in reproducing classical results of the King profiles for a self-gravitating fermion gas, were actually extremely restrictive, and not representative of the general solutions for a relativistic self-gravitating system of massive fermions. These restrictions correspond to three different constraints: 1) $\theta_0 = 0$; 2) the application of a cut-off in the phase space which implies the elimination of an important family of solutions; and 3) the use of a non relativistic Newtonian approach.
Figure 2.1: Normalized density profiles for different values of $W_0$ and fixed $\theta_0 = 0$. The dotted curve corresponds to the analogous King profiles while the dashed curve represents the degenerate limit (taken from [23] with permission).
2.1. First approaches: from King to quantum (fermionic) profiles

A fundamental step was made by Gao, Merafina and Ruffini (1990) [24], to include special relativity effects in the phase space of the distribution function, as well as general relativity. Thus, they considered the relativistic Fermi-Dirac distribution function for the ‘inos’\(^1\) without any cut-off in their momentum space, i.e. 
\[
f(\epsilon(p)) = (\exp[(\epsilon(p) - \mu)/(kT)] + 1)^{-1},
\]
where \(\epsilon(p) = \sqrt{c^2p^2 + m^2c^4} - mc^2\) is the particle kinetic energy and \(\mu\) the chemical potential with the particle rest-energy subtracted off.

They wrote the system of Einstein equations in the spherically symmetric metric 
\[
g_{\mu\nu} = \text{diag}(e^{\nu}, -e^{\lambda}, -r^2, -r^2\sin^2 \theta),
\]
where \(\nu\) and \(\lambda\) depend only on the radial coordinate \(r\), together with the thermodynamic equilibrium conditions of Tolman [76], and Klein [77],
\[
e^{\nu/2}T = \text{const.}, \quad e^{\nu/2}(\mu + mc^2) = \text{const.}
\]
in the following dimensionless way,
\[
\frac{d\hat{M}}{d\hat{r}} = 4\pi\hat{r}^2\hat{\rho}, \quad \frac{d\theta}{d\hat{r}} = \frac{\beta_0(\theta - \theta_0) - 1}{\hat{r}^2(1 - 2\hat{M}/\hat{r})} \hat{M} + 4\pi\hat{P}\hat{r}^3, \quad \frac{d\nu}{d\hat{r}} = \frac{\hat{M} + 4\pi\hat{P}\hat{r}^3}{\hat{r}^2(1 - 2\hat{M}/\hat{r})}, \quad \beta_0 = \beta(r)e^{(\nu(r) - \nu_0)/2}. \tag{2.6}
\]

There, the following dimensionless quantities were introduced: \(\hat{r} = r/\chi, \hat{M} = GM/(c^2\chi), \hat{\rho} = G\chi^2\rho/c^2\) and \(\hat{P} = G\chi^2P/c^4\), where \(\chi = 2\pi^{3/2}(\hbar/mc)(m_p/m)\) is the dimensional factor which has unit of length and scales as \(m^{-2}\); with \(m_p = \sqrt{\hbar c/G}\) the Planck mass, and the temperature and degeneracy parameters, \(\beta = kT/(mc^2)\) and \(\theta = \mu/(kT)\), respectively. I do not include the presence of antifermions, i.e. I consider temperatures that satisfy \(T \ll mc^2/k\).

The mass density \(\rho\) and pressure \(P\) are expressed in terms of the standard infinite integrals in momentum space weighted with the \(f(\epsilon(p))\) already given,

\(^1\)‘Ino’ refers to any beyond standard model neutrino that is suitable within the astrophysical application of the model presented here: for example the sterile neutrino (see e.g. [21]).
and reads (with the particle helicity $g = 2$)

$$\rho = m_0^2 \frac{2}{\hbar^3} \int f(p) \left[ 1 + \frac{\epsilon(p)}{m c^2} \right] d^3 p, \quad (2.7)$$

$$P = \frac{1}{3} \frac{2}{\hbar^3} \int f(p) \left[ 1 + \frac{\epsilon(p)}{m c^2} \right]^{-1} \left[ 1 + \frac{\epsilon(p)}{2m c^2} \right] \epsilon d^3 p, \quad (2.8)$$

In that work, they solved the initial condition problem for the variables of the system, $\theta(r)$, $\beta(r)$, $\nu(r)$, and $M(r)$, by giving at $r = 0$ (and indicated by a subscript '0'), $M_0 = \nu_0 = 0$, while giving arbitrary values for the temperature and degeneracy parameters $\beta_0$ and $\theta_0$, respectively. In Figs. 2.2–2.3, different normalized mass density solutions for different $\theta_0 < 0$ and $\theta_0 \geq 0$ respectively are shown, for a fixed non-relativistic central temperature parameter $\beta_0$.

Figure 2.2: Different density profiles for different $\theta_0 < 0$ and fixed $\beta_0$ in dimensionless variables. To note the simple cored plus $r^{-2}$ morphology (taken from [24] with permission).
2.1. First approaches: from King to quantum (fermionic) profiles

Figure 2.3: Different density profiles for different $\theta_0 \geq 0$ and fixed $\beta_0$ in dimensionless variables. To note the more complex core plus ‘plateau’ plus $r^{-2}$ morphology (taken from [24] with permission).

It is important to note that the system (2.3–2.6) has no particle mass $m$ dependence when solved in the dimensionless variables, while instead the physical magnitudes such as $r$ and $\rho$ have an explicit dependence on $m$ through the dimensional factor $\chi(m)$. The fact that they were mainly interested in the general properties of the solutions without going through the physical magnitudes, no particle mass constraints were put there.

2.1.2 Massive fermions within GR in normal galaxies I

I have recently returned to the Gao et. al. work, and propose a completely different way for solving the boundary condition problem for the system (2.3–2.6), in order to fulfill the observationally inferred values of typical dark matter halos in spiral galaxies as given in [9]. Namely, for given initial conditions $M_0 = \nu_0 = 0,$
arbitrary \( \theta_0 \) (depending on the chosen central degeneracy), and defining the halo radius \( r_h \) at the onset of the flat rotation curve, I solve an eigenvalue problem for the central temperature parameter \( \beta_0 \), until the observed halo circular velocity \( v_h \) is obtained. After this, I solve a second eigenvalue problem for the particle mass \( m \) until the observed halo mass \( M_h \) is reached at the radius \( r_h \).

The quest has been to use all these information in order to put constraints on the mass of the ‘ino’ in galactic halos by introducing the observational properties possibly to be utilized in this research.

Interestingly enough, as detailed in the next section (2.2) and based on [32], it turns out that only for an specific range of \( \theta_0 > 0 \) these two eigenvalue problems can be solved together, implying as a consequence, a novel reach morphology for the density profiles as well as a novel particle mass bound associated with it. The density profiles presents a quantum degenerate core, followed by a low degenerate plateau until it reach the \( r^{-2} \) Boltzmannian regime corresponding to the flat part in the rotation curve. In Fig. 2.4 I show a family of density profiles for different values of \( \theta_0 \) which fulfills the mentioned halo constraints. I also plotted for comparison the purely Boltzmannian curve which agrees with the same observed halo magnitudes.

As can be seen from Fig. 2.4, I obtain from this novel analysis a lower mass bound for the ‘ino’ mass \( m \geq 0.42 \text{ keV}/c^2 \) in typical spiral galaxies, which is larger (but comparable) than the corresponding sub-keV bounds found by Gunn & Tremaine (1.3), and by Ruffini & Stella in [23], for the same halo observables.

It is interesting that the quantum and relativistic treatment of the configurations considered here are characterized by the presence of central cored structures unlike the typical cuspy configurations obtained from a classic non-relativistic approximation, such as the ones of numerical N-body simulations in [14]. This naturally leads to a possible solution to the well-known core-cusp discrepancy [60].
2.1. First approaches: from King to quantum (fermionic) profiles

Figure 2.4: Physical density profiles for specific ino masses \( m \) and central degeneracies \( \theta_0 \) fulfilling the observational constraints \( M_h = 1.6 \times 10^{11} M_\odot \) and \( v_h = 168 \text{ km/s} \) at \( r_h = 25 \text{ Kpc} \) (as taken from [9] and detailed in [32]). In dot-dashed line the purely Boltzmannian profile for comparison.
2.2 A novel core-halo distribution of DM in dwarf and spiral galaxies

In this section I solve in the more general way, the TOV equations for hydrostatic equilibrium of a thermal and semi-degenerate fermion gas including relativistic effects, following the same boundary condition analysis as done already in section 2.1. In this case, I present in a more explicit manner the general characteristic of the density profiles, which shows a segregation of three physical regimes: 1) an inner core of almost constant density governed by degenerate quantum statistics; 2) an intermediate region with a sharply decreasing density distribution followed by an extended plateau, implying quantum corrections; 3) a decreasing density distribution \( \rho \propto r^{-2} \) leading to flat rotation curves fulfilling the classical Boltzmann statistics. I further show that this model can be applied to be consistent with a wide range of typical DM halo parameters (halo mass \( M_h \) and rotation velocity \( v_h \) at the halo radius \( r_h \)) ranging from dwarf galaxies up to big spirals. It is further demonstrated that the particle mass \( m \) is an eigenfunction of the mass of the inner quantum core, such that for an ‘ino’ mass \( m \) 10 keV, it is remarkably shown how typical dwarf halos harbor a dense quantum core of \( 10^4 M_\odot \) at their center, while typical spiral halos harbors compact quantum cores of \( 10^6 - 10^7 M_\odot \) well below sub-parsec scales.

2.2.1 Massive fermions within GR in normal galaxies II

Following Gao et al. [24], I consider the model described by the system of equations (2.3–2.6) already given in section 2.1. This is, I consider a system of general relativistic self-gravitating bare massive fermions in thermodynamic equilibrium. No additional interactions are assumed for the fermions besides their fulfillment of quantum statistics and the relativistic gravitational equations. I refer to this bare particles more generally as ‘inos’, leaving the consideration of possible additional fundamental interactions to be determined by further requirements to be fulfilled by the model. Already this treatment of bare fermions leads to a new class of configurations of equilibrium and, correspondingly, to new limits to the ‘ino’ mass. This is a necessary first step in view of a final treatment involving additional interactions to be treated self-consistently.

Recalling the system variables \([M(r), \theta(r), \beta(r), v(r)]\), I integrate Eqs. (2.3–2.6) for given initial conditions at the center, \( r = 0 \), in order to be consistent with
the observed dark matter halo mass \( M(r = r_h) = M_h \) and radius \( r_h \), defined in our model at the onset of the flat rotation curves. The circular velocity is \( v(r) = \sqrt{GM(r)/[r - 2GM(r)/c^2]} \) which at \( r = r_h \), is \( v(r = r_h) = v_h \).

I first apply this model to typical spiral galaxies, similar to our own galaxy, adopting dark matter halo parameters \([2, 9]\):

\[
r_h = 25 \text{ kpc}, \quad v_h = 168 \text{ km/s}, \quad M_h = 1.6 \times 10^{11} M_\odot.
\]  

(2.9)

I integrate then Eqs. (2.3–2.6) for selected values of \( \theta_0 \) and \( m \), corresponding to different degenerate states of the gas at the center of the configuration. The value of \( \beta_0 \) is actually an eigenvalue which is found by a trial and error procedure until the observed values of \( v_h \) and \( M_h \) at \( r_h \) are obtained. I show in Figs. 2.5–2.6 the density profiles and the rotation curves as a function of the distance for a wide range of parameters \( (\theta_0, m) \), for which the boundary conditions in (2.9) are exactly fulfilled.

Regarding the underlying physical characteristics of the equilibrium solutions of this model, is of central importance to understand the role of the full Fermi-Dirac DF when coupled to gravity: the phase-space distribution encompasses both the classical and quantum regimes. Correspondingly, the integration of the equilibrium equations leads to three marked different regimes (see Fig. 2.5): a) the first consisting in a core of quantum degenerate fermions. These cores are characterized by having \( \theta(r) > 0 \). The core radius \( r_c \) is defined by the first maximum of the velocity curve. A necessary condition for the validity of this quantum treatment of the core is that the interparticle mean-distance, \( \lambda_c \), be smaller or of the same order, of the thermal de Broglie wavelength of the inos, \( \lambda_B = h/\sqrt{2\pi mkT} \). As we show below (see Fig. 2.7), this indeed is fulfilled in all the cases here studied. b) A second regime where \( \theta(r) \) goes from positive to negative values for \( r > r_c \), all the way up to the so called classical domain where the quantum corrections become negligible. This transition region consists in a sharply decreasing density followed by an extended plateau. c) The classical regime described by Boltzmann statistics and corresponding with \( \theta(r) \ll -1 \) (for \( r \gtrsim r_h \)), in which the solution tends to the Newtonian isothermal sphere with \( \rho \sim r^{-2} \), where the flat rotation curve sets in.

I define the core mass, the circular velocity at \( r_c \), and the core degeneracy as \( M_c = M(r_c), \quad v_c = v(r_c) \) and \( \theta_c = \theta(r_c) \), respectively. In Table 2.1 I show the core
Figure 2.5: Mass density and degeneracy parameter profiles for specific ‘ino’ masses $m$ and central degeneracies $\theta_0$ fulfilling the observational constraints (2.9). The density solutions are contrasted with a Boltzmannian isothermal sphere with the same halo properties. It is clear that all the configurations, for any value of $\theta_0$ and corresponding $m$, converge for $r \gtrsim r_h$ to the classical Boltzmannian isothermal distribution.
Figure 2.6: Rotation velocity curves for ‘ino’ masses $m$ and central degeneracies $\theta_0$ fulfilling the observational constraints (2.9). The Boltzmannian rotation curve subject to the same halo boundary conditions is plotted for comparison. It is clear how the Boltzmann distribution, is as it should be, independent of $m$; while the determination of the mass of the ‘ino’ is a function of the mass and radius of the degenerate core. All the configurations for $r \sim r_h$ asymptotically approaches the Boltzmann distribution. Interestingly, when the value $M_c(r < 10^{-2} \text{ pc}) \sim 10^6 M_\odot$ is adopted, at distances of $r \gtrsim \text{ few } 10^2 \text{ pc}$ the onset of the classical Boltzmann regime takes place. This is consistent with the observed cored nature of the innermost resolved regions in spiral galaxies as analyzed in [9].
properties of the equilibrium configurations in spiral galaxies, for a wide range of \((\theta_0, m)\). For any selected value of \(\theta_0\) we obtain the correspondent ‘ino’ mass \(m\) to fulfill the halo properties (2.9), after the above eigenvalue problem of \(\beta_0\) is solved.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\theta_0 & m \text{ (keV/c}^2\text{)} & r_c \text{ (pc)} & M_c(M_\odot) & v_c \text{ (km/s)} & \theta_c \\
\hline
11 & 0.420 & 3.3 \times 10^1 & 8.5 \times 10^8 & 3.3 \times 10^2 & 2.1 \\
25 & 4.323 & 2.5 \times 10^{-1} & 1.4 \times 10^7 & 4.9 \times 10^2 & 5.5 \\
30 & 10.540 & 4.0 \times 10^{-2} & 2.7 \times 10^6 & 5.4 \times 10^2 & 6.7 \\
40 & 64.450 & 1.0 \times 10^{-3} & 8.9 \times 10^4 & 6.2 \times 10^2 & 8.9 \\
58.4 & 2.0 \times 10^3 & 9.3 \times 10^{-7} & 1.2 \times 10^2 & 7.5 \times 10^2 & 14.4 \\
98.5 & 3.2 \times 10^6 & 3.2 \times 10^{-13} & 7.2 \times 10^{-5} & 9.8 \times 10^2 & 21.4 \\
\hline
\end{array}
\]

Table 2.1: Core properties for different equilibrium configurations fulfilling the halo parameters (2.9) of spiral galaxies.

It is clear from Table 2.1 and Figs. 2.5–2.6 that the mass of the core \(M_c\) is strongly dependent on the ‘ino’ mass, and that the maximum space-density in the core is considerably larger than the maximum value considered in [62] for a Maxwellian distribution. Interestingly, as can be seen from Fig. 2.5, the less degenerate quantum cores in agreement with the halo observables (2.9), are the ones with the largest sizes, of the order of halo-distance-scales. In this limit, the fermion mass acquires a sub-keV minimum value which is larger, but comparable, than the corresponding sub-keV Gunn & Tremaine bound (1.3), for the same halo observables. Indeed, Eq. (1.3) gives a lower limit \(m \approx 0.05 \text{ keV/c}^2\) using the proper value for the King radius, \(r_K \approx 8.5 \text{ kpc}\), for \(\sigma = \sqrt{2/5}v_h\) and \(\rho_0 = 2.5 \times 10^{-2}M_\odot/\text{pc}^3\), associated to the Boltzmannian density profile in Fig. 2.5.

In the case of a typical spiral galaxy, for an ‘ino’ mass of \(m \sim 10 \text{ keV/c}^2\), and a temperature parameter \(\beta_0 \sim 10^{-7}\), obtained from the observed halo rotation velocity \(v_h\), the de Broglie wavelength \(\lambda_B\) is higher than the interparticle mean-distance in the core \(l_c\), see Fig. 2.7, safely justifying the quantum-statistical treatment applied here.

I turn now to the issue of an alternative interpretation to the black hole in SgrA*. From the results presented in Table 2.1 and Figs. 2.5–2.6 it is possible to conclude that a compact degenerate core mass \(M_c \sim 4 \times 10^6M_\odot\) is definitely possible corresponding to an ‘ino’ of \(m \sim 10 \text{ keV/c}^2\). However, the core radius...
of the configuration is larger by a factor $\sim 10^2$ than the one obtained with the closest observed star to Sgr A*, i.e. the S2 star [78]. Therefore, this means that the semi-degenerate quantum core in the density profile of a MW like galaxy is not compact enough. This important issue about the compactness of the quantum core in relation to the observations in SgrA*, is still under investigation by myself and co-authors in [74], and the main results of that research are presented in Chapter 4. My coauthors and I are attacking this problem through the introduction of additional interactions between the ‘inos’, which will affect the mass and radius of the new dense quantum cores depending on the interaction adopted. This is analogous for instance to the case of neutron stars, where nuclear fermion interactions strongly influence the mass-radius relation (see, e.g., [79]) . This may well make the mass and radius of this dark matter quantum core to fulfill the observational constraints imposed by the S2 star.

In next I compare and contrast in Fig. 2.8 the theoretical curves of Fig. 2.5 with the observationally inferred ones. Besides the fact that the presence of cored structures within this model, may naturally leads to a solution to the well-known core-cusp discrepancy; it is interesting to further notice that the difference between the ‘inos’ central core, the cuspy NFW profile, as well as the possible black
hole nature of the compact source in SgrA*, will certainly reactivate the development of observational campaigns in the near future. There is the interesting possibility, in view of the BlackHoleCam Project based on the largest Very Long Baseline Interferometry (VLBI) array\(^2\), to verify the general relativistic effects expected in the surroundings of the central compact source in SgrA*. Such effects depend on whether the source is modeled in terms of the RAR model presented here (with the possible inclusion of fermion interactions [74]), or as a black hole. To compare and contrast these two alternatives is an observational challenge now clearly open.

![Figure 2.8](image.png)

Figure 2.8: The cored behavior of the dark matter density profile from the Ruffini-Argüelles-Rueda (RAR) model is contrasted with the cuspy Navarro-Frenk-White (NFW) density profile [14], and with a cored-like Einasto profile [80, 81]. The free parameters of the RAR model are fixed as \( \beta_0 = 1.251 \times 10^{-7} \), \( \theta_0 = 30 \) and \( m = 10.54 \text{ keV}/c^2 \). The corresponding free parameters in the NFW formula \( \rho_{\text{NFW}}(r) = \rho_0 r_0 / [r(1 + r/r_0)^2] \) are chosen as \( \rho_0 = 5 \times 10^{-3} \text{M}_\odot \text{pc}^{-3} \) and \( r_0 = 25 \text{ kpc} \), and for the Einasto profile \( \rho_{\text{E}}(r) = \rho_{-2} \exp\left[-2n(r/r_{-2})^{1/n} - 1\right] \), \( \rho_{-2} = 2.4 \times 10^{-3} \text{M}_\odot \text{pc}^{-3} \), \( r_{-2} = 16.8 \text{ kpc} \), and \( n = 3/2 \). In the last two models, the chosen free parameters are typical of spiral galaxies according to [9, 82].

\(^2\)http://horizon-magazine.eu/space


### 2.2.2 Application to dwarf and spiral galaxies

Following the analysis developed here for a typical spiral, I have also considered two new different sets of physical dark matter halos: \( r_h = 0.6 \) kpc; \( v_h = 13 \) km/s; \( M_h = 2 \times 10^7 M_\odot \) for typical dwarf spheroidal galaxies, e.g. \[64\]; and \( r_h = 75 \) kpc; \( v_h = 345 \) km/s; \( M_h = 2 \times 10^{12} M_\odot \) for big spiral galaxies, as analyzed in \[83\]. For big spirals, \( \lambda_B/l_c = 5.3 \), while for typical dwarfs galaxies \( \lambda_B/l_c = 4.1 \), justifying the quantum treatment in both cases.

A remarkable outcome of the application of our model to such a wide range of representative dark halo galaxy types, from dwarfs to big spirals, is that for the same ‘ino’ mass, \( m \sim 10 \) keV/c\(^2\), we obtain respectively core masses \( M_c \sim 10^4 M_\odot \) and radii \( r_c \sim 10^{-1} \) pc for dwarf galaxies, and core masses \( M_c \sim 10^7 M_\odot \) and radii \( r_c \sim 10^{-2} \) pc for big spirals. This leads to a possible alternative to intermediate (\( \sim 10^4 M_\odot \)) and more massive (\( \sim 10^{6-7} M_\odot \)) black holes, thought to be hosted at the center of the galaxies. In the case of the most DM dominated objects (i.e. dwarf galaxies), our approach with fermion masses of \( m \sim 10 \) keV/c\(^2\) naturally explains why the observations are never compatible with central massive dark objects of the order \( M > 10^5 M_\odot \), but instead more likely of \( 10^4 M_\odot \), as found here. Interestingly enough, in the nearby dwarf NGC 205, a massive central dark object with an upper bound of \( \sim 2 \times 10^4 M_\odot \) was inferred from the observations at sub-parsec scales in \[84\], there interpreted as a black hole. This last issue is treated in details in section 3, in presence of baryonic matter.

As a by-product of the results given in the above paragraph, I have obtained out of first principles, a possible universal relation between the dark matter halos and the super massive dark central objects. For a fixed ‘ino’ mass \( m = 10 \) keV/c\(^2\), I found the \( M_c-M_h \) correlation law

\[
\frac{M_c}{10^6 M_\odot} = 2.35 \left( \frac{M_h}{10^{11} M_\odot} \right)^{0.52},
\]  

(2.10)

valid for core masses \( \sim [10^4, 10^7] M_\odot \) (corresponding to dark matter halo masses \( \sim [10^7, 10^{12}] M_\odot \)). Regarding the observational relation between massive dark compact objects and bulge dispersion velocities in galaxies (the \( M_c-\sigma \) relation \[85\]), it can be combined with two observationally inferred relations such as the \( \sigma-V_c \) and the \( V_c-M_h \) correlations, where \( V_c \) is the observed halo circular velocity and \( M_h \) a typical halo mass. This was done in \[35\] to find, by transitivity, a new correlation between central mass concentrations and halo dark masses (\( M_c-M_h \)).
Interestingly, such a correlation matches the one found above in Eq. (2.10) in the range $M_c = [10^6, 10^7] M_\odot$, without assuming the black hole hypothesis.

Such an approach of a core surrounded by a non-relativistic halo, is a key feature of the configurations presented here. It cannot however be extended to quantum cores with masses of $\sim 10^9 M_\odot$. Such core masses, observed in Active Galactic Nuclei (AGN), overcome the critical mass value for gravitational collapse $M_{cr} \sim M_{pl}^3/m^2$ for keV fermions, and therefore these cores have to be necessarily black holes. Indeed, this important issue is treated in detail in next section, and is based on the papers done by myself and coauthors in [30, 31]. The characteristic signatures of such supermassive black-holes, including jets and X-ray emissions, are indeed missing from the observations of the much quiet SgrA* source, or the centers of dwarf galaxies.

2.2.3 Conclusions

In conclusion:

I) A consistent treatment of self-gravitating fermions within general relativity has been here introduced and solved with the boundary conditions appropriate to flat rotation curves observed in galactic halos of spiral and dwarf galaxies. A new structure has been identified: 1) a core governed by quantum statistics; 2) a velocity of rotation at the surface of this core which is bounded independently of the mass of the particle and remarkably close to the asymptotic rotation curve; 3) a semi-degenerate region leading to an asymptotic regime described by a pure Boltzmann distribution, consistent with the flat rotation curves observed in galaxies.

II) The treatment here presented is by construction different from the one of Tremaine and Gunn [62]. It is interesting to notice that their conclusions are reached by adopting the maximum phase-space density, $Q_{\text{max}}^h \sim \rho_0^h m^{-4} \sigma_h^{-3}$, at the center of a halo described by a Maxwellian distribution, while in this model the maximum phase-space density is reached at the center of the dense quantum core described by Fermi-Dirac statistics, $Q_{\text{max}}^c \sim \rho_0^c m^{-4} \sigma_c^{-3}$. An entire new family of solutions exists for larger values of central phase-space occupation numbers, always in agreement with the halo observables (see Fig. 2.5). Now, since these phase-space values, by the Liouville’s theorem, can never exceed the maximum primordial phase-space density at decoupling, $Q_{\text{max}}^d$, we have necessarily $Q_{\text{max}}^{h,c} < Q_{\text{max}}^d$. Then, considering that all our quantum solutions satisfy $Q_{\text{max}}^c > Q_{\text{max}}^h$, it
directly implies larger values of our ‘ino’ mass with respect to the Tremaine and Gunn limit (1.3). As I have quantitatively shown, e.g. for the case of typical spiral galaxies, the two limits become comparable for our less degenerate ($\theta_0 \approx 10$) quantum cores in agreement with the used halo observables (2.9).

III) For $m \sim 10 \text{ keV}/c^2$ a universal relation between the mass of the core $M_c$ and the mass of the halo $M_h$ has been found. This universal relation applies in a vast region of galactic systems, ranging from dwarf to big spiral galaxies with core masses $\sim [10^4, 10^7] M_\odot$ (corresponding to dark matter halo masses $\sim [10^7, 10^{12}] M_\odot$).

Regarding the possible nature of the fermionic DM candidate here presented, a very interesting possibility is the right-handed neutrino in the minimal (non-supersymemtric) extension of the SM ($\nu$MSM) proposed in [18, 20, 21]. This standard model extension involves three right-handed neutrino states, in addition to the three left-handed active neutrinos of the standard model (SM) sector, of which the lightest, of mass at most a few tens of keV, can live longer than the age of the Universe, thus constituting a viable dark matter candidate. However, is important to point out that the conclusions given in this section regarding the mass of the dark matter fermion of about $10 \text{ keV}/c^2$, do not imply the necessity to use the sterile neutrino as the unique candidate. Nevertheless, if this would be the case, it is important to recall that cosmological models with right handed neutrinos of $m \sim 10 \text{ keV}/c^2$ within the nuMSM extension, accounting either for the totality or for a fraction the total dark matter budget, are in agreement with all the up to date astrophysical constraints including Lyman alpha forest, as detailed in [86, 87].

2.3 DM in big elliptical galaxies and their central super massive dark objects

In this section I present a detailed analysis of the model presented in the former sections in an extreme relativistic regime, which will allow to deal with the most massive central dark objects of $M_c \sim 10^9 M_\odot$ at miliparsec scales. For this, by fixing the fermion mass $m$ in the keV regime, the different solutions depending on the free parameters of the model, are systematically constructed along the one-parameter sequences of equilibrium configurations up to the critical point, which is represented by the maximum in a central density ($\rho_0$) Vs. core mass.
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$(M_c)$ diagram. In this context, I present the density profiles and rotation curves for a fermion mass $m \sim 10 \text{ keV}/c^2$, for the case in which the compact cores just reach the Oppenheimer-Volkoff limit. Clearly, this approach makes the use of a General Relativistic treatment to be mandatory. In exploring the full range of the free model parameters such as temperature and degeneracy, I conclude that if a super massive dark object of $M_c \sim 10^9 M_\odot$ is formed at the center, no astrophysical DM halo structure should be present simultaneously in that system.

The question of a simultaneous co-existence of super-massive dark objects and DM halos in big elliptical galaxies gives an extra motivation for the issue of this section. Elliptical and early type galaxies, are all expected to harbor super-massive dark objects (presumably SMBHs) at their center (see, e.g., [88], and [10] for extended review on this topic). Nevertheless, observational and technical obstacles as the intricate relation involving the spatial distribution of its density, mass-to-light ratios, the estimation of the very massive dark central mass; and also the low surface brightness beyond $R_e$, makes as a consequence a hard task to provide a definitive argument for the existence of DM halos in this biggest galaxies. For example, no evidence for DM halos in a sample of elliptical galaxies has been reported in [11, 12], by the use of (mass tracers) kinematical methods. Even if there are cases which present compelling evidence for the presence of DM halos, such as the case of the big elliptical M87 containing a super-massive dark object of $7 \times 10^9 M_\odot$, in general, the more interesting constraints on DM in early-type galaxies are restricted to the more massive systems, which are placed near the center of group or clusters. This implies to be a difficult task to confirm whether the existence of extensive halos are an inherent property of the galaxy itself, or whether corresponds to the group-scale (see e.g. [5] chapter 4.9.2). Some discussion to this respect in relation with observations and formation history at different cosmological redshift $z$ is given at the end of this section.

2.3.1 Critical configurations in semi-degenerate fermionic systems

Using the model presented in the two former sections, and fixing the fermion mass $m$ in the keV regime, different solutions depending on the free parameters of the model, $\theta_0$ and $\beta_0$, are systematically constructed along the one-parameter sequences of equilibrium configurations up to the critical point, which is repre-
2.3. DM in big elliptical galaxies and their central super massive dark objects

represented by the maximum in a central density \((\rho_0)\) Vs. core mass \((M_c)\) diagram.

For sake of definiteness, the system of Eqs. (2.3–2.6) is solved for a fixed particle mass \(m = 8.5\) keV/c², with initial conditions \(M(0) = \nu(0) = 0\), and given parameters \(\theta_0 > 0\) (depending on the chosen central degeneracy), and \(\beta_0\). I thus construct a sequence of different thermodynamic equilibrium configurations where each point in the sequence has different central temperatures \(T_0\) and central chemical potential \(\mu_0\), so that satisfy the \(\theta_0\) fixed condition.

Defining the core radius \(r_c\) of each equilibrium system at the first maximum of its rotation curve, I represent the results obtained for each sequence in a central density \((\rho_0)\) vs. core mass \((M_c)\) diagram (see Fig.2.9). It is shown that the critical core mass \(M_{c\tau}\) is reached at the maximum of each \(M_c(\rho_0)\) curve.

Figure 2.9: Different sequences of equilibrium configurations plotted in a central density \((\rho_0)\) Vs. core mass \((M_c)\) diagram. The critical core mass is reached at the maximal value of \(M_c\). Each sequence is built for selected values of \(\theta_0 = \mu_0/kT_0\) and different values of \(T_0, \mu_0\) varying accordingly.

It is important to emphasize that the method followed here to define the critical points along each family of thermodynamic equilibrium configurations, fulfills the turning point definition given in [89], which allows them to formally
demonstrate Sorkin’s theorem [90], showing the existence of a thermodynamic instability on one side of the turning point. 

In Table 2.2 I show a set of central critical parameters of the model together with the correspondent critical core masses, for a very wide range of fixed central degeneracy parameters \( \theta_0 \), and fixed particle mass \( m = 8.5 \text{ keV} / c^2 \).

<table>
<thead>
<tr>
<th>( \theta_0 )</th>
<th>( \beta_{0}^{\text{cr}} )</th>
<th>( \mu_{0}^{\text{cr}} / mc^2 )</th>
<th>( M_{c}^{\text{cr}} (M_\odot) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 6.45 \times 10^{-2} )</td>
<td>( 6.45 \times 10^{-2} )</td>
<td>( 1.59 \times 10^{10} )</td>
</tr>
<tr>
<td>5</td>
<td>( 2.23 \times 10^{-2} )</td>
<td>( 1.11 \times 10^{-1} )</td>
<td>( 7.91 \times 10^{9} )</td>
</tr>
<tr>
<td>40</td>
<td>( 8.33 \times 10^{-3} )</td>
<td>( 3.33 \times 10^{-1} )</td>
<td>( 7.44 \times 10^{9} )</td>
</tr>
<tr>
<td>55</td>
<td>( 6.06 \times 10^{-3} )</td>
<td>( 3.33 \times 10^{-1} )</td>
<td>( 7.44 \times 10^{9} )</td>
</tr>
<tr>
<td>100</td>
<td>( 3.33 \times 10^{-3} )</td>
<td>( 3.33 \times 10^{-1} )</td>
<td>( 7.44 \times 10^{9} )</td>
</tr>
</tbody>
</table>

Table 2.2: Critical temperature parameter and normalized chemical potential at the center of each different critical configuration, for different fixed central degeneracies.

Defining the halo radius of each configuration at the onset of the flat rotation curve, I show in Table 2.3 the critical halo magnitudes \( r_{h}^{\text{cr}} \), \( M_{h}^{\text{cr}} \) and \( v_{h}^{\text{cr}} \) corresponding to the same set of critical parameters as given in Table 2.2.

<table>
<thead>
<tr>
<th>( \theta_0 )</th>
<th>( r_{h}^{\text{cr}} ) (pc)</th>
<th>( M_{h}^{\text{cr}} / mc^2 (M_\odot) )</th>
<th>( v_{h}^{\text{cr}} ) (km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 4.4 \times 10^{-1} )</td>
<td>( 5.7 \times 10^{11} )</td>
<td>( 7.5 \times 10^{4} )</td>
</tr>
<tr>
<td>5</td>
<td>( 4.0 \times 10^{-1} )</td>
<td>( 4.3 \times 10^{11} )</td>
<td>( 6.2 \times 10^{4} )</td>
</tr>
<tr>
<td>40</td>
<td>( 4.3 \times 10^{3} )</td>
<td>( 1.1 \times 10^{15} )</td>
<td>( 3.3 \times 10^{4} )</td>
</tr>
<tr>
<td>55</td>
<td>( 2.9 \times 10^{5} )</td>
<td>( 6.0 \times 10^{16} )</td>
<td>( 2.9 \times 10^{4} )</td>
</tr>
<tr>
<td>100</td>
<td>( 2.0 \times 10^{11} )</td>
<td>( 2.3 \times 10^{22} )</td>
<td>( 2.2 \times 10^{4} )</td>
</tr>
</tbody>
</table>

Table 2.3: Critical halo magnitudes of different critical configurations, for different fixed central degeneracies as given in Table I.

The results obtained in Tables 2.2–2.3 imply a marked division in two different families depending on the value of \( M_{c}^{\text{cr}} \).

\(^3\)In reality, to properly implement the formal concept of turning point as used for example in [89], the total mass of the system \( M_t \) (previous choice of a cut-off in the momentum space at i.e. \( r_i \sim 10r_h \) to define it) should be used in the central density (\( \rho_0 \)) Vs. mass (\( M \)) diagram, instead of the core mass \( M_c \). Nonetheless, considering that in fully degenerate systems the critical masses \( M_{c}^{\text{cr}} \) are basically equal to the OV mass, it should imply that the extended and diluted halo plays no relevant role in the physics near the critical point, in some analogy to the case of Super-Nova core collapse where only the fully degenerate core is considered in the process.
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i) The first family: the critical mass has roughly a constant value $M_{cr}^c = 7.44 \times 10^9 M_\odot$. This family corresponds to large values of the central degeneracy ($\theta_0 \geq 40$), where the critical temperature parameter is always lower than $\beta_0^{cr} \lesssim 8 \times 10^{-3}$ and the critical chemical potential $\mu_0^{cr} \approx \text{const}$. Physically, these highly degenerate cores are entirely supported against gravitational collapse by the degeneracy pressure. In this case the critical core mass is uniquely determined by the particle mass according the relation $M_{cr}^c \propto M_{pl}^3/m^2$ (see also subsection 2.3.3).

ii) The second family: the critical core mass increases from $M_{cr}^c = 7.44 \times 10^9 M_\odot$ up to $M_{cr}^c \sim 10^{10} M_\odot$. This case corresponds to critical cores with a lower central degeneracy compared with the former family ($1 < \theta_0 < 40$). Here the critical temperature parameter ($\beta_0 \sim 10^{-2}$), is closer to the relativistic regime with respect to the first family. This result physically indicates that the thermal pressure term has now an appreciable contribution to the total pressure, which supports the critical core against gravitational collapse. In this case $M_{cr}^c$ is completely determined by the particle mass $m$, the central temperature $T_0^{cr}$ and the central chemical potential $\mu_0^{cr}$ (see subsection 2.3.3).

In Figs. (2.10) and (2.11) I show a critical metric factor $e^{\nu/2}$ and a critical temperature $kT$ as a function of the radius for the two different families mentioned above.

2.3.2 Application to DM in big elliptical galaxies & SMBHs

I will now attempt to use the critical configurations obtained above to explain the DM distribution in big elliptical galaxies, as well as providing an alternative candidate to the standard central black hole paradigm. From now on I will use, as before, the fixed particle mass of $m = 8.5 \text{ keV}/c^2$, being this choice motivated by the fact I want to deal with super massive dark objects having critical core masses of the order $M_{cr}^c \propto m_{pl}^3/m^2 \sim 10^9 M_\odot$. Moreover, such a relativistic object would have an OV radius $R_{OV}$ very near the Schwarschild radius $R_s$ ($R_{OV} \sim 3R_s$), and then practically indistinguishable from a BH of the same mass.

In Figs. (2.12), (2.13) and (2.14) I show different critical density profiles, critical rotation curves and critical mass profiles respectively, for a wide range of different central degeneracy parameters.

In Fig. 2.13 and Table 2.3 it is clearly shown that a critical configuration of self-gravitating fermions with central degeneracies ranging from $\theta_0 = 1$ (i.e. mainly thermal pressure supported cores) up to $\theta_0 = 100$ (i.e. mainly degener-
Figure 2.10: The critical temperature profile of the system (in keV) and the critical metric, for $\theta_0 = 5$ and $\beta_0^{cr} = 2.23 \times 10^{-2}$. The dashed line corresponds to the isothermality condition, $Te^{v/2} = \text{const}$.
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Figure 2.11: The critical temperature of the system (in keV) and the critical metric, for \( \theta_0 = 55 \) and \( \beta_0^\gamma = 6.06 \times 10^{-3} \). The dashed line corresponds to the isothermality condition, \( T e^{\nu/2} = \text{const} \).
Figure 2.12: Critical density profiles for different values of $\theta_0$ with the corresponding critical temperature parameters $\beta_{0\text{cr}}$. 

\[ \rho (M_\odot \cdot pc^{-3}) \]

\[ r (pc) \]

$\theta_0=1, (\beta_{0\text{cr}}=6.4 \times 10^{-2})$

$\theta_0=5, (\beta_{0\text{cr}}=2.2 \times 10^{-2})$

$\theta_0=40, (\beta_{0\text{cr}}=8.3 \times 10^{-3})$

$\theta_0=100, (\beta_{0\text{cr}}=3.3 \times 10^{-3})$
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Figure 2.13: Critical rotation curves for different values of $\theta_0$ as given in Fig. 2.12. To note the high values of $v_c(r) \sim 10^4$ km/s in the flat parts of each curve due to the high critical temperature parameters $\beta_0^{cr}$.

acy pressure supported cores), have flat rotation velocities $v_h^{cr} \sim 10^4$ km/s. Even if for $\theta_0 \sim 50$ the halo sizes could match observations, the halo masses are well above any observed value and so the circular velocities; further implying that none of these mathematical solutions are compatible with any observable astrophysical systems. However, since larger values of $\theta_0$ imply lower values for $\beta_0^{cr}$, as shown in Table I, there is a point (at $\theta_0 \sim 10^6$ and so $\beta_0^{cr} \sim 10^{-7}$) where the halo rotation curves reach the typical observed values as in spiral or elliptical galaxies of $v_h^{cr} \sim 10^2$ km/s. Nonetheless, since already at $\theta_0 \gtrsim 80$ (see for example $\theta_0 = 100$ in Fig. 2.12) the plateau region of the density profile is lower than the mean DM density of the Universe ($\rho_{uni} \sim 3H_0^2 / (4\pi G)$), then higher central degeneracies will imply even more diluted halos (i.e. already disappeared), never reaching typical flat rotation curves at $10^{1-2}$ kpc halo distance-scales. Thus, the

\[\text{These values of rotation curves were used in the very low (spetial) relativistic regime version of this model (i.e. } \theta_0 \sim 10^1), \text{ as presented in [29, 32, 91], and in that case leading to the correct halo masses and sizes in galaxies.}\]
2.3.3 An analytical expression for the critical mass

It is useful to find an analytical formula for the critical mass to try to understand the physics behind it. For this I will use the Newtonian hydrostatic equilibrium equation corresponding to the last stable configuration, where the pressure due to gravity is balanced by a high relativistic semi-degenerate Fermi gas:

\[
\frac{P_g(r)}{P_{urT}(r)} = \frac{GM(r)\rho(r)}{r} \approx \frac{\mu^4}{12\pi^2(hc)^3} + \frac{\mu^2(kT)^2}{6\sqrt{\pi}(hc)^3},
\] (2.11)

where \(P_{urT}(r)\) is the ultra relativistic approximation of a highly relativistic Fermi gas (\(\mu \gg mc^2\)), which has been expanded up to second order in temperature around \(\mu/kT \gg 1\) (see, e.g., [92]). I have used in 2.11 the fermi energy (\(\epsilon_f = \mu\)) with the rest energy substracted-off in consistency with the theoretical formula-
tion of our model. Considering that the density in the cores is nearly constant (see Fig. 2.12), i.e., \( \rho = \rho_c^r \approx \text{const.}, \forall r \leq r_c^r \), I can write the core radius as

\[
 r_c^r = \left( \frac{3M_c^r}{4\pi \rho_c^r} \right)^{1/3}
\]

With this, I can rewrite (2.11) as follows,

\[
 \left( \frac{4\pi}{3} \right)^{1/3} G(M_c^r)^{2/3} (\rho_c^r)^{4/3} \approx \frac{\mu^4}{12\pi^2 (\hbar c)^3} \left( 1 + \frac{2\pi^2}{\theta_0^2} \right),
\] (2.12)

Finally, I write the central mass density as \( \rho_c^r \approx n_{ur} m \), where \( n_{ur} = \frac{\mu^3}{3\pi^2 (\hbar c)^3} \) is the ultra-relativistic particle number density. With this expression for \( \rho_c^r \) in (2.12) I can directly give \( M_c^r \) in terms of \( \theta_0 \) as:

\[
 M_c^r \approx \frac{3\sqrt{\pi} M_{pl}}{16 \cdot m^2} \left( 1 + \frac{2\pi^2}{\theta_0^2} \right)^{3/2}.
\] (2.13)

It is clear from this equation that for high central degenerate systems (\( \theta_0 \gg \sqrt{2\pi} \)), the critical core mass \( M_c^r \) is independent of \( \theta_0 \) and then proportional to \( M_{pl}^3 / m^2 \). However, for low values of the central degeneracy (\( \theta_0 \sim \sqrt{2\pi} \)) the second term in (2.13) starts to be relevant, showing the finite temperature effects.

In fact, using \( \theta_0 = 40 \), we have \( M_c^r = 7.62 \times 10^9 M_\odot \), just a 2% difference with the numerical result of \( 7.44 \times 10^9 M_\odot \). However, for \( \theta_0 = 5 \), we have \( M_c^r = 1.79 \times 10^{10} M_\odot \), almost a factor 2 above the numerical value of \( 7.91 \times 10^9 M_\odot \) obtained in Table I. This shows that our approximation of an ultra-relativistic fermi gas in newtonian equilibrium breaks down and a fully relativistic treatment is needed.

### 2.3.4 Discussion and conclusions

In the light of the present analysis I then conclude that there is no critical core-halo configuration of self-gravitating DM fermions, able to explain both the most super-massive dark object at the center together with an outer DM halo simultaneously.

The concept of simultaneous co-existence of super-massive dark objects and DM halos at some cosmological epoch \( z \) is of central importance to better understand the structure growth, galaxy formation history and evolution. An important observational result aiming in these directions has been reported in [93]. In that work there is clear evidence for evolution of early-type galaxies evolving from \( z \sim 2.5 \) (consistently studied only in terms of light profiles) up to now \( (z \sim 0) \) enlarging their sizes and masses (lowering in density), which imply necessarily subsequent gathering of matter from larger-scale environments in their
complex evolutionary history (probably in the form of dark and/or baryonic matter). Thus, contrasting the results obtained in this work with observational results for big ellipticals as the ones already recalled in [93], [11], [12], [13] but also in [94, 95] for the more relevant case of the giant elliptical M87 with the detected super-massive dark object of \( M_c \sim 7 \times 10^9 M_\odot \); could well imply that even if there is clear evidence in some cases of co-existence of central dark massive objects and dark halos at \( z \sim 0 \), this could not be the case at early stages i.e. at \( z \sim 3 \). The connection between the formation of super-massive compact dark objects at early epochs, the relation with the host halo and possible subsequent accretion form their larger-scale environments are still important open questions in astrophysics.

### 2.4 DM halo Universality laws within the model

In this section I continue with the applications of the model presented in sections 2.1 and 2.2, being now interested only in the diluted outer part of the fermionic configurations. In particular, I will be interested here to contrast the model against an observational and Universal empirical correlation between the surface density of DM halos and their one-halo scale-lengths as found in [72]. Specifically speaking, the so-called ‘central’ surface density of galaxy DM halos \( \Sigma_{0D} = r_0 \rho_{0h} \), where \( r_0 \) and \( \rho_{0h} \) are the core radius of the halo (or halo-scale-length) and central (halo) density respectively, was found to be roughly a constant independently of the galaxy luminosity [72] (see Fig. 2.15). In that significant paper, a sample of several hundreds of rotation curves allowing for mass models in a very wide range of galaxy types from dwarf to elliptical galaxies, was analyzed under the assumption that each DM halo follows a Burkert profile \( \rho_B(r) \) (see [96]), to obtain the following relation

\[
\Sigma_{0D} = r_0 \rho_{0h} = 140^{+80}_{-30} M_\odot \text{pc}^{-2},
\]  

(2.14)

where the one-halo scale length \( r_0 \) is now the Burkert radius and \( \rho_{0h} \) the central halo density. The empirical relation (2.14) clearly implies that the bigger the halo size of the galaxy, the lower its central halo density, following a more or less a precise recipe. Therefore, an evident question to be answered is if the model of self-gravitating keV fermions introduced in sections 2.1 and 2.2 is able to
Figure 2.15: The ‘central’ surface density of galaxy DM halos $\Sigma_{0D} = r_0 \rho_0 h$ is approximately constant for many different galaxy types, independently of its luminosity. See [72] for the source of the different data sample.
follow the same kind of relation (2.14), e.g. for all the different galaxy samples considered in 2.2. The answer is yes, as it will be clear from the analysis done below. Moreover I recall in the conclusions of this section why this trend is an exclusive property of self-gravitating fermions, and can not be obtained for systems of self-gravitating DM particles modeled as boson condensates.

Before starting a detailed numerical analysis to better understand this problem, I show in next a direct compatibility between the theoretical model of semi-degenerate fermions and the observed relation (2.14), by taking the galaxy parameters (halo mass and circular velocity at the halo radius) for the typical dwarfs, spirals and big spirals as considered in section 2.2 and in [32]. Thus, I take the following physical dark matter halos parameters: \( r_h = 0.6 \text{ kpc}, v_h = 13 \text{ km/s}, M_h = 2 \times 10^7 M_\odot \) for typical dwarf spheroidal galaxies from [64]; \( r_h = 25 \text{ kpc}, v_h = 168 \text{ km/s}, M_h = 1.6 \times 10^{11} M_\odot \) for typical spiral galaxies from [2, 9]; and \( r_h = 75 \text{ kpc}, v_h = 345 \text{ km/s}, M_h = 2 \times 10^{12} M_\odot \) for big spiral galaxies, as analyzed in [83].

Then, for a mass of \( m = 10 \text{ keV}/c^2 \), in Table 2.4, I present the set of remaining free parameters of the model (\( \theta_0 \) and \( \beta_0 \)), such that all the different galaxy parameters given above (\( v_h \), and \( M_h \) at \( r_h \)) are fulfilled in each representative case. I also calculate in Table 2.4 the corresponding Burkert-like radius \( r_0 \) of each theoretical profile which according to the Burkert prescription \( \rho_B(r_0) = \rho_{0h}/4 \). The very close behaviour between our cored density profiles and the Burkert ones can be seen in next section, fully justifying this procedure. It is important to notice that our definition of halo radius \( r_h \) at the onset of the rotation curve (second maximum of it), implies \( r_h > r_0 \), and then, it is necessary to give precisely \( r_0 \) in each case to properly compare the theoretical model with (2.14). Finally, in the last column of the aforementioned Table, I present the surface DM density \( \Sigma_{0D} \), while in Fig. 2.16 I show each typical overall DM profile corresponding with each last raw of the Table. The surface DM density averaged over all the galaxy types (\( \langle \Sigma_{0D} \rangle \)) is also calculated in Fig. 2.16, clearly showing the good agreement with the empirical Universal law (2.14).

### 2.4.1 Core and halo scaling laws

It is possible to use the ‘observed’ Universality law (\( \Sigma_{0D} \approx \text{const} \)) presented above, to constrain the free parameters of our theoretical model by the use of scaling laws, which tells us how those parameters scales with the physical prop-
2.4. DM halo Universality laws within the model

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>$\theta_0$</th>
<th>$\beta_0$</th>
<th>$r_0$ (kpc)</th>
<th>$\rho_{0h}$ ($M_\odot/pc^3$)</th>
<th>$\Sigma_{0D}$ ($M_\odot/pc^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dwarfs</td>
<td>15.3</td>
<td>$7.5 \times 10^{-10}$</td>
<td>0.2</td>
<td>0.4</td>
<td>80</td>
</tr>
<tr>
<td>Spirals</td>
<td>29.7</td>
<td>$1.2 \times 10^{-7}$</td>
<td>10.</td>
<td>0.02</td>
<td>200</td>
</tr>
<tr>
<td>Big Spirals</td>
<td>33.7</td>
<td>$5.2 \times 10^{-7}$</td>
<td>26.</td>
<td>0.01</td>
<td>260</td>
</tr>
</tbody>
</table>

Table 2.4: The first two columns corresponds to the free parameters of the model which fulfills the Halo observables $v_h$ and $M_h$ at $r_h$ for typical dwarfs to big spirals as given above. In the last three columns are given, the Burkert radius, central density and DM surface density for each type of galaxies.

Figure 2.16: Density profiles from the theoretical model of semi-degenerate self-gravitating ‘inos’ with mass $m = 10$ keV/c$^2$ and free parameters given in Table 2.4. Each profile represents a typical galaxy type, starting from dwarfs following with spirals and up to big spirals. The observables have been taken from [64], [2, 9] and [83] respectively. On the top of the figure it is shown the (averaged) surface DM density at $r_0$ in good agreement with the observed Universality law (2.14).
erties of a given configuration, i.e., with the mass and radius of the core and mass and radius of the halo.

Before doing that, I show in Fig. 2.17 the density profiles and velocity rotation curves for different values of the degeneracy parameter and the temperature parameter, to have a visual picture of the scalings.

Figure 2.17: Upper figure: Density profiles for different values of the central degeneracy. Lower figure: rotation curves for different values of the temperature parameters. Both plots are done for an ‘ino’ mass of 10 keV/c^2.
2.4. DM halo Universality laws within the model

We can see that, despite the wide range of the parameters, the shape of the density profile is universal and composed of a central semi-degenerate dense core, a plateau of almost constant density and a tail that scales with $r^{-2}$, as already detailed in sections 2.2. Also, from fig. 2.17 it is clear that, for smaller values of $\theta_0 > 0$ we have smaller plateaus, as well as less pronounced drops from the central core to the plateau; and vice-versa for large values of $\theta_0 > 0$. In the more extreme case of very large values of the degeneracy parameter ($\theta_0 \gg 1$), the core-plateau drop is so sharp that the plateau is nearly lost (see also section 2.3).

The rotation curves are also universal and composed of four parts (see Fig. 2.17)

I) The core with constant density, where $v \propto r$;

II) The first part of the inner halo, where the mass of the core prevails over the mass of the halo and $v \propto r^{-1/2}$;

III) Second part of the inner halo, where now the mass of the halo prevails and again $v \propto r$;

IV) The outer halo, where the velocity tends to a constant value $v_0$ after some oscillations of diminishing magnitude.

It is important to notice also that the velocity tends to a constant value as required by observations and it depends only on the temperature parameter at the center:

$$\log \frac{v_0}{\text{km/s}} = 5.63 + 0.5 \log \beta_0.$$ (2.15)

This means I can uniquely determine $\beta_0$ for any system using only the asymptotic rotation velocity.

In order to use observations to constrain our parameters, I first build a set of scaling laws for the physical properties of a given system, i.e., the core radius and mass and the halo mass and radius, where the radius of the core $r_c$ is defined as the first maximum of the rotation curve and the radius of the halo $r_h$ as the second maximum, with respective masses:
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Core scaling laws

\[ M_c = 8.53 \times 10^{11} (\beta_0 \theta_0)^{3/4} \left( \frac{m}{\text{kev}/c^2} \right)^{-2} M_\odot, \quad (2.16) \]

\[ r_c = 0.19 (\beta_0 \theta_0)^{-1/4} \left( \frac{m}{\text{kev}/c^2} \right)^{-2} \text{pc}, \quad (2.17) \]

Halo scaling laws

\[ M_h = 8.95 \times 10^{12} \beta_0^{3/4} (1.445)^{\theta_0} \left( \frac{m}{\text{kev}/c^2} \right)^{-2} M_\odot, \quad (2.18) \]

\[ r_h = 0.22 \beta_0^{-1/4} (1.445)^{\theta_0} \left( \frac{m}{\text{kev}/c^2} \right)^{-2} \text{pc}, \quad (2.19) \]

It is interesting to note that, for the core, the important quantity is \( \beta_0 \theta_0 = \mu / (mc^2) \), and not the parameters themselves\(^5\). Also, the halo has a much stronger dependence on \( \theta_0 \) than the core. This scaling laws are exact for the particle mass \( m \) and are valid for \( \log \beta_0 \in [-11, -5] \) and \( \theta_0 \in [20, 200] \). The fact that these scaling laws do not allow to go through lower central degeneracies (i.e. become inexact), implied in section 2.1, the necessity to make the phenomenological analysis to constraint the ‘ino’ mass, in a full numerical way. In other words, these scaling laws are not useful to constraint the particle mass.

2.4.2 The model Vs. halo observables: A constant acceleration due to DM

As discussed above, Donato et. al.[72], fitting DM halos with the Burkert profile, found out that the surface density at the Burkert radius is constant for a wide number of galaxies, with different masses and magnitudes as mentioned above. Moreover, it can be shown, see e.g. M. Walker et. al. (2010) [98]), that this Universal result is consistent (or implies) a constant acceleration due to DM at the

\(^5\)For radii \( r < r_c \) the configurations corresponds to region I, where \( \rho(r) \approx \text{const} \) and then from the core scaling laws (2.16–2.17), I can write \( \rho_c \propto M_c / r_c^2 \) in terms of the chemical potential and particle mass \( (\beta_0 \theta_0 = \mu_0 / mc^2) \) as \( \rho_c \propto \mu_0^{3/2} m^{3/2} \). This dependence is precisely the one of a fully degenerate non-relativistic Fermi gas in presence of an external gravitational field, which is further coinciding with a polytrope of index \( n = 3/2 \) (see e.g. [97])
2.4. DM halo Universality laws within the model

(projected) half-light radius,

$$a_{DM} \approx 0.9 \text{ km}^2 \text{s}^{-2} \text{pc}^{-1}, \tag{2.20}$$

which holds ‘Universally’, i.e. for a very wide range of galaxy types. Therefore, under the assumption that this relation is nearly unchanged for the new halo-scale length $$r_h$$ here considered ($$r_h \sim 2.5 r_{half}$$), and using the Halo scaling laws (2.18–2.19) plus the fact that $$a_{DM} = G M_h / r_h^2$$, we have the scaling law for the acceleration due to DM:

$$\log \frac{a_{DM}}{\text{km}^2 \text{s}^{-2} \text{pc}^{-1}} = 11.91 + 1.25 \log \beta_0 - 0.16 \theta_0 + 2 \log \frac{m}{\text{keV}/c^2}. \tag{2.21}$$

In next I use the full range of circular velocities considered in [98], for the total sample of galaxies there studied: $$v \in (10, 300) \text{ km/s}$$, from dwarfs to big spirals respectively. Therefore, considering an ‘ino’ mass of 10 keV/c², and using Eqs. (2.15) and (2.21) together with the Universality $$a_{DM} \approx 0.9 \text{ km}^2 \text{s}^{-2} \text{pc}^{-1}$$, it is possible to restrict the physical parameters of the model as

$$\beta_0 \in [4.4 \times 10^{-10}, 4 \times 10^{-7}], \tag{2.22}$$

$$\theta_0 \in [14.1, 37.2]. \tag{2.23}$$

If this allowed range of $$\theta_0$$ is now analyzed for the effects in the quantum central regions of the configurations (see also Table 2.4 and Fig. 2.16 at the beginning of the section), it can be seen again, how the halo observables provide an extremely valuable constraint for the central degeneracy of the quantum core.

**Halo circular velocity Universality law**

Now, in next I will use another Universal correlation law obtained from observed rotation curves, together with the theoretical scaling laws obtained here, to derive the observational DM acceleration law (2.20). This is another way to show the consistency of the scaling laws and the model with the observations.

In [98], a novel approach was used to subtract the baryonic component of the full sample of observed rotation curves there considered, allowing for a new important correlation for the DM circular velocity at typical one-halo scale-lengths:

$$\log \frac{v}{\text{km/s}} = 1.47 + 0.5 \log \frac{r}{\text{kpc}}. \tag{2.24}$$
This law (plus-minus 10% of spread around the mean value) is an Universal correlation which holds for dwarfs up to big spiral galaxies [98]. Now, if Eq. 2.24 is combined with the Halo scaling laws (2.15 and 2.18–2.19) it is possible to obtain the following scaling between the halo mass $M_h$ and halo radius $r_h$:

$$M_h = 1.92 \times 10^2 \left( \frac{r_{\text{pc}}}{1 \text{pc}} \right)^2 M_\odot. \tag{2.25}$$

Then, if this correlation is combined with the expression for the acceleration due to DM $a_{DM} = GM_h/r_h^2$, it is obtained the following Universal law

$$a_{DM} = 0.86 \text{ km}^2\text{s}^{-2}\text{pc}^{-1}, \tag{2.26}$$

thus, in excellent agreement with Eq. 2.20, obtained purely from empirical arguments in [98].

An interesting consequence of the observational Universal law (2.24), is that it can be combined with Eqs. (2.15) plus the halo radius scaling (2.19), to obtain an approximate relation between the (central) temperature and degeneracy parameters of the model for $m = 10 \text{ keV/c}^2$:

$$\beta_0 \approx 6.8 \times 10^{-12} (1.342)^{\theta_0}. \tag{2.27}$$

However the validity to this equation has to be taken with caution, due to the more than 20% of full spread around the mean value in the observationally inferred Universal law used to derive it.

### 2.4.3 Conclusions

In conclusion, in this section it was demonstrated that the model of self-gravitating systems of keV fermions presented in sections 2.1 and 2.2, is perfectly able to reproduce the Universality law of constant surface density of DM halos. It was further shown how these Universal laws can be used to put constraints the free parameters ($\beta_0$ and $\theta_0$) within a definite range, as well as to provide an approximate correlation between these both free parameters in that range of validity. As a consequence of this phenomenological approach, the range of validity of $\theta_0$ can be used now to study the density behaviour of the central degenerate core regions as done in section 2.2. When this is done appears the remarkable
conclusion that the fulfillment of the DM Universality law, valid for dwarf up to big spirals, is at the same time consistent with dense compact DM objects (below sub-parsec scales) with masses $\sim 10^4$ and $\sim 10^7 M_\odot$ respectively (see also Fig. 2.16).

An important final remark proper of this section, is the fact that if the Universality law given in Eq. (2.14) is pretended to be fulfilled by the fuzzy galaxy DM model (see e.g. [99]), defined by a system of self-gravitating system of bosons (in a condensate phase) of $m \sim 10^{-24}$ eV, the result is negative. This is because the (galactic halo) mass-size relation for this bosons has the wrong behaviour, being the halo mass $M_h$ inversely proportional to the halo scale-length, in evident disagreement with (2.25). This result therefore gives more strength to the possible fermionic nature of DM, in contrast to the bosonic one (See also in [100] and refs. therein for an independent analysis in which DM boson condensates seems to be excluded as possibles DM candidates).

### 2.5 Statistical analysis with the THINGS sample

In 2008, a sample of 34 nearby (closer than 15 Mpc) spiral and irregular galaxies (Sb to Im) were observed with The HI Nearby Galaxy Survey (THINGS) [101]. These observations allowed to obtain the highest quality rotation curves available to date due to the high spatial and velocity resolution of THINGS. Then a sub-sample of these rotation curves, corresponding to 19 rotationally dominated and undisturbed galaxies, were combined with information on the distribution of gas and stars by [9] to construct mass models for the dark matter component of the sample.

After that, using these rotation curves from THINGS, the Einasto dark matter halo model (see Eq. 2.28) has been proposed as the standard model for dark matter halos by [82], as it provides both cored and cusped distributions for different values of model parameters (see Fig. 2.18). In that work, the fundamental core-cusp discrepancy is analyzed in detail for the whole sample of galaxies under study. It is clearly shown that for the majority of the galaxies considered in the sample, the cored halos (compatible with near unity Einasto indexes) are preferred over the cuspy ones (these instead compatible with higher Einasto in-


\[ \rho_E(r) = \rho_{-2} \exp \left( \frac{-2}{n} \left[ \left( \frac{r}{r_{-2}} \right)^n - 1 \right] \right), \tag{2.28} \]

where \( \rho_{-2} \) and \( r_{-2} \) are the density and radius at which \( \rho(r) \propto r^{-2} \), and \( n \) is the Einasto index which determines the shape of the profile.

Figure 2.18: Comparison of the density profiles for different phenomenological models of dark matter distribution.

I present here a different approach focusing on the galactic structures and the underlying microphysical component of Dark Matter. From a point of view of the variables of the constituents, the Einasto profile (see Eq. 2.28) is a function three parameters, \( r_{-2}, \rho_{-2} \) and \( n \), and no specific physical meaning is applied to them besides the \( \chi^2 \) extremization when compared with the data. In the fermionic model presented in this Thesis, the number of free parameters is also three (\( \beta_0, \theta_0 \) and \( m \)), being in many cases the extremization of \( \chi^2 \) better than the one of Einasto. But more important is the fact that they are linked to the physics and microphysics determining the reason of the equilibrium configurations.

The objective of this section is therefore to fit rotation curves of the THINGS sample with the semi-degenerate model introduced in sections 2.1 and 2.2, to further compare and contrast the results of the fitting procedures with differ-
2.5. Statistical analysis with the THINGS sample

ent dark matter phenomenological models used in the literature. In particular I compare the best-fitting results with respect to the Navarro–Frenk–White (NFW) 2-parameter model [14], and with respect to the Einasto 3-parameter model ([80] and [81]). As it is shown here, the comparison among different models shows that the fermionic structures can present a better fit when contrasted with the rotation curve data of THINGS. More relevant is the fact that the overall fermionic model solutions, in contrast with the other models analyzed here, are associated with important predicting power regarding the innermost dark matter distribution due to the quantum nature of their sub-parsec cores. In consistency with section 2.2, the best-fits obtained from the solutions of the fermionic model, predict the presence of a novel compact core structure with masses of $\sim 10^6 M_\odot$ at $\sim 10$ mpc distance-scales from the center; leaving an interesting challenge for its direct detection in future high angular resolution observations.

2.5.1 Properties of semi-degenerate configurations

Galactic halos have to be necessarily composed from cold particles, so that astrophysically relevant solutions will have temperature parameters $\beta \ll 1$. In this case, as detailed in section 2.2, the general solution for semi-degenerate configurations ($\theta_0 \gtrsim 10$) present three different regions: an inner degenerate compact core, an extended low-degenerate inner halo of almost constant density and a non-degenerate outer halo with characteristic slope $\rho \propto r^{-2}$. The infinite mass of the configuration extended up to spatial infinity is not a problem, because in reality it is limited by tidal interactions with other galaxies, which introduce an energy cutoff into the distribution function, see e.g. [102] and more recently in [26]. However this is not important for the inner parts of the configurations.

The characteristic and different physical regimes present in the rotation curves of the semi-degenerate model were already shown in the former section. Nevertheless, I recall here the overall behaviour of the rotation curves, and also re-derive the scaling laws of the physical variables in dimensionless units for future convenience.

I define the physical characteristics of each configuration as follows:

- The characteristic radius of the core $r_c$ is given by $v_{\text{circ}}(r_c) = \max$ in region I.
2. MASSIVE FERMIONS IN GR & THE DISTRIBUTION OF DM IN GALAXIES

$\rho(r)$

Figure 2.19: Dependence of $v_{\text{circ}}$ in km/s on the radius $r$ in pc, with $\beta_0 = 10^{-7}$, $\theta_0 = 24$ and a given fermion mass of $m = 10 \text{ keV}/c^2$.

- $M_c$ is the mass of the core given by $M_c = v_{\text{circ}}^2 r$ in the region II.$^6$

- The characteristic radius of the inner halo $r_h$ correspond to $v_{\text{circ}}(r_h) = \max$ in region III.

- The characteristic mass of the inner halo $M_h$ is given by $M_h = M(r_h)$ just between the regions III and IV.

For the parameters in the region of $\theta_0 \in [0, 200], \log \beta_0 \in [-10, -5]$, I calculate a grid of models and extracted numerically the physical characteristics mentioned above. Then I fit the obtained values by different double parametric functions and find out the best fitting formulae with the correspondent $(\beta_0, \theta_0)$ dependence for the range of $\theta_0 \in [20, 200]$. An interesting fact is that in that region of parameters, the circular velocity $v_{\text{circ}}$ in the flat part of region IV (i.e. $v_{\text{circ}}$) is defined by temperature $\beta_0$ only. In the range of astrophysically relevant parameters $\theta_0 \in [20, 200], \log \beta_0 \in [-10, -5]$, the scaling relation between circular velocity and $\beta_0$ corresponds to the Boltzmannian relation between $v_{\text{circ}}$ and one-dimensional dispersion velocity $\sigma = \sqrt{kT/m}$ (see e.g. [103])

$^6$Due to the sharp decrease of $\rho(r)$ from I until the beginning of III (for $\theta_0 > 20$), the contribution to the core mass $M(r_c)$, as defined in section 2.2, is negligible in that region (see also Fig. 2.14)
2.5. Statistical analysis with the THINGS sample

\[ \frac{v_\infty}{\text{km/s}} = \sqrt{2c} \sqrt{\beta_0}. \] (2.29)

I have here neglected general relativistic corrections which are very small in these ranges of parameters, i.e. \( e^{v(\infty) - v(0)} \approx 1 \), and then \( \beta(\infty) \approx \beta_0 \) by equation (2.6).

For the temperature and degeneracy free parameters in the range \( \log \beta_0 \in [-10, -5], \theta_0 \in [20, 200] \), respectively, I obtain the following dimensionless scaling laws for core radius and mass (for the relation with the dimensionfull magnitudes through \( \chi \), see section 2.1)

\[ r_c = 0.226 (\beta_0 \theta_0)^{-1/4}, \] (2.30)
\[ M_c = 0.234 (\beta_0 \theta_0)^{3/4}. \] (2.31)

However, the core region is typically very small and is not constrained by empirical data of the THINGS sample considered here. Moreover, the mass contribution of regions I and II to the total mass \( M_h \) at the end of region III is \( \lesssim 10^{-2} \) in the parameter range here considered, indicating that only regions III and IV are the relevant ones to be used in the fitting procedure against the data.

Dimensionless halo radius and mass have different scalings, they are proportional not to \( \theta_0^\alpha \), but to \( n^\beta \) as already shown in section 2.4

\[ r_h = 0.953 \beta_0^{-1/4} (1.445)^{\theta_0}, \] (2.32)
\[ M_h = 2.454 \beta_0^{3/4} (1.445)^{\theta_0}. \] (2.33)

Formulas (2.30–2.33) represent perfect scalings in the region of parameters considered above, which involves also the scaling of the whole rotational curves in regions III and IV. Formula (2.29) shows a perfect scaling in the flat part of the rotation curve for region IV. Moreover, the Newtonian expression for the dimensionless circular velocity \( \theta_0^2 \circ_c (\hat{r}) = \hat{M}/\hat{r} \) is perfectly suitable in the physical region under consideration. The expression for maximal rotation velocity in the
halo is thus obtained from (2.32) and (2.33) and reads

\[ \hat{\sigma}^2_{\hat{h}}(\hat{r}_h) = 2.51\beta_0. \]  

(2.34)

In next section I explain the fitting procedure with the use of the halo scaling laws for regions III and IV obtained here.

### 2.5.2 Observed rotation curves and fitting procedure

The mass models constructed from the THINGS and SINGS data samples in [101] and [9] were used to quantify the dark matter contribution for each galaxy by using the following formula

\[ V_{\text{obs}}^2 = V_{\text{gas}}^2 + \Upsilon_* V_*^2 + V_{\text{DM}}^2, \]  

(2.35)

which relates the observed input curves of \( V_{\text{obs}} \), \( V_{\text{gas}} \) and \( V_* \), defined below, with the dark matter rotation curve \( V_{\text{DM}} \) to be determined from the known input data once the the mass-to-light ratio \( \Upsilon_* \) is provided.

The total and gas observed rotation curves \( V_{\text{obs}} \) and \( V_{\text{gas}} \), respectively, were both obtained from the THINGS data: the first was obtained from velocity fields analysis and the second from the neutral hydrogen (HI) distribution maps, as described in [9]. Instead, each stellar (light) rotation curve \( V_* \) is obtained from the corresponding stellar distribution observed in the K band (i.e. at 3.6 \( \mu \)m) by the Spitzer Infrared Nearby Galaxy Survey (SINGS), independent of THINGS, and described in [9] and references therein. Finally, the mass-to-light ratio \( \Upsilon_*^K \) was used to determine the rotation curve associated with the stellar mass distribution from that of the measured light.

At this point it is relevant to further emphasize the underlying hypothesis to which equation (2.35) is subject to. This is, each baryonic rotation velocity \( V_{\text{gas}} \) and \( V_* \) was calculated from the correspondent observed baryonic mass density distribution, and was defined as the velocity that each matter component would induce on a test particle in the galactic plane as if they were isolated of any external influence. In section 3 I develop an extended theoretical approach to deal with this problematic in a more consistent way.

In [9] equation (2.35) was applied to test the cuspy Navarro-Frenk-White and cored pseudo-ISO dark matter models against data as follows: the (squared) ro-
2.5. Statistical analysis with the THINGS sample

tation curves of the baryonic components (after appropriate scaling with $\Upsilon_\ast^K$) were subtracted from the (squared) observed rotation curve $V_{\text{obs}}^2$ to apply a reduced $\chi^2$ fitting procedure in order to find the best fitting free parameters for each dark matter model. Soon after, the same analysis was extended further to Einasto dark matter profiles by [82], concluding that the Einasto model provides the best match to the observed rotation curves when compared with NFW and pseudo-ISO models with empirical fixed values for $\Upsilon_\ast^K$ for two different stellar initial mass functions (IMFs).

Here I propose a different dark matter halo model, which is neither based on numerical N-body simulations nor on phenomenological model proposals, but relies on the underlying microphysical composition of the dark matter candidate, as explained in former sections. Thus, analogously to [9] and [82], I use the HI high resolution observations of galaxies from THINGS survey [101]. I analyze here a sample of 16 rotationally dominated and undisturbed galaxies presented both in [9] and [82]. The galaxies are: NGC2366, NGC2841, NGC2903, NGC2976, NGC3521, NGC2403, NGC3031, IC2574, NGC3621, NGC4736, DDO154, NGC5055, NGC6946, NGC7331, NGC7793, NGC3198.

Regarding the rotation curves data of the baryonic components, I consider the contributions of the gas, the stellar disk and a spherical stellar bulge $V_b$ as given in [82]. The halo rotation velocity corresponding to the spherical dark matter component is taken from the semi-degenerate model, and given by Eq. 2.34. From now on this velocity will be called the fermionic dark matter velocity profile $V_f(r)$.

Once each component is provided, I make use of the equation analogous to (2.35)

$$V_{\text{obs}}^2 = V_{\text{gas}}^2 + \Upsilon_\ast V_\ast^2 + V_b^2 + V_f^2,$$  \hspace{1cm} (2.36)

With all the baryonic velocity terms ($V_{\text{gas}}^2$, $\Upsilon_\ast V_\ast^2$ and $V_b^2$) as observational inputs, I fit the HI observed rotation curve $V_{\text{obs}}^2$ by Levenberg–Marquardt nonlinear least-squares algorithm, in complete analogy as done in [82].

I did not take into account the contribution of molecular gas because the total gas surface densities are dominated by atomic gas for the majority of the sample, as explained in [82] and references therein. Total rotation curve was taken from [9]. I have not considered models with free mass-to-light ratios. Instead following [82] I have adopted the fixed mass-to-light ratios of stellar populations with
a bursty star formation history with a Kroupa IMF. I choose this IMF instead of the diet-Salpeter IMF (also considered in [82] and in [9]), as it generally provides better agreement with observations for rotation curves (see Fig. 5 of [82]), and in some cases the Salpeter IMF leads to rotational velocities due to stellar component only already in pronounced excess over observed total rotational velocity (see, for example, cases of NGC3521 and NGC5055 at Fig. 3 of [82]).

2.5.3 Comparison with previous results and discussion

Besides the semi-degenerate model, the whole fitting procedure explained before will be also applied to other different DM halo models used in the literature:

- Cored profiles with central density $\rho_0$ and characteristic radius $r_0$:

  - pseudo-isothermal sphere profile

    \[ \rho_{DM}(r) = \rho_0 \frac{r_0^2}{r^2 + r_0^2}, \]  
    \[ (2.37) \]

  - Burkert profile

    \[ \rho_{DM}(r) = \rho_0 \frac{r_0^3}{(r_0 + r)(r_0^2 + r^2)}. \]  
    \[ (2.38) \]

- Cusped profiles with characteristic radius $r_{-2}$ where the density profile has a (logarithmic) slope of $-2$ (the "isothermal" value) and $\rho_{-2}$ as the local density at that radius. In the case of Einasto profiles a third parameter is needed, the Einasto index $n$ which determines the shape of the profile.

  - Navarro–Frenk–White profile

    \[ \rho_{DM}(r) = 4\rho_{-2} \frac{r_{-2}}{r} \left( \frac{r_{-2}}{r + r_{-2}} \right)^2, \]  
    \[ (2.39) \]
2.5. Statistical analysis with the THINGS sample

- Einasto profile

\[
\rho_{\text{DM}}(r) = \rho_{-2} \exp \left\{ -2n \left[ \left( \frac{r}{r_{-2}} \right)^{1/n} - 1 \right] \right\}.
\] (2.40)

As I have to compare models with different number of parameters, which are not nested into each other, we use Bayesian Information Criterion (BIC) introduced by [104]. It provides a penalty to models with larger number of parameters to check what of them is more likely to be correct. Model with minimum BIC value is preferred. For the models with the same number of parameters, BIC is equivalent to \(\chi^2\) criterion.

Regarding the semi-degenerate model, the set of the three free parameters which minimizes the reduced \(\chi^2\) function results in a degeneracy between the \((\theta_0, m)\) parameters, all subject to small statistical errors. The physical meaning of this result can be understood as follows: as well as the experimental data extends up to regions III and IV (see Fig. 2.19), it does not provide definite information about regions I and II (i.e. the core). On the other hand, the fitting procedure for the dark matter rotation velocity is described by formula (2.34), which contains only information of the halo scaling laws. In that formula is explicitly shown how \(\beta_0\) fixes the height of the halo rotation curve, while for determining the halo radius \(r_h\) (see also (2.32) and the relation \(\hat{r} = r/\chi\) with \(\chi \propto m^{-2}\)) we have two extra free parameters \((\theta_0, m)\) for this unique value. Consequently, this degeneracy will imply different possibilities for the characteristics of the quantum core mass \(M_c\) and radius and \(r_c\), due to the different form of the core scaling laws (2.30–2.31) with respect to the halo ones. This physical concept was already known from section 2.2 and [32], where it was clearly shown that the mass of the ‘inos’ can only be determined as an eigenfunction of the mass of the inner quantum cores. In particular, there it was shown that in order to obtain compact cores of \(M_c \sim 10^6 M_\odot\) at sub-parsec scales (for Milky Way-like asymptotic rotation velocities), the mass of the ‘ino’ has necessarily to be \(m \sim 10 \text{ keV}/c^2\).

Thus, I will use the results described in the above paragraph based on the physical insight gained in section 2.2 and in [32]. With this, and by taking \(m = 10 \text{ keV}/c^2\) we break the aforementioned \((\theta_0, m)\) degeneracy, being then able to

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7The family of Einasto profiles with relatively large indices \(n > 4\) are identified with cuspy halos, while low index values \(n < 4\) presents a cored-like behaviour [82]. The lower the \(n\) the more cored-like the halo profile.
predict an specific quantum core in each galaxy of the sample.

In Tables 2.6–2.10 I present the results of the fitting procedures, showing the best fit parameters of each model, in the cases where the semi-degenerate model is statistically preferred over Einasto one. In Figs. 2.20–2.21 I give, as typical examples, two plots which explicitly show the \((\theta_0, m)\) degeneracy, corresponding to the minimization of the \(\chi^2\) function respect to \(\beta_0\). The first example corresponds to the dark matter dominated galaxy NGC2366 (see also Fig. 2.22 for the detailed halo matter component), and the second corresponds to NGC3521 which is mainly baryonic dominated (see also Fig. 2.23 for details). In particular, these plots show that the above mentioned degeneracy is given according to the relation

\[
\theta_0(m) = \theta^*_0 + 12.52 \log \frac{m}{10 \text{ keV}/c^2}. \tag{2.41}
\]

Each of these statistical results showing the \((\theta_0, m)\) degeneracy, allows to find a minimum for the ‘ino’ mass in each case (at 1\(\sigma\) confidence level), below which the halo circular velocity data is no longer well fitted (see Figs. 2.20–2.21). Interestingly, this minimum is in all cases of the order of \(\sim 10^{-1} \text{ keV}/c^2\), in compatibility with the results found for typical spiral galaxies in section 2.2 by a different numerical method.

In Table 2.5 and Figs. (2.24–2.29) I show the predictions within the semi-degenerate model, for the dark matter distribution in the inner core regions (down to 100 pc), all composed by ‘ino’ masses of \(m = 10 \text{ keV}/c^2\). The fact that the baryonic component (i.e. disk, bulge, gas) is expected to be dominant at \(\sim 10^{1–2}\) pc distances, it clearly implies an extremely difficult task to put observational constraints on the dark matter component there. Nevertheless, at much shorter (sub-parsec) scales the semi-degenerate model predicts the appearance of a compact quantum core with typical densities several orders of magnitude higher than halo densities, leaving a good possibility for its detection in future high angular resolution astronomical observations.

At this point is important to emphasize that in each of the degenerate cores belonging to the best fitting profiles here analyzed (for \(m \sim 10 \text{ keV}/c^2\), and \(\beta_0 \lesssim 10^{-7}\)), the de-Broglie wavelength \((\lambda_B = h/(2\pi mkT)^{1/2})\) is higher than the inter-particle mean-distance \(l_c\), safely justifying the quantum-statistical treatment applied here.
Figure 2.20: Typical \((\theta_0, m)\) degeneracy resulting from the minimization of the reduced \(\chi^2\), when applied for the fitting procedure of the semi-degenerate model to the NGC2366 case. Each contour corresponds to confidence levels of \(1\sigma, 2\sigma\) and \(3\sigma\) respectively. To note the allowed region for the fermion mass above a few keV/c^2.
Figure 2.21: Typical ($\theta_0, m$) degeneracy resulting from the minimization of the reduced $\chi^2$, when applied for the fitting procedure of the semi-degenerate model to the NGC3521 case. Each contour corresponds to confidence levels of $1\sigma$, $2\sigma$ and $3\sigma$ respectively. To note the allowed region for the fermion mass above a few keV/$c^2$. 
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Figure 2.22: Halo rotation curve for NGC2366 from THINGS. In blue and continuous line, the dark matter best fit of the semi-degenerate model. In green and dashed line the dark matter best fit of the Einasto model. In red and dotted line the total baryonic component. The lower black continuous line, is the pure dark matter component of the fermionic model.

<table>
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<tr>
<th>Galaxy</th>
<th>CORE PREDICTIONS</th>
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<td></td>
<td>( M_c, 10^6 M_\odot )</td>
<td>( r_c, 10^{-2} \text{ pc} )</td>
<td></td>
</tr>
<tr>
<td>NGC2366</td>
<td>0.449 ± 0.009</td>
<td>8.86 ± 0.06</td>
<td></td>
</tr>
<tr>
<td>NGC2841</td>
<td>3.32 ± 0.10</td>
<td>4.54 ± 0.05</td>
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<tr>
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<td>4.281 ± 0.007</td>
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<td>NGC3521</td>
<td>2.61 ± 0.16</td>
<td>4.92 ± 0.10</td>
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Table 2.5: Predicted values for the compact degenerate core mass and radius according to the semi-degenerate model.
Figure 2.23: Halo rotation curve for NGC3521 from THINGS. The distinction between the lines in relation with the matter component and best fits is the same than in Fig. 2.22. To notice the dark matter dominance only in the outermost part at difference with NGC2366, being this last overall dark matter dominated.

Table 2.6: Best fit parameters for the semi-degenerate model with a fermion mass of $m = 10 \text{ keV}/c^2$. In the six cases here presented the semi-degenerate model is preferred over the Einasto model.
2.5. Statistical analysis with the THINGS sample

Figure 2.24: Double log plot for the full dark matter rotation curve rotation corresponding to the best fit of the semi-degenerate model (in blue and continuous line), compared with the Einasto best fit (in dashed). To notice the predicted dark matter component which becomes relevant at sub-parsec scales, and may work as alternative to massive black holes thought to be hosted at the center, as detailed in section 2.2. Similar results are shown in Figs 2.25–2.29 for the other cases.
2. MASSIVE FERMIONS IN GR & THE DISTRIBUTION OF DM IN GALAXIES

**NGC2841**

![Graph of v_circ vs r for NGC2841](image)

Figure 2.25:

**NGC2903**

![Graph of v_circ vs r for NGC2903](image)

Figure 2.26:
2.5. Statistical analysis with the THINGS sample

NGC2976

Figure 2.27:

NGC3198

Figure 2.28:
2. MASSIVE FERMIONS IN GR & THE DISTRIBUTION OF DM IN GALAXIES

NGC3521

Figure 2.29:

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>$r_{-2}$, kpc</th>
<th>$\rho_{-2}$, $10^{-3}$</th>
<th>$n$</th>
<th>$\chi^2$</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC2366</td>
<td>2.9 $\pm$ 0.4</td>
<td>6.86 $\pm$ 0.04</td>
<td>0.9 $\pm$ 0.3</td>
<td>0.13</td>
<td>39</td>
</tr>
<tr>
<td>NGC2841</td>
<td>24.5 $\pm$ 0.6</td>
<td>1.091 $\pm$ 0.006</td>
<td>0.54 $\pm$ 0.08</td>
<td>2.5</td>
<td>367</td>
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<tr>
<td>NGC2903</td>
<td>5.33 $\pm$ 0.15</td>
<td>28.983 $\pm$ 0.005</td>
<td>2.9 $\pm$ 0.2</td>
<td>1.6</td>
<td>326</td>
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<td>NGC2976</td>
<td>70000 $\pm$ ND</td>
<td>0.019 $\pm$ ND</td>
<td>4.0 $\pm$ 70</td>
<td>0.49</td>
<td>100</td>
</tr>
<tr>
<td>NGC3198</td>
<td>11.5 $\pm$ 0.4</td>
<td>3.029 $\pm$ 0.006</td>
<td>1.80 $\pm$ 0.17</td>
<td>1.1</td>
<td>257</td>
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<tr>
<td>NGC3521</td>
<td>9.3 $\pm$ 0.5</td>
<td>6.27 $\pm$ 0.02</td>
<td>1.2 $\pm$ 0.3</td>
<td>4.3</td>
<td>443</td>
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</table>

Table 2.7: Best fit parameters for the Einasto model. It is important to notice that in all the cases of the THINGS sample in which the Einasto model provides cored-like profiles (i.e. $n \lesssim 4$), the semi-degenerate is preferred in six cases (shown in table), and Einasto model is preferred in other six cases presented in the text. The density parameter $\rho_{-2}$ is given in units of $M_\odot/pc^3$. ND means not constrained model parameters.
2.5. Statistical analysis with the THINGS sample

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>NAVARRO–FRENK–WHITE</th>
<th></th>
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<tbody>
<tr>
<td></td>
<td>$r_{-2}$, kpc</td>
<td>$\rho_{-2}$, $10^{-3}$</td>
<td>$\chi^2_r$</td>
<td>BIC</td>
<td></td>
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<td>NGC2366</td>
<td>$200 \pm 1100$</td>
<td>$0.02 \pm 0.24$</td>
<td>1.1</td>
<td>117</td>
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</tr>
<tr>
<td>NGC2841</td>
<td>$150 \pm 30$</td>
<td>$0.05 \pm 0.02$</td>
<td>3.8</td>
<td>402</td>
<td></td>
</tr>
<tr>
<td>NGC2903</td>
<td>$4.75 \pm 0.16$</td>
<td>$34 \pm 2$</td>
<td>1.8</td>
<td>334</td>
<td></td>
</tr>
<tr>
<td>NGC2976</td>
<td>$900 \pm 40000$</td>
<td>$0.009 \pm 1$</td>
<td>2.1</td>
<td>158</td>
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<tr>
<td>NGC3198</td>
<td>$16.5 \pm 0.9$</td>
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<td>NGC3521</td>
<td>$18 \pm 3$</td>
<td>$1.5 \pm 0.4$</td>
<td>5.1</td>
<td>457</td>
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</table>

Table 2.8: Best fit parameters for the NFW model. This model is not preferred against any of the other models for the whole THINGS sample here considered, confirming the preference of cored dark matter halos against cuspy ones for the THINGS sample as presented in [82]. The density parameter $\rho_{-2}$ is given in units of $M_\odot/pc^3$.

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>PSEUDO–ISO</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
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<tr>
<td></td>
<td>$r_0$, kpc</td>
<td>$\rho_0$, $10^{-3}$</td>
<td>$\chi^2_r$</td>
<td>BIC</td>
<td></td>
</tr>
<tr>
<td>NGC2366</td>
<td>$1.29 \pm 0.17$</td>
<td>$40 \pm 11$</td>
<td>0.15</td>
<td>42</td>
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</tr>
<tr>
<td>NGC2841</td>
<td>$12.5 \pm 0.7$</td>
<td>$4.6 \pm 0.6$</td>
<td>2.8</td>
<td>374</td>
<td></td>
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<tr>
<td>NGC2903</td>
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<td>$2300 \pm 500$</td>
<td>3.9</td>
<td>406</td>
<td></td>
</tr>
<tr>
<td>NGC2976</td>
<td>$9 \pm 18$</td>
<td>$30 \pm 180$</td>
<td>0.49</td>
<td>98</td>
<td></td>
</tr>
<tr>
<td>NGC3198</td>
<td>$2.7 \pm 0.14$</td>
<td>$51 \pm 5$</td>
<td>1.2</td>
<td>266</td>
<td></td>
</tr>
<tr>
<td>NGC3521</td>
<td>$2.4 \pm 0.3$</td>
<td>$78 \pm 19$</td>
<td>4.2</td>
<td>439</td>
<td></td>
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</table>

Table 2.9: Best fit parameters for the Pseudo-iso model. This model is not preferred against any of the other models for the whole THINGS sample here considered. The density parameter $\rho_0$ is given in units of $M_\odot/pc^3$. 

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2. MASSIVE FERMIONS IN GR & THE DISTRIBUTION OF DM IN GALAXIES

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>$r_0$, kpc</th>
<th>$\rho_0$, $10^{-3}$</th>
<th>$\chi^2$</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC2366</td>
<td>2.2 ± 0.2</td>
<td>43 ± 10</td>
<td>0.12</td>
<td>35</td>
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<tr>
<td>NGC2841</td>
<td>20.6 ± 0.9</td>
<td>5.2 ± 0.5</td>
<td>2.7</td>
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<tr>
<td>NGC2903</td>
<td>2.89 ± 0.06</td>
<td>388 ± 18</td>
<td>1.1</td>
<td>283</td>
</tr>
<tr>
<td>NGC2976</td>
<td>20 ± 20</td>
<td>40 ± 150</td>
<td>0.49</td>
<td>97</td>
</tr>
<tr>
<td>NGC3198</td>
<td>6.32 ± 0.18</td>
<td>36 ± 2</td>
<td>0.99</td>
<td>248</td>
</tr>
<tr>
<td>NGC3521</td>
<td>5.4 ± 0.4</td>
<td>60 ± 8</td>
<td>4.2</td>
<td>437</td>
</tr>
</tbody>
</table>

Table 2.10: Best fit parameters for the Burkert model. This model model provides the lowest BIC number in the case of NGC3198. The density parameter $\rho_0$ is given in units of $M_\odot/pc^3$.

In general, from the 16 galaxies analyzed, our model has minimum BIC value in 5 cases (NGC2366, NGC2841, NGC2903, NGC2976, NGC3521), Einasto model in 10 cases (NGC2403, NGC3031, IC2574, NGC3621, NGC4736, DDO154, NGC5055, NGC6946, NGC7331, NGC7793), and in the case of NGC3198 Burkert model is the best one (but only marginally with respect to the fermionic and Einasto models). Besides this general comparison in which apparently Einasto model is preferred against our model, there is a more relevant comparison which must be made considering that the semi-degenerate model provides cored halos only. For this, I compare the Einasto model against the semi-degenerate one for the sub-set of galaxies which are cored-like (i.e. with Einasto index $n \lesssim 4$), and then the same comparison is made for the sub-set of galaxies which are cuspy-like (i.e. with Einasto index $n > 4$). The important outcome of this new BIC comparison is that the semi-degenerate model is equivalently as good as Einasto for the cored-like sub-sample. Specifically, for cored-like galaxies in 6 cases the fermionic model has lower BIC number than Einasto model (NGC2366, NGC2841, NGC2903, NGC2976, NGC3198, NGC3521), plotted in Figs.2.24–2.29. Inversely in other 6 cases Einasto is better (NGC3031, IC2574, NGC4736, DDO154, NGC5055, NGC7793). Instead, for the cuspy-like sub-sample (NGC2403, NGC3621, NGC6946, NGC7331), in all the cases Einasto model has lower BIC numbers as logically one may expect due to the cored nature of the semi-degenerate halos.
2.6 Conclusion

It follows from the results of fitting that the semi-degenerate fermionic distributions can fit dark matter in the THINGS sample of galaxies at least as well as other profiles considered in the literature, with the important ‘revenue’ that this profile is theoretically motivated, and is not phenomenological as most of the others.

Thus, while the Einasto profile is a pure phenomenological one based on best fit of the observational and numerical simulation data, the fermionic profile is derived from the first principles and based on the General Relativistic treatment of self-gravitating semidegenerate neutral fermions. Therefore, there is the distinct possibility that the treatment here presented gives the conceptual physical motivation for the existence of the cored Einasto profile directly from the structure of the microphysical constituents of dark matter.

In particular, with the fermionic model and for each galaxy of the sample, it is possible to predict very massive compact cores of dark matter dominated by quantum pressure at sub-parsec scales, with different masses and radii (in addition to the results of section 2.2). Eventually, future observations with better angular resolution than present ones, will lead to the identification of such central cores and the precise determination of the particle mass.
2. MASSIVE FERMIONS IN GR & THE DISTRIBUTION OF DM IN GALAXIES
Chapter 3

MULTI-COMPONENT
SELF-GRAVITATING SYSTEMS

In the realm of galactic dynamics, baryonic and Dark Matter (DM) components are usually treated in the literature in terms of the Jeans equations (see e.g. [5] for a full development of the theory and next section for a brief introduction). When leading with dwarf galaxies it is usually assumed that the underlying gravitational potential $\Phi(r)$ in the halo regions is dominated by the DM component. This ansatz together with the assumptions of time-independent systems in spherical symmetry with no angular momentum dependence and constant line-of-sight velocity dispersions $\sigma_{\text{los}}$ (LOSVD), allows to break the Jeans degeneracy appearing in anisotropic systems with $\sigma_\theta^2 \neq \sigma_r^2$ (see e.g. [5]). In this case it is possible to fully solve the Jeans equations to express the DM density profile in terms of the observables: $\sigma_{\text{los}}$ and $\Sigma(R)$, the last being the surface brightness (see e.g. [105] for a theoretical approach on this matter, and [64] for a phenomenological approach).

The main motivation of this chapter is to establish a connection between the observations of the well resolved and nucleated dwarf galaxies as observed and studied in [106], with the model of semi-degenerate self-gravitating system of fermions introduced in sections 2.1 and 2.2. The fact that phase-space densities ($Q \sim \rho_0 m^{-4} \sigma^{-3}$) through the center of the configurations can rise to values much higher than its outer halo counterparts (see also conclusions in Chapter 2.2), suggests a natural way to explain the observed nucleated regions at the center of the dwarf galaxies. Indeed, this nucleated structure arising in the majority of dwarfs galaxies below pc scales is a non-well understood issue, many times associated with complex merging processes (see e.g. a discussion on this matter...
3. MULTI-COMPONENT SELF- GRAVITATING SYSTEMS

in [106], and also [107] for a recent work supporting the idea here presented). Thus, the final objective of this more general approach, even under the simplifying symmetries here adopted, is to give more light on this matter by providing an underlying fermionic phase-space distribution for the dark component which naturally condenses through the center due to the quantum pressure.

In next I provide a theoretical background based on the Jeans equations to deal, in a more consistent way, with the baryonic and DM components from the center up to the halo of well resolved dwarf galaxies. In this new picture, the assumption of considering an overall gravitational potential dominated by the DM component has to be relaxed to properly account for the gravitational effect of the baryons towards the center.

3.1 Standard Jeans equations & galactic DM

The Jeans equations corresponds to the theoretical framework of major importance when studying the problem of equilibrium distribution of matter in galaxies. These formalism, being written in terms of the collisionless Boltzmann equation, allow to study ideal systems of N collisionless identical point masses in dynamical equilibrium, which may represent stars or even dark matter particles. This treatment involves a smoothly distributed system’s mass (or particle number, or even luminosity) in space, expressed in terms of an underlying phase-space distribution function \( f(x, v, t) \). It has been shown that the assumption of a smooth, rather than discretized, distribution of matter in space is a very good approximation in order to give a complete description of the dynamics of the system. In fact, when N is a large number (\( \gtrsim 10^4 \)), the possible deviation from the true trajectory of the particle in this idealized model, is very small even for times scales of the order of the age of the Universe (see e.g [5] chapter 7).

For time-independent systems, each of these spatial density distributions: of mass (\( \rho(x) \)), particle number (\( n(x) \)) or luminosity (\( j(x) \)), is obtained by simply multiplying the probability density of finding a particular component of the system at \( x \),

\[
v(x) \equiv \frac{g}{h^3} \int d^3v f(x, v),
\]

by M, N or L respectively, where M is the total mass, N the total number of particles and L the total luminosity. With \( g \) the particle state degeneracy, and \( h \)
3.1. Standard Jeans equations & galactic DM

the Planck constant.

Since the astrophysical systems I am interested here are mainly isolated dwarf spheroidal galaxies (dSph), I will work with the specific Jeans equations under the general assumptions of time-independent systems in spherical symmetry with no angular momentum dependence\(^1\), which reads (see e.g. [5] chapter 4):

\[
\frac{d}{dr} \left( \bar{v}_r^2 \right) = - \nu \frac{d}{dr} \Phi(r). \tag{3.2}
\]

Where \( \Phi(r) \) is the gravitational potential and \( \bar{v}_r^2 \) is the mean square radial velocity which, in this special symmetry, coincides with the radial dispersion velocity:

\[
\bar{v}_r^2 = \sigma_r^2 = 4\pi / (3\nu(r)) \int d^4v f(r, v^2). \tag{3.3}
\]

The probability density (3.1) in this case is simply given by,

\[
\nu(r) = 4\pi (g/h^3) \int_0^\infty d\sigma \sigma^2 f(r, \sigma^2). \tag{3.4}
\]

The equation (3.2) is an hydrostatic equilibrium-like equation, and differs only in that \( \nu(r) \) represents a probability density instead of a mass density, and that the mean particle velocity replaces the fluid velocity; being therefore \( \nu(r)\bar{v}_r^2 \) the pressure-like term.

The application of this Jeans analysis to the spatial distribution of matter in galaxies, depends on which kinds of matter components are pretended to be studied in relation with the observables. For example, in studying galactic halos of dSph galaxies, is usually assumed in the literature that \( \nu(r) \) represents the distribution of luminous matter (i.e. stars), while the underlying gravitational potential \( \Phi(r) \) is dominated by the dark matter component (see e.g. [64, 105]), allowing thus to write the Poisson equation uniquely in terms of the dark matter density \( \rho_{DM}(r) \), this is,

\[
\nabla^2 \Phi(r) = 4\pi G \rho_{DM}(r) \tag{3.5}
\]

\(^1\)This systems are characterized by a distribution function just depending on the total energy \( f(H = 1/2v^2 + \Phi(r)) \), and then having zero anisotropy (i.e. \( \bar{v}_r^2 = \bar{v}_\theta^2 = \bar{v}_\phi^2 \))
The observables are the surface brightness $\Sigma(R)$ (where $R$ is the projected radius perpendicular to the line of sight), and the line-of-sight velocity dispersion $\sigma_{\text{los}}^2(R)$.

The connection between the observables and the theoretical functions in (3.2) are expressed in terms of the following formulas:

$$j(r) = -\frac{1}{\pi} \int_r^\infty \frac{d\Sigma}{dR} \frac{dR}{\sqrt{R^2 - r^2}}, \quad \rho_L(r) = Y_L j(r), \quad (3.6)$$

where $j(r)$ is the three-dimensional luminosity density, related with the probability density by,

$$j(r) = L \nu(r). \quad (3.7)$$

This function is related with the measured surface brightness $\Sigma(R)$ through the Abel de-projection formula as shown in (3.6). The factor which relates the mass density of the luminous material $\rho_L(r)$ and the luminous density $j(r)$ is the mass-to-light ratio of the given stellar population $Y_L$, which in general is assumed to be a constant. The line-of-sight velocity dispersion $\sigma_{\text{los}}^2(R)$ relates with the radial velocity dispersion $\sigma^2_r$ by the corresponding Abel de-projection formula (see e.g. [103]):

$$\sigma^2_r(r) = -\frac{1}{\pi j(r)} \int_r^\infty \frac{d}{dR} \left( \Sigma \sigma_{\text{los}}^2 \right) \frac{dR}{\sqrt{R^2 - r^2}} \quad (3.8)$$

Once the link between the observables and the theoretical variables has been established, and using the simplifying assumption of constant dispersion velocity $\sigma_{\text{los}}^2(R) = \text{const}.$ (as suggested by observations in most dSphs [64]), it is possible to obtain an explicit expression for the dark matter density profile purely in terms of the observables. First of all, it is important to notice that if $\sigma_{\text{los}}^2(R)$ is a constant, then from (3.8) one directly obtains ($\sigma^2_r = \sigma_{\text{los}}^2$). Then, taking equation (3.2) in terms of the luminous mass density $\rho_L(r) = Y j(r)$ ($Y = \text{const}.$), and equation (3.5), we have the system:

$$\sigma_{\text{los}}^2 \frac{d}{dr} j(r) = -j(r) \frac{d}{dr} \Phi(r), \quad (3.9)$$

$$\nabla^2 \Phi(r) = 4\pi G \rho_{\text{DM}}(r). \quad (3.10)$$
3.2 Generalized formalism for a system of DM plus baryons

These two equations can be easily combined to yield

$$\rho_{DM}(r) = -\frac{\sigma_{los}^2}{4\pi G} \nabla^2 \ln j(r),$$  \hspace{1cm} (3.11)

with $j(r)$ given by equation (3.6). Moreover, it is even possible to have analytic expressions for $\rho_{DM}(r)$ if standard light profiles as King, Plummer, Sérsic, etc. are obtained using parametric best fits from the observed surface brightness $\Sigma(R)$. The analysis presented above has been taken from [105], and I have just shown here the part relevant to this Chapter.

It is interesting to notice that with this kind of Jean analysis it is possible to make use of the observables (expressed in terms of the stellar density $\nu(x)$), without the need of knowing explicitly the underlying phase-space distribution function $f(x,v)$.

### 3.2 Generalized formalism for a system of DM plus baryons

The objective of this section is to generalize the hydrostatic equilibrium-like equation (3.2) to the more general case of a multiple-component system of point masses in dynamical equilibrium. For definiteness, I will consider a self-gravitating system composed by $N_1$ identical collisionless dark matter particles and $N_2$ identical collisionless stars, neglecting any possible interaction (other than gravitational) between both kinds of matter. Therefore, within this more general effective treatment, I can write the analogous of Eq. (3.2) but in terms of the mass density and pressure terms as follows:

$$\frac{d}{dr} P_T = -\rho_T \frac{d}{dr} \Phi(r),$$  \hspace{1cm} (3.12)

where $P_T$ and $\rho_T$ are the total pressure and total mass density of the multi-component system composed by DM particles and stars.

There are many motivations for this generalization. I present in what follows three main important reasons:

1) I want to build a model to account in a more consistent way for the dynamical equilibrium of systems with two different matter components, luminous objects (i.e. stars), and dark matter particles.
2) Being interested in the overall distribution of matter in dSph galaxies, from sub-parsec up to kilo-parsec scales, I will relax the hypothesis of $\rho_{DM}(r) \gg \rho_L(r)$ usually considered in the literature (see e.g. [64, 105]).

3) Have a formal approach to analyze the fundamental problem of understanding the detailed spatial distribution of dark matter relative to the baryonic one in relation with the observed central nucleated regions in dwarfs galaxies.

I will work here only under the simplifying assumptions of time-independent systems in spherical symmetry and with no angular momentum dependence.

The following step is to write Eq. (3.12) in terms of each pressure and density components. This is, by assuming the following decompositions $P_T = P_L + P_{DM}$ and $\rho_T = \rho_L + \rho_{DM}$, Eq. (3.12) reads

$$\frac{dP_L}{dr} + \frac{dP_{DM}}{dr} + \rho_L \frac{d\Phi_T}{dr} + \rho_{DM} \frac{d\Phi_T}{dr} = 0 . \quad (3.13)$$

Because I am here assuming a possible linear independence between the gravitational effects of each component, plus the non-interacting (other than gravity) nature between the two matter components, I can write (3.13) as a coupled system of two ordinary differential equations as follows,

$$\frac{d}{dr} (j(r)\sigma_r^2) = -j(r) \frac{d}{dr} \Phi_T(r) , \quad (3.14)$$
$$\frac{d}{dr} P_{DM}(r) = -\rho_{DM}(r) \frac{d}{dr} \Phi_T(r) . \quad (3.15)$$

I have expressed the luminous pressure term above in terms of the luminosity density, which has been also introduced in the right side of the equation for compatibility. Notice moreover that this is possible because I am considering here constant (luminous) mass-to-light ratios.

The above system equations is considered together with the Poisson equation,

$$\nabla^2 \Phi_T(r) = 4\pi G \rho_T(r) . \quad (3.16)$$

With the aim of obtaining an unique equation which contain the information of the coupled system (3.14–3.15), I divide both equations to eliminate the gravitational gradient, and separate each matter component functions at each side of

---

\textsuperscript{2}Even if in [8] has been demonstrated that the dSphs are the most dark matter dominant galaxies in the Universe, this result corresponds only for halo regions.
3.2. Generalized formalism for a system of DM plus baryons

the new equation, to obtain,

$$\frac{1}{\rho_{DM}(r)} \frac{d}{dr} P_{DM}(r) = \frac{1}{J(r)} \frac{d}{dr} J(r) \sigma_r^2.$$  \hspace{1cm} (3.17)

Equation (3.17) will be considered from now on as a Jean master equation, containing the information of both kinds of matter which self-gravitates in an unique system.

I turn now to deal with the equation of state of the dark matter model. Here I will limit to deal with the parametric equation of state of a non-relativistic self-gravitating Fermi gas, being this physical regime more than sufficient when dealing with normal galaxies, and in particular as in this work, dwarf galaxies. Thus we have (the spin degeneracy has been taken $g = 2$),

$$\rho_{DM} = \frac{m^4}{\pi^2 \hbar^3} \int_0^\infty v^2 f_{DM}(r, v^2) dv,$$  \hspace{1cm} (3.18)

$$P_{DM} = \frac{1}{3} \frac{m^4}{\pi^2 \hbar^3} \int_0^\infty v^4 f_{DM}(r, v^2) dv,$$  \hspace{1cm} (3.19)

where $f_{DM}(r, v^2)$ is given by

$$f_{DM}(r, v^2) = \frac{1}{\exp \left[ \left( \frac{mv^2}{2kT} - \theta(r) \right) + 1 \right]}.$$  \hspace{1cm} (3.20)

with $m$ the fermion mass, $T = constant$, the temperature of the isothermal dark matter component, $\theta(r) = \mu(r)/kT$ is the degeneracy parameter defined in terms of the gravitationally coupled chemical potential $\mu(r)$, and $k$ the Boltzmann constant. The infinite integrals in (3.18–3.19) can be expressed in terms of the Polylogarithmic special functions $Li_s(z)$ of order $s$ and argument $z$. Considering that $Li_s(-z) = -\Gamma(s)^{-1} \int_0^\infty dt t^{s-1} / [\exp(t)/z + 1]$, with $t = mv^2/(2kT)$, $z = \exp(\theta(r))$, and $s = 3/2$ or $s = 5/2$ in correspondence with (3.18) or (3.19) respectively. If the following property for the derivative of the Polylogarithm $d[Li_s(z(r))] / dr = z'(r) / z(r) Li_{s-1}(z(r))$ is used, we directly have for the left side in Eq. (3.17)

$$\frac{1}{\rho_{DM}(r)} \frac{d}{dr} P_{DM}(r) = \frac{kT}{m} \frac{d}{dr} \theta(r).$$  \hspace{1cm} (3.21)

Equation (3.21) will be considered from now on as a dark matter master equa-
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tion, containing only information about the equation of state of the dark matter component.

Thus, I now combine the two master equations (3.17) and (3.21) in one unique equation given by,

$$\frac{1}{j(r)} \frac{d}{dr}(j(r)\sigma_r^2) = \frac{kT}{m} \frac{d}{dr} \theta_j(r).$$

(3.22)

It is important to notice that equation (3.22) is an ordinary linear differential equation in $\theta_j(r)$, this last being interpreted as the degeneracy parameter affected by the gravitational effect of the baryonic distribution $j(r)$. Notice that on the left side of Eq. (3.22) we have the observables, and on the right side we have the parameters of the dark matter component (T, $\theta$, m), with T the DM temperature which must be found to fully solve the equations.

Once the solution for the degeneracy parameter $\theta_j(r)$ is obtained from the observables $\sigma \equiv \sigma_r$ and $j(r)$, this must be replaced in the Polylogarithm variant of equation (3.18) to yield the following important expression for the dark matter density function,

$$\rho_{DM}(r) = -\frac{m^{5/2}(kT)^{3/2}}{\sqrt{2}\pi^{3/2}\bar{h}^{3}}Li_{3/2}[-\exp[\theta_j(r)]]$$

(3.23)

The importance of Eq. 3.23 lays in the fact that if analytic expressions can be obtained for the luminosity density $j(r)$ and $\sigma$ from observations, then we can have semi-analytic dark matter density profiles from it. This would further imply a valuable possibility to constraint the dark matter candidate mass $m$, when the observationally inferred magnitudes as core and halo mass are provided. In next section I proceed to apply this generalized treatment to a sample of well resolved nucleated dwarf galaxies, for which either spectrometric and photometric measurements has been obtained from parsec distance-scales up to $\sim 10^2$ pc.

3.3 Application to nucleated dwarf galaxies

The dwarf galaxies are an excellent astrophysical laboratory to study the distribution and nature of the dark matter particles because they belong to the most dark matter-dominated objects in the Universe as demonstrated in [8]. Recently, in [106] a big sample of about 70 dwarf galaxies in the Coma cluster (of distance...
3.3. Application to nucleated dwarf galaxies

$D = 100 Mpc$ were analyzed from high-resolution spectroscopic and photometric data, evidencing a nucleated luminosity profile through the center in the majority of the cases. It is believed that this central light excess is an imprint of the formation history of these galaxies, but there is no closed explanation of the causes and processes which leads to this new structure at pc distance-scales or below. The application of the generalized approach here introduced pretends to give more light to this important issue.

The nucleated surface brightness profiles observed in dwarf galaxies [106], are typically modeled by a Sérsic+Gaussian model of the form $^3$ (see also Fig. 3.1)

$$\frac{\Sigma(R)}{L_\odot / pc^2} = \Sigma_0 e^{-0.5(R/R_c)^2} + \Sigma_e e^{-b(R/R_e-1)},$$

(3.24)

where $\Sigma_0$ is the central observed value of the surface brightness and $\Sigma_e$ the effective surface brightness, while $R_c$ and $R_e$ are the central core scale radius and the effective radius respectively. It is important to notice that the Sérsic index $n$ in (3.24) has been taken equal to unity as it is representative of the majority of the sample considered in [106]. The value of $b$ depends on $n$ (see e.g. [108]), and in the cases analyzed here (i.e. $n = 1$) it is $b \approx 1.66$. The three dimensional luminosity density profile $j(r)$ is obtained through the Abel de-projection formula to yield the following analytic expression,

$$\frac{j(r)}{L_\odot / pc^3} = \frac{1.25 \Sigma_0}{\pi R_c} e^{-0.5r/R_c^2} + \frac{7.92 \Sigma_e}{\pi R_e} K_0(1.6r/R_e),$$

(3.25)

where $K_0(x)$ is the modified Bessel function of second kind and of order 0. We adopt typical values of luminosity and scale-radii in dwarfs as shown in [106]: $\Sigma_0 = 560 L_\odot / pc^2$, $\Sigma_e = 40 L_\odot / pc^2$, $R_c = 25$ pc, $R_e = 850$ pc. The constant line-of-sight velocity dispersion adopted here is $\sigma_{los} \equiv \sigma_r = 9$ km/s, according to [106], thus implying a total (integrated) mass-to-light ratio of $\Upsilon = 1.6$ as obtained from [106] (see Fig. 12 of that paper).

The DM temperature $T$ needed to finally solve Eqtn. (3.22) is obtained by assuming DM predominance in the halo region, where clearly the Mawellian regime in the Fermi-Dirac distribution function is reached (i.e. $\mu(r)/kT << -1$, as demonstrated in sections 2.1 and 2.2). Therefore we must have necessarily

---

$^3$I take the nucleated dwarf GMP3080 as a prototype galaxy of the sample studied in [106], from which typical photometric and spectroscopic observed values are taken as a reference.
\[ T \approx \frac{\text{m} \sigma_{DM}^2}{k}, \] where \( \sigma_{DM} \) is the DM one-dimensional dispersion velocity. Now, \( \sigma_{DM} \) can be obtained from the flat part of the DM rotation curve proper of this classical regime, where the following relation holds (see e.g. \([5]\)) \( v_{\text{circ}} = \sqrt{2\sigma_{DM}} \). With this we have the desired DM temperature as \( T = \frac{mv_{\text{circ}}^2}{2} \), leading to \( kT/m = \frac{\sigma_{DM}^2}{2} (\text{km/s})^2 \), most important in order to solve Eq. (3.22).

The function \( \theta_j(r) \) is obtained by integration of the equation (3.22) between \( r_0 \) and \( r \), with \( \sigma_r^2 = 81 \text{ (km/s)}^2 \) and \( kT/m = 84.5 \text{ (km/s)}^2 \), to yield \( \theta_j(r) = 0.95 \ln \left[ \frac{j(r)}{j(r_0)} \right] + \theta_j(r_0) \), with \( j(r_0 = 4 \text{ pc}) = 12.3 \text{ L}_\odot/\text{pc}^3 \), being \( r_0 \) the innermost resolved radius for a typical dwarf galaxy as studied in \([106]\).

Once with the solution for \( \theta_j(r) \), and by using Eq. (3.23) together with the total mass density \( \rho_T(r) = \Upsilon_j(r) \), it is possible to obtain the ratio between them. This ratio is calculated in dimensionless units to obtain an expression only in terms of the free parameter \( \theta^0_j \) (i.e. independently of the fermion mass). For this, Eq. (3.23) is normalized dividing by \( \rho_{DM}^* = \frac{m^4 v_{\text{circ}}^3}{(4\pi^3/2 \bar{h}^3)} \), while \( \rho_T^*(r) \) is normalized dividing by the central total mass density \( \rho_0^T \approx 20 M_\odot/\text{pc}^3 \). Therefore the new normalized formulas are

\[
\rho_{DM}^n = -L_{3/2} \left[ -\left( \frac{j(r)}{j(0)} \right)^{0.95} e^{\theta^0_j} \right]; \quad \rho_T^n = \frac{\Upsilon_j(r)}{\rho_0^T}. \tag{3.26}
\]

In Fig. 3.2 I show the (normalized) total mass density profile typical of a nucleated dwarf galaxy in the Coma cluster \( \rho^n_T \) together with two different \( \rho^n_{DM} \) for two different values of \( \theta^0_j \). The value of \( \theta^0_j = -0.4 \) is selected assuming a dark matter dominance of \( \sim 94\% \) at \( R_e \) (and a dominance of \( \sim 55\% \) at \( r_0 \)), while \( \theta^0_j = -0.7 \) corresponds for a dark matter dominance of \( \sim 70\% \) at \( R_e \) (with a \( \sim 42\% \) at \( r_0 \)). Values of \( \theta^0_j > 0 \) are prohibited because otherwise the dark matter density would overcome the total mass density.

Once the precise DM dominance at the center of the configuration \( r_0 \) is known, the ‘ino’ mass \( m \) can be obtained from the DM density equation \( \rho^n_{DM} \) together with the normalization factor \( \rho_{DM}^* \). The calculations in the two cases here assumed \( \theta^0_j = -0.4 \) and \( \theta^0_j = -0.7 \) leads respectively, to rest fermion masses of \( m = 1.15 \text{ keV}/c^2 \) and \( m = 1.14 \text{ keV}/c^2 \); implying an small effect of few \( 10^4 \text{ eV}/c^2 \) due to the different dark matter halo dominance adopted. These mass values has to be seriously considered only as order of magnitude due to the many different simplifying assumptions adopted such as spherical symmetry and constant \( \sigma_{\text{los}} \).
3.4 Conclusions

In conclusion, from this two-component (DM plus stars) dynamic approach, and due to the semi-degenerate nature of the dark matter phase-space adopted here, it is possible to better understand the so-called central light excess observed in the light profiles of many dwarf galaxies. This is, the fact that the dominating DM component condenses through the center due the (fermionic) quantum pressure, it generates a deepen in the gravitational potential well in which the baryonic component naturally falls in, generating as a response a nucleated behaviour in the light profile we observe at pc distance-scales or below. The second, and most important outcome of this approach, is that once the dark matter dominance is known at the effective radius $R_e$, the ‘ino’ mass value can be obtained from the equations, falling in the keV region. Nevertheless, a subtle point remains regarding the relation between the DM temperature and the observed dispersion velocity. It is important to notice that the DM density solution (see Eq. (3.26)) is very sensitive to the rate $\sigma_{los}^2/(kT/m)$, which enters as a power in the argument of the Polylogarythmic function $Li_{3/2}$. The fact that the DM tem-

Figure 3.1: A double-component Sérsic-Gaussian model fit to a typical observed surface brightness $\Sigma(R)$ in $(L_\odot/pc^2)$, as considered in [106].

and Y, being not necessarily the case in real dwarf galaxies.
3. MULTI-COMPONENT SELF-GRAVITATING SYSTEMS

Figure 3.2: Two different dimensionless dark matter density profiles in correspondence with the free parameters $\theta_j^0 = -0.4$ and $\theta_j^0 = -0.7$ implying a dark matter dominance of $\sim 94\%$ and $\sim 70\%$ at $R_c$ respectively. These dark profiles are obtained from the dynamical multi-component approach here developed and are contrasted against the total mass density profile as obtained directly from the observables, nicely showing how light follows dark matter all along the configuration.

Temperature is calculated through the observed circular velocity at halo scales ($v_{\text{circ}}$) as explained above, while $\sigma_{\text{los}}$ comes from the distribution of light, makes crucial to obtain accurate data in both cases. In any case, if we center the attention in dwarf spheroidal galaxies (in better consistency with the symmetries adopted) as studied for example in [64], we see that $\sigma_{\text{los}}$ is always around 10 km/s, while typical $v_{\text{circ}}$ are about 13 km/s for cored DM profiles, exactly as considered here.

Moreover, I want to emphasize the potential importance of this approach in views of future high-resolution observations through the center of nearby dwarfs which will reach sub-pc distance-scales; leading to a better understanding in the role of dark matter in connection with massive dark central objects generally interpreted as intermediate massive black holes.
Chapter 4

SELF-INTERACTING STERILE DM NEUTRINOS WITHIN GR

The self-interacting nature of the DM particles at cluster and galaxy scales, has been seriously considered after the discovery of the ‘bullet cluster’, where different upper bounds on the cross-section per unit mass of the order $\sigma/m \lesssim 1 \text{ cm}^2/\text{g}$, could be estimated based on merging and weak-lensing analysis (see [36] for a review on these results). Moreover, under the assumption of self-interaction among the DM particles, possible consequences of this effect in the inner DM halo regions of galaxies have been studied, concluding that shallower inner DM profiles with a consequent dump in the amount of sub-structures should occur. In some analogy to the last case, In this Chapter I will analyze the effects and consequences on the DM density profiles due to an assumed self-interacting nature of DM, but instead of doing so at halo distance-scales, I will focus the attention in the (sub-pc) central regions; motivated by the quantum nature of the DM fermionic phase-space density proven to exist within the model presented in chapter 2 of this Thesis.

In Chapter 2.2 and [32] I evaluate the possibility of an alternative interpretation to the black hole in SgrA*, in terms of the high concentration of dark matter present in the inner quantum core which arises below sub-parsec scales at the center of the DM density profile. However, it is important to recall that although a compact degenerate core mass $M_c \sim 4 \times 10^6 M_\odot$ is definitely possible with an ‘ino’ of $m \sim 10 \text{ keV}/c^2$, the core radius is larger by a factor $\sim 10^2$ than the one obtained from the observational limits imposed by the S-star trajectories such as S1 and S2 orbiting around SgrA* [78, 109].

In Chapter 2.2, we saw also that two possible extensions of the semi-degenerate
fermionic model, in turn, could solve the problem of the quantitative inconsistency of the radius of the quantum core with the observations of SgrA*: 1) the identification of the sterile right-handed neutrinos appearing in extensions of the SM (see, e.g., [21]), as a viable candidate for the ‘ino’ particles in our new scenario, and 2) the inclusion of interactions between the inos in order to be able to reach higher central degeneracies of the DM particles, with consequent higher compactness of the inner quantum core.

Indeed, right-handed neutrinos of warm dark matter (WDM) type, with masses less than 50 keV may still play a role in particle physics today, as conjectured in the so-called right-handed neutrino minimal (non-supersymemtric) extension of standard model ($\nu$MSM) proposed in [18, 20, 21]. This standard model extension involves three right-handed neutrino states, in addition to the three left-handed active neutrinos of the standard model (SM) sector, of which the lightest, of mass at most a few tens of keV, can live longer than the age of the Universe, thus constituting a viable dark matter candidate. Such relatively light right-handed neutrinos appear compatible with cosmological dark matter and Big-Bang-Nucleosynthesis constraints, provided their mixing angles with the active neutrinos of the SM sector are sufficiently small, as shown in fig. 4.1.

The aim of this Chapter is to elaborate further on the above two issues. Before developing the theoretical extension of the original model introduced in Chapter 2, in section 4.1 I first present a motivation which deals with the successful relativistic mean field theory applied to compact objects. Then, in section 4.2 I present the model of self-interacting right-handed neutrinos within the RMF approach, and study its properties under the conditions of finite temperature and density relevant for galactic structures. In section 4.3 I apply the new extended model to the description of the Milky way and finally I discuss the results in section 4.4.

4.1 Motivations

The Relativistic Mean Field (RMF) approach (see e.g. [110]), in particular the $\sigma - \omega - \rho$ meson model formulated to deal with the interactions among the baryons beyond nuclear density, has been successfully applied to Neutron Stars (NS) physics (see e.g. [111]). A central characteristic of these kind of systems is the extreme degenerate state of the large number of nucleons inside the core of the star.
4.1. Motivations

Figure 4.1: Cosmological constraints on the mass ($M_1$) and mixing ($\theta_1$) parameters of the lightest sterile neutrino state $N_1$ of the $\nu$MSM model, in order for it to be a dark matter candidate, consistent with all the current astrophysical and cosmological data [18, 20, 21]. The strongest constraints are imposed by the (non observation) of X-ray lines arising from the decay $N_1 \rightarrow \gamma \nu$, to active neutrinos $\nu$, which implies a lifetime for the $N_1$ states exceeding the Age of the Universe by at least nine orders of magnitude, making them an excellent dark matter candidate.
I will consider here a very different astrophysical system much less dense than a NS, i.e. the sub-miliparsec central region of our galaxy: SgrA*. Despite being talking about two such dissociated environments, if SgrA* is modeled as a self-gravitating system of semi-degenerate fermions at finite temperature, it can be shown that this last shares a common point with the former kind of compact objects: the aforementioned key feature of being a large system of fermionic particles in a highly degenerate state.

At this point, it is important to recall the important work of E. Maoz (1998) [112] regarding the possible candidates constituting a dark cluster which may work as viable alternative to a SMBH. He made an analysis based on dynamical constraints upon the lifetime of a dark cluster formed by different nonluminous (or under-luminous) objects such as, brown dwarfs, neutron stars, stellar BHs or elementary particles. He made his study based on two central ansatz: i) clusters with lifetimes much shorter than 10 Gyr are unacceptable due to the very low probability of being observed at the present time; ii) lowest possible concentration and equal-mass objects. Therefore, making a lifetime analysis of a cluster against evaporation and collisions (see e.g. [5]), he ended up with the conclusion that only elementary particles and primordial BHs are the only possible candidates that are not ruled out by that lifetime considerations when contrasted with the phenomenology in the central region of galaxies. Now, if we recall that primordial BHs can only constitute \( \lesssim 20\% \) of the galactic DM according to the MACHOs project [41] (see also Chapter 1, section 2.1), and that self-gravitating bosonic DM particles (or fuzzy DM) are excluded at halo distance-scales (see Chapter 2, section 2.4) but also may be excluded in the central regions [100]; it implies a fermionic candidate as a promising choice, as considered here.

Indeed, the fermion ‘ball’ idea applied to work as an alternative to the massive black hole thought to be at the center of SgrA*, had been already implemented in the ‘90s ([113–115]). The results there presented indicate that for a ‘heavy-neutrino’ mass of \( m \sim \text{few } 10^1 \text{ keV}/c^2 \), the required compact dark object mass of \( \sim \text{few } 10^6 M_\odot \) within some miliparsecs was achieved, in successful enough agreement with SgrA* observations considering the reduced amount of data of that time. Some years later in [25] this idea was extended to try to model in an unified manner the distribution of dark matter in the galaxy, from the very center up to the outer halo. By that time a model of this kind with keV fermions was enough accurate to yield the required central compactness dictated by the data of the nearest revolving S-stars [78], together with an outer halo in fare
agreement with observations. However, in the last few years those results were
superseded by new constraints imposed by further observational limits on the
trajectory of S-stars such as S1 and S2 [78, 109], indicating a compactness of the
fermion ball not large enough to be in consistency with the observed orbits.

Very recently, as shown in Chapter 2 and in [28, 32], this problem has been
reconsidered presenting a good agreement with galactic observables, ranging
from dwarfs to big spirals for an ‘ino’ mass of 10 keV; with the inclusion of new
interactions (other than gravity) among the ‘inos’.

Therefore, for the first time I extend the above mentioned fermionic model
by including interactions (other than gravity) between the fermions via vector-
meson mediators, in the context of a RMF approach. Thus, in some analogy with
NS physics, we will use the high degenerate self-gravitating system of many-
body interacting-fermions, as a motivation for the starting point of this work.

An important clarification is needed before proceed with the description of
such analog. To start, the effective $\sigma - \omega - \rho$ mesons within SM are mediators of
strong-like forces in baryonic environments above nuclear density, and then ex-
clusive of quark-composite matter. This establish a fundamental difference with
respect to our super-cooled degenerate cores, where the only real constituents
we will use are beyond SM Majorana leptons, described here as right handed
sterile neutrinos $N_{RI}$ of $m \sim 10 \text{ keV}/c^2$; and then, only sensitive to gravity and
possibly other extra effective interactions. Thus, as we will deal here only with
an extra interaction based only on chargeless, spin one massive vector-meson
effective fields $V_\mu$, which are its own anti-particle and possibly carrier of a weak-
like force; a better SM analog (qualitatively speaking only) are the $Z^0$ bosons. In
addition, different effective interaction strengths will be also analyzed.

In this sense, the exchanging of the new $V_\mu$ bosons between the inos, are
thought as neutral-current interactions. This dark sector meson $V_\mu$, (with some
analogy to the $Z^0$ boson), must be thought according to the elastic-scattering
channel (analogous to $\nu_e e^- \rightarrow \nu_e e^-$), and not via neutrino annihilations; leaving
so the interacting sterile neutrinos unaffected except for the fact of momentum
transfer.

Even if a dark sector vector-meson $V_\mu$ analogous of the SM $Z^0$ boson seems a
natural one, it is important to have in mind that in our case the proposed weak-
like interaction is an effective one mediating between single ino-condensate states
$\langle N_{RI} \gamma^\mu N_{RI} \rangle_k$. As will be clear in the next paragraph, this effective force
acts at distance-ranges some orders of magnitude higher than the standard short-
range magnitude of the true weak interaction.

Other interesting parallels can be made between these two systems where the RMF treatment is applied. While in NS the typical mean particle distances ($l$) are of order $1 \text{ fm} = 10^{-13} \text{ cm}$, in galactic cores (i.e. SgrA* as modeled here) and somehow contrary which could be expected, they are not that far from the former value reaching inter-particle mean distances as small as $l = 10^{-9} \text{ cm}$. Of course, central densities are extremely different, being of the order of nuclear density for NS (i.e. $10^{14} \text{ g/cm}^3$), and as big as just $10^{-3} \text{ g/cm}^3$ for the galactic core. The key difference to understand these numbers relays in the relation between the particle number and the radius of each compact object. In NS we have of the order of $N \sim 10^{57}$ neutrons roughly in a radius of $\sim 10 \text{ km}$, while in a much more extended galactic core of order $10^{10} \text{ km}$, we have a huge number of fermions of about $N \sim 10^{70}$. These numbers implies that in NS, the need to use fully degeneracy pressure to maintain the relativistic star in equilibrium, is assured by the fact that the Compton wavelength of a neutron is few times higher than the mean particle distance in its core (i.e. 1 fm). Instead, in a much more dilute environment as the galactic core as modeled here, the gas of thermalized ‘inos’ are in a non-relativistic state, implying very low kinetic energies of the order $kT \sim 10^{-6} \text{ keV}$. So, in this case the necessity of invoke a semi-degenerate equation of state for the keV inos (to maintain the gravitational equilibrium) is assured by the fact that the thermal De-Broglie wavelength $(\hbar/(2\pi m kT)^{1/2})$ is several times higher than the mean particle distance.

The existence of fermionic-condensates has been found already in the laboratory. More specifically, in some systems with large number of ultra-cold atoms (well below Kelvin temperatures) which behave as effective fermions (i.e. $^6\text{Li}$), it has been shown how (see e.g. [116] and refs therein) the data taken from ultra-cold atomic collisions is very well explained in terms of a N-body Hamiltonian with fermion-fermion interactions, similarly as done here. An important difference between both physical contexts, is that in laboratory experiments there is no gradient of the gravitational potential, but the need of an external trapping potential such as a magnetic field; while in the quantum core of the galaxies, the trapping is assured by gravity.
4.2 Self-interacting sterile neutrino DM within RMF

The extension of the model, starts with a minimal but important extension of the lightest-right-handed-fermion sector of νMSM [18, 20, 21] which plays the role of dark matter, by introducing on the basis of a phenomenological effective picture, self-neutrino interactions through a massive vector meson $V_\mu$ mediator.

The Lagrangian of the right-handed neutrino sector, including gravity, reads (we adopt hereafter units with $\hbar = c = 1$):

$$\mathcal{L} = \mathcal{L}_{GR} + \mathcal{L}_{N_{R1}} + \mathcal{L}_V + \mathcal{L}_I \quad (4.1)$$

where

$$\mathcal{L}_{GR} = -\frac{R}{16\pi G} \quad (4.2)$$

$$\mathcal{L}_{N_{R1}} = i \bar{N}_{R1} \gamma^\mu \nabla_\mu N_{R1} - \frac{1}{2} m N_{R1} \bar{N}_{R1} \quad (4.3)$$

$$\mathcal{L}_V = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{1}{2} m_V^2 V_\mu V^\mu \quad (4.4)$$

Here $R$ denotes the Ricci curvature scalar, $m$ the mass of the sterile neutrino and $\nabla_\mu = \partial_\mu - \frac{i}{8} \omega_\mu^{ab} [\gamma_a, \gamma_b]$ is the gravitational covariant derivative acting on a Majorana spinor, with $\omega_\mu^{ab}$ the spin connection and $[,]$ the commutator; Latin indexes denote flat tangent space indexes and are raised and lowered with the Minkowski $\eta_{ab}$ metric.

The vector-meson mass is $m_V$ and $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, where the Lorentz gauge has been applied for the vector-field $V_\mu$. For the purposes of our study here, the microscopic origin of $m_V$ is not specified. It may well come from an appropriate Higgs mechanism in the dark sector (with a Higgs field that is not necessarily the one in the SM sector). The right-handed sterile neutrinos $N_{R1}$ satisfy the Majorana four-spinor condition, $\Psi^c = \Psi$, together with $\Psi = \Psi^T C$, where the conjugate spinor field $\Psi^c = C\Psi^T$ and $C$ is the unitary ($C^+ = C^{-1}$) charge conjugation operator, flipping the fermion chirality, i.e. $(\Psi_L)^c = (\Psi^c)_R$ is right-handed (R), whilst $(\Psi_L)^c = (\Psi^c)_L$ is left-handed (L). The definition of chirality (handedness) is the standard one, $\Psi_{L,R} = \frac{1}{2} \left( 1 \mp \gamma^5 \right) \Psi$, with the + (-) sign denoting Right-(Left)handed spinors, and $\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$, with $\gamma^\mu$ the $4 \times 4$ Dirac matrices, satisfying $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$, where $g^{\mu\nu}$ is the (inverse)
space-time metric in the galactic core, which is taken to be static and spherically symmetric

\[ g_{\mu\nu}dx^\mu dx^\nu = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \]  

(4.5)

where the metric functions \( e^\nu \) and \( e^\lambda \) depend only on the radial coordinate, \( r \).

The interactive part \( L_I \) of the Lagrangian density, adopting a minimal coupling, is given by

\[ L_I = -g_{\mu V} V_{\mu} J_{V}, \]  

(4.6)

with

\[ J_{V}^\mu = \overline{N}_{R1} \gamma^\mu N_{R1}, \]  

(4.7)

the sterile neutrino current which is conserved, if decays of sterile neutrinos are ignored. Even if this is an important approximation done here, in general, one may add to (4.1) a Yukawa term coupling the right-handed neutrino to the active neutrino sector (see, e.g., [18, 20, 21])

\[ L_{Yuk} = F_{a1} \ell_{a} \overline{N}_{R1} \phi^c + h.c. \]  

(4.8)

where \( \ell_a \) are the lepton doublets of the SM, \( \alpha = e, \mu, \tau \), \( F_{a1} \) are appropriate Yukawa couplings, and \( \phi^c \) is the SM conjugate Higgs field, \( i.e. \phi^c = i \tau_2 \phi^* \), with \( \tau_2 \) the \( 2 \times 2 \) Pauli matrix. Upon considering such a coupling, one obtains the stringent X-ray constraints of the mixing angle and mass of \( N_{R1} \) depicted in fig. 4.1. In such a case the sterile neutrino current (4.7) is not conserved in time. For our purposes we shall ignore such a mixing with the SM sector, setting \( F_{a1} = 0 \). The important feature here are the self-interactions of the right-handed neutrino in ensuring phenomenologically correct values for the radius and mass of the galactic core. Since, as we shall see, the mass range we obtain is compatible with the one in figure 4.1, one may switch on the term (4.8) in a full phenomenological study, including the SM sector, and in particular neutrino oscillations and Early Universe physics (e.g. leptogenesis [18, 20, 21]), without affecting our conclusions.

From (4.1) one obtains the following equations of motion for the various
4.2. Self-interacting sterile neutrino DM within RMF

fields:

\[ G_{\mu \nu} + 8 \pi G T_{\mu \nu} = 0, \]  \hspace{1cm} (4.9)
\[ \nabla_\mu V^{\mu \nu} + m_V^2 V^\nu - g_V J^\nu_V = 0, \]  \hspace{1cm} (4.10)
\[ \bar{N}_{R1} i \gamma^\mu \overleftrightarrow{D}_\mu + \frac{1}{2} m \bar{N}_{R1} = 0, \]  \hspace{1cm} (4.11)

where \( G_{\mu \nu} \) is the Einstein tensor and \( T_{\mu \nu} \) is the total energy-momentum tensor of the free-fields composed by two terms: \( T_{N_{R1}}^{\mu \nu} \) and \( T_{V}^{\mu \nu} \), each of which satisfies the perfect fluid prescription

\[ T^{\mu \nu} = (E + P) u^\mu u^\nu - Pg^{\mu \nu}, \]  \hspace{1cm} (4.12)

with \( E \) and \( P \) the energy-density and pressure which we define below.

I now introduce the relativistic mean-field (RMF) approximation. In this approach, the system can be considered as a static uniform matter distribution in its ground state. Thus, the vector meson field as well as the the source currents are replaced by their mean values in this state implying independence of the spatial coordinates \( x \) (space translational invariance); consequently no spatial current exists, i.e. \( V_\mu \rightarrow \langle V_0 \rangle \) and only the temporal component of (4.7) is non zero \( J^\mu_V \rightarrow \langle J^0_V \rangle = \langle \bar{N}_{R1} \gamma^0 N_{R1} \rangle = \langle N_{R1}^\dagger N_{R1} \rangle \), where the last expression within brackets denotes the finite number density of right-handed neutrino matter times the temporal component of the pertinent (average) velocity.

The mean-field approximation allows one to solve the coupled system of differential equations (4.11) rather straightforwardly, to obtain directly the mean-field vector meson as

\[ V_0 = \frac{g_V}{m_V^2} J^V_0 \]  \hspace{1cm} (4.13)

with the notation \( \langle V_0 \rangle \equiv V_0 \) and

\[ \langle J^0_V \rangle \equiv J^V_0 = n u_0, \]  \hspace{1cm} (4.14)

where \( u_0 = e^\nu/2 \) is the time-component of the (average) future-directed four velocity vector, and I have used the normalization condition \( u^\mu u_\mu = 1 \).

The Majorana spinors in the RMF approximation can be simply expressed

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as the corresponding momentum (Fourier) eigen-states with no $x-$dependent terms (see, e.g., [111]) $\Psi(x) = \Psi(k) e^{-ik \cdot x}$. Recalling that I am working here with a system comprising of a very large number $N$ of fermions in thermodynamic equilibrium at finite temperature $T$, I can assume that the fermion number density is expressed in terms of the Fermi-Dirac distribution function $f(k)$

$$n = e^{-v/2} \langle N_{R1}(k) \gamma^0 N_{R1}(k) \rangle = \frac{8}{(2\pi)^3} \int d^3k f(k). \quad (4.15)$$

where $g$ is a spin-degeneracy factor for the Majorana spinors, the momentum integration is extended over all the momentum space, and $f(k) = (\exp[(\epsilon(k) - \mu)/(k_B T)] + 1)^{-1}$. Here $\epsilon(k) = \sqrt{k^2 + m^2} - m$ is the particle kinetic energy, $\mu$ is the chemical potential with the particle rest-energy subtracted off, $T$ is the temperature of the heat bath, and $k_B$ is the Boltzmann constant. It is important to notice that we are working with the right-handed component of the full Majorana spinor $\Psi$, and so, although a full Majorana spinor (left plus right chiral states) is its own antiparticle implying a spin degeneracy $g = 4$, this is not the case for the singlet right-handed component $N_{R1}$ (thought of as a spin +1/2 fermion of one helicity state), for which $g = 1$ and that we adopt hereafter.

The contributions to the energy-density and pressure from fermions, in the RMF approximation, are $\langle T^0_0 \rangle_{N_{R1}} = \mathcal{E}_C$ and $\langle T^1_1 \rangle_{N_{R1}} = \mathcal{P}_C$ respectively. They are fully determined by the distribution function $f(k)$ (with particle helicity $g = 1$):

$$\mathcal{E}_C = m \frac{1}{(2\pi)^3} \int f(k) \left[ 1 + \frac{\epsilon(k)}{m} \right] d^3k, \quad (4.16)$$

$$\mathcal{P}_C = \frac{1}{3} \frac{1}{(2\pi)^3} \int f(k) \left[ 1 + \frac{\epsilon(k)}{m} \right]^{-1} \left[ 1 + \frac{\epsilon(k)}{2m} \right] \epsilon d^3k, \quad (4.17)$$

where the integration is over all the momentum space, while the ones from the vector field are

$$\mathcal{E}_V = \frac{1}{2} e^{-v} m_V^2 V_0^2 = \frac{1}{2} C n^2, \quad (4.18)$$

$$\mathcal{P}_V = \frac{1}{2} e^{-v} m_V^2 V_0^2 = \frac{1}{2} C n^2, \quad (4.19)$$

Alternatively, this contribution to the energy can be expressed as the expectation value of the energy $\langle \Psi \gamma_0 k_0 \Psi \rangle$, where $E(k) \equiv k_0$ are the energy eigenvalues of the corresponding Majorana Hamiltonian (see, e.g., [111]).

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1Alternatively, this contribution to the energy can be expressed as the expectation value of the energy $\langle \Psi \gamma_0 k_0 \Psi \rangle$, where $E(k) \equiv k_0$ are the energy eigenvalues of the corresponding Majorana Hamiltonian (see, e.g., [111]).
I have used Eqs. (4.13) and (4.14) in the last term of the above equations, and I have introduced the parameter $C_V \equiv g_V^2 / m_V^2$, which encodes information about the strength of the coupling of the effective interactions of the fermions and the mass of the vector meson mediator.

Therefore, I can finally write the total energy-density and pressure as

\begin{align}
\mathcal{E} &= \mathcal{E}_C + \mathcal{E}_V , \\
\mathcal{P} &= \mathcal{P}_C + \mathcal{P}_V .
\end{align}

I now introduce the thermodynamic equilibrium conditions. In the case of a self-gravitating system of semi-degenerate fermions at finite temperature in general relativity, in absence of any self-interactions (other than gravity) such conditions read: $e^{\nu/2}T =$constant and $e^{\nu/2}(\mu + m) =$constant, exactly as given in Chapter 2. The first equation corresponds to the Tolman condition [76], and the second to the Klein condition [77]. In the presence of the vector-meson mediator interaction (4.6), it can be shown that only the Klein condition is modified with the generalized thermodynamic equilibrium conditions now reading (see, e.g., [117], for details)

\begin{align}
e^{\nu/2}T &= \text{constant} , \\
e^{\nu/2}(\mu + m) + g_V V_0 &= e^{\nu/2}(\mu + m + C_V n) = \text{constant},
\end{align}

where the term $g_V V_0$ is interpreted as a potential energy associated to the new meson field $V_\mu$, and we have used in the second equation of (4.22), Eqs. (4.13) and (4.14).

We can then finally write the full system of Einstein equations (4.9) together with the thermodynamic equilibrium conditions (4.22) in the following dimen-
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The following dimensionless quantities were introduced (as in the original version of the model): \( \hat{r} = r / \chi, \hat{h} = Gm\chi^2, \) \( \hat{M} = GM / \chi, \hat{E} = G\chi^2 E, \) \( \hat{P} = G\chi^2 P, \) with \( m_p = \sqrt{1/G} \) the Planck mass, and \( \chi = 2\pi^3/(1/m)(m_p/m) \) the dimensional factor with units of length, scaling as \( m^{-2}. \) The temperature and degeneracy parameters \( \beta = k_B T / m, \) and \( \theta = \mu / (k_B T) \) respectively, are the same as in the original model (Chapter 2). The constants of the equilibrium conditions of Tolman and Klein has been evaluated at the center \( r = 0, \) which we indicate with a subscript ‘0’. I discuss in the next section the boundary conditions under which the system of equilibrium equations (4.23–4.26) must be integrated when applied to model the distribution of DM in our galaxy.

4.3 The Milky Way: from SgrA* to the DM halo

The variables of the system (4.23–4.26) are \( M(r), \theta(r), \beta(r), \) and \( v(r). \) The initial conditions are set as follows. \( M(0) = 0, \theta(0) = \theta_0, \beta(0) = \beta_0, \) while \( v(0) \) can be any value since the system (4.23–4.26) depends only on the radial derivative of \( v \) and not explicitly on the function itself. Namely, the equations are invariant under the change \( v \rightarrow v + \) constant. Therefore, the correct value of \( v \) can be obtained after the numerical integration by shifting the function by an appropriate constant in order to match the gravitational potential to the one given by the exterior Schwarzschild solution at the outer boundary of the galaxy. We thus choose \( v(0) = 0 \) for simplicity. The boundary conditions are given by the request of the observational agreement of the inner DM core and halo with: 1) the

\[ \begin{align*}
\frac{d\hat{M}}{d\hat{r}} &= 4\pi\hat{r}^2 \hat{E}, \\
\frac{dv}{d\hat{r}} &= 2\hat{M} + 4\pi\hat{P}\hat{r}^3 \\
\frac{d\theta}{d\hat{r}} &= -\frac{1}{2\beta} \frac{dv}{d\hat{r}} \left(1 + \frac{Cm^2}{4\pi^2} \hat{h} - \frac{Cm^2}{4\pi^2} \beta \frac{d\hat{h}}{d\hat{r}}\right) \left(1 + \frac{Cm^2}{4\pi^2} \frac{1}{\beta} \frac{d\theta}{d\hat{r}}\right), \\
\beta &= \beta_0 e^{\nu - \nu(r)}.
\end{align*} \]

For \( C_V = 0, \) the coupled system of differential equations (4.23–4.26) reduces to the form presented in Chapter 2 and in [28, 32].
4.3. The Milky Way: from SgrA* to the DM halo

compactness of the sub-mpc region of the galaxy centered in SgrA*, 2) the dark matter outer halo mass $M_h$ at the halo $r_h$, and 3) the flat galactic rotation curves with the specific value at the outer halo radius, $v_h$. The details of how to solve the boundary condition problem is given in the core-halo transition sub-section below. It is important to recall that we define the radius of the inner quantum core $r_c$ as the distance at which the rotation curve reaches its first maximum, and the outer halo radius $r_h$ at the onset of the flattening of the rotation curve, which occurs at the second maximum. The rotation curve is given by the circular velocity

$$v(r) = \sqrt{\frac{GM(r)}{r - 2GM(r)}}. \quad (4.27)$$

Following this procedure, I will constrain the physical conditions $\beta_0$ and $\theta_0$, together with the physical parameters, such as the sterile neutrino mass $m$, as well as the coupling parameter $C_V$. It is important to recall, as shown in Chapter 2, sections 2.1, 2.2, and detailed for this case in the appendix, the specific value of the circular velocity in the flat region is intimately related to the temperature parameter $\beta$.

The non-interacting case $C_V = 0$ of the model (4.1) was solved in Chapter 2 and in [28, 32], to give as a result a dark matter density profile with already mentioned characteristics: 1) an inner core of almost constant density governed by degenerate quantum statistics; 2) an intermediate region with a sharply decreasing density distribution followed by an extended plateau, implying quantum corrections; 3) a decreasing density distribution $\rho \propto r^{-2}$ leading to flat rotation curves fulfilling the classical Boltzmann statistics. The mass of the inos was determined as an eigenfunction of the mass of the inner quantum cores.

It is important to recall that in Fig. 2.8 (Chapter 2) a solution with $m \sim 10$ keV/$c^2$ of the $C_V = 0$ non-interacting model, was already compared and contrasted with different DM halo profiles used in the literature, showing good results. While, in the sub-parsec core region and for core masses of $\sim 10^6 M_\odot$ typical of (Milky Way-like) galaxies, for an ino mass $\sim 10$ keV/$c^2$, the thermal de-Broglie wavelength, $\lambda_B = h / \sqrt{2\pi m k_B T}$, is larger than the inter-particle mean distance $l$ of the inos, justifying the quantum-statistical nature of the core.

In this line, I adopt here the ansatz that the self-interactions occur only in the core region when the condition $\lambda_B / l > 1$ is satisfied. Therefore, I introduce the
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following piecewise form of the coupling parameter:

\[ C_V(r) = \begin{cases} 
C_0 & \text{when } \lambda_B/l > 1 \text{ at } r < r_c + \delta r, \\
0 & \text{when } \lambda_B/l < 1 \text{ at } r \geq r_c + \delta r,
\end{cases} \quad (4.28) \]

where \( C_0 \) is a positive constant, and \( \delta r \ll r_c \) is a thin layer whose physical importance will become clear later, when the core-halo matching conditions are specified. As \( r \) approaches \( r_c + \delta r \) the dark matter distribution undergoes a (first-order) phase-transition from the quantum degenerate state to the Boltzmannian one (i.e. a transition from \( \theta > 0 \) to \( \theta < 0 \) implying the physical transition from \( \lambda_B/l > 1 \) to \( \lambda_B/l < 1 \)).

Hereafter I normalize the coupling constant \( C_0 \) to the Fermi constant \( C_F \approx 10^{-5} \) GeV\(^{-2}\) of the SM weak-interaction, i.e. I introduce the dimensionless constant \( \overline{C}_0 = C_0/C_F \).

4.3.1 SgrA* region

The mass \( M_c \) of the degenerate quantum core must agree with the mass enclosed within the region bounded by the pericenter of the S2 star. At the same time, I use the S2-pericenter as an upper limit to the core radius \( r_c \), i.e. [78]

\[ M_c = 4.4 \times 10^6 M_\odot; \quad r_c = 6 \times 10^{-4} \text{ pc}. \quad (4.29) \]

There is an error of 8% in the above value of \( M_c \) due to the uncertainties in the measurement of the distance to the galactic center \( R_0 = 8.33 \pm 0.35 \) kpc, while the error in the pericenter of the S2 star is of about 4% [78]. The above parameters imply a central density of order \( \sim 10^{16} M_\odot/\text{pc}^3 \), which is almost five orders of magnitude larger than the one obtained for the model without self-interactions (see Fig. 2.8) with the same core mass.
4.3. The Milky Way: from SgrA* to the DM halo

4.3.2 Dark matter halo region

For the observables in this region, and as done in Chapter 2, section 2.2 and [32], I adopt the following dark matter halo parameters [118]:

\[ r_h = 32.4 \text{ kpc}; \quad v_h = 155 \text{ km/s}; \quad M_h = 1.75 \times 10^{11} M_\odot, \]

(4.30)

where the subscript \( h \) indicates quantities at the halo radius. All the halo parameters are subject to a \( \sim 10\% \) of error [118].

4.3.3 Core-Halo transition

The piecewise treatment for the self-interactions proposed in (4.28), implies two set of initial conditions for the variables of the system \( v(r), \beta(r), \theta(r), M(r) \). One set such denoted by: \( v_0, \beta_0, \theta_0, M_0 \) (starting at \( r_0 \)), such that \( C_0 \neq 0 \) in fulfilment with the SgrA* region observables; and another set denoted by: \( v_h^0, \beta_h^0, \theta_h^0, M_h^0 \) (starting at \( r_m = r_c + \delta r \)), such that \( C_0 = 0 \) in fulfilment with DM halo observables given above. A matching condition must thus be found just after the transition layer, at the matching radius \( r_m \), in order to be in agreement with all the required observables and at the same time ensuring the continuity of the metric function \( v(r) \) at that radius. The detailed procedure is given in four steps,

1. The generalized Tolman+Klein conditions (4.22) have to be fulfilled at any radius. Therefore, I first ask for the fulfilment of these generalized thermodynamic potentials conditions between \( r_0 \) and \( r_m = r_c + \delta r \), to have the right physical relation between the two corresponding set of free parameters. When this is done, and after some algebra, the following implicit formula for \( \beta_0 \) must hold (recalling that the interactions are ‘turned-off’ at \( r_h^0 )

\[ \beta_0 = \frac{-(C_0 n_0/m + 1)}{(\theta_0 - \theta_h^0 - 1/\beta_h^0)}, \quad n_0 = \frac{\sqrt{2} m^3 \beta_0^{3/2}}{\pi^2} (F_{1/2} + \beta_0 F_{3/2}), \]

(4.31)

\(^3\)According to [118], the dark matter best-fit distribution for the Milky Way, is provided from the two parametric cored Burkert profile with a specific central density parameter \( \rho_B^0 = 2 \times 10^{-2} M_\odot/\text{pc}^3 \), and a dark halo length scale parameter \( h = 10 \ \text{kpc} \). This corresponds to a halo radius \( r_h \), as we define at the maximum of the rotation curve at the onset of the flat behavior at the dark matter dominant part, with an associated halo velocity \( v_h \) and mass \( M_h \) as given in (4.30).
where \( n_0 \) is the central particle number density (see Eq. 4.15) expressed in terms of the generalized Fermi-Dirac integrals evaluated at the center \( r_0 \):

\[
F_{j} = \int_{0}^{\infty} dx x^{j}(1 + 1/2\beta_{0}x)^{1/2}/(1 + e^{x-\theta_{0}}).
\]

2. After obtaining the value of \( \beta_{0}^{h} = 1.065 \times 10^{-7} \) as detailed in appendix, a set of parameters \((\theta_{0}^{h}, m)\) is found in agreement with the halo observables (4.30). Only ‘ino’ masses in the keV region are considered.

3. Once \((\theta_{0}^{h}, m)\) together with \( \beta_{0}^{h} = 1.065 \times 10^{-7} \) are obtained in agreement with (4.30), a value for the interaction constant \( C_{0} \) is given, to find pair of parameters \((\beta_{0}, \theta_{0})\) from (4.31)^4, which are in agreement with the observables (4.29)^5.

4. The fact that the core observables are fulfilled at \( r_{c} < r_{m} \), implies that the matching with the halo region starting at \( r_{c} + \delta r = r_{m} \) is still needed. As said already in the first step, the Klein + Tolman condition as to be fulfilled at \( r_{m} \). But at this transition radius, the continuity of the metric function \( \nu \) (or equivalently the continuity of \( \beta \) due to Tolman condition) is needed. I thus, ask for the fulfilment of (4.22) at \( r_{c} + \delta r \) such that \( \beta(r_{c} + \delta r) = \beta_{0}^{h} = 1.065 \times 10^{-7} \). With this condition (4.22) now reads (recalling that \( \theta(r_{m}) = \theta_{0}^{h} \))

\[
\theta(r_{c} + \delta r) = \theta_{0}^{h} - \frac{C_{0}n(r_{c} + \delta r)}{m\beta_{0}^{h}}, \quad (4.32)
\]

\[
n(r_{c} + \delta r) = \frac{\sqrt{2}m^{3}(\beta_{0}^{h})^{3/2}}{\pi^{2}}(F_{1/2} + \beta_{0}^{h}F_{3/2}), \quad (4.33)
\]

where now the generalized Fermi-Dirac integrals are evaluated at \( r_{c} + \delta r \):

\[
F_{j} = \int_{0}^{\infty} dx x^{j}(1 + 1/2\beta_{0}x)^{1/2}/(1 + e^{x-\theta(r_{c} + \delta r)}). \]

Therefore, in each interaction regime, the value of \( \delta r \) has to be found such that the condition (4.33) is fulfilled.

The results respecting the above procedure are presented in next section.

---

^4Due to the implicit nature of Eq. (4.31), the values \((\beta_{0}, \theta_{0})\) are found numerically with at least three digit precision.

^5Notice that the observable \( r_{c} \) in (4.29) is an upper limit, and does not need to be fulfilled precisely because any lower value is allowed by the the observational S-star orbit-constraints.
4.3.4 Numerical integration results

Following the above steps, in Tables (4.1–4.2) I summarize the results of pair solutions \((m, \theta^h_0)\), and \((\beta_0, \theta_0)\) which fulfils the core and halo observables (4.30 and 4.29) respectively, for different physically relevant sets of interaction constants \(C_0\). In particular, I will deal only with interaction strengths ranging from weak-like (i.e. \(C_0 = 1\)), up to strong-like (i.e. \(C_0 = 10^{14}\)). For all cases we use the aforementioned value of \(\beta_0 = 1.065 \times 10^{-7}\).

Even if the upper limit in the sterile neutrino mass \((m \lesssim 50 \text{ keV}/c^2)\) is an outcome imposed by cosmological and astrophysical constraints under the assumption of mixing with the SM sector (4.8) (cf. fig. 4.1); we will also explore larger (phenomenologically) possible mass values considering that the ‘inos’ at galactic centers does not interact with the active sector.

\[
\begin{array}{|c|c|c|}
\hline
C_0 & m(\text{keV}) & \theta^h_0 \\
\hline
0 & 10 & -24.1 \\
0 & 47 & -29.3 \\
0 & 350 & -37.3 \\
\hline
\end{array}
\]

Table 4.1: Pair of \((m, \theta^h_0)\) parameters in agreement with the halo observables (4.30), in absence of self-interactions.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
C_0 & m (\text{keV}) & \beta_0 & \theta_0 & \delta r (\text{pc}) & \theta(r_c + \delta r) \\
\hline
1 & 47 & 1.0657 \times 10^{-7} & 3.70 \times 10^3 & 2.1 \times 10^{-4} & -29.3 \\
& 350 & 1.4312 \times 10^{-7} & 2.40 \times 10^6 & 6.7 \times 10^{-7} & -37.3 \\
10^{14} & 47 & 1.0657 \times 10^{-7} & 3.63 \times 10^3 & 2.2 \times 10^{-4} & -29.3 \\
& 350 & 1.1046 \times 10^{-7} & 1.27 \times 10^5 & 9.4 \times 10^{-7} & -37.3 \\
\hline
\end{array}
\]

Table 4.2: Set of model parameters \((m, \theta_0, \beta_0)\) in fulfilment with (4.31) and for different interaction regimes (from weak-like to strong-like) in agreement with the SgrA* observables (4.29). The right-handed neutrino masses are limited in the range \((47,350) \text{ keV}\), where 350 keV is the value at which the critical core mass \(M^*_c \approx 4.4 \times 10^6 M_\odot\) is approximately reached. The value of \(\theta(r_c + \delta r) < 0\) in each case corresponds to the last core degeneracy parameter, for which the continuity of \(\beta\) (and \(\nu\)) is ensured at that radius, after which the transition to the halo region \((C_0 = 0)\) is achieved.

\(\dagger\) The value of \(\theta(r_c + \delta r)\) calculated from (4.33) is in all cases here studied.
lower than $\theta_0^h$ by a negligible amount.

Two important conclusions can be drawn from the numerical analysis presented in Tables 4.1–4.2:

I) In the weak-like interaction regime ($\overline{C}_0 = 1$), it exists a lower and upper bound for the ino mass. While $m = 47$ keV/$c^2$ sets the lowest admissible particle mass up to which the core observational constraints are fulfilled (within observational errors); the uppermost bound of $m = 350$ keV/$c^2$ is due to first principle physics. At this fermion mass the critical core mass $M_{cr} \propto M_{pl}^3/m^2 \approx 4.4 \times 10^6 M_\odot$ is achieved$^6$, where $M_{pl}$ is the Planck mass. This formula is valid in the case when the core becomes fully-degenerate ($\theta_0 \gg 1$), where the equilibrium solutions arrives to the critical point as numerically demonstrated for keV ‘ino’ masses in [31].

II) As the interaction parameter $C_V \equiv C_0$ increases, the contribution to the total energy and pressure from the meson-vector field $(1/2 C n^2)$ becomes more and more relevant, such that when the onset of the strong-like interaction regime ($\overline{C}_0 = 10^{14}$) is reached, some appreciable effects in the core mass appears for the same set of remaining parameters with respect to the weak-like interaction regime. Precisely speaking, at this strong-interaction regime, as can be seen from Table 4.2 second row, for $m = 47$ keV/$c^2$ a slightly lower value for the central degeneracy is needed to compensate for the same core mass as compared with the weak regime; or by the inverse reasoning, if the same central degeneracy as in the former weak-interaction case is used, an increase of $\sim$ few % in $M_c$ appears. More evident is the case when the ino mass reaches $m = 350$ keV/$c^2$, where the desired core mass is achieved already at a central degeneracy about one order of magnitude lower with respect to the weak-interacting case.$^7$

In Fig. 4.2 I present the overall density distribution $\rho(r)$ together with the

---

$^6$Strictly speaking this formula is valid only in the case of absence of self-interactions, but, at this weak-like regime, the contribution to the total energy and pressure from the meson-vector field is such that $E_V \ll E_C$, and $P_V \ll P_C$; and therefore no appreciable effect in the critical point is present.

$^7$It is important to mention that at this much higher (strong-like) interaction regime, the mass $m = 350$ keV/$c^2$ does not exactly implies the arrival to the critical core mass through the formula $M_{cr} \propto M_{pl}^3/m^2$. For $\overline{C}_0 = 10^{14}$, the critical point as to be found following the track of the one-parameter sequence of equilibrium configurations up to the critical point represented by the maximum in a central density $\rho_0$ Vs. core mass $M_c$ diagram. Therefore the critical core mass $M_{cr}$ will be reached at a different ‘ino’ mass around 350 keV/$c^2$. Similar consequences will occur for even higher values of $\overline{C}_0$, which I am not going to investigate here.
Figure 4.2: Mass density and degeneracy parameter profiles for \( m = 47 \text{ keV}/c^2 \) in the weak-interaction regime \( C_0 = 1 \). These physical parameters fulfill the core and halo observational constraints (4.29–4.30).
corresponding behaviour of the degeneracy parameter $\theta(r)$ for a specific sterile neutrino mass ($m = 47$ keV/c$^2$) satisfying all the observables. It is important to notice that the density profile in the observationally well constrained halo region of Fig. 4.2, coincides with the one solved in the absence of self-interactions in Chapter 2, section 2.2 (see Fig. 2.8) and in [32], where it is shown the good agreement with other phenomenological profiles used in the literature.

Appendix

Determination of $\beta_0^h$

A quite precise determination of $\beta_0^h$, i.e. at $r_m = r_c + \delta r$, can be understood through the following two concepts (the value of the speed of light $c$ is here given in km/s):

1) Boltzmann regime at $r \sim r_m$: Because at $r \gg r_m$ the degeneracy parameter fulfills $\theta(r) \ll -1$, the Fermi-Dirac statistics necessarily entered already in a pure Boltzmann regime. The Boltzmann distribution function has associated the familiar formula for the one dimensional dispersion velocity independent of the radius

$$
\sigma^2 = \frac{k_B T}{m} \quad (4.34)
$$

2) Classical isothermal-sphere condition: A classical self-gravitating system of Boltzmannian particles in hydrostatic equilibrium is described by the isothermal-sphere model. The relation between the circular velocity $v_c(r)$ and $\sigma$ for an isothermal-sphere model is $v_c^2(r) = -\sigma^2 (d \ln \rho(r) / d \ln r)$, where $\rho(r)$ is the mass density (see e.g. [5]). Different cored solutions to $\rho(r)$ depends only in the constant initial condition $\rho_0^h$ and $\sigma$, implying an universal behaviour (scaling) in the morphology of the profile. Thus, the logarithmic derivative evaluated at the halo radius $r_h$ (defined at the maximum of the velocity curve, i.e. the onset of the flat part) is $(d \ln \rho(r) / d \ln r)|_{r_h} = -2.51$. This implies the following relation $v_h^2 = 2.51 \sigma^2$, and consequently from the equations (??–4.34) we have

$$
\beta_0^h = \frac{1}{2.51} \left( \frac{v_h}{c} \right)^2 \quad (4.35)
$$
which for $v_h = 155$ km/s (and $c = 2.9979 \times 10^5$ km/s) it gives

$$\beta^h_0 = 1.065 \times 10^{-7},$$

(4.36)

as I wanted to prove.

### 4.4 Summary and Conclusions

In summary, motivated by the RMF theories successfully applied to the self-interacting fermions in astrophysical compact objects, a model of self-interacting right-handed neutrinos (more specifically sterile-neutrinos within the νMSM) was developed by introducing a vector boson field $V_\mu$, interacting (in the minimal coupling choice) with the fermionic field.

On astrophysical grounds, a dense DM cluster scenario was assumed as an alternative to the usual central massive BH scenario, in order to apply the above extended model to be in fulfillment with the minimal required compactness in the central region of our galaxy, in addition to the DM halo. The central compactness value is defined by the observations of the S2 star orbit, being the minimum required density of the DM cluster ($\rho_0 \sim 10^{16} M_\odot / pc^3$) enclosed below the mpc distance-scale. The resulting very large number of self-interacting-fermions which fulfill those constraints, are in a highly degenerate state, allowing to consider the system as an uniform and static (space translation invariance) field in its ground state. This implies the possibility to replace the vector boson field as well as the sterile-neutrino current by its mean values (i.e. $V_\mu \rightarrow V_0$, in the mean field approach), and easily solve the equations of motion of the problem. Once this is done, and after adopting a perfect fluid prescription for the energy momentum tensor of the fermions and the vector bosons, the (parametric) equation of state of the whole system is provided. This equation includes now new energy and pressure terms coming from the boson mediators, generalizing the original equation of state of the free fermions (see Eqs. 4.16–4.17), and is written in terms of the new free interaction parameter: $C_0 = (g_V / m_V)^2$, where $g_V$ is the coupling constant of the vector interaction and $m_V$ the vector boson mass, which don’t need to be specified. The inclusion of the vector boson field introduces a new potential energy $g_V V_0$ implying a generalization of the thermodynamic equilibrium conditions (Klein potentials), which, consequently, generalizes the system of equilibrium (differential) equations of the free fermions into the sys-
Finally, the generalized system of equilibrium equations accounting for the self-interactions, are applied to be in agreement with the galactic center observables (where different interaction regimes, from weak-like to strong-like are analyzed), as well as with the DM halo observables (in the absence of self-interactions). The density profile solutions are consistent with a core-halo (first-order) phase-transition happening within a very thin layer, where the system goes from a self-interacting fermionic condensed phase, to a Boltzmannian and diluted one. The physics of the transition is treated effectively through the ratio between the thermal de-Broglie wavelength and the inter-particle mean distance $\lambda_B/l$, which is higher than one in the quantum core, and much lower than unity in the halo region. Interestingly, for any interaction regime studied, a sterile-neutrino mass range between few $10^1$ keV till few $10^2$ keV is allowed by the core-halo observables. The fact that an sterile neutrino with few tens of keV is selected independently from both galactic and DM astro-particle physics studies, points towards an important role of the right-handed neutrinos in the cosmic structure.
Chapter 5

DARK EINSTEIN CLUSTERS AS ALTERNATIVES TO SMBHs

In 1939 Einstein [119] provided a model of self-gravitating masses, each moving along geodesic circular orbits under the influence of the gravitational field of the rest of the particle’s system. This model allowed him to argue that ‘Schwarzschild singularities’ do not exist in physical reality because a cluster with a given number of masses cannot be arbitrarily concentrated. And this is due to the fact that otherwise the particles constituting the cluster would reach the speed of light. Of course, actually this model can only be considered as an interesting possibility to try to provide a counterexample of a singularity within gravity Einstein’s theory, being nonetheless the Black Holes, a physical reality within the theory of General Relativity.

In this chapter I will first present the theoretical formalism of Einstein Clusters (EC), and secondly I will use the model under the special assumption of a constant density distribution to model the central (sub-miliparsec) region of our galaxy, in order to provide an alternative for the SMBH of $M = 4.4 \times 10^6 M_\odot$ thought to be hosted at very center [78, 109]. The matter content will be treated as dark matter, i.e. I assume a dark EC composed by dark matter particles of mass $m$ (regardless of its nature), and therefore no contribution to the pressure in form of radiation is assured as the cluster shrinks till relativistic regimes. I will first analyze the stability condition in the specific case of a regular and relativistic energy density EC, contained marginally inside the S2 star peri-center ($r_{p(S2)}$) as observed in [78]. Secondly, and for an EC with fixed particle number $N$, I will explicitly show through the $R$ vs. $M$ relation, and for particle velocities ranging from zero up to the speed of light, up to which point an EC can be
contracted before losing its global stability.

5. DARK EINSTEIN CLUSTERS AS ALTERNATIVES TO SMBHs

5.1 Einstein clusters: a brief review of the theoretical formalism

The full theoretical formalism of ‘Einstein Clusters’ (EC) and their different stability analysis has been extensively studied in [120–125]. I give in next a short summary of the most important outcomes of this theory, pointing out the principal formulas which will allow us to deal with the astrophysical application object of this work. Thus, consider a static spherically symmetric distribution of particles all with rest mass \( m \) which are moving along circular geodetic orbits about the center of symmetry. The associated line element \( ds^2 \) is written in terms of a Schwarzschild metric of the form \( g_{\mu\nu} = \text{diag}(-e^\nu, e^\lambda, r^2, r^2\sin^2 \theta) \), where \( \nu \) and \( \lambda \) depend only on the radial coordinate \( r \). From now and on I will work in the geometric unit system \((G = c = 1)\).

The stress-energy tensor in the laboratory frame is assumed to take the form

\[
T^{\mu\nu} = m \, n_0 \, U^\mu \, U^\nu, \quad U = \gamma [e_t + v^\theta e_\theta + v^\phi e_\phi],
\] (5.1)

which is just the Einstein’s ansatz [119] (or a dust-like ansatz). \( n_0 \) is the proper particle number density (i.e. defined at rest w.r.t a coordinate system of special relativity), \( U \) is the particle 4-velocity satisfying the circular geodetic equations in the laboratory frame with \( v^\theta \) and \( v^\phi \) the linear velocities along the angular directions, \( \gamma = (1 - \nu^2)^{-1/2} \) \((\nu^2 = \delta_{\theta\phi} v^\theta v^\phi)\), and the unitary vectors introduced in \( U \) corresponds to the following orthonormal frame (adapted to the static observers)

\[
e_t = e^{-\nu/2} \partial_t, \quad e_r = e^{-\lambda/2} \partial_r, \quad e_\theta = \frac{1}{r} \partial_\theta, \quad e_\phi = \frac{1}{r^2 \sin \theta} \partial_\phi.
\] (5.2)

In the laboratory frame, and after applying the killing vector formalism to this specific spacetime (see [125]) naturally appears the two constants of motion associated with each trajectory, the energy \( E \) and the angular momentum \( L \) which reads

\[
E = m \gamma \nu^{\nu/2}, \quad L^2 = L^2_\theta + L^2_\phi \sin^2 \theta = m^2 \gamma^2 r^2 \nu^2.
\] (5.3)
The angular momentum formula in (5.3) together with the definition of \( \gamma \) directly implies \( \gamma = (1 + \tilde{L}^2/r^2)^{1/2} \) which will be very useful in what follows, with \( \tilde{L} = L/m \).

By writing the conserved \( L^2 \) in terms of each angular component \( L_\theta = m \gamma rv_\theta \) and \( L_\phi = m \gamma rv_\phi \sin \theta \) as done in (5.3), implies the following relation

\[
1 = \left( \frac{L_\theta}{L} \right)^2 + \left( \frac{L_\phi}{(L \sin^2 \theta)} \right)^2.
\]

This last equation further implies that the possible values of \( L_\theta/L \) and \( L_\phi/(L \sin^2 \theta) \) lie on a circle of unit radius, and then each angular component can be written in terms of an angle \( \alpha \) respect to the \( e_\theta \)-axis. This decomposition allow us to make an average of the angular momentum components in the \( e_\theta - e_\phi \) plane (i.e. around each orbit with \( \alpha \in [0, 2\pi] \)), as originally proposed by Einstein [119]. The averaged variables reads\(^1\).

\[
\langle L_\theta \rangle = \langle L_\phi \rangle = 0, \quad \langle L^2_\theta \rangle = \langle L^2_\phi / \sin^2 \theta \rangle = L^2/2.
\]

The above averaging allows to express the averaged stress-energy components without any angular dependence, and reads

\[
\langle T^t_t \rangle = -mn_0 \left( 1 + \frac{\tilde{L}^2}{r^2} \right) \equiv -\rho, \quad \langle T^\theta_\theta \rangle = \langle T^\phi_\phi \rangle = \frac{mn_0 \tilde{L}^2}{2r^2} \equiv p_t,
\]

where \( \rho \) is the energy density of the system and \( p_t \) the tangential pressure. It can be easily verified that the divergence of the stress-energy tensor vanishes identically.

The relevant Einstein equations are

\[
\frac{1}{r^2} [r(1 - e^{-\lambda})]' = 8\pi \rho,
\]
\[
v' = \frac{1}{r}(e^\lambda - 1),
\]
\[
\frac{e^{-\lambda}}{2} \left[ v'' + \frac{v'^2}{2} + \frac{v' - \lambda'}{r} - \frac{v'\lambda'}{2} \right] = 8\pi p_t,
\]

where a prime denotes differentiation with respect to \( r \). By using the standard definition of the mass function in terms of \( \lambda \), \( e^\lambda = (1 - 2M(r)/r)^{-1} \), the system

\(^1\)The average or mean value is defined by \( 2\pi \langle L_a(\alpha) \rangle = \int_0^{2\pi} L_a(\alpha) d\alpha \), with \( a \) either \( \theta \) or \( \phi \).
5. DARK EINSTEIN CLUSTERS AS ALTERNATIVES TO SMBHs

(5.6–5.8) is solved to give:

\[ M(r) = 4\pi \int_0^r \rho r^2 dr, \quad e^\nu = (1 - 2M/R)e^{-2\Phi(r)}, \quad (5.9) \]

where

\[ \Phi(r) = \int_r^R \frac{v_k^2}{r} dr, \quad \frac{v_k^2}{r} = \frac{M(r)}{r - 2M(r)}, \quad \tilde{L} = \gamma v_k r \quad (5.10) \]

where \( v_k \) is the Keplerian speed. Thus, in order to completely solve the problem I have to provide a the energy density \( \rho(r) \) (or equivalently the mass profile).

Another needed relevant quantity is the total particle number \( N \). It can be easily shown that the averaged 4-current \( \langle J^\mu \rangle = -n_0 \langle U^\mu \rangle \) is divergence-free. The associated conserved particle number is thus given by [126]

\[ N = \int_{\Sigma} \langle J^\mu \rangle d\Sigma_\mu, \quad (5.11) \]

where \( \Sigma \) denotes a spacelike hypersurface with infinitesimal element \( d\Sigma^\nu = n^\nu d\Sigma \) and unit timelike normal \( n \). By choosing \( \Sigma \) to be a \( t = \text{const} \) hypersurface with unit normal \( n = e_t \) and \( d\Sigma = e^{\lambda/2}r^2 \sin \theta dr d\theta d\phi \), Eq. (5.11) gives

\[ N = 4\pi \int_0^R n_0(r)\gamma e^{\lambda/2}r^2 dr. \quad (5.12) \]

I now want to obtain the total gravitational mass \( M \) in terms of the particle number distribution \( N(r) \) and the energy of the orbits \( E(r) \), determined by an observer in the laboratory frame (i.e.: any inertial observer at any finite \( r \) from the source), and also by a very distant one. I start this analysis with the expression of the total mass as seen by a very distant observer at rest in \( r \to \infty \)

\[ M(r) = 4\pi \int_0^r \rho r^2 dr, \quad (5.13) \]

Or, using \( \rho = mn_0\gamma_k^2 \) I can rewrite the above equation by

\[ M(r) = 4\pi \int_0^r mn_0\gamma_k^2 r^2 dr, \quad (5.14) \]

This kind of observers at \( r \to \infty \) are in an asymptotically flat space-time, thus, if
we want the gravitational mass $\tilde{M}$ respect to any inertial observer at any finite radius (in the curved space-time), we need simply to consider the proper volume element corresponding to a $t = \text{const}$ hypersurface, $\sqrt{-g} dV = e^{\nu/2} e^{\lambda/2} r^2 \sin \theta \, dr \, d\theta \, d\phi$ instead of $4\pi r^2 \, dr$, giving

$$\tilde{M}(r) = 4\pi \int_0^r mn_0 \gamma^2 e^{\nu/2} e^{\lambda/2} r^2 \, dr ,$$

(5.15)

Then, as from the equation of $N(r)$ I get

$$\frac{dN(r)}{dr} = 4\pi n_0 \gamma K e^{\lambda/2} r^2 ,$$

(5.16)

and from the equation for the conserved energy of the orbits

$$E(r) = m\gamma K e^{\nu/2} ,$$

(5.17)

Thus, from direct comparison of this last two with (5.15), I finally get

$$\tilde{M} = \int_0^r \frac{dN(r)}{dr} E(r) \, dr .$$

(5.18)

Which is the desired expression for the total gravitational mass of the cluster in the laboratory frame for any inertial observer at finite radius from the source. While, the total mass $M(r)$ given by (5.13) determined now by a very distant observer in an asymptotically flat space-time is, (from (5.16) and (5.17))

$$M(r) = \int_0^r \frac{dN(r)}{dr} E(r) e^{-(\lambda + \nu)/2} \, dr .$$

(5.19)

Noting that the two metric factors of difference between the last two expressions are obviously due to the relativity of the observers. Being the last expression for the mass usually called the 'physical' mass of the system.
5. DARK EINSTEIN CLUSTERS AS ALTERNATIVES TO SMBHs

5.2 Stability analysis for exterior orbits in Schwarzschild metric

The discussion about the stability of exterior circular orbits in a Schwarzschild metric can be found in any text-book of GR (see e.g. [127]). I give here a quick review on this topic for matter of context, to then move in next section to the interior case, whose solutions must match the exterior ones in the boundary region while providing all the novel and rich physics to be applied to the SgrA* region.

So, for the exterior case, in geometrical units \( c = G = 1 \), I start with the following Schwarzschild metric

\[
ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega^2,
\]

with \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \) the angular surface element and \( e^\nu \) and \( e^\lambda \) the metric functions given by (see e.g. [127]),

\[
e^\nu = \left(1 - \frac{2M}{R}\right)^{-1}, \quad e^\lambda = \left(1 - \frac{2M}{r}\right)^{-1}.
\] (5.20)

In every centrally symmetric field, the motion occurs in a single plane, so I will choose the plane \( \theta = \pi/2 \).

In order to make the required analysis in a brief manner, I use the Hamilton-Jacobi equation in General Relativity, recalling we are using \( c = G = 1 \),

\[
\mathcal{G}^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m^2 = 0
\] (5.21)

This is no other than the known relation for the contraction of the four velocity \( u^\mu u_\mu = -1 \), with the standard conjugate momenta definition \( p_\mu = \frac{\partial S}{\partial x^\mu} \), where \( S(q_i, t) \) is the action following the Hamilton-Jacobi equation \( \frac{\partial S}{\partial t} + H = 0 \) with \( H(q_i, p_i, t) \) the Hamiltonian of the theory.

So, from (5.21) and with the metric functions \( g^{\mu\nu} \) given above together with (5.20) I obtain,

\[
- \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{\partial S}{\partial t}\right)^2 + \left(1 - \frac{2M}{r}\right) \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \phi}\right)^2 + m^2 = 0.
\] (5.22)

A particle moving in the vacuum of a Schwarzschild metric the action must
5.2. Stability analysis for exterior orbits in Schwarzschild metric

have the form

\[ S = -E_0 t + L\phi + S_r(r), \]  

(5.23)

where \( \frac{\partial S}{\partial t} = -E_0 \) and \( \frac{\partial S}{\partial \phi} = L \) are the constant energy \( E_0 \) and angular momentum \( L \).

Substituting (5.23) in (5.22) I find the expression for \( \frac{\partial S}{\partial r} \) and so

\[ S_r(r) = \int \left[ E_0^2 \left( 1 - \frac{2M}{r} \right)^{-2} - \left( \frac{L^2}{r^2} + m^2 \right) \left( 1 - \frac{2M}{r} \right)^{-1} \right]^\frac{1}{2} dr, \]  

(5.24)

Once having \( S_r \) in (5.24) and proceeding according the Hamilton-Jacobi formalism, I can use the equation \( \frac{\partial S}{\partial E_0} = \text{const} = 0 \) for this case, which, also with (5.23) gives

\[ t = \frac{E_0}{m} \int \frac{dr}{(1 - \frac{2M}{r}) \left[ \left( \frac{E_0}{m} \right)^2 - \left( 1 + \frac{L^2}{r^2 m^2} \right) \left( 1 - \frac{2M}{r} \right) \right]^\frac{1}{2}}. \]  

(5.25)

Which, in terms of differentials takes the form

\[ \frac{r}{(1 - \frac{2M}{r})} \frac{dr}{dt} = \frac{1}{E_0} \left[ E_0^2 - U(r)^2 \right]^\frac{1}{2}, \]  

(5.26)

where I have defined \( U(r) \) as

\[ U(r) = m \left[ \left( 1 + \frac{L^2}{r^2 m^2} \right) \left( 1 - \frac{2M}{r} \right) \right]^\frac{1}{2}, \]  

(5.27)

playing this expression the role of an ‘effective potential energy’ in analogy to the nonrelativistic theory. Equivalently, I can express (5.27) in a more convenient form as a function of \( x = r/r_s \), as shown in Fig. (5.1) for different values of \( L/mr_s \), being \( r_s = 2M \) the Schwarzschild radius.

\[ U(x) = m \left[ \left( 1 + \frac{L^2}{m^2 r_s^2 x^2} \right) \left( 1 - \frac{1}{x} \right) \right]^\frac{1}{2}. \]  

(5.28)

I calculate now the possible stable and unstable circular orbits by asking for
Figure 5.1: Different ‘effective potential energies’ for different values of $L/mr_s$, where it can be see that it is needed $\approx 5.6\%$ of the rest-mass energy of the particle to take it out from the closest stable orbit to infinity (at rest).
5.2. Stability analysis for exterior orbits in Schwarzschild metric

the simultaneous solutions of the equations $E_0 = U(r)$ and $U(r)' = 0$, which gives:

$$\frac{r}{r_s} = \frac{L^2}{m^2 r_s^2} \left( 1 \pm \sqrt{1 - \frac{3m^2 r_s^2}{L^2}} \right),$$  \hspace{1cm} (5.29)$$

The stability of the orbits can be also shown by plotting the (normalized) angular momentum versus the normalized radius ($r/r_s$ vs $L/mr_s$), as given in Fig. 5.2,

Figure 5.2: An stable region can be seen for $L/mr_s \geq \sqrt{3}$ and $r \geq 6M$, while an unstable one is seen in the region given by $L/mr_s \in \left( \sqrt{3}, \infty \right)$ and $r/r_s \in [1.5, 3]$, i.e. between the event horizon and the ISCO, where the smallest unstable orbit is at $r = 3M$. 

5.3 Regular Einstein clusters centered in SgrA*: stability analysis

A constant energy density $\rho = 3M/(4\pi R^3)$ implies a radial distribution mass $M(r) = Mr^3/R^3$, and consequently through second and third Eqs. in (5.10), an angular momentum per unit mass $\tilde{L}$ with the corresponding number distribution of the particles $mn_0$ given by

$$\tilde{L} = \sqrt{\frac{M r^2}{R} \left(1 - \frac{3Mr^2}{R^3}\right)}^{-1/2}, \quad mn_0 = \frac{3M}{4\pi R^3} \frac{R^3 - 3Mr^2}{R^3 - 2Mr^2}. \quad (5.30)$$

where $0 \leq r \leq R$. Thus, the full solution of the Einstein equations (5.6–5.8) gives for the metric functions

$$e^\nu = \left(1 - \frac{2M}{R}\right)^{3/2} \left(1 - \frac{2Mr^2}{R^3}\right)^{-1/2}, \quad e^\lambda = \left(1 - \frac{2Mr^2}{R^3}\right)^{-1}, \quad (5.31)$$

The stability conditions for particles moving along a circular geodetic orbit on the equatorial plane is studied in next for the specific case of an EC of constant energy density, in terms of the effective potential $V_{eff} = e^{\nu/2}(1 + \tilde{L}^2/r^2)^{1/2}$ (see e.g. [125] for a general discussion of Stability). The existence of circular orbit at $r_0$ is calculated through the necessary condition $V_{eff}'(r_0) = 0$, while the the necessary condition for stability is $V_{eff}''(r_0) > 0$. In general (for any given EC) both necessary conditions reads respectively

$$r > 3M(r), \quad \frac{d(\ln M(r))}{d(\ln r)} + 1 - \frac{6M(r)}{r} > 0, \quad (5.32)$$

In particular, for $\rho = 3M/(4\pi R^3) = \text{const}$, the stability analysis directly implies that stable circular orbits exist within the cluster in the range

$$r < R \sqrt{\frac{R}{3M}}. \quad (5.33)$$

Note that there is no upper limit on $r$ if $R > 3M$, implying that circular orbits are stable all the way up to the boundary of the configuration.

For outer particles $r > R$, the stability conditions in (5.32) makes possible
to distinguish the following classes: models with $R > 6M$ and models with $3M < R < 6M$. If $R > 6M$ the cluster is always stable, because circular orbits are always stable both inside and outside the configuration (see also Fig. 5.3, top).

If $3M < R < 6M$ all particles constituting the cluster move on stable orbits, but in the adjacent exterior region of the configuration there is a region of instability $(R < r < 6M)$, so that the cluster is metastable (see also the Fig. 5.3, bottom). This stability criterion was first applied in [124].

Another formal criterion which will be also used in next to classify an EC regarding the stability, is the one adopted in [120] based on the behaviour of the gravitational binding energy of the system. Where the fractional binding energy of the cluster is defined by

$$E_b^f = \frac{mN - M}{mN}.$$  \hfill (5.34)

A regular and relativistic EC marginally inside the pericenter of the S2 star, has to fulfill the following observational constraints for its boundary $R$ and total mass $M$

$$R = r_{p(S2)} = 6 \times 10^{-4} \text{ pc}, \quad M = 4.4 \times 10^6 M_\odot,$$  \hfill (5.35)

where both values are subject to some $\sim$ few % of error due to propagated error in the distance from the sun to the galactic center $R_0 \approx 8.3$ kpc (see e.g. [78]). These constraints implies (in geometrical units) a ratio $R/M = 2840.9 \gg 6$, which safely indicates global stability, i.e. both inside and outside the cluster according with the criterion presented above with reference [124].

In next I analyze, for an EC of constant number particles $N$, up to which extent it can be shrunk inside $r_{p(S2)}$ without becoming metastable, and moreover, what happens when the particles approach the ultra-relativistic regime. For this I first calculate the relation between $M$ and $R$ for fixed values of the rest mass $mN$ of the system, taking the velocity $0 < v_k \le 1$ as a parameter. By use of Eq. (5.12) we have

$$mN = M [1/v_k^2 + 2]^{3/2} F(v_k),$$  \hfill (5.36)
with $F(v_k)$ given by the following complicated formula

$$F(v_k) = -3/4[\arctan(x(v_k) - 3)^{-1/2} + (x(v_k) - 3)^{1/2}/x(v_k)] + (3/4)^{1/2}\arctan(3/(x(v_k) - 3))^{1/2},$$

being $x(v_k) = R/M(v_k) = 1/v_k^2 + 2$. The direct relation between $R/M$ and $v_k$ is easily understood from the Keplerian velocity formula in (5.10) evaluated at $r = R$. Eq. (5.36) together with $x(v_k)$ automatically leads to the following total mass and radius normalized variables

$$\frac{R}{mN} = \frac{1}{F(v_k)[1/v_k^2 + 2]^{1/2}}, \quad \frac{M}{mN} = \frac{1}{F(v_k)[1/v_k^2 + 2]^{3/2}}.$$ (5.38)

In Fig. (5.3) I explicitly show the $R$ vs. $M$ relation (normalized with the constant rest mass) with $v_k$ taken as a free parameter. Regions of stability and metastability are differentiated depending on the value of the rotation velocity ($v_k$) at the boundary of the EC (see caption for details). Instead, in Fig. (5.4) I show the behaviour of the binding energy as a function of the velocity $v_k$, this is, showing the fraction of the total mass that turns into binding energy when the cluster is contracted from $R \gg 1$ to a given $R$. After the maximum a change of stability takes place and the cluster itself becomes unstable according to this criterion (see caption for details).

Even though these systems reach meta-stability (according to the classification given in [124]) or become unstable (according the binding energy analysis), well before the velocity $v_k$ reaches the ultra-relativistic regime; it is interesting to note that these tangential pressure supported self-gravitating systems, never reaches a critical mass as in the case of a radial pressure supported self-gravitating systems, being the neutron stars a typical example of this last case.

In Fig. (5.5) I present two examples of constant energy density EC, the case of $R/M = 10$ (top Fig.) where circular stable orbits exist either for particles forming the EC but also for outside ones, and the case of $R/M = 3.1$ (bottom Fig.) where unstable orbits (i.e. a maximum in $V_{eff}$) appears for outer particles located in the outer vicinity of the border of the EC. In the second case the EC is called metastable according to the characterization given in [124].

The fact of being working with a fixed rest mass energy $mN$ which can be calculated with the observational constraints (5.38), implies that the constant
Figure 5.3: Gravitational mass vs. boundary radius relation (in units of rest mass) for an EC with constant energy density. The velocity at the boundary $0 < v_k < 1$ is taken as a parameter. In the Fig. on the top: for $v_k \to 0$ the total mass approaches the rest mass at $R/(mN) \to \infty$. At $v_k = 0.5$ the EC becomes meta-stable (i.e. $R/M = 6$), while $v_k = 0.6$ corresponds to the minimum in $M/(mN)=0.955$ which further implies the maximum bounded state for the cluster (see Fig. 5.4 for comparison). At $v_k = 0.903$ the gravitational mass equals the rest mass, and at $v_k = 0.98$ a turning point in the radius appears (see the bottom Fig. for a zoom). Finally, the onset of instability $R/M = 3$ (according to the classification given in [124]) is asymptotically approached when $v_k \to 1$. 
Figure 5.4: The behaviour of the fractional binding energy (5.34) as a function of $v_k$ with fixed particle number. The maximum corresponds to an EC which has shrunk to $R/(mN) = 4.5$ where $E_b^f \approx 0.045$. The latter vanishes at $R/(mN) = 3.22$ for a velocity $v_k = 0.903$ where the cluster is considered unstable according to this criterion. At $v_k = 0.5$ the radius of the cluster is $R/(mN) = 5.75$ implying $R/M = 6$ (see Fig. 5.3 for comparison).
5.3. Regular Einstein clusters centered in SgrA*: stability analysis

Figure 5.5: Behaviour of the effective potential as a function of $r/M$ for an EC with constant energy density. In the Fig. on the top I show a globally stable cluster with $R/M = 10$, and in the bottom Fig. a meta-stable one with $R/M = 3.1$. Only in the last case, and for relatively high values of $\tilde{L}$ an external maximum appears showing the existence of unstable orbits in the outer vicinity of the cluster. The dots corresponds to the point a maximum angular momentum at the border of the cluster.
energy density $\rho = 3M/(4\pi R^3)$ increases more and more according the velocity $v_k$ increases. From Eqs. (5.36) and $R/M(v_k) = 1/v_k^2 + 2$ is is possible to give an explicit expression for $\rho(v_k) = 3/(4\pi)(F(v_k)/(mN))^2$, from which I can give the uppermost limits for the density of a dark EC inside SgrA*. Changing units to $M_\odot/pc^3$, the first upper limit corresponds to $\rho(v_k = 0.5) \approx 5.5 \times 10^{23} M_\odot/pc^3$, below which the EC is always globally stable. The second limit is given by $\rho(v_k = 0.6) \approx 1.1 \times 10^{24} M_\odot/pc^3$, and will be considered as the uppermost limit for a regular and relativistic EC inside S2 and centered in SgrA*, due to the fact that above this velocity the value of the binding energy (5.34) starts to decrease from its maximum, undergoing a change of stability (see also Fig. 5.4). These results are in consistency with the bound obtained in [128] from a different stability analysis, based of lose of isotropy of the fluid due to non-radial perturbations; since anisotropy serves as a source of instability (see [129] and refs. therein).

5.4 Conclusions

The theory of ECs was reviewed and applied to work as alternative to the SMBH thought to be at the center of our galaxy. The particles composing the cluster were assumed as DM particles, implying therefore no radiation pressure from the cluster as it shrinks to relativistic regimes; and also representing an interesting complementary idea to semi-degenerate fermionic model given in the other Chapters. The main conclusion of this Chapter is that it is possible to find stable dark ECs working as an alternative to the SMBH thought to be hosted in the center of SgrA*, with $\rho = const.$ in the range $10^{16} M_\odot/pc^3 \lessgtr \rho \lessgtr 5. \times 10^{23} M_\odot/pc^3$. While the lower limit is comes from the minimum required compactness dictated by the S2 star orbit, the upper limit of $\rho(v_k = 0.5) \approx 5.5 \times 10^{23} M_\odot/pc^3$ comes from EC stability analysis given above. It is interesting to notice that this maximum possible density is already about one order of magnitude higher than the lowest required limit for the mass density of SgrA* as imposed in [130] through the 1.3 mm Very Large Baseline Interferometry observations, but below the critical density required for a black hole of $4.4 \times 10^6 M_\odot$. 

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Conclusion

The central conclusion of this Thesis is the prediction of a novel core-halo distribution of DM in galaxies based on fermionic quantum statistics and gravitational interactions. More specifically, by solving in the more general way the TOV equations for hydrostatic equilibrium of a thermal and semi-degenerate fermion gas including relativistic effects, I proved the existence of a core-halo DM profile composed by: 1) an inner core of almost constant density governed by degenerate quantum statistics; 2) an intermediate region with a sharply decreasing density distribution followed by an extended plateau, implying quantum corrections; 3) a decreasing density distribution $\rho \propto r^{-2}$ leading to flat rotation curves fulfilling the classical Boltzmann statistics. The ‘ino’ mass is determined as an eigenfunction of the mass of the inner quantum cores, such that, when this mass is compared with the lower limit by Tremaine and Gunn (1979), it is shown that the latter is approached for the less degenerate quantum cores in agreement with the fixed halo observables. This model, based on the above physical pillars such as quantum physics and GR, has been contrasted phenomenologically with galactic observables ranging from dwarf to big spiral galaxies, evidencing that a fermion mass of about 10 keV, implies fermionic DM profiles in agreement with the observed constant and Universal DM halo surface density; as well as their dense central cores provide an alternative to IMBHs to SMBHs with masses $M_c \sim 10^4$ to $M_c \sim 10^7$ respectively. This model was also applied to big elliptical galaxies harboring SMBHs of $\sim 10^9 M_\odot$ at their centers, by analyzing the critical points of the last equilibrium configurations; concluding that no critical core-halo configurations are able to explain both: the most super-massive dark-compact-objects at the center of elliptical galaxies, together with the DM halo simultaneously. An interesting possibility of prior SMBH formation at early redshift stages (i.e. $z \sim 3$), with subsequent gathering of matter from larger-scale environments in the complex galactic evolutionary history was also discussed. I have also shown the very good agreement between the theoretical curves and
the high quality observationally inferred ones based on statistical analysis. The fact that the inner halo resolution power will increase in the near future, opens the interesting possibility for high angular resolution astronomical observations to resolve sub-parsec scales. The importance of the baryonic matter component through the central region of galaxies, certainly imposes a serious difficulty to disentangle the potential DM counterpart. Nevertheless, the fact that the model here presented predicts the appearance of a compact quantum core below sub-parsec scales, with typical densities several orders of magnitude higher than halo densities, leaves a good possibility for its indirect detection. In particular, I have shown the central role of nearby dwarf galaxies, where kinematical and photometric data was obtained at parsec scales, and therefore predictions of possible central DM profiles in connection with the observed (light) nucleated regions were presented. Finally, an extension of the model described above was developed, by adding new effective interactions (other than gravity) among the keV fermions. By applying the RMF tractable theoretical framework to deal with the many body interacting particles, novel effects of the new effective interactions in the distribution of DM in galaxies were presented; and in particular, it was shown that the dense quantum core provides a good alternative to the massive BH thought to be hosted in Sagittarius A*. Together with this SMBH alternative, another interesting proposal was provided by the application of the theory of Einstein clusters, where a regular and (possibly) relativistic dark EC can answer for the right compactness when harbored within the S2 orbit around the galactic center.

‘As well as dark matter permeates the whole Universe delineating its underlying mass distribution, in the realm of life, it may well exist some unknown underlying matter-energy field we all are interacting with; but, as we are part of it, our actual awareness cannot disentangle its behaviour...’

C. R. Argüelles, October (2014)
Bibliography


BIBLIOGRAPHY


